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Sufficient conditions for prisms to produce orthogonal image orientation functions without spectral dispersion

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One of the main functions of a prism is to produce an image orientation function without spectral dispersion. The present study extends the work previously reported in [JOSAA **33**(7), 2016, pp. 1257 –1266] to analyze the sufficient conditions for a prism to produce two particular image orientation functions (Φ_2 and Φ_5) without spectral dispersion. It is shown that there exist two sufficient conditions under which spectral dispersion can be avoided: (1) the rays enter and exit the prism perpendicularly, or (2) the prism comprises two halves which are mirror images of one another. The present findings provide a useful basis for the design of prisms without spectral dispersion.

Keywords: prism, image orientation, spectral dispersion.

1. Introduction

Prisms are widely used in optical applications for reflecting/refracting light or dispersing light into its spectral components (p. 91 of [1], and [2]). The exact angles between the flat boundary surfaces of a prism depend on the particular application. The colloquial use of "prism" generally refers to a triangular prism with a triangular base and rectangular sides. However, in fact, prisms may actually have many different geometric forms, including square prism, triangular prism and pentagonal prism.

While traditionally used for the purposes indicated above, prisms have been more recently employed as a means of changing the polarization direction of the incident light [3]. For example, Appel and Dyer [4] changed the polarization direction of the incoming light by 90° by rotating the optical beam via total internal reflection in three fused-silica glass components. Moreno [5] employed a Jones matrix formulation and an exact ray-tracing method to investigate the polarization-transforming properties of rotational prism. It was shown that the theoretical predictions for the output states of polarization of a linearly polarized beam incident on a Dove prism were in good agreement with the experimental observations.

One of the most common uses of prisms is to output an image with an orientation changed in a prescribed manner relative to the object. The prism design process is generally performed using the trial-and-error method proposed by SMITH (pp. 100–121 of [1]). GALVEZ and HOLMES [6] proposed a method based on the concept of geometric phase for analyzing the image orientation produced by optical rotators. GINSBERG [7] presented a simple equation for predicting the rotation of an image by a mirror or prism without the need for vectorial analysis. WENWEI MAO [8] examined the effect of small rotations of a reflecting prism about an arbitrary axis on the orientation of the output image. NING LIN *et al*. [9] used a second-order approximation to investigate the orientation conjugation between the object and the image in a system of reflecting rotating prisms. However, all of these studies consider the performance of existing prisms or mirror systems, rather than the design of new prisms. Accordingly, in [10, 11], the present group proposed a numerical approach for determining the minimum number of reflecting surfaces required in a single prism to produce an output image with a specific orientation.

Refraction processes at the first and last boundary surfaces (*i.e.*, r_1 and r_n) of a prism often cause spectral dispersion. Such an effect is undesirable in many applications for a prism to change the orientation of an image. Studies have shown that spectral dispersion can be avoided if ℓ_0 and ℓ_n enter/exit the boundary surface perpendicularly [11]. Thus, when using a prism to re-orientate the image, the entrance rays and exits rays are usually designed as normal incidence rays. However, Porro prisms and solid glass corner-cubes avoid spectral dispersion even when the entrance and exits rays are not normal to their respective boundary surfaces (pp. 109 and 113 of [1]). Accordingly, taking two orthogonal image orientation functions (IOFs) (Φ_2 and Φ_5) for illustration purposes, this study performs a further investigation into the sufficient conditions under which spectral dispersion can be avoided in a prism.

Throughout the remainder of this paper, the i-th unit directional vector ${}^{\mathrm{g}}\ell_i$ is written as a column matrix ${}^{\mathrm{g}}\ell_i = [l_{ix} \;\; l_{iy} \;\; l_{iz}]^{\mathrm{T}}$, in which the pre-superscript "g" of the leading symbol ${}^{\mathrm{g}}\ell_i$ indicates that the vector is defined with respect to coordinate frame $(xyz)_{\mathrm{g}}$. Furthermore, given a vector ${}^{\mathrm{g}}\ell_i$, its transformation ${}^{\mathrm{h}}\ell_i$ is represented by the matrix product ${}^{\mathrm{h}}\ell_i = {}^{\mathrm{h}}A_{\mathrm{g}}{}^{\mathrm{g}}\ell_i$, where ${}^{\mathrm{h}}A_{\mathrm{g}}$ is a 3×3 orientation matrix which defines the orientation of coordinate frame $(xyz)_{\mathrm{g}}$ with respect to coordinate frame $(xyz)_{\mathrm{h}}$. If a vector ${}^{0}\ell_i$ is referred to the world coordinate frame $(xyz)_0$, then its pre-superscript "0" is omitted for simplicity.

2. Reflector matrix and orthogonal image orientation function

Consider a prism with n flat boundary surfaces labeled sequentially from $i = 1$ to $i = n$. Assume that the i-th boundary surface, denoted as r_i, has a unit normal vector of $\mathbf{n}_i$

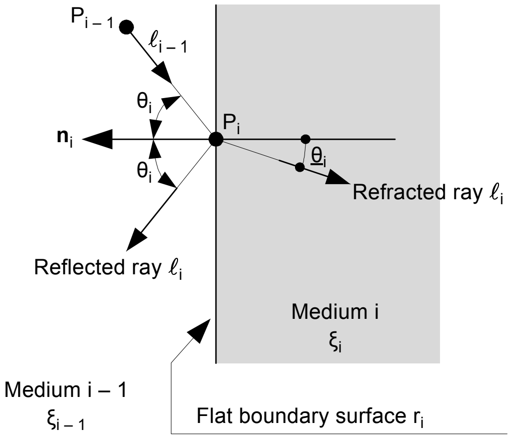

Fig. 1. Unit directional vectors of the incoming ray and refracted (or reflected) ray.

(see Fig. 1) and the first and last boundary surfaces are denoted as r_1 and r_n, respectively. To determine the image orientation change produced by the prism, it is first necessary to establish the world coordinate frame $(xyz)_0$. The image orientation $(xyz)'_0$ relative to the object $(xyz)_0$ can then be given by the following IOF:

$$\Phi = \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \tag{1}$$

where **a**, **b** and **c** are the unit directional vectors of the x'_0, y'_0 and z'_0 axes of $(xyz)'_0$, respectively, with respect to $(xyz)_0$ (see Fig. 2).

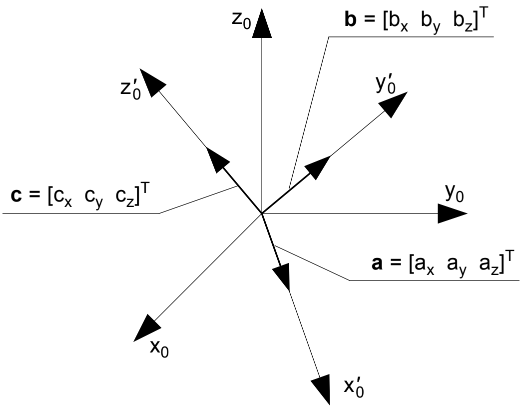

Fig. 2. Orientation of $(xyz)'_0$ with respect to $(xyz)_0$.

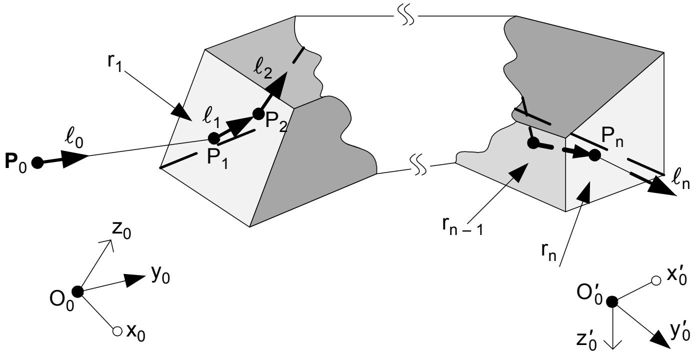

Fig. 3. Image orientation of object $(xyz)_0$ imaged by a prism with n boundary surfaces.

When the angles between any two axes of frames $(xyz)'_0$ and $(xyz)_0$ are equal to 0°, 90°, 180° or 270°, Φ in Eq. (1) is referred to as an orthogonal IOF. According to Eq. (2) of [11], Φ can be obtained from the following matrix manipulation:

$$\Phi = \frac{\partial \boldsymbol{\ell}_n}{\partial \boldsymbol{\ell}_0} = \frac{\partial \boldsymbol{\ell}_n(\mathbf{n}_n, N_n)}{\partial \boldsymbol{\ell}_{n-1}} \frac{\partial \boldsymbol{\ell}_{n-1}(\mathbf{n}_{n-1})}{\partial \boldsymbol{\ell}_{n-2}} \cdots \frac{\partial \boldsymbol{\ell}_i(\mathbf{n}_i)}{\partial \boldsymbol{\ell}_{i-1}} \cdots \frac{\partial \boldsymbol{\ell}_2(\mathbf{n}_2)}{\partial \boldsymbol{\ell}_1} \frac{\partial \boldsymbol{\ell}_1(\mathbf{n}_1, N_1)}{\partial \boldsymbol{\ell}_0} \tag{2}$$

where $\partial \boldsymbol{\ell}_i / \partial \boldsymbol{\ell}_{i-1}$ is the first-order derivative matrix (*i.e.*, Jacobian matrix) of the unit directional vector $\boldsymbol{\ell}_i$ of the reflected/refracted ray with respect to the unit directional vector $\boldsymbol{\ell}_{i-1}$ of the incoming ray (see Fig. 3). For a reflection process, $\partial \boldsymbol{\ell}_i / \partial \boldsymbol{\ell}_{i-1}$ (from $i = 2$ to $i = n - 1$) is given by

$$\frac{\partial \boldsymbol{\ell}_i}{\partial \boldsymbol{\ell}_{i-1}} = \frac{\partial \boldsymbol{\ell}_i(\mathbf{n}_i)}{\partial \boldsymbol{\ell}_{i-1}} = \begin{bmatrix} 1 - 2n_{ix}n_{ix} & -2n_{ix}n_{iy} & -2n_{ix}n_{iz} \\ -2n_{iy}n_{ix} & 1 - 2n_{iy}n_{iy} & -2n_{iy}n_{iz} \\ -2n_{iz}n_{ix} & -2n_{iz}n_{iy} & 1 - 2n_{iz}n_{iz} \end{bmatrix} = I_{3\times 3} - 2\mathbf{n}_i\mathbf{n}_i^{\mathrm{T}} \tag{3}$$

In other words, for a reflection process, $\partial \boldsymbol{\ell}_i(\mathbf{n}_i) / \partial \boldsymbol{\ell}_{i-1}$ is a function only of the unit normal vector $\mathbf{n}_i = [n_{ix} \; n_{iy} \; n_{iz}]^{\mathrm{T}}$ of the flat boundary surface r_i. For a refraction process, $\partial \boldsymbol{\ell}_i / \partial \boldsymbol{\ell}_{i-1}$ ($i = 1$ and $i = n$) is given as

$$\frac{\partial \boldsymbol{\ell}_i}{\partial \boldsymbol{\ell}_{i-1}} = \frac{\partial \boldsymbol{\ell}_i(\mathbf{n}_i, N_i)}{\partial \boldsymbol{\ell}_{i-1}} = N_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + N_i B_i \begin{bmatrix} n_{ix}n_{ix} & n_{ix}n_{iy} & n_{ix}n_{iz} \\ n_{iy}n_{ix} & n_{iy}n_{iy} & n_{iy}n_{iz} \\ n_{iz}n_{ix} & n_{iz}n_{iy} & n_{iz}n_{iz} \end{bmatrix}$$

$$= N_i(I + B_i\mathbf{n}_i\mathbf{n}_i^{\mathrm{T}}) \tag{4}$$

with

$$B_i = \frac{N_i \cos(\theta_i)}{\sqrt{1 - (N_i \sin(\theta_i))^2}} - 1 \tag{5}$$

It is noted from Eq. (5) that for a refraction process, $\partial \ell_i(\mathbf{n}_i, N_i)/\partial \ell_{i-1}$ is a function not only of the unit normal vector $\mathbf{n}_i$, but also of the relative refractive index N_i and incidence angle θ_i.

In general, the refraction processes at r_1 and r_n of Eq. (4) have a direct effect on the orientation of the output image. However, under certain conditions, the two refraction events have no effect on the image orientation produced by the other $(n-2)$ reflectors in the prism. In other words,

$$\Phi = \frac{\partial \ell_n(\mathbf{n}_n, N_n)}{\partial \ell_{n-1}} \Phi \frac{\partial \ell_1(\mathbf{n}_1, N_1)}{\partial \ell_0} \tag{6}$$

The aim of this paper is to determine the sufficient conditions under which the two refraction processes at r_1 and r_n can be ignored. These sufficient conditions can be obtained by expanding Eq. (6) using $\partial \ell_n/\partial \ell_{n-1}$ (Eq. (4) with $i = n$) and $\partial \ell_1/\partial \ell_0$ (Eq. (4) with $i = 1$) to give

$$B_n(\mathbf{n}_n \mathbf{n}_n^{\mathrm{T}})\Phi + B_1 \Phi(\mathbf{n}_1 \mathbf{n}_1^{\mathrm{T}}) + B_n B_1(\mathbf{n}_n \mathbf{n}_n^{\mathrm{T}})\Phi(\mathbf{n}_1 \mathbf{n}_1^{\mathrm{T}}) = \mathbf{0} \tag{7}$$

As discussed in [12], there exist six categories of orthogonal IOF, namely Φ_1, Φ_2, Φ_3, Φ_4, Φ_5 and Φ_6. However, Φ_3, Φ_4 and Φ_6 can be transformed to Φ_2 and Φ_5 by referring them to another coordinate frame (Eqs. (24), (B6) and (B7) of [12]). As a result, there actually exist only three main types of orthogonal IOF, namely

$$\Phi_1 = \pm \begin{bmatrix} a_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c_z \end{bmatrix} \quad \text{with} \quad a_x = \pm 1 \text{ and } c_z = \pm 1 \tag{8}$$

$$\Phi_2 = \pm \begin{bmatrix} 0 & 0 & c_x \\ 0 & 1 & 0 \\ a_z & 0 & 0 \end{bmatrix} \quad \text{with} \quad a_z = \pm 1 \text{ and } c_x = \pm 1 \tag{9}$$

$$\Phi_5 = \pm \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & c_y \\ a_z & 0 & 0 \end{bmatrix} \quad \text{with} \quad a_z = \pm 1 \text{ and } c_y = \pm 1 \tag{10}$$

The necessary conditions to avoid spectral dispersion when producing Φ_1 have been addressed in [12]. Accordingly, this study investigates the sufficient conditions

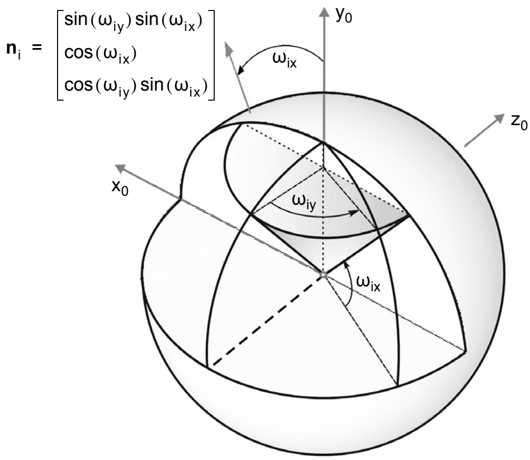

Fig. 4. Unit normal vector $\mathbf{n}_i$ of r_i ($i = 1$ and n).

to avoid spectral dispersion for the other two categories of orthogonal IOF, namely Φ_2 and Φ_5.

3. Sufficient conditions for avoiding spectral dispersion for Φ_2

The matrix Φ_2 has the an anti-diagonal matrix form, *i.e.*, the elements are all equal to zero other than those on the diagonal running from the lower-left corner to upper-right corner. By contrast, Φ_1 is a diagonal matrix (see Eq. (8)). It can be easily proven that Φ_2 cannot be transformed to Φ_1. In other words, Φ_1 and Φ_2 belong to different categories of orthogonal IOF. However, as shown in the following, even though Φ_2 and Φ_1 have completely different structures, the sufficient conditions for avoiding spectral dispersion are similar in both cases.

To derive the sufficient conditions from Eq. (7), one first needs the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_n$. Mathematically, a unit normal vector can be defined by two independent angles. Therefore, the required unit normal vectors can be obtained as (see Fig. 4)

$$\mathbf{n}_1 = \pm\begin{bmatrix}\sin(\omega_{1y})\sin(\omega_{1x})\\ \cos(\omega_{1x})\\ \cos(\omega_{1y})\sin(\omega_{1x})\end{bmatrix}, \quad -90^\circ \le \omega_{1x} \le 90^\circ, \quad 0^\circ \le \omega_{1y} < 360^\circ \tag{11a}$$

$$\mathbf{n}_n = \pm\begin{bmatrix}\sin(\omega_{ny})\sin(\omega_{nx})\\ \cos(\omega_{nx})\\ \cos(\omega_{ny})\sin(\omega_{nx})\end{bmatrix}, \quad -90^\circ \le \omega_{nx} \le 90^\circ, \quad 0^\circ \le \omega_{ny} < 360^\circ \tag{11b}$$

Substituting Eqs. (11a) and (11b) into Eq. (7) with $\Phi = \Phi_2$ yields the following equations:

$$\left[a_z B_n \sin(\omega_{ny}) \cos(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[1 + B_1 \sin^2(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ \left[B_n \sin(\omega_{ny}) \sin(\omega_{nx}) \cos(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x}) \sin(\omega_{1y}) \sin(\omega_{1x})\right]$$
$$+ \left[c_x B_n \sin^2(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y}) \sin(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ c_x B_1 \cos(\omega_{1y}) \sin(\omega_{1y}) \sin^2(\omega_{1x}) = 0 \tag{12a}$$

$$\left[a_z B_n \sin(\omega_{ny}) \cos(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x})\right]$$
$$+ \left[B_n \sin(\omega_{ny}) \sin(\omega_{nx}) \cos(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1x})\right]$$
$$+ \left[c_x B_n \sin^2(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x})\right]$$
$$+ c_x B_1 \cos(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x}) = 0 \tag{12b}$$

$$\left[a_z B_n \sin(\omega_{ny}) \cos(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y}) \cos(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ \left[B_n \sin(\omega_{ny}) \sin(\omega_{nx}) \cos(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x}) \cos(\omega_{1y}) \sin(\omega_{1x})\right]$$
$$+ \left[c_x B_n \sin^2(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ c_x B_1 \cos^2(\omega_{1y}) \sin^2(\omega_{1x}) = 0 \tag{12c}$$

$$\left[a_z B_n \cos(\omega_{nx}) \cos(\omega_{ny}) \sin(\omega_{nx})\right]\left[1 + B_1 \sin^2(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ \left[B_n \cos^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x}) \sin(\omega_{1y}) \sin(\omega_{1x})\right]$$
$$+ \left[c_x B_n \cos(\omega_{nx}) \sin(\omega_{ny}) \sin(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y}) \sin(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ B_1 \cos(\omega_{1x}) \sin(\omega_{1y}) \sin(\omega_{1x}) = 0 \tag{12d}$$

$$\left[a_z B_n \cos(\omega_{nx}) \cos(\omega_{ny}) \sin(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x})\right]$$
$$+ \left[B_n \cos^2(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1x})\right]$$
$$+ \left[c_x B_n \cos(\omega_{nx}) \sin(\omega_{ny}) \sin(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x})\right]$$
$$+ B_1 \cos^2(\omega_{1x}) = 0 \tag{12e}$$

$$
\begin{aligned}
&\left[a_z B_n \cos(\omega_{nx})\cos(\omega_{ny})\sin(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y})\cos(\omega_{1y})\sin^2(\omega_{1x})\right] \\
&\quad + \left[B_n \cos^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x})\cos(\omega_{1y})\sin(\omega_{1x})\right] \\
&\quad + \left[c_x B_n \cos(\omega_{nx})\sin(\omega_{ny})\sin(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1y})\sin^2(\omega_{1x})\right] \\
&\quad + B_1 \cos(\omega_{1x})\cos(\omega_{1y})\sin(\omega_{1x}) \ = \ 0
\end{aligned} \tag{12f}
$$

$$
\begin{aligned}
&\left[a_z B_n \cos^2(\omega_{ny})\sin^2(\omega_{nx})\right]\left[1 + B_1 \sin^2(\omega_{1y})\sin^2(\omega_{1x})\right] \\
&\quad + \left[B_n \cos(\omega_{ny})\sin(\omega_{nx})\cos(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x})\sin(\omega_{1y})\sin(\omega_{1x})\right] \\
&\quad + \left[c_x B_n \cos(\omega_{nx})\sin(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y})\sin(\omega_{1y})\sin^2(\omega_{1x})\right] \\
&\quad + a_z B_1 \sin^2(\omega_{1y})\sin^2(\omega_{1x}) \ = \ 0
\end{aligned} \tag{12g}
$$

$$
\begin{aligned}
&\left[a_z B_n \cos^2(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x})\right] \\
&\quad + \left[B_n \cos(\omega_{ny})\sin(\omega_{nx})\cos(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1x})\right] \\
&\quad + \left[c_x B_n \cos(\omega_{ny})\sin(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x})\right] \\
&\quad + a_z B_1 \sin(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x}) \ = \ 0
\end{aligned} \tag{12h}
$$

$$
\begin{aligned}
&\left[a_z B_n \cos^2(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y})\cos(\omega_{1y})\sin^2(\omega_{1x})\right] \\
&\quad + \left[B_n \cos(\omega_{ny})\sin(\omega_{nx})\cos(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x})\cos(\omega_{1y})\sin(\omega_{1x})\right] \\
&\quad + \left[c_x B_n \cos(\omega_{ny})\sin(\omega_{ny})\sin^2(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1y})\sin^2(\omega_{1x})\right] \\
&\quad + a_z B_1 \sin(\omega_{1y})\cos(\omega_{1y})\sin^2(\omega_{1x}) \ = \ 0
\end{aligned} \tag{12i}
$$

Equations (12a)–(12i) can be solved numerically to determine the required unit normal vectors (*i.e.*, $\mathbf{n}_1$ and $\mathbf{n}_n$) which produce Φ_2 given a particular refractive index ξ_i of the prism and a particular unit directional vector ℓ_0 of the source ray. However, for most applications, the source rays are usually converging or diverging, *i.e.*, the rays have different unit directional vectors ℓ_0. Consequently, an analytical (rather than numerical) approach is required to determine the conditions under which spectral dispersion can be avoided. In practice, Eqs. (12a)–(12i) should be investigated for four particular cases, *i.e.*,

$$
\begin{aligned}
&\omega_{1y} = 0^\circ \ \text{ and } \ \omega_{ny} = 90^\circ \\
&\quad (\mathbf{n}_1 = \pm[0 \ \ \cos(\omega_{1x}) \ \ \sin(\omega_{1x})]^{\mathrm{T}}, \ \ \mathbf{n}_n = \pm[\sin(\omega_{nx}) \ \ \cos(\omega_{nx}) \ \ ,0]^{\mathrm{T}})
\end{aligned} \tag{13a}
$$

$$\omega_{1y} = 90° \text{ and } \omega_{ny} = 0°$$
$$(\mathbf{n}_1 = \pm[\sin(\omega_{1x}) \ \cos(\omega_{1x}) \ 0]^{\mathrm{T}}, \quad \mathbf{n}_n = \pm[0 \ \cos(\omega_{nx}) \ \sin(\omega_{nx})]^{\mathrm{T}}) \tag{13b}$$

$$\omega_{1y} = \omega_{ny} = 0°$$
$$(\mathbf{n}_1 = \pm[0 \ \cos(\omega_{1x}) \ \sin(\omega_{1x})]^{\mathrm{T}}, \quad \mathbf{n}_n = \pm[0 \ \cos(\omega_{nx}) \ \sin(\omega_{nx})]^{\mathrm{T}}) \tag{13c}$$

$$\omega_{1y} = \omega_{ny} = 90°$$
$$(\mathbf{n}_1 = \pm[\sin(\omega_{1x}) \ \cos(\omega_{1x}) \ 0]^{\mathrm{T}}, \quad \mathbf{n}_n = \pm[\sin(\omega_{nx}) \ \cos(\omega_{nx}) \ 0]^{\mathrm{T}}) \tag{13d}$$

It is noted that $\mathbf{n}_n = \pm\Phi_2 \mathbf{n}_1$ is satisfied if the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_i$ are given as shown in Eq. (13a) or (13b). Consequently, as shown in Example 3 of [12], spectral dispersion is avoided if the source ray and exit ray are both normal incidence rays. However, $\mathbf{n}_n = \pm\Phi_2 \mathbf{n}_1$ is not satisfied if $\mathbf{n}_1$ and $\mathbf{n}_n$ have the directions shown in Eq. (13c) or (13d). In other words, spectral dispersion occurs even if the source and exit rays are normal incidence rays. Accordingly, the following results consider the sufficient conditions under which spectral dispersion can be avoided for the four cases shown in Eqs. (13a)–(13d).

Table 1 summarizes all the relevant solutions of Φ_2 for the case with $\omega_{1y} = 0°$ and $\omega_{ny} = 90°$. Consequently, Table 1 is also applicable to Φ_2 for unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_n$ with $\omega_{1y} = 90°$ and $\omega_{ny} = 0°$ provided that c_x in Table 1 is replaced by a_z. Table 2 presents all of the solutions of Φ_2 for the cases with $\omega_{1y} = \omega_{ny} = 0°$ and $\omega_{1y} = \omega_{ny} = 90°$.

4. Sufficient conditions for avoiding spectral dispersion for Φ_5

It is noted that Φ_5 is neither a diagonal matrix nor an anti-diagonal matrix. Consequently, its derivations and sufficient conditions for avoiding spectral dispersion are different from those of Φ_1 and Φ_2. Furthermore, $\mathbf{n}_n = \pm\Phi_5 \mathbf{n}_1$ is not satisfied when the directions

Table 1. Sufficient conditions to avoid spectral dispersion for IOF Φ_2 when $\mathbf{n}_1$ and $\mathbf{n}_n$ are defined in Eq. (13a) or (13b).

Cases			Solutions
$c_x = 1$	$\omega_{nx} = \omega_{1x}$	$\omega_{nx} = \omega_{1x} = 0°$	$B_n + B_n B_1 + B_1 = 0$
		$\omega_{nx} = \omega_{1x} \neq 0°$	$B_n + B_n B_1 + B_1 = 0$
	$\omega_{nx} = -\omega_{1x} \neq 0°$		$B_n = B_1 = 0$
$c_x = -1$	$\omega_{nx} = \omega_{1x}$	$\omega_{nx} = \omega_{1x} = 0°$	$B_n + B_n B_1 + B_1 = 0$
		$\omega_{nx} = \omega_{1x} \neq 0°$	$B_n = B_1 = 0$
	$\omega_{nx} = -\omega_{1x} \neq 0°$		$B_n + B_n B_1 + B_1 = 0$

Table 2. Sufficient conditions to avoid spectral dispersion for IOF Φ_2 when $\mathbf{n}_1$ and $\mathbf{n}_n$ are defined in Eq. (13c) or (13d).

Cases		Solutions
$\omega_{1x} = 0°$	$\omega_{nx} = 0°$	$B_n + B_n B_1 + B_1 = 0$
	$\omega_{nx} \neq 0°$	$B_n = B_1 = 0$
$\omega_{1x} \neq 0°$		$B_n = B_1 = 0$

of $\mathbf{n}_1$ and $\mathbf{n}_n$ are defined in any of the forms shown in Eqs. (13a)–(13d). Consequently, spectral dispersion occurs even if the source ray and exit ray are normal incidence rays. Accordingly, the following discussions derive the sufficient conditions under which spectral dispersion can be avoided. Substituting Eqs. (11a) and (11b) into Eq. (7) with $\Phi = \Phi_5$, the following nine equations are obtained:

$$\begin{aligned}&\left[a_z B_n \sin(\omega_{ny})\cos(\omega_{ny})\sin^2(\omega_{nx})\right]\left[1 + B_1\sin^2(\omega_{1y})\sin^2(\omega_{1x})\right]\\&\quad+\left[B_n\sin^2(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1\cos(\omega_{1x})\sin(\omega_{1y})\sin(\omega_{1x})\right]\\&\quad+\left[c_y B_n\sin(\omega_{ny})\sin(\omega_{nx})\cos(\omega_{nx})\right]\left[B_1\cos(\omega_{1y})\sin(\omega_{1y})\sin^2(\omega_{1x})\right]\\&\quad+B_1\cos(\omega_{1x})\sin(\omega_{1y})\sin(\omega_{1x}) = 0\end{aligned} \tag{14a}$$

$$\begin{aligned}&\left[a_z B_n \sin(\omega_{ny})\cos(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1\sin(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x})\right]\\&\quad+\left[B_n\sin^2(\omega_{ny})\sin^2(\omega_{nx})\right]\left[1 + B_1\cos^2(\omega_{1x})\right]\\&\quad+\left[c_y B_n\sin(\omega_{ny})\sin(\omega_{nx})\cos(\omega_{nx})\right]\left[B_1\cos(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x})\right]\\&\quad+B_1\cos^2(\omega_{1x}) = 0\end{aligned} \tag{14b}$$

$$\begin{aligned}&\left[a_z B_n \sin(\omega_{ny})\cos(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1\sin(\omega_{1y})\cos(\omega_{1y})\sin^2(\omega_{1x})\right]\\&\quad+\left[B_n\sin^2(\omega_{ny})\sin^2(\omega_{nx})\right]\left[B_1\cos(\omega_{1x})\cos(\omega_{1y})\sin(\omega_{1x})\right]\\&\quad+\left[c_y B_n\sin(\omega_{ny})\sin(\omega_{nx})\cos(\omega_{nx})\right]\left[1 + B_1\cos^2(\omega_{1y})\sin^2(\omega_{1x})\right]\\&\quad+B_1\cos(\omega_{1x})\cos(\omega_{1y})\sin(\omega_{1x}) = 0\end{aligned} \tag{14c}$$

$$\begin{aligned}&\left[a_z B_n \cos(\omega_{nx})\cos(\omega_{ny})\sin(\omega_{nx})\right]\left[1 + B_1\sin^2(\omega_{1y})\sin^2(\omega_{1x})\right]\\&\quad+\left[B_n\cos(\omega_{nx})\sin(\omega_{ny})\sin(\omega_{nx})\right]\left[B_1\cos(\omega_{1x})\sin(\omega_{1y})\sin(\omega_{1x})\right]\\&\quad+\left[c_y B_n\cos^2(\omega_{nx})\right]\left[B_1\cos(\omega_{1y})\sin(\omega_{1y})\sin^2(\omega_{1x})\right]\\&\quad+c_y B_1\cos(\omega_{1y})\sin(\omega_{1y})\sin^2(\omega_{1x}) = 0\end{aligned} \tag{14d}$$

$$\begin{aligned}&\left[a_z B_n \cos(\omega_{nx})\cos(\omega_{ny})\sin(\omega_{nx})\right]\left[B_1\sin(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x})\right]\\&\quad+\left[B_n\cos(\omega_{nx})\sin(\omega_{ny})\sin(\omega_{nx})\right]\left[1 + B_1\cos^2(\omega_{1x})\right]\\&\quad+\left[c_y B_n\cos^2(\omega_{nx})\right]\left[B_1\cos(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x})\right]\\&\quad+c_y B_1\cos(\omega_{1y})\sin(\omega_{1x})\cos(\omega_{1x}) = 0\end{aligned} \tag{14e}$$

$$\left[a_z B_n \cos(\omega_{nx}) \cos(\omega_{ny}) \sin(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y}) \cos(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ \left[B_n \cos(\omega_{nx}) \sin(\omega_{ny}) \sin(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x}) \cos(\omega_{1y}) \sin(\omega_{1x})\right]$$
$$+ \left[c_y B_n \cos^2(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ c_y B_1 \cos^2(\omega_{1y}) \sin^2(\omega_{1x}) = 0 \tag{14f}$$

$$\left[a_z B_n \cos^2(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[1 + B_1 \sin^2(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ \left[B_n \cos(\omega_{ny}) \sin(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x}) \sin(\omega_{1y}) \sin(\omega_{1x})\right]$$
$$+ \left[c_y B_n \cos(\omega_{ny}) \sin(\omega_{nx}) \cos(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y}) \sin(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ a_z B_1 \sin^2(\omega_{1y}) \sin^2(\omega_{1x}) = 0 \tag{14g}$$

$$\left[a_z B_n \cos^2(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x})\right]$$
$$+ \left[B_n \cos(\omega_{ny}) \sin(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1x})\right]$$
$$+ \left[c_y B_n \cos(\omega_{ny}) \sin(\omega_{nx}) \cos(\omega_{nx})\right]\left[B_1 \cos(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x})\right]$$
$$+ a_z B_1 \sin(\omega_{1y}) \sin(\omega_{1x}) \cos(\omega_{1x}) = 0 \tag{14h}$$

$$\left[a_z B_n \cos^2(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \sin(\omega_{1y}) \cos(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ \left[B_n \cos(\omega_{ny}) \sin(\omega_{ny}) \sin^2(\omega_{nx})\right]\left[B_1 \cos(\omega_{1x}) \cos(\omega_{1y}) \sin(\omega_{1x})\right]$$
$$+ \left[c_y B_n \cos(\omega_{ny}) \sin(\omega_{nx}) \cos(\omega_{nx})\right]\left[1 + B_1 \cos^2(\omega_{1y}) \sin^2(\omega_{1x})\right]$$
$$+ a_z B_1 \sin(\omega_{1y}) \cos(\omega_{1y}) \sin^2(\omega_{1x}) = 0 \tag{14i}$$

T a b l e 3. Sufficient conditions to avoid spectral dispersion for IOF Φ_5 when $\mathbf{n}_1$ and $\mathbf{n}_n$ are defined in Eqs. (13c), (13b) and (13d) with $\omega_{1y} = \omega_{ny} = 0°$, $\omega_{1y} = 90°$ and $\omega_{ny} = 0°$, and $\omega_{1y} = \omega_{ny} = 90°$.

Cases		Solutions
$\omega_{1y} = \omega_{ny} = 0°$	$\omega_{1x} = 90°$ and $\omega_{nx} = 0°$	$B_n + B_n B_1 + B_1 = 0$
	$\omega_{1x} \neq 90°$ or $\omega_{nx} \neq 0°$	$B_n = B_1 = 0$
$\omega_{1y} = 90°$ and $\omega_{ny} = 0°$	$\omega_{1x} = \omega_{nx} = 90°$	$B_n + B_n B_1 + B_1 = 0$
	$\omega_{1x} \neq 90°$ or $\omega_{nx} \neq 90°$	$B_n = B_1 = 0$
$\omega_{1y} = \omega_{ny} = 90°$	$\omega_{1x} = 0°$ and $\omega_{nx} = 90°$	$B_n + B_n B_1 + B_1 = 0$
	$\omega_{1x} \neq 0°$ or $\omega_{nx} \neq 90°$	$B_n = B_1 = 0$

Table 4. Sufficient conditions to avoid spectral dispersion for IOF Φ_5 when $\mathbf{n}_1$ and $\mathbf{n}_n$ are defined in Eq. (13a) with $\omega_{1y} = 0°$ and $\omega_{ny} = 90°$.

Cases			Solutions
$c_x = 1$	$\omega_{nx} = \omega_{1x}$	$\omega_{nx} = \omega_{1x} \neq 45°$	$B_n = B_1 = 0$
		$\omega_{nx} = \omega_{1x} = 45°$	$B_n + B_n B_1 + B_1 = 0$
	$\omega_{nx} = -\omega_{1x}$		$B_n = B_1 = 0$
$c_x = -1$	$\omega_{nx} = \omega_{1x}$		$B_n = B_1 = 0$
	$\omega_{nx} = -\omega_{1x}$	$\omega_{nx} = -\omega_{1x} \neq 45°$	$B_n = B_1 = 0$
		$\omega_{nx} = -\omega_{1x} = 45°$	$B_n + B_n B_1 + B_1 = 0$

As in the previous section, Eqs. (14a)–(14i) are investigated for the four cases given in Eqs. (13a)–(13d). The solutions of Φ_5 for the four different cases are summarized in Tables 3 and 4.

5. Conclusions

One of the main functions of a prism is that of re-orienting the image in such a way as to produce the required orthogonal image orientation function Φ. This study has investigated the sufficient conditions for producing two particular categories of orthogonal image orientation function (*i.e.*, Φ_2 and Φ_5) without spectral dispersion. Assuming that $\mathbf{n}_1$ and $\mathbf{n}_n$ are the unit normal vectors of the first r_1 and last r_n boundary surfaces of the prism, respectively, the present results have shown that either one of the following two conditions should be satisfied in order to avoid spectral dispersion for a given Φ:

1) The entrance and exit rays of the prism are both normal incidence rays (*i.e.*, the rays enter and exit the respective boundary surfaces perpendicularly) and equation $\mathbf{n}_n = \pm\Phi\,\mathbf{n}_1$ should be satisfied.

2) The incidence angle θ_1 at the first boundary surface r_1 is equal to the refraction angle $\underline{\theta}_n$ at the last boundary surface r_n in order to fulfill the sufficient $B_n + B_n B_1 + B_1 = 0$. It is noted that this condition is usually satisfied by mirror-symmetry prisms (*i.e.*, prisms in which the two halves are mirror images of one another).

In general, the findings presented in this study provide a useful basis for analytical methods aimed at designing prisms which produce the required orthogonal image orientation function without spectral dispersion.

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