

# ESTIMATING A GRADUAL PARAMETER CHANGE IN AN AR(1)-PROCESS\*

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**Abstract:** The authors discuss the estimation of a change-point  $t_0$  at which the parameter of a (non-stationary) AR(1)-process possibly changes in a *gradual* way. Making use of the observations  $X_1, \dots, X_n$ , we study the least squares estimator  $\hat{t}_0$  for  $t_0$ , which is obtained by minimizing the sum of squares of the residuals with respect to the given parameters. As the first result it can be shown that, under certain regularity and moment assumptions,  $\hat{t}_0/n$  is a consistent estimator for  $\tau_0$ , where  $t_0 = \lfloor n\tau_0 \rfloor$ , with  $0 < \tau_0 < 1$ , i.e.,  $\hat{t}_0/n \rightarrow^P \tau_0$  ( $n \rightarrow \infty$ ). Based on the rates obtained in the proof of the consistency result, a rough convergence rate statement can also be given. Some possible further investigations are briefly discussed, including the weak limiting behaviour of the (suitably normalized) estimator.

**Keywords:** gradual change, change-point estimation, least squares estimator, AR(1)-process

## 1. Introduction and statistical framework

This work studies the estimation of a change-point at which the parameter of a (non-stationary) AR(1)-process possibly changes in a *gradual* way. More precisely, the authors observe a time series possessing the structure

$$X_t = (\beta_0 + \beta_1 g(t, t_0))X_{t-1} + e_t \quad (t = 1, 2, \dots), \quad \text{with } X_0 = e_0, \quad (1)$$

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where  $\{e_t\}_{t=0,1,\dots}$  is a sequence of independent, identically distributed random variables with  $Ee_0 = 0, 0 < \sigma^2 = Ee_0^2 < \infty, Ee_0^4 < \infty, \beta_0, \beta_1$  are unknown parameters satisfying

$$|\beta_0| < 1, \beta_1 = \beta_{1,n} \rightarrow 0, |\beta_1| \sqrt{n} \rightarrow \infty (n \rightarrow \infty), \quad (2)$$

and  $g(\cdot, t_0)$  is a (known) real function such that

$$g(t, t_0) = 0 (t \leq t_0) \text{ and } g(t, t_0) \neq 0 (t > t_0). \quad (3)$$

That is, we assume that the parameter  $\beta_0$  of the AR(1)-process changes *gradually* at an unknown time-point  $t_0 = [n\tau_0]$ , with  $0 < \tau_0 < 1$ , where  $[\cdot]$  denotes the integer part, and the aim is to provide an estimator for  $t_0$  resp.  $\tau_0$ , making use of observations  $X_1, \dots, X_n$  and under certain assumptions on function  $g(\cdot, t_0)$  in (3) to be specified below.

Note that if  $g(\cdot, t_0)$  is a bounded function, then in view of (2),  $b := \sup_{t \geq 1} |\beta_0 + \beta_1 g(t, t_0)| < 1$  for sufficiently large  $n$  and, by a repeated application of (1),

$$X_t = e_t + \sum_{j=1}^t e_{t-j} \prod_{i=0}^{j-1} (\beta_0 + \beta_1 g(t-i, t_0)) \\ (t = 1, 2, \dots).$$

Let us study the least squares estimator  $\hat{t}_0$  for  $t_0$ , which is obtained by minimizing

$$(b_0, b_1, t_*) = \sum_{t=1}^n [X_t - (b_0 + b_1 g(t, t_*)) X_{t-1}]^2$$

with respect to  $b_0, b_1 \in \mathbb{R}, t_* = 0, 1, \dots, [n(1 - \delta)], \delta > 0$  arbitrarily small, i.e.

$$S(\hat{b}_0, \hat{b}_1, \hat{t}_0) = \min_{b_0, b_1, t_*} S(b_0, b_1, t_*) = \\ \min_{t_*} \min_{b_0, b_1} S(b_0, b_1, t_*). \quad (4)$$

Via partial derivatives, it is not difficult to show that, for fixed  $t_*$ ,

$$\hat{b}_0(t_*) = \frac{\sum_{t=1}^n X_t X_{t-1}}{\sum_{t=1}^n X_{t-1}^2} - \hat{b}_1(t_*) \frac{\sum_{t=1}^n g(t, t_*) X_{t-1}^2}{\sum_{t=1}^n X_{t-1}^2} \quad (5)$$

and

$$\hat{b}_1(t_*) = \frac{\sum_{t=1}^n X_t X_{t-1} g(t, t_*) - \frac{\sum_{j=1}^n X_j X_{j-1}}{\sum_{j=1}^n X_{j-1}^2} \sum_{t=1}^n X_{t-1}^2 g(t, t_*)}{\sum_{t=1}^n X_{t-1}^2 g^2(t, t_*) - \frac{(\sum_{t=1}^n g(t, t_*) X_{t-1}^2)^2}{\sum_{t=1}^n X_{t-1}^2}}. \quad (6)$$

On plugging (5) to (6) into (4), we obtain

$$S(\hat{b}_0, \hat{b}_1, \hat{t}_0) = \min_{t_*} \left[ \sum_{t=1}^n \left( X_t - \frac{\sum_{j=1}^n X_j X_{j-1}}{\sum_{j=1}^n X_{j-1}^2} X_{t-1} \right)^2 - \hat{b}_1^2(t_*) \sum_{t=1}^n X_{t-1}^2 \left( g(t, t_*) - \frac{\sum_{j=1}^n g(j, t_*) X_{j-1}^2}{\sum_{j=1}^n X_{j-1}^2} \right)^2 \right]. \tag{7}$$

Since the first term in (7) does not depend on  $t_*$ , a combination of (4) – (7) results in

$$\hat{t}_0 = \arg \max_{t_*} \frac{\left( \sum_{t=1}^n X_t X_{t-1} g(t, t_*) - \frac{\sum_{j=1}^n X_j X_{j-1}}{\sum_{j=1}^n X_{j-1}^2} \sum_{t=1}^n X_{t-1}^2 g(t, t_*) \right)^2}{\sum_{t=1}^n X_{t-1}^2 g^2(t, t_*) - \frac{(\sum_{t=1}^n g(t, t_*) X_{t-1}^2)^2}{\sum_{t=1}^n X_{t-1}^2}}. \tag{8}$$

For the theoretical studies of  $\hat{t}_0$  below, it is convenient to make use of the model equation (1) and rewrite (8), after a multiplication with  $1/n$ , as

$$\hat{t}_0 = \arg \max_{t_*} \frac{\left[ \beta_1 \left( \frac{1}{n} \sum_{t=1}^n g(t, t_0) g(t, t_*) X_{t-1}^2 - \frac{\frac{1}{n} \sum_{j=1}^n g(j, t_0) X_{j-1}^2 \frac{1}{n} \sum_{j=1}^n g(j, t_*) X_{j-1}^2}{\frac{1}{n} \sum_{j=1}^n X_{j-1}^2} \right) + \frac{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2 g^2(t, t_*) - \left( \frac{1}{n} \sum_{t=1}^n g(t, t_*) X_{t-1}^2 \right)^2}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2} \right]^2}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2 g^2(t, t_*) - \frac{\left( \frac{1}{n} \sum_{t=1}^n g(t, t_*) X_{t-1}^2 \right)^2}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2}} + \frac{\frac{1}{n} \sum_{t=1}^n g(t, t_*) e_t X_{t-1} - \frac{\frac{1}{n} \sum_{j=1}^n e_j X_{j-1} \frac{1}{n} \sum_{j=1}^n g(j, t_*) X_{j-1}^2}{\frac{1}{n} \sum_{j=1}^n X_{j-1}^2}}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2 g^2(t, t_*) - \frac{\left( \frac{1}{n} \sum_{t=1}^n g(t, t_*) X_{t-1}^2 \right)^2}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2}} \right]^2. \tag{9}$$

The main result showed that  $\hat{t}_0/n$  is a consistent estimator for  $\tau_0$ , where  $t_0 = [n\tau_0]$ , with  $0 < \tau_0 < 1$ , i.e.  $\hat{t}_0/n \xrightarrow{P} \tau_0$  ( $n \rightarrow \infty$ ).

## 2. Assumptions and the main result

For the asymptotic results below, we assume that the gradual change function  $g(\cdot, t_*)$  satisfies the following assumptions:

(A.1) For every  $t_* = 0, 1, \dots, n$ , the function  $g(\cdot, t_*)$  is of the form

$$g(t, t_*) = g_0 \left( \frac{t - t_*}{n} \right), t = 0, 1, \dots, n,$$

where  $g_0: (-\infty, 1] \rightarrow \mathbb{R}$  is a real function satisfying.

(A.2) It holds that

$$g_0(x) = 0 \quad (x \leq 0) \text{ and } g_0(x) \neq 0 \quad (0 < x \leq 1).$$

(A.3) The function  $g_0: (-\infty, 1] \rightarrow \mathbb{R}$  is bounded and Lipschitz continuous, i.e.

$$|g_0(x)| \leq C \text{ and } |g_0(x) - g_0(y)| \leq D|x - y|, \quad x, y \leq 1,$$

with some positive constants  $C$  and  $D$ .

**Remark 1.** Function  $g_0$ , for example, could be such that  $g_0(x) = \pm x_+^\kappa$ , where  $x_+$  denotes the positive part of  $x$  and  $\kappa \geq 1$  is a fixed constant.

**Theorem 1.** Let Assumptions (A.1) to (A.3) be satisfied. Then under model (1) and the corresponding conditions formulated above, estimator  $\hat{t}_0$  from (8) and (9), respectively, is consistent, i.e.

$$\frac{\hat{t}_0}{n} \xrightarrow{P} \tau_0 \quad (n \rightarrow \infty). \quad (10)$$

### 3. A rough rate of consistency

On checking the estimates in the proof of Theorem 1 more carefully, a rough rate of consistency for estimator  $\hat{t}_0$  can be obtained as follows:

**Theorem 2.** Under the conditions of Theorem 1, assume that the limit function  $\tau_* \mapsto f(\tau_*)$  in (13) below is twice continuously differentiable in a small neighbourhood of  $\tau_0$ , with  $f''(\tau_*) > D$  for some  $D > 0$ . Then, with  $\hat{t}_0 = \lfloor n\hat{t}_0 \rfloor$ , for every sequence  $\{\varepsilon_n\}$  with  $\varepsilon_n \rightarrow 0$ ,

$$|\hat{t}_0 - \tau_0| = O_P(|\beta_1|^{1/2}) + o_P\left(\frac{1}{|\beta_1|^{1/2} \varepsilon_n n^{1/4}}\right) \quad (n \rightarrow \infty). \quad (11)$$

**Remark 2.** If, for example,  $\beta_1 = n^{-\alpha}$ , with  $0 < \alpha < 1/2$ , then  $\varepsilon_n$  could be chosen as  $(\log n)^{-p}$ , with  $p > 0$ , therefore one would have the polynomial consistency rate

$$|\hat{t}_0 - \tau_0| = \begin{cases} O_P\left(\frac{1}{n^{\alpha/2}}\right), & \text{if } 0 < \alpha < 1/4, \\ O_P\left(\frac{\log^p n}{n^{1/4-\alpha/2}}\right), & \text{if } 1/4 \leq \alpha < 1/2, \end{cases} \quad (n \rightarrow \infty). \quad (12)$$

### 4. Some remarks on the proof

Here the authors only outline some key steps of the proof. For details refer to Hušková et al. (2019).

**Proof of Theorem 1.** To prove the convergence  $\hat{t}_0/n \rightarrow^P \tau_0$  ( $n \rightarrow \infty$ ), the idea is to show uniform convergence of the terms in (9) to a non-random limit function possessing a unique maximum at  $\tau_* = \tau_0$ , where  $t_* = \lfloor n\tau_* \rfloor$ . It turns out that the terms in the first line of (9) are the dominating ones and the terms in the second line are of a lower order.

**Step 1.** Replace the random variables  $X_{t-1}^2$  by their expectations  $EX_{t-1}^2$ , for example, as  $n \rightarrow \infty$ , for every sequence  $\{\varepsilon_n\}$  with  $\varepsilon_n \rightarrow 0$ ,

$$\frac{\varepsilon_n}{\sqrt{n}} \max_{t_*} \left| \sum_{t=1}^n g(t, t_0)g(t, t_*) (X_{t-1}^2 - EX_{t-1}^2) \right| = o_P(1).$$

**Step 2.** Show that the corresponding terms with expectations converge, for example, as  $n \rightarrow \infty$ ,

$$\max_{t_*} \left| \frac{1}{n} \sum_{t=1}^n g(t, t_0)g(t, t_*) EX_{t-1}^2 - \frac{\sigma^2}{1 - \beta_0^2} \int_0^1 g_0(x - \tau_0)g_0(x - \tau_*) dx \right| = O_P(|\beta_1|).$$

**Step 3.** Estimate the negligible terms, for example, as  $n \rightarrow \infty$ , for every sequence  $\{\varepsilon_n\}$  with  $\varepsilon_n \rightarrow 0$ ,

$$\frac{\varepsilon_n}{\sqrt{n}} \max_{t_*} \left| \sum_{t=1}^n e_t X_{t-1} g(t, t_*) \right| = o_P(1).$$

**Step 4.** Show convergence of the dominating term in (9), that is, as  $n \rightarrow \infty$ , with  $t_* = \lfloor n\tau_* \rfloor$ ,

$$f_n(\tau_*) := \frac{\left[ \frac{1}{n} \sum_{t=1}^n g(t, t_0)g(t, t_*) X_{t-1}^2 - \frac{\frac{1}{n} \sum_{j=1}^n g(j, t_0) X_{j-1}^2 \frac{1}{n} \sum_{j=1}^n g(j, t_*) X_{j-1}^2}{\frac{1}{n} \sum_{j=1}^n X_{j-1}^2} \right]^2}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2 g^2(t, t_*) - \left( \frac{\frac{1}{n} \sum_{t=1}^n g(t, t_*) X_{t-1}^2}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2} \right)^2} \rightarrow^P \frac{\sigma^2}{1 - \beta_0^2} \frac{\left[ \int_0^1 g_0(x - \tau_0)g_0(x - \tau_*) dx - \int_0^1 g_0(x - \tau_0) dx \int_0^1 g_0(x - \tau_*) dx \right]^2}{\int_0^1 g_0^2(x - \tau_*) dx - \left( \int_0^1 g_0(x - \tau_*) dx \right)^2} =: f(\tau_*), \quad (13)$$

uniformly in  $\tau_* \in [0, 1 - \delta]$ , i.e.  $\max_{\tau_* \in [0, 1 - \delta]} |f_n(\tau_*) - f(\tau_*)| \rightarrow^P 0$  as  $n \rightarrow \infty$ .

**Step 5.** Show that the limit function  $\tau_* \mapsto f(\tau_*)$  in (13) is continuous on  $[0, 1 - \delta]$  and has a unique maximum at  $\tau_* = \tau_0$ .

**Step 6.** The proof of (10) can then be completed by applying the following lemma from real analysis:

**Lemma.** Let  $f$  be a continuous real function on compact set  $K$  and  $x_0$  be a unique maximizer of  $f$ , i.e.  $x_0 = \arg \max_x f(x)$ . Furthermore assume that for sequence  $\{f_n\}$  of real functions on  $K$ ,  $\lim_{n \rightarrow \infty} \max_{x \in K} |f_n(x) - f(x)| = 0$ , and let  $\hat{x}_n = \arg \max_x f_n(x)$  be a maximizer of  $f_n$  (not necessarily unique). Then

$$\hat{x}_n \rightarrow x_0 \text{ as } n \rightarrow \infty.$$

**Proof of Theorem 2.** With the notations in (13), we have in a close neighbourhood of  $\tau_0$ ,

$$\begin{aligned} f_n(\hat{\tau}_0) - f(\tau_0) &= \max_{\tau_*} f_n(\tau_*) - \max_{\tau_*} f(\tau_*) = \\ &= [f_n(\hat{\tau}_0) - f(\hat{\tau}_0)] + [f(\hat{\tau}_0) - f(\tau_0)] = \\ &= [f_n(\hat{\tau}_0) - f(\hat{\tau}_0)] + \left[ f'(\tau_0)(\hat{\tau}_0 - \tau_0) + f''(\tau_n) \frac{(\hat{\tau}_0 - \tau_0)^2}{2} \right], \end{aligned}$$

where  $\tau_n$  is between  $\hat{\tau}_0$  and  $\tau_0$ . Since  $f'(\tau_0) = 0$  and  $|f''(\tau_n)| \geq D$ , for some  $D > 0$ , this results in the estimate

$$\begin{aligned} |\hat{\tau}_0 - \tau_0|^2 &\leq \frac{2}{D} \left[ |\max_{\tau_*} f_n(\tau_*) - \max_{\tau_*} f(\tau_*)| + \max_{\tau_*} |f_n(\tau_*) - \right. \\ &\quad \left. f(\tau_*)| \right] \leq \frac{4}{D} \max_{\tau_*} |f_n(\tau_*) - f(\tau_*)|. \end{aligned}$$

Now, on carefully checking the convergence rates obtained in the proof of Theorem 1, it is obvious that

$$\max_{\tau_*} |f_n(\tau_*) - f(\tau_*)| = O_P(|\beta_1|) + o_P\left(\frac{1}{|\beta_1| \varepsilon_n \sqrt{n}}\right),$$

which completes the proof of (11).

## 5. Concluding remarks

It is very likely that, in addition to the consistency result in Theorem 1, the rough convergence rate statement of Theorem 2 can be improved. For instance, some rather technical investigations in Hušková et al. (2019) indicate that, under additional conditions on the smoothness of the change function  $g_0$ , the change-point estimator  $\hat{\tau}_0$  resp.  $\hat{\tau}_0$  from Theorems 1 and 2 has a normal limiting distribution after suitable normalization.

Moreover, a small simulation study presented in Hušková et al. (2019), demonstrates that if the change does not occur too early or too late and if the change parameter  $\beta_1$  is not too small, then the suggested least squares estimator performs reasonably well in finite samples.

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## References

Hušková M., Prášková Z., Steinebach J.G., 2019, *Estimating a gradual parameter change in an AR(1)-process*. Preprint (2019), Charles University in Prague and the University of Cologne.

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