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Vinod Kumar MISHRA[1]
Kripa SHANKER[2]

OPTIMAL ORDERING QUANTITIES FOR SUBSTITUTABLE ITEMS UNDER JOINT REPLENISHMENT WITH COST OF SUBSTITUTION

An inventory system of two mutually substitutable items has been studied where an item is out of stock, demand for it is met by the other item and any part of demand not met due to unavailability of the other item is lost. In the event of substitution, there is an additional cost of substitution involved for each unit of the substituted item. The demands are assumed to be deterministic and constant. Items are ordered jointly in each ordering cycle, in order to take advantage of joint replenishment. The problem is formulated and a solution procedure is suggested to determine the optimal ordering quantities that minimize the total inventory cost. The critical value of the substitution rate is defined to help in deciding the optimal value of decision parameters. Extensive numerical experimentation is carried out, which shows that prior knowledge of the critical value of the substitution rate helps to minimize the total inventory cost. Sensitivity analysis is carried out for the improvement in the optimal total cost with substitution as compared to the case without substitution to draw insights into the behaviour of the model.

Keywords: *inventory, substitutable item, joint replenishment, cost of substitution, critical substitution rate*

1. Introduction

We consider an inventory system where a portion of the demand for an item is fulfilled by some other item. This phenomenon of demand substitution has been studied by researchers in great detail. Substitution can happen under a variety of conditions. In stochastic conditions, an item is substituted by another item to avoid or minimize the

[1]Department of Computer Science and Engineering, Bipin Tripathi Kumaon Institute of Technology, Dwarahat, Almora, Uttarakhand 263653, India, e-mail address: vkmishra2005@gmail.com

[2]Department of Industrial and Management Engineering, Indian Institute of Technology, Kanpur, 208016, UP, India.

effects of shortages occurring due to uncertainty in the system, whereas in deterministic cases, a portion of demand is substituted by another item in a planned manner. In both cases, when an item which is primarily demanded is substituted by some other item, an additional cost is incurred, known as substitution cost. Such substitution costs may arise due to a variety of reasons: the cost of the reworking required on an item to make it substitutable for the other, loss of a customer's goodwill due to substitution, etc. Generally, it will be based on the number of units substituted. Further, it is also noted that an item could either be completely substituted or only partially substituted by another item. Accordingly, Kim and Bell [11] categorize substitution into symmetrical substitution and asymmetrical substitution. Besides cost related reasons, there could sometimes be some marketing motives that may also involve item substitutions.

As can be easily seen, the phenomenon of demand substitution involves multiple items (at least two). Also, once an inventory system involves more than one item, then the issue of replenishment policy arises, in the sense that items can be procured independently, jointly or in a coordinated manner.

This paper formulates a model of a two-item inventory system with partial substitution, together with any inventory related decisions, and studies the impact of the cost of substitution on the performance of the system. The system parameters are assumed to be deterministic. Further, a joint replenishment policy for the procurement of the items is used, where both items are jointly replenished in every cycle. For the case of a two-item inventory system with demand substitution and joint replenishment policies (JRP), Drezner et al. [3] developed an EOQ model and compared the results with the case of no substitution and showed that full substitution is never optimal. Gurnani and Drezner [6] extended the model of Drezner et al. [3] to more than two items, where the unmet demand for one item is fully converted to demand for the other. Salameh et al. [21] extended the model of Drezner et al. [3] by considering partial substitution. Recently, Rasoulia and Nakhai Kamalabadi [20] and Krommyda et al. [12] developed an inventory model similar to that of Salameh et al. [21] by considering the demand to be dependent on price, as well as on stock.

This paper extends the work of Salameh et al. [21] and Krommyda et al. [12] in three directions: (i) introduction of a substitution cost, (ii) analytical derivation of the optimal ordering quantities and (iii) studying the effect of substitution on system performance and exploration of the critical value of the substitution rate.

Regarding replenishment policies for inventory systems dealing with multiple items, a considerable amount of work has been reported in the literature. Khouja and Goyal [10] provide a detailed review of joint replenishment policies. In the context of formulating joint replenishment policies, Porras and Dekker [18, 19] developed models for determining optimal order quantities under various ordering conditions. An efficient method for deriving optimal (or near-optimal) solutions has always been an issue for researchers. In most of the previous studies related to joint replenishment policies, the optimal ordering quantities have been obtained by some heuristic search algorithm.

Hong and Kim [7] developed a closed form formula to obtain optimal order quantities using an unbiased estimator and genetic algorithm. Further, it has been shown by Schulz and Telha [22] that in the case of deterministic demand, the complexity of obtaining the optimal quantities increases exponentially with respect to time.

The issue of demand substitution under stochastic demand possesses a very interesting dimension, where substitution becomes an alternative in situations of uncertainty. Inventory models of substitution with stochastic demand have been studied by many researchers, some major contributions are due to Parlar and Goyal [16], Pasternack and Drezner [17], Ernst and Kouvelis [4], Gerchack and Grosfeld [5], Mishra and Raghunathan [15]. Zhao et al. [25] studied systems with two items, while Ye [24], Huang and Ke [9], Li et al. [14], Li and You [13], Hsieh [8], Xue and Song [23] developed inventory policies for multiple substitutable items.

In this article, we develop an inventory model for two mutually substitutable items by taking into account the cost of substitution. If one of the items is out of stock, then its demand is partially fulfilled by the second item and the remaining unmet demand is lost. The demand for each item is assumed to be deterministic and static. The two items are ordered jointly and thus the replenishment cycle for both items is the same. The rest of the paper is organized as follows. In section 2, we describe the assumptions and notation used. Section 3 gives the details of the mathematical formulation of the model, while Section 4 describes in detail the solution procedure with a proof of the pseudo-convexity of the total cost function and derivation of the critical value of the substitution rate. Section 5 provides some theoretical analysis. Section 6 presents some numerical examples and sensitivity analysis. Section 7 presents the conclusions.

2. Assumptions and notation

2.1. Assumptions

A. Demand. The demand rates for the two items are known and constant (deterministic and static).

B. Joint procurement policy. The two items are procured jointly in every ordering cycle.

C. Lead time and rate of replenishment. The procurement lead times are zero and replenishment rates for both items are infinite.

D. Rate of substitution. When an item becomes out of stock and there is on-hand inventory of the second item available, then the second item, while still being used to satisfy primary demand, is also used to substitute demand for the first item during its stock out period. The remaining un-substituted demand for the first item is lost. In the case of partial substitution, even when the stocks of the substitute are unlimited, this substitution need not be able to satisfy the whole demand for the stocked out item. The amount of substituted demand might be limited to a fraction of the total excess demand for an item during its stock out period. The maximum proportion of the number of units of demand for an item that can be substituted by another item and the total excess demand for the first item is known as the substitution rate. If the substitution rate is 1, this means that all of the excess demand for an item can potentially be substituted by the other item and a substitution rate of 0 indicates that no substitution is possible.

E. Mutual substitutability. Both items are mutually substitutable, that is, each one can substitute the other in the case of a lack of stock. However, both the cost and rate of substitution may differ.

2.2. Notation

Parameters

D_1, D_2 – demand rates for items 1 and 2, respectively
a_1, a_2 – fixed ordering cost per order for items 1 and 2, respectively
c_1, c_2 – unit procurement cost for items 1 and 2, respectively
π_1, π_2 – lost sales cost per unit for items 1 and 2, respectively
c_{s12} – unit substitution cost for item 1 if substituted by item 2
c_{s21} – unit substitution cost for item 2 if substituted by item 1
i – inventory holding cost (rate per unit of good per unit time)

Decision variables

Q_1, Q_2 – ordering quantities for item 1 and 2, respectively
α_1, α_2 – substitution rates for replacing item 1 by item 2 and for replacing item 2 by item 1, respectively
α_1^e, α_2^e – critical values of substitution rate for item 1 and item 2, respectively

Objective functions

$TC_{WO}(Q_1, Q_2)$ – total average annual cost without substitution
$TC1(Q_1, Q_2)$ – total average annual cost with substitution for the case when item 1 is substituted by item 2
$TC2(Q_1, Q_2)$ – total average annual cost with substitution for the case when item 2 is substituted by item 1

3. Formulation

We consider an inventory system with two mutually substitutable items. The inventory diagrams for the three possible situations (item 1 substituted by item 2, item 2 substituted by item 1 and no substitution) are shown in Figs. 1–3.

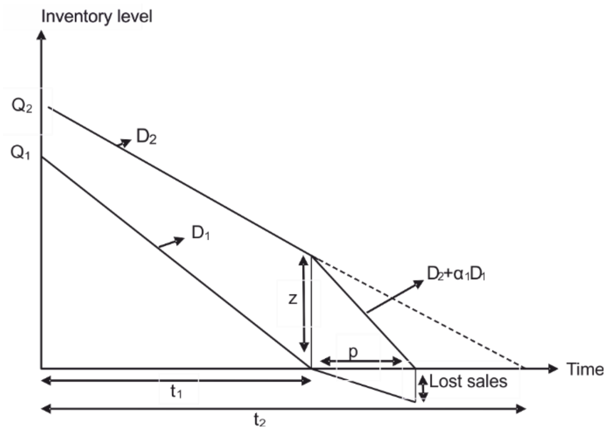

Fig. 1. Case 1: item 1 substituted by item 2 ($t_1 < t_2$)

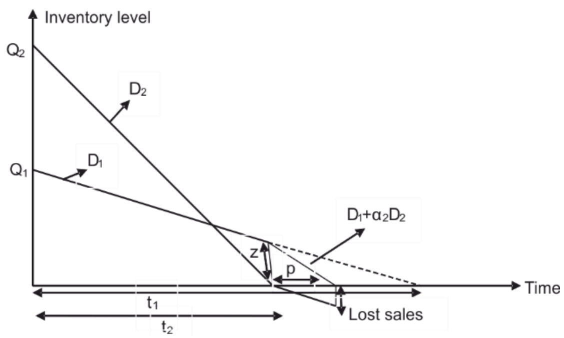

Fig. 2. Case 2: item 2 substituted by item 1 ($t_2 < t_1$)

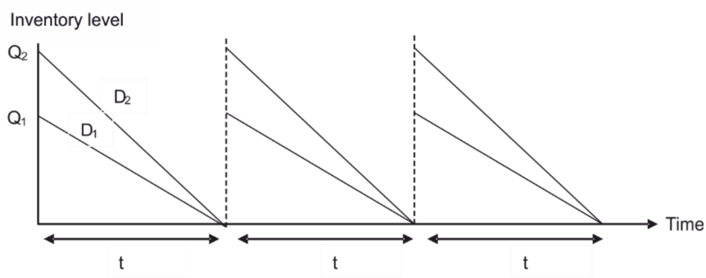

Fig. 3. Inventory levels without substitution under joint replenishment

At the beginning of the replenishment cycle, Q_1 units of item 1 and Q_2 units of item 2 are received. The on-hand inventories deplete at the constant rates of D_1 and D_2. There are three possible cases:

- Q_1 depletes before Q_2 (as depicted in Fig. 1), i.e., item 1 becomes out of stock first, and item 2 starts partially substituting the demand for item 1 with substitution rate α_1. Thus a total of $\alpha_1(D_1p)$ units are substituted at a rate of D_1 and consequently the inventory of item 2 experiences a total demand rate of $(D_2 + \alpha_1 D_1)$ during the entire period p. The remaining $(1 - \alpha_1)(D_1p)$ units of demand that are not substituted are lost.
- Q_2 depletes before Q_1 (as depicted in Fig. 2), i.e., the item 2 becomes out of stock first, and item 1 starts partially substituting the demand for item 2 with the substitution rate α_2. Thus a total of $\alpha_2(D_2p)$ units are substituted at a rate of D_2 and consequently the inventory of item 1 experiences a total demand rate of $(D_1 + \alpha_2 D_2)$ during the entire period p. The remaining $(1 - \alpha_2)(D_2p)$ units of demand that are not substituted are lost.
- Q_1 and Q_2 deplete simultaneously (as depicted in Fig. 3), i.e., items 1 and 2 can never go out of stock individually and thus there is no substitution of any of the items.

In the event of substitution, i.e., case 1 or case 2, a cost of substitution is incurred at the rate of C_{S12} per unit of item 1 substituted by item 2 (Fig. 1) and at the rate of C_{S21} per unit of item 2 substituted by item 1 (Fig. 2).

The derivations of the total annual costs for the three cases are discussed below.

Case 1. Item 1 is substituted by item 2 ($t_1 < t_2$)

For case 1, the average total cost is derived in the usual manner, i.e., by summing the various cost components per cycle and then by multiplying it by the average number of cycles per year. For this purpose, the following relationships can easily be established from Fig. 1:

- The inventory level z of item 2 at the instant when the inventory of item 1 depletes to zero

$$z = Q_2 - D_2 t_1 = Q_2 - D_2 \frac{Q_1}{D_1}$$

- The length of time p for which item 1 is substituted by item 2

$$p = \frac{z}{D_2 + \alpha_1 D_1} = \frac{Q_2 - D_2 \frac{Q_1}{D_1}}{D_2 + \alpha_1 D_1} = \frac{D_1 Q_2 - D_2 Q_1}{D_1 \left(D_2 + \alpha_1 D \right)_1}$$

- Total number of units of item 1 substituted by item 2 per cycle – $\alpha_1(D_1p)$

$$\alpha_1 \left(D_1 p \right) = \frac{\alpha_1 \left(D_1 Q_2 - D_2 Q_1 \right)}{D_2 + \alpha_1 D_1}$$

- Total amount of lost sales of item 1 per cycle (number of units) $(1-\alpha_1)D_1p$

$$(1-\alpha_1)D_1p = \frac{(1-\alpha_1)(D_1Q_2 - D_2Q_1)}{D_2 + \alpha_1 D_1}$$

- Length of cycle $t_1 + p$

$$t_1 + p = \frac{Q_1}{D_1} + \frac{D_1Q_2 - D_2Q_1}{D_1(D_2 + \alpha_1 D_1)} = \frac{\alpha_1 Q_1 + Q_2}{D_2 + \alpha_1 D_1}$$

- Average number of cycles per unit time (say year) – 1/(length of cycle)

$$\frac{1}{\text{length of cycle}} = \frac{D_2 + \alpha_1 D_1}{\alpha_1 Q_1 + Q_2}$$

The cost components per cycle consist of (A) costs related to item 1 (B) costs related to item 2 (C) costs of lost sales and (D) substitution costs.

The total cost associated with item 1 per ordering cycle consists of the fixed ordering cost, purchase cost and holding cost, and can be expressed as

$$\text{Total cost associated with item 1} = A_1 + C_1Q_1 + \frac{1}{2}\frac{iC_1Q_1^2}{D_1} \tag{1}$$

The total cost associated with item 2 per ordering cycle consists of the fixed ordering cost, purchase cost and holding cost and can be expressed in terms of a given Q_1 as

$$\text{Total cost associated with item 2} = A_2 + \mathrm{C}_2\mathrm{Q}_2$$
$$+\frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2 - D_2Q_1^2) + \frac{1}{2}\frac{iC_2(\mathrm{D}_1\mathrm{Q}_2 - D_2Q_1)^2}{D_1^2(\mathrm{D}_2 + \alpha_1 D_1)} \tag{2}$$

The cost of lost sales is incurred due to the demand for item 1 which is not satisfied: a total number of $(1-\alpha_1)(D_1p)$ units at a cost of π_1 per unit lost, which can be expressed as

$$\text{Costs of lost sales} = \pi_1\left((1-\alpha_1)\frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right) \tag{3}$$

The substitution cost is incurred according to the number of units of item 1 substituted by item 2 at the rate of C_{S12} per unit substituted and can be expressed as

$$\text{Substitution cost} = C_{S12}\left(\alpha_1 \frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right) \tag{4}$$

Thus, the total cost per ordering cycle $TC(Q_1, Q_2)$, from Eqs. (1)–(4) is given as

$$TC(Q_1,Q_2) = A_1 + A_2 + C_1Q_1 + C_2Q_2 + \frac{1}{2}\frac{iC_1Q_1^2}{D_1} + \frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2 - D_2Q_1^2)$$
$$+\frac{1}{2}\frac{iC_2(D_1Q_2 - D_2Q_1)^2}{D_1^2(D_2 + \alpha_1 D_1)} + \pi_1\left((1-\alpha_1)\frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right)$$
$$+ C_{S12}\left(\alpha_1 \frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right) \tag{5}$$

Finally, for case 1 (when $t_1 < t_2$), $TC1(Q_1, Q_2)$, the average total cost per unit time (say a year) is obtained by multiplying the total cost per ordering cycle by the average number of cycles per year and is given as

$$TC1(Q_1,Q_2) = \frac{D_2 + \alpha_1 D_1}{Q_1\alpha_1 + Q_2}\Bigg(A_1 + A_2 + C_1Q_1 + \mathrm{C}_2\mathrm{Q}_2 + \frac{1}{2}\frac{iC_1Q_1^2}{D_1}$$
$$+\frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2 - D_2Q_1^2) + \frac{1}{2}\frac{iC_2(D_1Q_2 - D_2Q_1)^2}{D_1^2(D_2 + \alpha_1 D_1)}$$
$$+\pi_1\left((1-\alpha_1)\frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right) + C_{S12}\left(\alpha_1 \frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right)\Bigg) \tag{6}$$

Case 2. Item 2 is substituted by item 1 ($t_2 < t_1$)

Following an approach analogous to case 1, for case 2 (when $t_1 > t_2$), $TC_2(Q_1, Q_2)$, the average total cost per unit time (say a year) is

$$TC2(Q_1,Q_2)=\frac{D_1+\alpha_2 Q_2}{Q_2\alpha_2+Q_1}\left(A_1+A_2+C_1Q_1+C_2Q_2+\frac{1}{2}\frac{iC_2Q_2^2}{D_2}\right.$$
$$+\frac{1}{2}\frac{iC_1}{D_2^2}(2D_2Q_1Q_2-D_1Q_2^2)+\frac{1}{2}\frac{iC_1(D_2Q_1-D_1Q_2)^2}{D_2^2(D_1+\alpha_2D_2)}$$
$$\left.+\pi_2\left((1-\alpha_2)\frac{(D_2Q_1-D_1Q_2)}{(D_1+\alpha_2D_2)}\right)+C_{S21}\left(\alpha_2\frac{(D_2Q_1-D_1Q_2)}{(D_1+\alpha_2D_2)}\right)\right) \tag{7}$$

Case 3. No substitution

Figure 3 illustrates the inventory levels for the case of no substitution. Under a joint replenishment policy, the inventories of both items deplete to zero simultaneously, i.e., $Q_1/D_1 = Q_2/D_2$.The average total cost per unit time for an inventory system without substitution under joint replenishment, $TC_{WO}(Q_1, Q_2)$,consists only of setup costs, purchase costs and holding costs and is given as

$$TC_{WO}(Q_1,Q_2)=\frac{D_2}{Q_2}\left(A_1+A_2+C_1Q_1+\mathrm{C}_2\mathrm{Q}_2+\frac{1}{2}\frac{iC_1Q_1^2}{D_1}+\frac{1}{2}\frac{iC_2Q_2^2}{D_2}\right) \tag{8}$$

4. Solution procedure

For the case of an inventory model without substitution under joint replenishment when $Q_1/D_1 = Q_2/D_2$ (case 3, Fig. 3), the optimal order quantities are obtained by standard calculus using the method of minimizing a function of two variables and are given as

$$Q_{1WO}^* = \sqrt{\frac{2D_1^2(A_1+A_2)}{i(C_1D_1+C_2D_2)}} \tag{9}$$

$$Q_{2WO}^* = \sqrt{\frac{2D_2^2(A_1+A_2)}{i(C_1D_1+C_2D_2)}} \tag{10}$$

By substituting the values of these optimal order quantities into Eq. (8), we get the minimum total cost under the condition of no substitution, which is given as

$$TC_{WO}(Q_{1WO}^*,Q_{2WO}^*)=\left(C_1D_1+C_2D_2+\sqrt{2i(A_1+A_2)(C_1D_1+C_2D_2)}\right) \tag{11}$$

As can be seen, the expressions for the average cost per period given by Eqs. (6) and (7) for the two cases of substitution, i.e., case 1 and case 2, are rather complex. To obtain the optimal ordering quantities using calculus, first we study the behaviour of the expressions for the average total cost with respect to the two decision variables Q_1 and Q_2. We check whether the expressions given by Eqs. (6) and (7) possess some sort of convexity under certain conditions. Using the properties related to pseudo-convexity [2], we show that the total cost functions given by Eqs. (6) and (7) are pseudo-convex functions under certain condition and thus possess a unique minimum.

4.1. Case 1 ($t_1 < t_2$)

4.1.1. Optimal order quantities and minimum total cost

Using the properties defined above, we show that the total cost function $TC1(Q_1, Q_2)$ given by Eq. (6) is a pseudo-convex function under certain conditions and thus possesses a unique minimum.

Theorem 1. The total cost $TC1(Q_1, Q_2)$ is pseudo-convex if $C_1 \geq \alpha_1 C_2$.

Proof. See Appendix A. Since the total cost function $TC1(Q_1, Q_2)$ is a pseudo-convex function of (Q_1, Q_2), the unique optimal ordering quantities (Q_1^*, Q_2^*) can be obtained by

$$\nabla TC1(Q_1, Q_2) = \left(\frac{\partial TC1}{\partial Q_1}, \frac{\partial TC1}{\partial Q_2} \right) = (0, 0)$$

For a detailed derivation, see appendix B.

The optimal ordering quantities are as follows:

$$Q_1^* = \frac{D_1(C_{S12}\alpha_1 + \pi_1(1-\alpha_1) - (C_1 - C_2\alpha_1))}{i(C_1 - C_2\alpha_1)} \tag{12}$$

$$Q_2^* = \frac{D_1\alpha_1 C_2((C_1 - C_2\alpha_1) - C_{S12}\alpha_1 - \pi_1(1-\alpha_1)) + \sqrt{C_2(C_1 - C_2\alpha_1)(D_2 + \alpha_1 D_1)}}{iC_2(C_1 - C_2\alpha_1)}$$
$$\times \big(C_{S12}D_1\alpha_1(2(C_1 - C_2\alpha_1) - C_{S12}\alpha_1) + 2\pi_1 D_1(C_1 - C_2\alpha_1)(1-\alpha_1)$$
$$- \pi_1^2 D_1(1-\alpha_1)^2 - 2C_{S12}\pi_1 D_1\gamma_1(1-\alpha_1) - D_1(C_1 - C_2\alpha_1)^2$$
$$+ 2i(A_1 + A_2)(C_1 - C_2\alpha_1)\big)^{1/2} \tag{13}$$

By substituting the above values of the optimal order quantities into Eq. (6), we obtain the optimal cost as

$$TC1(Q_1^*, Q_2^*) = C_2(D_2+\alpha_1 D_1) + C_{S12}\alpha_1 D_1 + \pi_1 D_1(1-\alpha_1) + \sqrt{\frac{C_2(D_2+\alpha_1 D_1)}{(C_1 - C_2\alpha_1)}}$$
$$\times\big(C_{S12}D_1\alpha_1(2(C_1-C_2\alpha_1)-C_{S12}\alpha_1)+2\pi_1 D_1(C_1-C_2\alpha_1)(1-\alpha_1)$$
$$-\pi_1^2 D_1(1-\alpha_1)^2 - 2C_{S12}\pi_1 D_1\alpha_1(1-\alpha_1) - D_1(C_1-C_2\alpha_1)^2$$
$$+2i(A_1+A_2)(C_1-C_2\alpha_1)\big)^{1/2} \tag{14}$$

4.1.2. Effect of the substitution rate and critical substitution rate

The total average annual cost given by Eq. (14) is a function of the substitution rate α_1. We wish to study the behaviour of this cost with respect to the rate of substitution to explore what value of the substitution rate results in the minimum total cost. This value of the substitution rate α_1 then becomes a policy decision and an indicator of how much substitution should be planned and allowed to gain the maximum economic advantage.

In order to obtain the optimal value of the substitution rate α_1 to obtain the minimum total cost, we need to observe that the cost expression given by Eq. (14) has been derived using the optimal order quantities given by Eqs. (12) and (13), which have been derived under the condition $C_1 \geq \alpha_1 C_2$. We call the value of α_1 which minimizes the average costs under this constraint the critical value of the substitution rate. In order to obtain the critical value of α_1, one may use standard calculus given that the average cost function is pseudo-convex, i.e., when $C_1 \geq \alpha_1 C_2$. Equivalently, one can consider the difference between the optimal total cost with substitution (Eq. (14)) and the optimal total cost without substitution (Eq. (11)). Using standard calculus, one can obtain the critical value of α_1 with respect to this difference between costs under the condition $C_1 \geq \alpha_1 C_2$. Considering the difference between these two costs is sometimes more appealing, as it indicates the economic benefits of the possibility of substitution.

The difference between costs, $D_F(\alpha_1)$, in terms of α_1, the substitution rate of item 1 is given as

$$D_F(\alpha_1) = TC_{WO}(Q_{1WO}^*, Q_{2WO}^*) - TC1(Q_1^*, Q_2^*) \tag{15}$$

where $TC_{WO}(Q_1, Q_2)$ is given by Eq. (8), which does not involve α_1, and $TC1(Q_1, Q_2)$ is given by Eq. (14).

Since $D_F(\alpha_1)$ as given by Eq. (15) upon substitution of the relevant expressions, or for that matter $TC1(Q_1, Q_2)$ as given by Eq. (14), is highly non-linear and complex, its concavity is established numerically (cf. next section). The critical value of α_1, the substitution rate of item 1, is obtained using calculus and is given as

$$\alpha_1^e = \left(\frac{iC_1\sqrt{2(A_1 + A_2)} + (C_1 - \pi_1)\sqrt{i(C_1D_1 + C_2D_2)}}{iC_2\sqrt{2(A_1 + A_2)} + (C_2 + C_{S12} - \pi_1)\sqrt{i(C_1D_1 + C_2D_2)}} \right) \tag{16}$$

4.2. Case 2 ($t_1 > t_2$)

4.2.1. Optimal order quantities and minimum total cost

Using a similar approach to the one in section 4.1.1, we show that the total cost function $TC2(Q_1, Q_2)$ given by Eq. (7) is a pseudo-convex function under certain conditions and thus possesses a unique minimum.

Theorem 2. The total cost $TC2(Q_1, Q_2)$ is pseudo-convex if $C_2 \geq \alpha_2 C_1$.

Proof. Analogous to the proof of Theorem 1.

The optimal ordering quantities are as follows:

$$\begin{aligned} Q_1^* = \Big(& D_2\alpha_2C_1((C_2 - C_1\alpha_2) - C_{S21}\alpha_2 - \pi_2(1-\alpha_2)) + \big(C_1(C_2 - C_1\alpha_2)(D_1 + \alpha_2D_2) \big)^{1/2} \\ & \times \big(C_{S21}D_2\alpha_2(2(C_2 - C_1\alpha_2) - C_{S21}\alpha_2) + 2\pi_2D_2(C_2 - C_1\alpha_2)(1-\alpha_2) \\ & - \pi_2^2D_2(1-\alpha_2)^2 - 2C_{S21}\pi_2D_2\alpha_2(1-\alpha_2) - D_2(C_2 - C_1\alpha_2)^2 \\ & + 2i(A_1 + A_2)(C_2 - C_1\alpha_2) \big)^{1/2} \Big) \big(iC_1(C_2 - C_1\alpha_2) \big)^{-1} \end{aligned} \tag{17}$$

$$Q_2^* = \frac{D_2(C_{S21}\alpha_2 + \pi_2(1-\alpha_2) - (C_2 - C_1\alpha_2))}{i(C_2 - C_1\alpha_2)} \tag{18}$$

By substituting the above values of the optimal order quantities into Eq. (7), we obtain

$$TC2(Q_1^*, Q_2^*) = C_1(D_1 + \alpha_2 D_2) + C_{S21}\alpha_2 D_2 + \pi_2 D_2(1-\alpha_2) + \left(\frac{C_1(D_1 + \alpha_2 D_2)}{(C_2 - C_1\alpha_2)} \right)^{1/2}$$

$$\times \big(C_{S21} D_2 \alpha_2 (2(C_2 - C_1\alpha_2) - C_{S21}\alpha_2) + 2\pi_2 D_2 (C_2 - C_1\alpha_2)(1-\alpha_2)$$

$$- \pi_2^2 D_2 (1-\alpha_2)^2 - 2C_{S21}\pi_2 D_2 \alpha_2 (1-\alpha_2) - D_2 (C_2 - C_1\alpha_2)^2$$

$$+ 2i(A_1 + A_2)(C_2 - C_1\alpha_2) \big)^{1/2} \tag{19}$$

4.2.2. Effect of the substitution rate and critical substitution rate

The arguments for this case are analogous to those given in Section 4.1.2.We thus obtain the critical value of α_2, the substitution rate of item 2, as

$$\alpha_2^e = \left(\frac{iC_2 \left(2(A_1 + A_2) \right)^{1/2} + (C_2 - \pi_2) \left(i(C_1 D_1 + C_2 D_2) \right)^{1/2}}{iC_1 \left(2(A_1 + A_2) \right)^{1/2} + (C_1 + C_{S21} - \pi_2) \left(i(C_1 D_1 + C_2 D_2) \right)^{1/2}} \right) \tag{20}$$

4.3. Solution procedure. A simple iterative process

We notice that the expressions for the optimal ordering quantities (Eqs. (12), (13), (17) and (18)) and critical values of the substitution rates (Eqs. (16) and (20)) are highly non-linear and implicit. In order to determine their values, the following simple iterative procedure is suggested and implemented for the numerical examples. The procedure is based on trying out both types of substitution in turn, i.e. first substituting item 1 by item 2 and then item 2 by item 1.

Step 0. Initialize all the parameters of the system.

Step 1. Obtain the critical value of the substitution rate α_1 from Eq. (16).

Step 2. Choose an appropriate substitution rate between 0 and the critical value obtained in Step 1.

Step 3. Find the optimal values of the order quantities and total cost corresponding to case 1, i.e., satisfying the condition $\frac{Q_1^*}{D_1} \le \frac{Q_2^*}{D_2}$ from Eqs. (12)–(14). If these conditions are found to be unsatisfied, go to Step 4.

Step 4. Obtain the critical value of the substitution rate α_2 from Eq. (20).

Step 5. Choose an appropriate substitution rate between 0 and the critical value obtained in Step 4.

Step 6. Find the optimal values of the order quantities and total cost corresponding to case 2, i.e. satisfying the condition $\frac{Q_1^*}{D_1} \geq \frac{Q_2^*}{D_2}$ from Eqs. (17)–(19).

Step 7. Exit the algorithm.

5. Analysis of the model

In order to get better economic insight into the benefit substitution, we study the behaviour of the optimal order quantities, substitution rates, total cost, etc. The analyses are presented in the form of theorems.

Theorem 3. At the critical value of the substitution rate (α_1^e), the optimal ordering quantities with substitution are equal to the optimal ordering quantities without substitution.

Proof. By substituting the value of α_1^e (Eq. (15)) into Q_1^* (Eq. (12)) and Q_2^* (Eq. (13)), we obtain

$$Q_1^* = D_1\left(C_{S12}\left(\frac{iC_1x + (C_1 - \pi_1)}{iC_2x + (C_2 + C_{S12} - \pi_1)} \right) + \pi_1(1 - \left(\frac{iC_1x + (C_1 - \pi_1)}{iC_2x + (C_2 + C_{S12} - \pi_1)} \right) \right.$$
$$\left. -(C_1 - C_2\left(\frac{iC_1x + (C_1 - \pi_1)}{iC_2x + (C_2 + C_{S12} - \pi_1)} \right)\right)\right)\left(i\left(C_1 - C_2\left(\frac{iC_1x + (C_1 - \pi_1)}{iC_2x + (C_2 + C_{S12} - \pi_1)} \right)\right)\right)^{-1}$$

where

$$x = \left(\frac{2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)} \right)^{1/2}$$

After simplification, we obtain (see Eq. (9))

$$Q_1^* = D_1x \Rightarrow Q_1^* = \left(\frac{2D_1^2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)} \right)^{1/2} = Q_{1WO}^*$$

Similarly, by substituting the value of α_1^e into Q_2^* (Eq. (13)), after simplification, we obtain (see Eq. (10))

$$Q_2^* = \left(\frac{2D_2^2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)} \right)^{1/2} = Q_{2WO}^*$$

These results simply state that at the critical value of the substitution rate, the optimal ordering quantities for the items with substitution are equal to the optimal ordering quantities for the items without substitution. Thus we may conclude that at the critical value of the substitution rate there is no advantage to be gained from substitution.

Theorem 4. The feasible region of the substitution rate for substituting item 1 with item 2 (case 1) is between 0 and α_1^e (i.e., $0 \le \alpha_1 \le \alpha_1^e$).

Proof. The critical substitution rate for case 1 (from Eq. (16))

$$\alpha_1^e = \frac{iC_1\left(2(A_1 + A_2)\right)^{1/2} + (C_1 - \pi_1)\left(i(C_1D_1 + C_2D_2)\right)^{1/2}}{iC_2\left(2(A_1 + A_2)\right)^{1/2} + (C_2 + C_{S12} - \pi_1)\left(i(C_1D_1 + C_2D_2)\right)^{1/2}}$$

by rearranging the terms we have

$$\alpha_1^e = \frac{iC_1\left(\dfrac{2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)}\right)^{1/2} + (C_1 - \pi_1)}{iC_2\left(\dfrac{2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)}\right)^{1/2} + (C_2 + C_{S12} - \pi_1)}$$

$$= \frac{iC_1\dfrac{\left(\dfrac{2D_1^2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)}\right)^{1/2}}{D_1} + (C_1 - \pi_1)}{iC_2\dfrac{\left(\dfrac{2D_2^2(A_1 + A_2)}{i(C_1D_1 + C_2D_2)}\right)^{1/2}}{D_2} + (C_2 + C_{S12} - \pi_1)}$$

from Eqs. (9) and (10)

$$\alpha_1^e = \frac{iC_1 \dfrac{Q_{1WO}^*}{D_1} + (C_1 - \pi_1)}{iC_2 \dfrac{Q_{2WO}^*}{D_2} + (C_2 + C_{S12} - \pi_1)}$$

from Theorem 2

$$\alpha_1^e = \frac{iC_1 \dfrac{Q_1^*}{D_1} + (C_1 - \pi_1)}{iC_2 \dfrac{Q_2^*}{D_2} + (C_2 + C_{S12} - \pi_1)}$$

Since $\dfrac{Q_1^*}{D_1} \leq \dfrac{Q_2^*}{D_2}$ in the case 1 (when $t_1 < t_2$), and the critical value of the substitution rate corresponds to $\dfrac{Q_1^*}{D_1} = \dfrac{Q_2^*}{D_2}$ (Theorem 3) thus for case 1

$$\left(iC_1 \frac{Q_1^*}{D_1} + (C_1 - \pi_1) \right) \leq \left(iC_2 \frac{Q_2^*}{D_2} + (C_2 + C_{S12} - \pi_1) \right) \Rightarrow 0 \leq \alpha_1 \leq \alpha_1^e$$

Alternatively since the value of substitution rate lies between 0 and 1, i.e., $0 \leq \alpha_1 \leq 1$ and α_1^e is the critical value of the substitution rate (α_1) of item 1 by item 2 (Theorem 3). Therefore,

$$\alpha_1 \leq \alpha_1^e \Rightarrow 0 \leq \alpha_1 \leq \alpha_1^e$$

Theorem 5. The optimal ordering quantities with substitution are equal to the optimal ordering quantities without substitution at the critical value of the substitution rate (α_2^e).

Proof. Analogous to the proof of Theorem 3.

Theorem 6. The feasible region of the substitution rate for substituting item 2 with item 1 is between 0 and α_2^e (i.e., $0 \leq \alpha_2 \leq \alpha_2^e$).

Proof. Analogous to the proof of Theorem 4.

6. Numerical example and sensitivity analysis

In this section, we provide a numerical example in order to illustrate the proposed model. The Maple mathematical modelling package was used. The values of the parameters given in Table 1 were selected to illustrate the effect of substitution and to provide a general explanation.

Table 1. Parameters used for numerical illustration

Parameters	Item 1	Item 2
Consumption rates (D_1, D_2)	100	20
Setup costs (A_1, A_2)	250	250
Purchase costs (C_1, C_2)	3	5
Lost sales costs (π_1, π_2)	6	7
Substitution costs (C_{S12}, C_{S21})	2	2
Holding cost rate (i)	2	2

We discuss the results for case 1. The results for case 2 are analogous. From Theorem 1, the total cost $TC1(Q_1, Q_2)$ is pseudo-convex if $C_1 \geq \alpha_1 C_2$, i.e., when $\alpha_1 \leq C_1/C_2 = 0.6$. In order to understand the behaviour of various quantities of interest, the following studies and analysis are carried out:

- Change in the optimal total costs according to the substitution rate α_1.
- Relationship between the total cost ($TC1$) and cost difference (D_F) according to the substitution rate α_1.
- Sensitivity analysis for the critical value of the substitution rate and optimal total cost.
- Sensitivity analysis for the critical value of the substitution rate of item 1 and the optimal total cost with a fixed substitution rate.

6.1. Change in the optimal total costs according to the substitution rate α_1

Following the procedure outlined in section 4.3, we calculate the critical value of the substitution rate (Step 1) and the optimal quantities (Step 3). For the given values of the parameters given in Table 1 and using the appropriate expressions given in Section 4.3, we obtain

- From Eq. (16), the critical value of the substitution rate $\alpha_1 = 0.3044$.
- From Eqs. (12) and (13), the optimal ordering quantities with substitution at the critical value of the substitution rate are 111.80 and 22.40 for items 1 item 2, respectively.

• From Eq. (14), the optimal total cost at the critical value of the substitution rate equal to 1294.43 per unit time.

• From Eqs. (9) and (10), the optimal ordering quantities without substitution are also 111.80 and 22.40 for items 1 and 2, respectively. From Eq. (10), the optimal total cost is 1294.43 per unit time. This result illustrates Theorem 3.

We tabulate the following for different values of the substitution rate α_1 within the range 0–0.3044.

• The optimal ordering quantities with substitution from Eqs. (12) and (13), and the optimal total cost from Eq. (14).

• The optimal ordering quantities without substitution from Eqs. (9) and (10), and the optimal total cost from Eq. (11).

The results are shown in Table 2. We notice that the improvement in the optimal total cost due to substitution decreases with increasing substitution rate (and becomes zero at the critical value of the substitution rate equal to 0.3044).

Table 2. Optimal ordering quantities and optimal total cost for different substitution rates α_1 (within feasible region)

α_1	Optimal total cost and optimal ordering quantities						Improvement in optimal total cost [%]
	with substitution			without substitution			
	Q_1^*	Q_2^*	$TC1$	Q_{1WO}^*	Q_{2WO}^*	TC_{WO}	
0.10	62.00	43.02	1202.28	111.80	22.40	1294.43	7.12
0.15	70.00	41.73	1237.32	111.80	22.40	1294.43	4.41
0.20	80.00	38.55	1265.52	111.80	22.40	1294.43	2.23
0.25	92.85	32.83	1285.53	111.80	22.40	1294.43	0.69

6.2. Relationship between the total cost (TC1) and cost difference (DF) according to the substitution rate α_1

Further, the relationships between the total cost ($TC1$) and substitution rate and between the difference between costs (D_F) and substitution rate ranging from 0 to 0.6 (i.e., up to $C_1/C_2 = 0.6$) are illustrated in Figs. 4 and 5, respectively.

The relationships in Figs. 4 and 5 are strictly concave and convex, respectively, which verifies that there exists a unique extreme value of the substitution rate. Figure 5 also shows that at the critical value of the substitution rate (α_1^e), the total optimal cost is independent of whether the possibility of substitution exists.

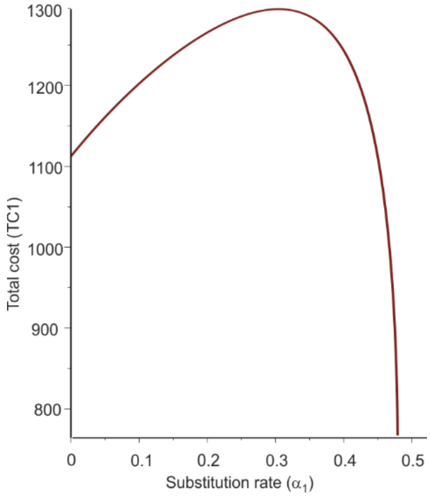

Fig. 4. Total cost (*TC*1) in function of the substitution rate

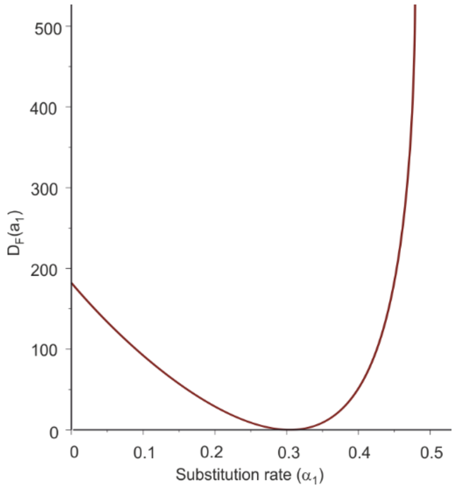

Fig. 5. Cost difference (D_F) in function of the substitution rate

6.3. Sensitivity analysis for the critical value of the substitution rate and optimal total cost

Next, we carry out the sensitivity analysis of the critical value of the substitution rate of item 1, and of the optimal total cost and optimal ordering quantities according to given values of the rate of substitution. The percentage improvements in the optimal total cost according to the values of various parameters are also presented. The numerical results are given in Table 3.

Table 3. Sensitivity analysis for the critical value of the substitution rate of item 1 and the optimal total cost with a fixed substitution rate, $\alpha_1 = 0.15$

Parameter		α_1^e	Optimal total cost and optimal ordering quantities						Improvement in optimal total cost [%]
			Substitution at $\alpha_1 = 0.15$			No substitution			
Symbol	Value		Q_1^*	Q_2^*	$TC1$	Q_{1WO}^*	Q_{2WO}^*	TC_{WO}	
C_1/C_2	1.0	0.802	13.52	56.66	1301.98	91.28	18.25	1695.44	0.232
	0.8	0.545	33.07	52.05	1285.18	100.00	20.00	1500.00	0.143
	0.6	0.304	70.00	41.73	1237.32	111.80	22.36	1294.42	0.044
	0.5	0.191	104.28	30.91	1180.59	119.52	23.90	1186.66	0.005
	0.4	0.083	166.00	8.09	1044.97	129.09	25.81	1074.59	infeasible

Parameter		α_1^e	Optimal total cost and optimal ordering quantities						Improvement in optimal total cost [%]
			Substitution at $\alpha_1 = 0.15$			No substitution			
Symbol	Value		Q_1^*	Q_2^*	$TC1$	Q_{1WO}^*	Q_{2WO}^*	TC_{WO}	
i	4	0.385	35.00	34.20	1504.00	79.05	15.81	1664.91	0.097
	3	0.354	46.66	37.61	1384.19	91.28	18.25	1495.94	0.075
	2	0.304	70.00	41.73	1237.32	111.80	22.36	1294.42	0.044
	1	0.195	140.00	41.55	1027.76	158.11	31.62	1032.45	0.005
	0.7	0.127	200.00	30.82	927.89	188.98	37.79	929.15	infeasible
π_1	3	0.441	13.33	56.92	1049.23	111.80	22.36	1294.42	0.189
	4	0.402	32.22	52.92	1122.62	111.80	22.36	1294.42	0.133
	5	0.357	51.11	47.90	1185.74	111.80	22.36	1294.42	0.084
	6	0.304	70.00	41.73	1237.32	111.80	22.36	1294.42	0.044
	9	0.07	126.66	12.19	1281.92	111.80	22.36	1294.42	infeasible
$A_1 = A_2$	450	0.375	70.00	63.85	1458.52	150.00	30.00	1600.00	0.088
	350	0.346	70.00	53.75	1357.51	132.28	26.45	1379.79	0.069
	250	0.304	70.00	41.73	1237.32	111.80	22.36	1294.42	0.044
	150	0.227	70.00	25.94	1079.95	86.60	17.32	1092.82	0.012
	75	0.09	70.00	6.18	881.80	61.23	12.24	889.89	infeasible
C_{S12}	1	0.331	66.66	42.91	1229.15	111.80	22.36	1294.42	0.050
	2	0.304	70.00	41.73	1237.32	111.80	22.36	1294.42	0.044
	3	0.281	73.33	40.50	1245.07	111.80	22.36	1294.42	0.038
	4	0.261	76.66	39.23	1252.37	111.80	22.36	1294.42	0.032
	20	0.122	130.00	9.452	1274.52	111.80	22.36	1294.42	infeasible

- The sensitivity analysis for the critical value of the substitution rate of item 1 is carried out with different relative purchasing costs (C_1/C_2), inventory cost rate (i), cost of lost sales per unit (π_1), fixed ordering costs ($A_1 = A_2$)and the unit cost of substitution (C_{S12}).
- The critical value of the substitution rate of item 1 decreases as C_1/C_2 decreases (i.e., the cost of item 1 decreases and/or the cost of item 2 increases), which is also evident from Eq. (16). This indicates that as the cost of item 1 decreases and/or the cost of item 2 increases, a smaller number of units of item 1 will be substituted with item 2, and vice versa. Similar observations are made when the cost of lost sales per unit (π_1) or unit cost of substitution (C_{S12}) increases, and so on.

• A sensitivity analysis for the optimal total cost and optimal ordering quantities with a fixed rate of substitution, $\alpha_1 = 0.15$, and with various cost parameters as described above. The last row corresponding to each parameter matches an infeasible solution, because with these values of the parameters, the critical value of the substitution rate is lower than the chosen substitution rate. The relationship $Q_1^*/D_1 \le Q_2^*/D_2$ is not satisfied, which is the necessary condition for case 1. Thus the solution is infeasible for the chosen substitution rate.

6.4. Sensitivity analysis of the decrease in total cost when substitution is possible compared to the case without substitution

The decrease in total cost when substitution is possible compared to the case without substitution for various parameters of the system was investigated (Figs. 6–8).

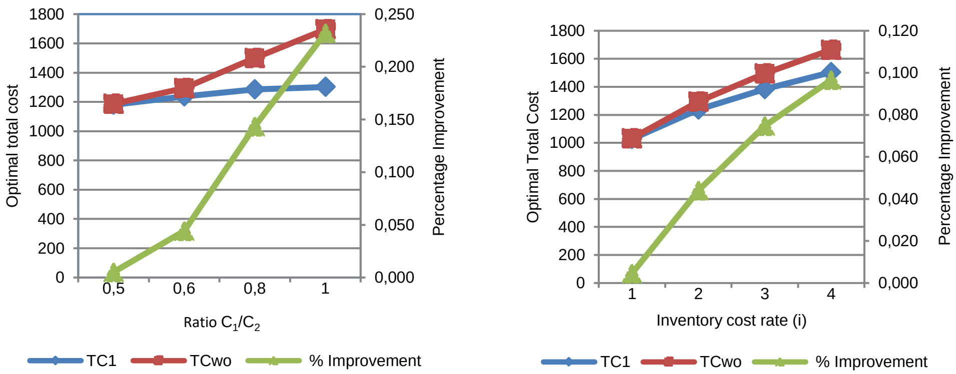

Fig. 6. Decrease in $TC1$ compared to TC_{WO} as a function of the ratio of purchasing costs (C_1/C_2) and inventory cost rate (i)

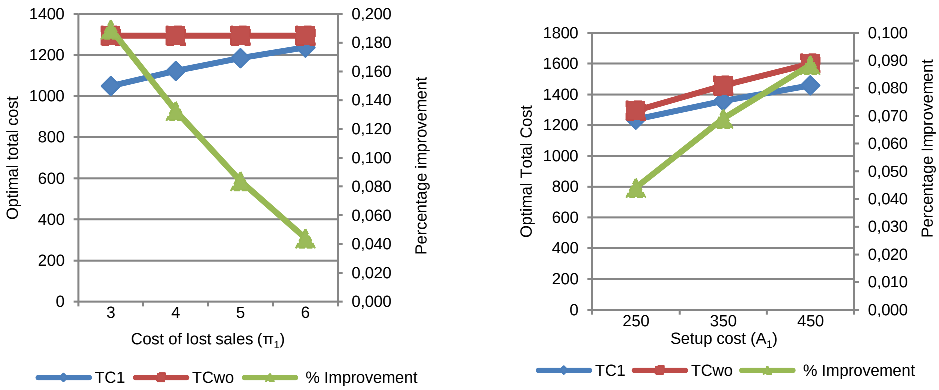

Fig. 7. Decrease in $TC1$ compared to TC_{WO} as a function of the cost of lost sales (π_1) and setup cost A_1 (equal to A_2)

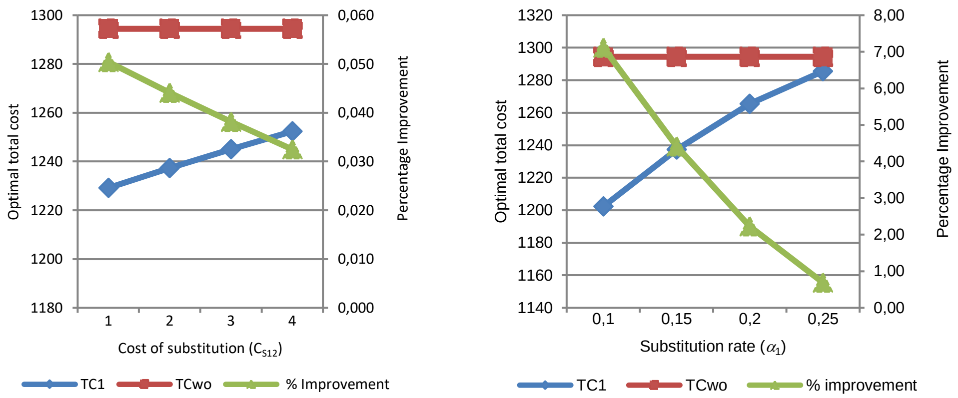

Fig. 8. Decrease in $TC1$compared to TC_{WO}
as a function of the cost of substitution (C_{S12}) and substitution rate (α_1)

While the extent and relative rates may vary, the general nature of the percentage improvement in $TC1$ compared to TC_{WO} follows intuitive reasoning. The findings are presented in Table 4.

Table 4. Total optimal cost when substitution is possible
compared to total cost without substitution

Parameter	Variation	TC_{WO}	$TC1$	Improvement
C_1/C_2	increases	increases	increases	increases
i				
π_1		constant		decreases
A_1		increases		increases
C_{S12}		constant		decreases
α_1				

7. Summary and conclusions

In this paper we have developed a model of an inventory for two mutually substitutable items by taking into account the cost of substitution and assuming jointly procurement items. Three mutually exclusive and exhaustive cases are formulated. Analysing the procedure for obtaining the minimum total cost via calculus, the concept of the critical value of a substitution rate is introduced. Prior knowledge of the critical value helps to minimize the total inventory cost. Numerical experimentation indicates that as we approach this critical substitution rate, the percentage improvement in the total optimal cost decreases and it is no longer beneficial to substitute items beyond this point. Sensitivity analysis for the percentage decrease in the total optimal cost was carried out with respect to important parameters of the system. It is believed that the treatment presented in this paper can be extended to multiple items and to the case of replenishment policies other than joint replenishment policies.

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Appendix A
Proof of the pseudo-convexity of the total cost function. Proof of Theorem 1

The total cost per unit time in the ordering cycle for case 1 (Fig. 1) is (Eq. (6))

$$TC1(Q_1, Q_2) = \frac{D_2 + \alpha_1 D_1}{Q_1\alpha_1 + Q_2}\left(A_1 + A_2 + C_1Q_1 + C_2Q_2 + \frac{1}{2}\frac{iC_1Q_1^2}{D_1}\right.$$
$$+ \frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2 - D_2Q_1^2) + \frac{1}{2}\frac{iC_2(D_1Q_2 - D_2Q_1)^2}{D_1^2(D_2 + \alpha_1 D_1)}$$
$$\left. + \pi_1\left((1-\alpha_1)\frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right) + C_{S12}\left(\alpha_1\frac{(D_1Q_2 - D_2Q_1)}{(D_2 + \alpha_1 D_1)}\right)\right)$$

This equation can be re-written as follows:

$$TC1(Q_1, Q_2) = \frac{TC1^*(Q_1, Q_2)}{\dfrac{Q_1\alpha_1 + Q_2}{D_2 + \alpha_1 D_1}}$$

where

$$TC1^*(Q_1,Q_2)=A_1+A_2+C_1Q_1+C_2Q_2+\frac{1}{2}\frac{iC_1Q_1^2}{D_1}+\frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2-D_2Q_1^2)$$
$$+\frac{1}{2}\frac{iC_2(D_1Q_2-D_2Q_1)^2}{D_1^2(D_2+\alpha_1D_1)}+\pi_1\left((1-\alpha_1)\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1D_1)}\right)$$
$$+C_{S12}\left(\alpha_1\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1D_1)}\right)$$

Clearly, the above equation is the ratio of the term $TC1^*(Q_1,Q_2)$ and a function which is linear in Q_1 and Q_2, $(Q_1\alpha_1+Q_2)/(D_2+\alpha_1D_1)$. Since the ratio of a positive convex function to a linear function is a pseudo-convex function [1], to prove the pseudo-convexity of $TC1(Q_1,Q_2)$, we need to prove that $TC1^*(Q_1,Q_2)$ is a convex function. To show this, we prove that all the principal minors of the Hessian matrix of $TC1^*(Q_1,Q_2)$ are non-negative. The **H**-matrix of the function $TC1^*(Q_1,Q_2)$ is defined as

$$\mathbf{H}=\begin{bmatrix}\dfrac{\partial^2TC1^*(Q_1,Q_2)}{\partial Q_1^2} & \dfrac{\partial^2TC1^*(Q_1,Q_2)}{\partial Q_1\partial Q_2}\\[2ex] \dfrac{\partial^2TC1^*(Q_1,Q_2)}{\partial Q_2\partial Q_1} & \dfrac{\partial^2TC1^*(Q_1,Q_2)}{\partial Q_2^2}\end{bmatrix}$$

Now

$$TC1^*(Q_1,Q_2)=A_1+A_2+C_1Q_1+C_2Q_2+\frac{1}{2}\frac{iC_1Q_1^2}{D_1}+\frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2-D_2Q_1^2)$$
$$+\frac{1}{2}\frac{iC_2(D_1Q_2-D_2Q_1)^2}{D_1^2(D_2+\alpha_1D_1)}+\pi_1\left((1-\alpha_1)\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1D_1)}\right)$$
$$+C_{S12}\left(\alpha_1\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1D_1)}\right)$$

$$\frac{\partial TC1^*(Q_1,Q_2)}{\partial Q_1}=C1+\frac{iC_1Q_1}{D_1}+\frac{iC_2}{D_1^2}(D_1Q_2-D_2Q_1)-\frac{iC_2(D_1Q_2-D_2Q_1)d_2}{D_1^2(D_2+\alpha_1D_1)}$$
$$-\pi_1\left(\frac{(1-\alpha_1)D_2}{(D_2+\alpha_1D_1)}\right)-\frac{C_{S12}\alpha_1D_2}{(D_2+\alpha_1D_1)}$$

$$\frac{\partial TC1^*(Q_1,Q_2)}{\partial Q_2}=C_2+\frac{iC_2Q_1}{D_1}+\frac{iC_2(D_1Q_2-D_2Q_1)}{D_1(D_2+\alpha_1D_1)}+\pi_1\left(\frac{(1-\alpha_1)D_1}{(D_2+\alpha_1D_1)}\right)+\frac{C_{S12}\alpha_1D_1}{(D_2+\alpha_1D_1)}$$

$$\frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_1^2}=\frac{iC_1}{D_1}-\frac{iC_2D_2}{D_1^2}+\frac{iC_2D_2^2}{D_1^2(D_2+\alpha_1D_1)}$$

$$=\frac{iC_1(D_1(D_2+\alpha_1D_1)-iC_2D_2(D_2+\alpha_1D_1)+iC_2D_2^2}{D_1^2(D_2+\alpha_1D_1)}$$

$$=\frac{iC_1D_1D_2+iC_1\alpha_1D_1^2-iC_2\alpha_1D_2D_1}{D_1^2(D_2+\alpha_1D_1)}$$

$$=\frac{iC_{1i}\alpha_1D_1^2+iD_1D_2(C_1-C_2\alpha_1)}{D_1^2(D_2+\alpha_1D_1)}>0$$

Therefore, if $C_1-C_2\alpha_1>0$ or $C_1>C_2\alpha_1$, then the above inequality is satisfied

$$\frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_2^2}=\frac{iC_2}{(D_2+\alpha_1D_1)}>0,\quad \frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_1\partial Q_2}=\frac{iC_2}{D_1}-\frac{iC_2D_2}{D_1(D_2+\alpha_1D_1)}$$

The determinant of the **H**-matrix of $TC1^*(Q_1,Q_2)$ is

$$\left(\frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_1^2}\right)\left(\frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_2^2}\right)-\left(\frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_1\partial Q_2}\right)^2$$

$$=\left(\frac{iC_1\alpha_1D_1^2+iD_1D_2(C_1-C_2\alpha_1)}{D_1^2(D_2+\alpha_1D_1)}\right)\left(\frac{iC_2}{(D_2+\alpha_1D_1)}\right)-\left(\frac{iC_2}{D_1}-\frac{iC_2D_2}{D_1(D_2+\alpha_1D_1)}\right)^2$$

$$=\left(\frac{iC_2}{D_1(D_2+\alpha_1D_1)}\right)\left(\left(\frac{iC_1\alpha_1D_1^2+iD_1D_2(C_1-C_2\alpha_1)}{D_1(D_2+\alpha_1D_1)}\right)-\left(\frac{i\alpha_1^2C_2D_1^2}{D_1(D_2+\alpha_1D_1)}\right)\right)$$

$$=\frac{i^2C_2}{D_1(D_2+\alpha_1D_1)}\left(C_1-C_2\alpha_1\right)>0$$

Therefore, if $C_1-\alpha_1C_2>0$ or $C_1>\alpha_1C_2$, then the **H**-matrix is positive definite. This completes the proof.

Appendix B

Upon partially differentiating the equation for the total cost per unit time, given by Eq. (6), with respect to Q_1 and Q_2, we obtain

$$\frac{\partial TC_1(Q_1,Q_2)}{\partial Q_1}=\frac{D_2+\alpha_1 D_1}{Q_1\alpha_1+Q_2}\left(C_1+\frac{iC_1Q_1}{D_1}+\frac{iC_2}{D_1^2}(D_1Q_2-D_2Q_1)-\frac{iC_2(D_1Q_2-D_2Q_1)d_2}{D_1^2(D_2+\alpha_1 D_1)}\right.$$

$$\left.-\pi_1\left(\frac{(1-\alpha_1)D_2}{(D_2+\alpha_1 D_1)}\right)-\frac{C_{S12}\alpha_1 D_2}{(D_2+\alpha_1 D_1)}\right)$$

$$-\frac{\gamma_1(D_2+\alpha_1 D_1)}{(Q_1\alpha_1+Q_2)^2}\left(A_1+A_2+C_1Q_1+C_2Q_2+\frac{1}{2}\frac{iC_1Q_1^2}{D_1}\right.$$

$$+\frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2-D_2Q_1^2)+\frac{1}{2}\frac{iC_2(D_1Q_2-D_2Q_1)^2}{D_1^2(D_2+\alpha_1 D_1)}$$

$$\left.+\pi_1\left((1-\alpha_1)\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1 D_1)}\right)+C_{S12}\left(\alpha_1\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1 D_1)}\right)\right)$$

$$\frac{\partial TC1(Q_1,Q_2)}{\partial Q_2}=\frac{D_2+\alpha_1 D_1}{Q_1\alpha_1+Q_2}\begin{pmatrix} C_2+\dfrac{iC_2Q_1}{D_1}+\dfrac{iC_2(D_1Q_2-D_2Q_1)}{D_1(D_2+\alpha_1 D_1)} \\ +\pi_1\left(\dfrac{(1-\alpha_1)D_1}{(D_2+\alpha_1 D_1)}\right)+\dfrac{C_{S12}\alpha_1 D_1}{(D_2+\alpha_1 D_1)}\end{pmatrix}$$

$$-\frac{(D_2+\alpha_1 D_1)}{(Q_1\alpha_1+Q_2)^2}\left(A_1+A_2+C_1Q_1+C_2Q_2+\frac{1}{2}\frac{iC_1Q_1^2}{D_1}\right.$$

$$+\frac{1}{2}\frac{iC_2}{D_1^2}(2D_1Q_1Q_2-D_2Q_1^2)+\frac{1}{2}\frac{iC_2(D_1Q_2-D_2Q_1)^2}{D_1^2(D_2+\alpha_1 D_1)}$$

$$\left.+\pi_1\left((1-\alpha_1)\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1 D_1)}\right)+C_{S12}\left(\alpha_1\frac{(D_1Q_2-D_2Q_1)}{(D_2+\alpha_1 D_1)}\right)\right)$$

By solving the simultaneous equation

$$\frac{\partial TC_1(Q_1, Q_2)}{\partial Q_1} = 0, \quad \frac{\partial TC_1(Q_1, Q_2)}{\partial Q_2} = 0$$

we obtain the following optimal values of Q_1 and Q_2

$$Q_1^* = \frac{D_1(C_{S12}\alpha_1 + \pi_1(1-\alpha_1) - (C_1 - C_2\alpha_1))}{i(C_1 - C_2\alpha_1)}$$

$$\begin{aligned} Q_2^* = \Big(\Big(& D_1\alpha_1 C_2 ((C_1 - C_2\alpha_1) - C_{S12}\alpha_1 - \pi_1(1-\alpha_1)) + \left(C_2 (C_1 - C_2\alpha_1)(D_2 + \alpha_1 D_1) \right)^{1/2} \\ & \left(C_{S12} D_1 \alpha_1 (2(C_1 - C_2\alpha_1) - C_{S12}\alpha_1) + 2\pi_1 D_1 (C_1 - C_2\alpha_1)(1-\alpha_1) \right. \\ & - \pi_1^2 D_1 (1-\alpha_1)^2 - 2C_{S12}\pi_1 D_1 \alpha_1 (1-\alpha_1) - D_1 (C_1 - C_2\alpha_1)^2 \\ & \left. + 2i(A_1 + A_2)(C_1 - C_2\alpha_1) \right)^{1/2} \Big) \left(iC_2 (C_1 - C_2\alpha_1) \right)^{-1} \end{aligned}$$

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