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TAIL INDEPENDENCE IN EXTREME VALUE MODELS – AN APPLICATION FOR EAST AND CENTRAL EUROPE STOCK EXCHANGE MARKETS[1]

Summary: The concept of tail dependence represents the current standard to describe the amount of extremal dependence. While *extreme value theory* allows for constructing estimators of the tail dependence coefficient, tests for tail independence are indispensable when working with tail dependence, since all estimators of the *tail dependence coefficient* are strongly misleading when the data does not stem from a tail dependent setting.The main aim of this paper is to compare the power of the extreme-value dependence tests which are based on extreme value theory (a log-likelihood ratio test and goodness of fit tests) on original time series from stock exchange markets from East and Central Europe.

Keywords: tests for independence, tail dependence coefficient, extreme-value theory (EVT), copula.

1. Introduction

Tail independence – which is also known as asymptotic independence or extreme independence – exists in many applications, especially in financial time series analysis. Not taking this dependence into account may lead to misleading results. Tail independence is described via the *tail-independence coefficient* introduced by Sibuya [1960]. Extreme value theory is the natural choice for inferences on extreme values.

In this paper only some basic concepts of tail independence, estimation of tail independence coefficient and copulas are presented assuming that these issues are described in details in recent papers [Embrechts et al 1997, 2003; Poon et al 2004; Frahm et al 2005]. The most important purpose of the paper is to emphasize the importance of testing tail-independence. On the basis of empirical data we present the result of examination of the extreme-value dependence tests which were presented by Falk and Michel [2006] and considered by them only on simulation data.

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2. Preliminary data analysis

In this study we examined the behavior of four indices from Central and East Europe stock exchange markets, namely Polish WIG20, Hungarian BUX, Russian RTS, Czech PX50.[2] There are a number of papers studying the co-movements of international equity markets (US, UK, Germany, Japan, France etc.), but there is little research that studies dependences in Central and Eastern Europe stock markets. The results of such research have important implications for both global investment management and asset pricing modeling. Central and Eastern European markets can become attractive option for global investors who want to diversify their portfolios internationally.

As previously found in other studies, returns exhibit excess kurtosis and negative skewness. Plots (1a-1d) presents indices prices and the original log-returns.

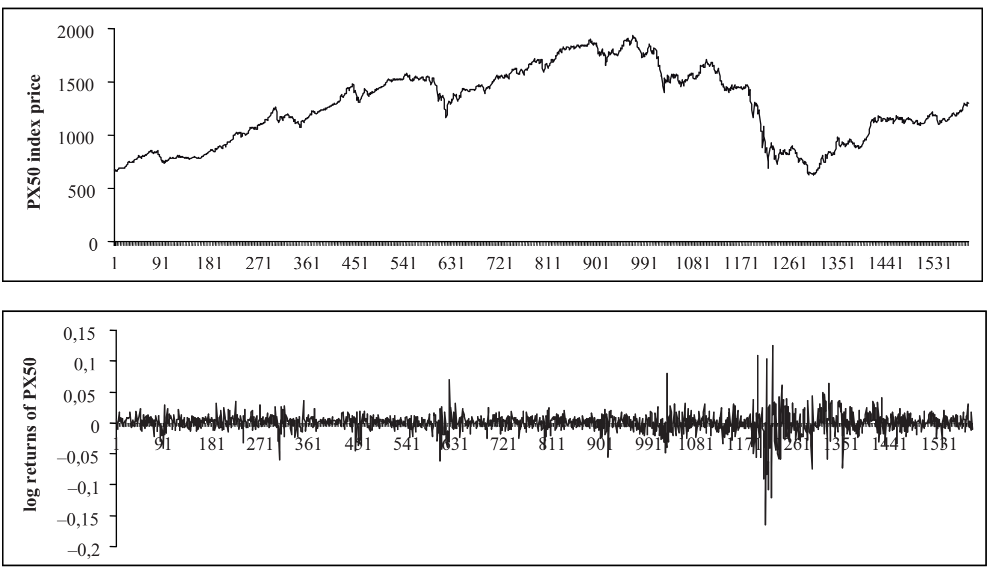


Figure 1a. PX50 Index price for period January 6th 2004 to April 29th 2010 (1600 data points) and the original log returns

Source: own calculations.

[2] For all indices, we compute daily log-returns data.

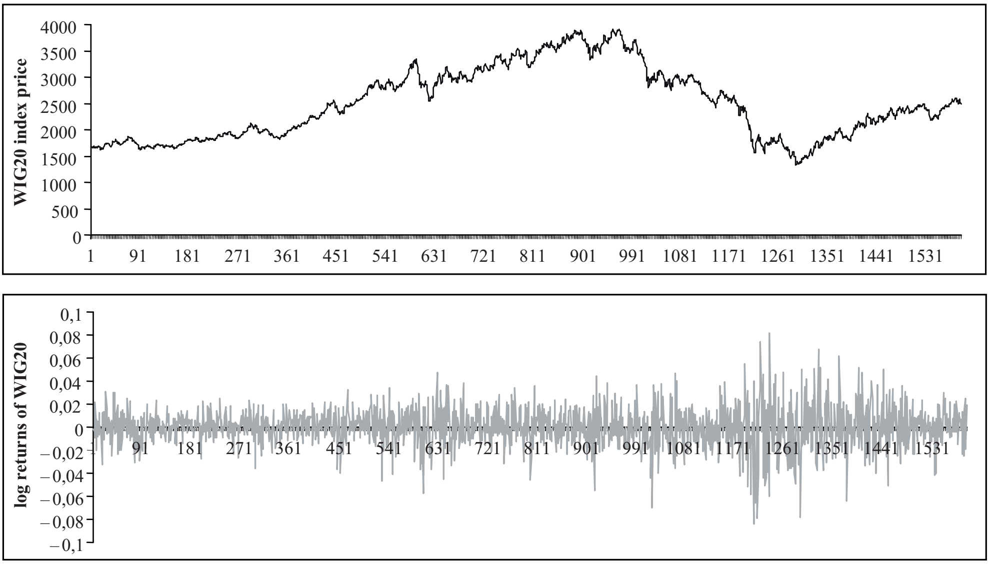


Figure 1b. WIG20 Index price for period January 6th 2004 to April 29th 2010 (1600 data points) and the original log returns

Source: own calculations.

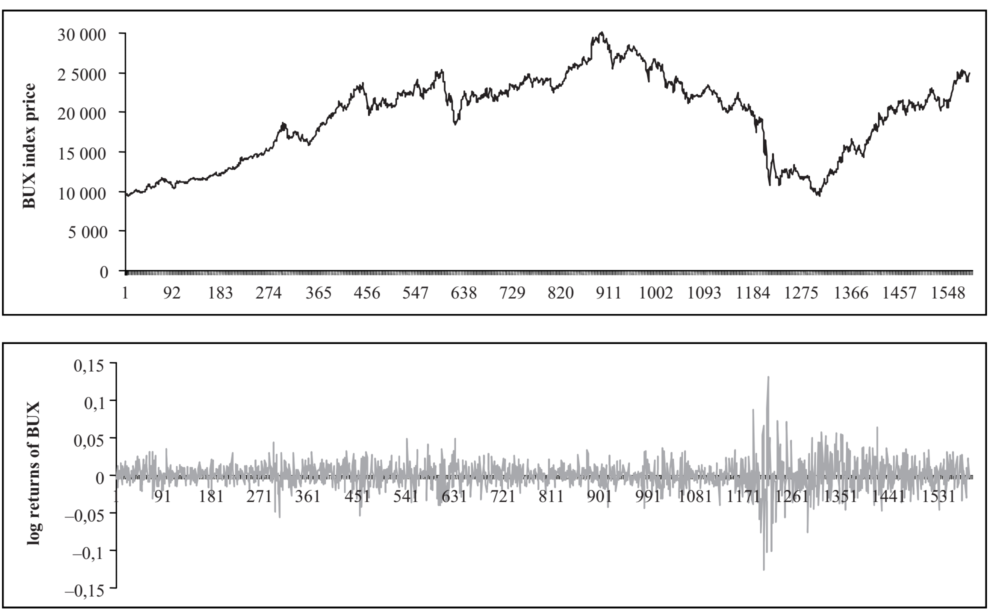


Figure 1c. BUX Index price for period January 6th 2004 to April 29th 2010 (1600 data points) and the original log returns

Source: own calculations.

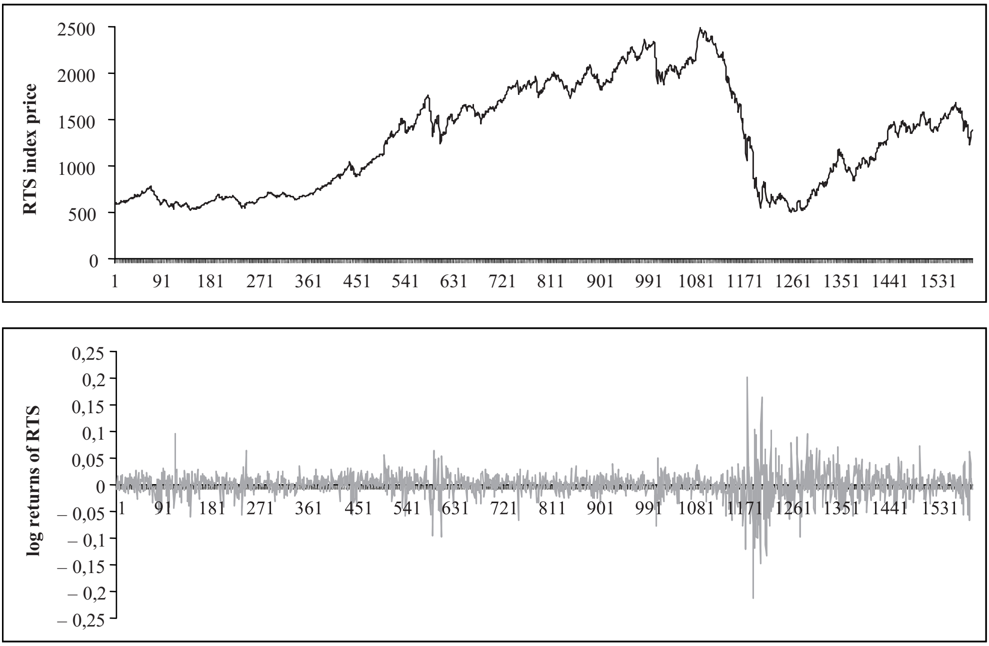


Figure 1d. RTS Index price for period January 6th 2004 to April 29th 2010 (1600 data points) and the original log returns

Source: own calculations.

3. Tail dependence basic concept (tdc)

Sibuya [1960] introduced tail independence between two random variables with identical marginal distributions. De Haan and Resnick [1977] extend it to the case of multivariate random variables. The TDC can be defined via the notion of copula, introduced by Sklar [1959]. A copula *C* is a cumulative distribution function whose margins are uniformly distributed on [0.1]. As shown by Sklar [1959], the joint distribution function *F* of any random pair (X, Y) with marginals F_x and F_y can be represented as $F(x,y)=C(F_x(x),F_y(y))$ in terms of a copula *C* which is unique when F_x and F_y are continuous [Nelsen 1999 or Joe 1997].

Therefore,, if C is the copula[3] of (X, Y), then the definition of tail independence and tail dependence between two random variables is given below.

The coefficient of upper tail-dependence of X and Y is defined as:

$$\lambda_u = \lim_{v\uparrow 1} P(Y > F_Y^{-1}(v) \mid X > F_X^{-1}(v)) = \lim_{v\uparrow 1} \frac{1-2v+C(v,v)}{1-v} \tag{1}$$

[3] The most frequently used copulas in finance field are: archimedian (Gaussian, t-student, Clayton).

Similarly, the coefficient of lower tail-dependence:

$$\lambda_l = \lim_{v \downarrow 0} P(Y \le F_Y^{-1}(v) \mid X \le F_X^{-1}(v)) = \lim_{v \downarrow 0} \frac{C(v,v)}{v}. \tag{2}$$

If λ_u $(\lambda_l \in (0,1])$ X and Y are said to be asymptotically dependent in the upper (lower) tail. If $\lambda_u = 0$ $(\lambda_l = 0)$ they are asymptotically independent in the upper (lower) tail.

Recent attention given to the statistical properties of asymptotically independent distributions is largely a result of a series of articles by Ledford and Tawn [1996, 1997, 1998]. Coles et al. [1999] give an elementary synthesis of the theory.

4. Estimation of the TDC basic concept

Frahm et al. [2005] give estimators for the TDC under different assumptions: using a specific distribution (e.g. t-distribution), within a class of distributions (e.g. elliptically contoured distributions), using a specific copula (e.g. Gumbel), within a class of copula (e.g. Archimedean) or a nonparametric estimation (without any parametric assumption). The authors compare the performance of different estimators for different cases: whether the assumption is true or wrong and whether there is tail dependence or not. It turns out that some of the estimators perform well if there is tail dependence, but bad if there is not. In practical applications, one will never know which copula model is the correct one. The estimation can only be under misspecification. In the present paper, no parametric assumptions are made for the copula and the marginal distribution functions. TDC estimates are obtained from the empirical copula $\hat{C}$. Empirical counterparts of λ_u and λ_l can be obtained by plugging the empirical copula into Eqs. (1) and (2):

$$\hat{C}\left(\frac{i}{n},\frac{j}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{u_k \le u(i), v_k \le v(j)\}};\ i,j = 1,2,\ldots,n, \tag{3}$$

where $u_{(1)} \le u_{(2)} \le \ldots \le u_{(n)}$ and $v_{(1)} \le v_{(2)} \le \ldots \le v_{(n)}$ are the order statistics.

Difficulties in selecting a proper estimation method brings us to the important issue of testing for the tail independence.

5. Tail independence tests

One of the most interesting approaches for testing for tail independence is given in Falk and Michel [2006]. They prove the following theorem:

With $c \to 0$*, we have uniformly for* $t \in [0,1]$:

$$P(X+Y>ct \mid X+Y>c)=\begin{cases} t^2; & \textit{there is no tail dependence} \\ t; & \textit{else} \end{cases}.$$

Using this theorem, Falk and Michel propose four different tests for tail independence, which can be grouped into two different classes: a Neymann-Pearson test (NP) and three goodness of fit tests: Fisher's κ, Kolmogorov-Smirnov and χ^2.

Suppose now that we have n independent copies $(X_1,Y_1), ..., (X_n,Y_n)$ of (X,Y). The marginal distribution is assumed to be reverse exponential (i.e. $F(x,0)=F(0,x)=\exp(x)$). Fix a threshold $c<0$ and consider $E=\{C_i=X_i+Y_i;\ C_i>c\}$. Let $k(n)=\#E$ and define $V_i=C_i/c \quad \forall i=1, ..., k(n)=m$. According to the above theorem we test the hypothesis:

$$H_0: F_0(t)=t^2 \text{ vs. } H_1: F_1(t)=t.$$

Neyman-Pearson test (NP)

The first test is suggested by the Neyman-Pearson lemma. The NP test considers the distribution function of V_i and tests whether it is more likely from $F_{(0)}(t)=t^2$ or $F_{(1)}(t)=t$. The test statistic for testing $F_{(0)}$ against $F_{(1)}$ is (for fixed n):

$$T_{NP} := \log\left(\prod_{i=1}^{k(n)} \frac{1}{2V_i}\right) = -\sum_{i=1}^{k(n)} \log(V_i) - k(n)\log(2).$$

$F_{(0)}$ is rejected when T_{NP} gets large, if the approximate p-value $p_{NP} := \Phi(k(n)^{-1/2}\sum_{i=1}^{k(n)}(2\log(V_i)+1))$ is too close to 0, typically if $p_{NP} \le .05$; Φ – standard normal df.

Kolmogorov-Smirnov test (KST)

A different possibility of using Falk and Michel [2006] theorem is to carry out a goodness-of-fit test, in this case using the Kolmogorov-Smirnov test. Therefore, define conditional on $K(n)=m$: $U_i := F_c(C_i/c) = \dfrac{1-(1-C_i)\exp(C_i)}{1-(1-c)\exp(c)}$, $\forall i \in \{1, ... m\}$.

Denote $\hat{F}_m(t) = \dfrac{1}{m}\sum I_{[0,t]} C_i$ the ecdf of U_i, $i=1, .., m$. The Kolmogorov test statistic is then: $T_{KS} := \dfrac{1}{m}\sup_{t\in[0,1]}\left|\hat{F}_m(t)-t\right|$.

The approximate p-value is $p_{KS} = 1 - K(T_{KS})$, where K is the cdf of the Kolmogorov distribution. According to a rule of thumb given by the authors: for $m > 30$, tail independence is rejected if $T_{KS} > c_{0,05} = 1,36$.

Fisher's *κ* test

Next, we consider Fisher's *κ* test based on C_i / c for tail independence of *X* and *Y*. The random variables $U_i \quad \forall i \in \{1, ..., m\}$ are independent and uniformly distributed on (0, 1), if *X* and *Y* are tail independent and *c* is close to 0.

Consider the corresponding order statistics $U_{1:m} \leq ... \leq U_{m:m}$ and denote $S_j := U_{j;m} - U_{j-1;m}, 1 \leq j \leq m+1$. κ_m is the Fisher's *κ* statistic, conditional on $K(n) = m > 0$:

$$\kappa_m := (m+1) M_n$$

where

$$M_m := \max_{j \leq m+1} S_j .$$

$F_{(0)}$ is rejected when $p_\kappa := 1 - G_{m+1}\left(\frac{\kappa_m}{m+1}\right) = 1 - G_{m+1}(M_m)$ with $G_{m+1}(x) = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (\max(0, 1 - jx))^m, \quad x > 0$ is small, typically if $p_\kappa \leq .05$.

Chi-square test

The last test is the chi-square goodness-of-fit test applied to $U_i \quad \forall i \in \{1, ..., m\}$ conditional on $K(n) = m > 0$. Divide the interval $[0, 1]$ into *k* consecutive and disjoint intervals $I_1, ..., I_k$. The test statistic is:

$$\chi^2_{m,k} := \sum_{i=1}^{k} \frac{(m_i - mp_i)^2}{mp_i},$$

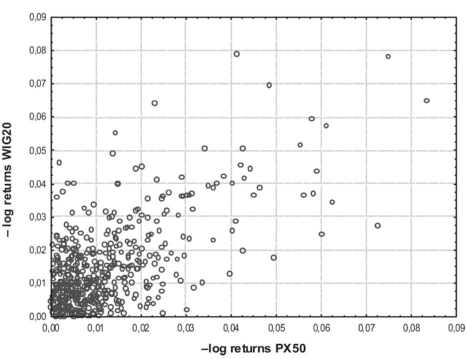
where m_i is the number of observation among $U_i, \quad i = 1, ..., m$ that falls into the interval I_i, p_i is the length of I_i $(1 \leq i \leq k)$. $F_{(0)}$ is rejected when $p_{\chi^2} := 1 - \chi^2_{k-1}(\chi^2_{m,k})$ is small.

The main problem is choosing the threshold *c*. Numerous simulations which Falk and Michel carried out indicate that the Neyman-Pearson test has the smallest type II error rate, closely followed by the Kolmogorov-Smirnov test and the chi-square test, whereas Fisher's κ almost fails. The NP does not, however, control the type I error rate if the c is too far away from 0. The other three tests control the type I error rate for any c.

6. Estimation of the TDC and testing for tail independence – empirical analysis

The figures below (Figures 1-6) graphically summarize the tail dependence properties of six financial data-sets. We provide the scatter plots of daily negative log-returns of the each pair of analyzed indices and compare them to the corresponding tail independence coefficient estimate of (2) for various k. For the purpose of estimation we used empirical copula. Both plots (for each pair of indices) give an intuition for the presence of tail dependence and the order of magnitude of the tail dependence coefficient. For modeling reasons we assume that the daily log-returns are *iid* observations. All plots related to the estimation of the tail dependence coefficient show the typical variance-bias problem for various k. In particular, a small k comes along with a large variance of the estimate, whereas an increasing k results in a strong bias. In the presence of tail dependence, such k is chosen that the tail independence coefficient estimate $\hat{\lambda}_l$ lies on a plateau between the decreasing variance and the increasing bias (results are presented in Table 1).

For the PX50 and WIG20 pair the estimates of lower-tail dependence coefficient is the highest – around 0.44 (as shown in Figure 2) for k between 40 and 43. The smallest lower-tail dependence coefficient is for BUX and WIG20 pair.



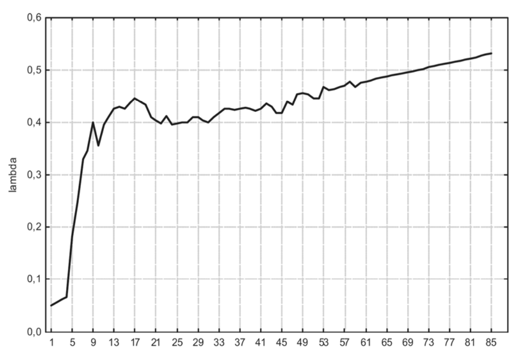


Figure 2. Scatter plot of PX50 versus WIG20 log-returns and the corresponding tail dependence coefficient estimate lambda for various k

Source: own calculations.

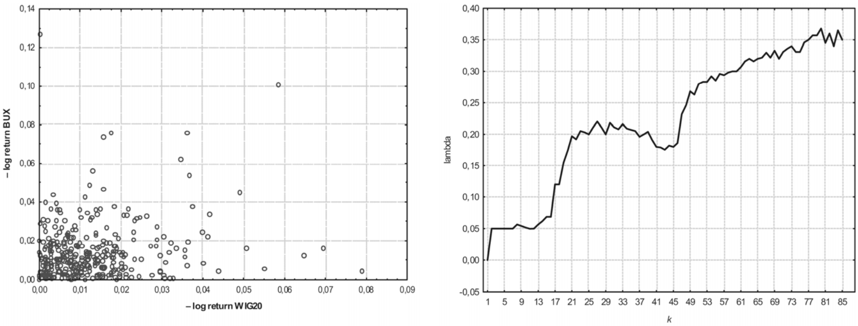


Figure 3. Scatter plot of BUX versus WIG20 log-returns and the corresponding tail dependence coefficient estimate lambda for various *k*

Source: own calculations.

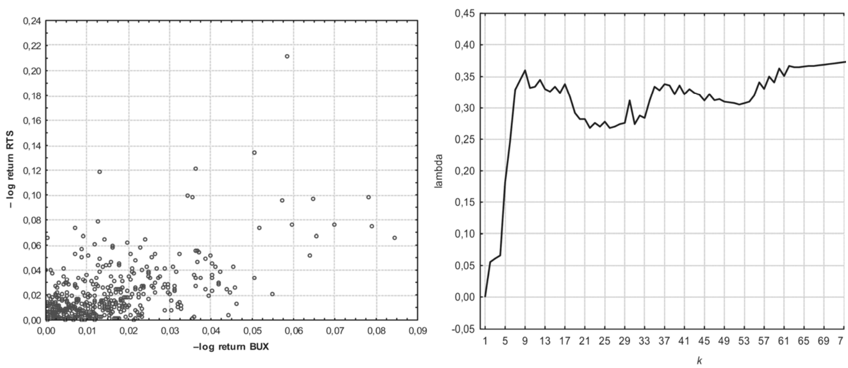


Figure 4. Scatter plot of BUX versus RTS log-returns and the corresponding tail dependence coefficient estimate lambda for various *k*

Source: own calculations.

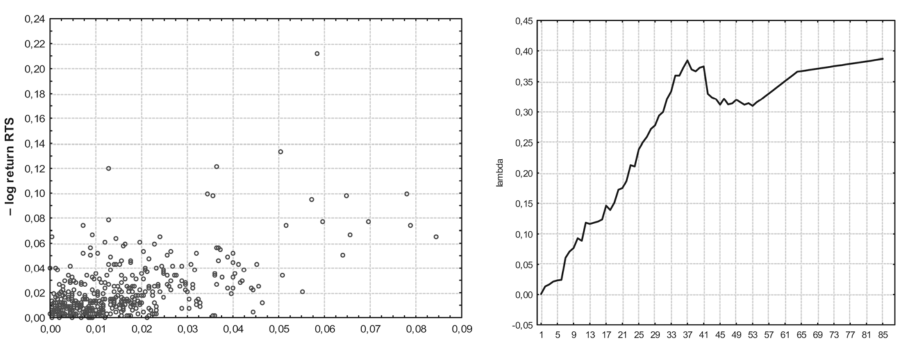


Figure 5. Scatter plot of WIG20 versus RTS log-returns and the corresponding tail dependence coefficient estimate lambda for various *k*

Source: own calculations.

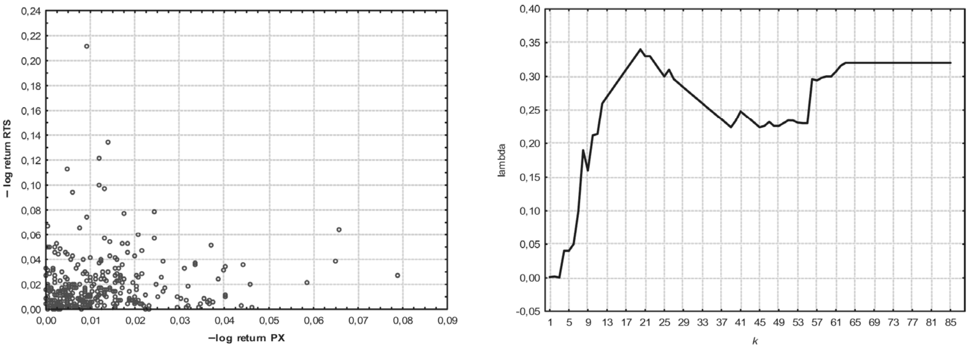


Figure 6. Scatter plot of PX50 versus RTS log-returns and the corresponding tail dependence coefficient estimate lambda for various *k*

Source: own calculations.

Table 1. Estimates of tail-dependence coefficient for each pair based on their empirical copulas with various number of threshold *k*

Paired return	Interval for *k*	λ_L
PX ver RTS	(50.55)	0.23
BUX ver RTS	(49.54)	0.30
WIG20 ver RTS	(52.55)	0.31
PX ver WIG20	(40.43)	0.44
BUX ver WIG20	(43.46)	0.17

Source: own calculations.

Table 2 reports the computation of all four described earlier tail-dependency tests for pairs of indices. As we see in Table 2 extreme-value independency is rejected for all pairs in left tail at the 5 percent significance level. These estimates of left-tail dependence coefficients are fairly close to those reported above in Table 1. Therefore, all tests suggest that there is tail dependence in pairs.

Our next step consists of fitting a suitable copula to the data. Table 3 shows the result of fitting the symmetrized Joe-Clayton, normal, Gumbel, Frank and t-student copula to all pairs of joint standardized data using maximum likelihood (as for each paired returns all tests suggest lower-tail dependency). Therefore, a t-student and symmetrized Joe-Clayton seem a suitable choice in our case.

The last step is an examination of the power of the extreme-value dependence tests. In order to examine this issue we carry out Monte Carlo experiments. Each of the experiments consists of generating two returns series of 1000 observations each from GARCH(1, 1) processes, whose joint behavior is assumed to be adequately represented by a SJC copula and t copula with 5 degrees of freedom. Each of the

Monte Carlo experiments is repeated 100 hundred times and tests: Neyman-Pearson, Kołmogorov, chi-square and Fisher are computed for the lower tails at each iteration. Our results are reported in Table 4.

Table 2. Extreme-value independency tests using 4 statistics for each pair

Paired return	Optimal threshold c	λ_L *	p-value			
			np	fisher	ks	chi-square
PX ver RTS	–0.05105	0.2658	0.001	0.021	0.002	0.000
BUX ver RTS	–0.04997	0.3726	0.024	0.037	0.042	0.031
WIG20 ver RTS	–0.05174	0.3029	0.033	0.038	0.018	0.029
PX ver WIG20	–0.04427	0.4917	0.038	0.005	0.044	0.015
BUX ver WIG20	–0.04023	0.1237	0.021	0.041	0.028	0.034

* The non-parametric way of choosing the optimal threshold level estimate the Generalized Pareto Distribution parameters corresponding to various threshold levels, representing respectively 1%, 2%, 3%,.....till 12% of the extreme observations.

Source: own calculations.

Table 3. Fitting copula to the original log-returns of indices

Paired return	Best copula**
PX ver RTS	t-student
BUX ver RTS	SCJ
WIG20 ver RTS	SCJ
PX ver WIG20	SCJ
BUX ver WIG20	t-student

** "Best" copula criterion: consider the distance (based on the discrete L^2 norm) between each considered copula (symmetrized Joe-Clayton, normal, Gumbel, Frank and t-student) and empirical copulas.

Source: own calculations.

Table 4. Simulation of rejection rate of H_0: *tail independence* (5% SIGNIFICANCE LEVEL)

Copula	Np test	Fischer test	Ks test	Chi-square test
sjc	95	56	98	86
t-student	100	67	100	90

Source: own calculations.

The results in Table 4 show that power of Neyman-Pearson and Kolmogorov-Smirnov tests based on a t-student copula approaches 1. That is, the false null hypothesis is virtually always rejected. The chi-square test exhibits the lowest power.

7. Conclusion

Testing tail independence is simple and transparent enough to be implemented and easily monitored. Omitting the test for tail independence would introduce a large bias in the estimation and make it difficult to decide whether there is just correlation or in fact tail dependence. One important feature of this paper is the implementation of the tests for tail independence, which is recognized to be indispensable but rarely utilized in a financial context. The most important conclusions are:
– existence of dependence between Poland–Czech, Poland–Hungary stock markets (the strongest between Poland–Czech),
– Polish, Czech and Hungarian equity markets are dependent on the Russian market (as the largest financial market in consideration),
– NP and KS correctly reject the null hypothesis of extreme-value independence in the left tail (one hundred percent of time for t-student copula).

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NIEZALEŻNOŚĆ W OGONACH W MODELACH WARTOŚCI EKSTREMALNYCH – ZASTOSOWANIE NA GIEŁDACH PAPIERÓW WARTOŚCIOWYCH WSCHODNIEJ I CENTRALNEJ EUROPY

Streszczenie: Koncepcja zależności w ogonach rozkładu stanowi obecny trend w ocenie siły ekstremalnych zależności. O ile teoria wartości ekstremalnych pozwala na konstruowanie estymatorów współczynnika zależności w ogonach, o tyle niezbędnym elementem jest testowanie tejże niezależności ze względu na fakt, iż estymatory współczynnika zależności w ogonach mogą prowadzić do błędnych wniosków. Celem artykułu jest analiza porównawcza wybranych testów niezależności w modelach rozkładów wartości ekstremalnych (test oparty na ilorazie wiarygodności oraz testy dopasowania dobroci). Na podstawie rzeczywistych stóp zwrotu wybranych indeksów z parkietów giełdowych centralnej i środkowej Europy sprawdzimy, który z rozważanych testów wykazuje najwyższą moc w testowaniu zależności ekstremalnych.

Słowa kluczowe: testowanie niezależności, współczynnik zależności w ogonie rozkładu, teoria wartości ekstremalnych, funkcja połączeń (copula).