IWAN G. TABAKOW

DISCRETE MATHEMATICAL STRUCTURES

PART I: PROPOSITIONS, PREDICATES AND SETS

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In memory of my parents:
Nikolina and Georgi.
In memory of my teachers:
prof. dr Ludwik Borkowski
and prof. dr Jerzy Bromirski

Contents

"There is no any mathematical abstraction which would not be applicable at an earlier time or later time in practice".
Nikolai Ivanovich Lobachevsky (1792 – 1856)

Preface

Knowledge is the most valuable attainment of the human civilisation. In fact, without knowledge there is no any science, and vice versa. And hence, knowledge and science can be considered as elements of some discrete sequential process. Obviously, the initial state of any such process is knowledge. *Wisdom* is a reasonable principle based on existing humans knowledge. A more formal treatment of the last notion should be very difficult, but in the broad sense it may be described e.g. as "a habit or disposition to perform the action with the highest degree of adequacy under any given circumstance" (*The Free Encyclopaedia, The Wikimedia Foundation, Inc*). However, wisdom should be associated with some good thing (i.e. disinterested goodness, etc.) in the same way as beauties should not be useful. Moreover, wisdom should not involve contradictory interpretations (e.g. such as: a peace war, peace bombardment, humanitarian intervention, twofold standards, privatisation or distortion of historical, scientific or other facts and so on) . It should be an alternative to "technical rationality" and hence, should involves knowledge of what is good or bad. And also, it should not be time restricted, as an example the following sentence: "Much do not means well, but well means much" (Socrates 469/70 – 399 b.c.) or also: " It is not sufficient to know very many for being wise" (Heraclitus of Ephesus, 535 – 475 b.c.). In particular, such notions as: wisdom and ethical goodness were also fundamental in Confucius' philosophy (Kǒng Fūzî, 551 – 479 b.c., near Qufu, China): "Do not do to others what you do not want done to yourself".

Logic is the science and art which so directs the mind in the process of reasoning and subsidiary processes as to enable it to attain clearness, consistency and validity in those processes (Turner W. 1999). The term '*logic*' can be considered into two aspects. The first one describes the use of valid reasoning in some activity while the second names the science or study of reasoning. In the latter sense, it features most prominently in the subjects of philosophy, mathematics, and computer science (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*).

The science of computers, computer systems and their applications (often called in short "*computer science*" or also "*theoretical and applied informatics*") is a very young discipline starting together with the development of the first electronic computers*. Sometimes computer science is erroneously interpreted as e.g. "computer hardware science" or also "computing (or computation) science". In fact, a computation in the broad sense can be realised without using any computers, e.g. the Euclid´s algorithm (computation of the greatest common divisor of two integers a and b), computation of $\sqrt{x}$ ($x \geq 0$), known in ancient times, etc. But in the narrow sense, any *computation* is introduced as a process, i.e. a strongly connected sequence of computer memory states realised under some subset of computer instructions. It can be observed that without computers there is no any sense of using the term '*informatics*'. On the other hand, today's computer systems became a very important and at the same time necessary condition for any scientific investigation.

The problem-solving emphasis of computer science borrows heavily from the areas of mathematics and logic. Faced with a problem, computer scientists must first formulate a solution. This method of solution, or *algorithm* as it is often called in computer science, must be thoroughly understood before the computer scientists make any attempt to implement the solution on the computer (Lambert K.A. et al. 1996). Any using of (digital) computers in resolving mathematical tasks would require a discretisation of the considered domain. And so, the involved computational process, i.e. a sequence of computer memory states, is discrete (and finite, if the corresponding algorithm would be convergent).

* Electronic computers were initially developed in the 1940s by the American physicist John Vincent Atanasoff, son of a Bulgarian immigrant (1903 – 1995). In fact, Atanasoff designed and built the first electronic, digital computer (non-programmable).

Discrete mathematical structures (in short: *discrete structures*), in particular such as mathematical logic and set theory, algebraic systems, formal languages, automata theory, graphs, number theory, coding theory, combinatorial analysis, discrete probability theory, Petri nets and so on, underpin a large amount of modern computer science[*]. Discrete structures became a very large and dynamic science. Unfortunately, the speedy developments and knowledge in this area makes impossible the presentation of all notions, definitions and applications used here.

The subjects of this part are *propositional* (or equivalently: *sentential*) *calculus*, *first-order predicate* (or equivalently: *quantifier*) *calculus*, and *set theory*. The last three topics can be considered as basic, having now sufficiently large applications in computer science theory. Without this “surgical instrumentarium” any more formal treatment of definitions or descriptions, various properties or theorem proofs would not be complete. Some elements of classical mathematical logic and set theory are first introduced. Then non-classical logic systems and sets are also presented.[†]

Some basic notions concerning: operations and algebraic systems, lattices, Boolean, multiple valued and fuzzy algebras, homomorphisms of algebraic systems (i.e. epimorphism, monomorphism, isomorphism, endomorphism and automorphism), congruencies, quotient algebraic systems, finite direct products of algebraic systems and free algebraic systems, grammars and sequential machines, algorithms, computability, recursion, graph theory, combinatorial analysis, probability theory, Markov’s chains, number theory, information, coding and algorithm complexity will be briefly considered in Part II of this book.

In general, the *natural deduction* in the broad sense can be considered as a way of the human thinking and inferenc. But in the narrow sense, i.e. in senso stricto, it corresponds to the assumptional system style (as a natural way of imitation of any such thinking and inference)[‡]. In fact, this is the most generally known and important approach and any other automated (algebraic or “artificial”) approach can be described by this one, but not vice versa. Moreover, the last approaches seem to be very difficult in using for non-classical systems. The natural deduction style is the most preferable and at the same time required in any scientific research. Moreover, the proofs in the above assumptional systems are nearly the same as in the case of usual mathematical proofs or other scientific reasoning. And hence, in accordance with the above advantages of these systems, they sometimes are known under the name “natural deduction”. In the case of propositional and predicate logic systems, the existing assumptional and axiomatic system styles are theoretically equivalent. But the main problem of any axiomatic system is related to the time and space effectiveness of the corresponding proofs. This is because they are used here only two rules, i.e. the rule of detachment for implication and the rule of replacement. And hence, from the practical point of view, we have a different provability effectiveness related to the above two styles. An additional advantage of the assumptional system style is the possibility of constructing different proofs for the same thesis and hence, the possibility of studying different proof techniques. Finally, the assumptional and the axiomatic system styles theoretically are having the same modelling power, but having different decision powers (related to concrete scientific research, e.g. as in Turing machines and high-level Petri nets[§]). And so, the assumptional system approach is used in this book. Usually a non-classical propositional logic involves a non-classical predicate logic, and hence – a non-classical set theory. Here, some such approaches are also presented. The material considered was selected mainly based on further investigations in computer science.

At the beginning in Chapter I, the classical propositional[**] calculus is presented. The logical calculi considered here are based on a system of rules, which define the methods used in proofs from assumptions. The methodological problems in the deductive sciences are given only an elementary treatment (Słupecki J. and Borkowski L. 1967). The last problems such as consistency, completeness, independence, etc. of a given system arise when a given axiomatic system is considered (in ways analogous to assumptional systems). Obviously, the most important of these properties is the consistency of a system. Next, a short review of selected axiomatic systems is presented and the proof of the

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[*] In contradistinction to the numerical analysis, i.e. the study of algorithms for the problems of continuous mathematics (as distinguished from discrete mathematics).

[†] In general, the existing logic systems may be classified as *formal* or *non-formal* (or “*intuitive*”: e.g. tracing somebody’s steps: a perfect system of the Australian Aborigines). We shall study only formal logic systems. The last ones may be classified as *classical* or *non-classical*. On the other hand, the non-classical systems can be classified as *partially classical* (or: “*anti-classical*”) and extended (or “*extra-classical*”), e.g. see: (Burgess J.P. 2009). Provided there is no ambiguity, the term ‘*mathematical logic*’ is sometimes used to denote the applications of formal logic in mathematics.

[‡] Natural deduction is a *deductive reasoning* as distinct from *abductive reasoning* (a form of logical inference based on observations).

[§] Alan Mathison Turing (1938: 1912 – 1954), Carl Adam Petri (1962: 1926 – 2010).

[**] From the Latin: ‘*propositio*’.

well-known deduction theorem is given (a summary presentation under Słupecki J. and Borkowski L. 1967). The last theorem can be considered as a basis of introducing assumptional system style. Some elements concerning direct reasoning and automated deduction methods are related to the sequent presentation. As it is shown, any Gentzen's rule can be represented as a corresponding thesis in the above-considered calculus. In the second section of this chapter some non-standard logic systems are briefly considered, such as many-valued, fuzzy, modal, deontic and temporal logics (Zadeh L.A. 1974, Manna Z. and Pnueli A. 1992 and 1995, Carson J. 1995,Gotwald S. 1995, Suber P. 1997, Hájek P. 1998, Navara M. 2000, etc.). Here, the natural deduction methods are extended to some non-classical systems, e.g. such as many-valued, fuzzy, modal, deontic and temporal logic systems. The Gentzen's sequent calculus is extended to Łukasiewicz's Ł-BL systems. In particular, it is shown that any fuzzy propositional formula provable under Hájek's axioms of the logic BL is also provable under the above-proposed approach, i.e. using proofs from assumptions. This approach seems to be more attractive, more simpler and natural in practical use than the axiomatic one. Moreover, there are introduced new t-norm and t-conorm and then it is defined a generalised Łukasiewicz's system, denoted by $Ł_{\alpha}$-BL ($\alpha > 0$), where the previous one becomes a particular case with $\alpha = 1$. The used fuzzy implication is specified as a residuum of the above t-norm (which is left-continuous) and hence, the last implication is unique. In particular, it is also shown that the generalised Łukasiewicz's fuzzy t-norm, given in (Tabakow I.G. 2010, 2014), can be also used for obtaining more universal fuzzy flip-flops and hence more flexible control systems (modeled with fuzzy interpreted Petri nets). It is also shown that this t-norm, redefined as generalised Łukasiewicz's intuitionistic fuzzy t-norm is a t-representable intuitionistic fuzzy t-norm. And hence, a generalised Łukasiewicz's fuzzy intuitionistic implication (satisfying all corresponding requirements) is also introduced. Some new inference rules are also given in the case of modal and temporal systems. Some introductory notions related to the dynamic logic, modal μ-calculus and their applications are also presented (Pratt V.R. 1981, Kozen D.C. 1983, Venema Y. 2008, Gurov D. and Huisman M. 2013, etc.). As a first pass, some notions concerning the last calculus and related to the next two chapters may be omitted. There are also briefly presented (or commented) some other non-classical systems, such as: epistemic, game, quantum dynamic-epistemic, intuitionistic and fuzzy intuitionistic, linear, (intuitionistic) computability, paraconsistent, relevant and non-monotonic logic systems. Some comments concerning fractal logic are also given.

Propositional logic is not sufficient for computer science. And hence, in the next Chapter II, the first-order predicate calculus is initially considered. Some well-known basic notions related to the classical predicate logic are first introduced. As an illustration, by using such formulae, some example mathematical and/or computer science definitions are also described and a set of primitive rules is then presented. Next, a carefully selected subset of theses is proved. The corresponding formal proofs are based on assumptions. The notion of the existential uniqueness quantifier is next presented and some properties are also given. The Gentzen's sequent calculus is also illustrated. As in the previous chapter, the corresponding rules are proved. Some new proofs and/or theses, mainly concerning bounded (or equivalently: restricted) quantifiers, are also given. In the next considerations the skolemisation, resolution and interpretation techniques are discussed (Chang C.-L. and Lee R.C.-T. 1973). The resolution technique is considered as a construction of ramified indirect proofs with joined additional assumptions and natural numbers are used in the formula interpretation rules. Next, the higher order predicate logic is briefly considered. Basic notions related to the genralised quantifier theory are also presented (Pogonowski J. and Smigerska J. 2008). Some considered here notions are related to the next Chapter III (e.g. the semantic values of quantifier expressions arer sets of sets). Other non-classical systems, such as fuzzy, modal, deontic and temporal predicate calculi are considered in the second section of Chapter II. Here, the intuitionistic and paraconsistent predicate logic systems are also briefly presented.

In the last Chapter III, elements of set theory are first presented. This theory is a basic tool in discrete mathematics and also in mathematical analysis (concerning infinite sets). Initially, there is given a historical outline related to the development of this theory (*The little encyclopaedia of logic* 1988). And next, starting with the axiomatic foundations, some well-known (set-algebraic) classical basic notions and definitions are briefly introduced (Słupecki J. and Borkowski L. 1967, Kerntopf P. 1967). In particular, there are presented relations which are binary*. The most of considered proofs are from assumptions. In the next considerations Kripke – Platek and other set theories are briefly presented. Several applications are also presented. Some comments concerning commonsense sets are also given. Next some elements of non-classical set theories are given, such as: multisets (or bags) and multirelations, fuzzy sets and fuzzy relations, rough sets or also non-standard approaches: fuzzy rough sets, interval type-2 fuzzy sets, near sets and nested sets, paraconsistent sets or other set models. Bunch theory is also briefly presented. It is shown that the generalised Łukasiewicz's fuzzy t-norm (introduced in Chapter I) may be considered

* In accordance with Part II: the presentation of abstract algebra that focuses on binary relations.

as an adequate t-norm in the case of obtaining a distance function of the Minkowski class. Some other properties and examples are also given. In particular, a possibility of a generalisation and improving of the notions of lower and upper approximations used in fuzzy rough sets is also presented. All the above set theories are a very important part of today's computer science. And hence, some applications are also considered, e.g. concerning multigraphs, high-level Petri nets, fuzzy graphs and nets, fuzzy clustering, fuzzy control, information systems and decision tables, data mining, theory of programming, etc.

The above basic theories may be used as a part of lectures, predestined at first for computer science students, however it can also be useful in other areas, e.g. such as system techniques and control, technical cybernetics, telecommunication, managing etc. And also, the considered here systems may be useful for any researcher who is interested in the above given area. The expected effects can be summarised as twofold. First, thorough knowledge of the sense of using natural deduction methods in computer science and second, a possibility of obtaining a knowledge for the purpose of efficient bibliographic search in this field of application and also with respect to future scientific investigations and/or practical applications.

Several parts of this work were presented during my lectures at the Institute for Mechanical and Electrical Engineering in Sofia, now known as TU Sofia, Bulgaria and also at the Wroclaw University of Technology in Wroclaw, Poland. A preliminary version of this study was realised in accordance with some research projects, e.g. such as *Z0802-331557-W0800*, *Z0802-341763-W0800*, etc., during my stay in Wroclaw.

In addition, I would like to thank my wife for her countless patience and love during the writing of the manuscript of this book.

This open access work is firstly addresses to the (advanced) computer science students, but may be useful for any researcher who is interested in the above given area. Any suggestions or other comments related to this work are well come. To all such remarks I would be grateful.

Iwan.G.Tabakow (retd. Professor)

The used designations

The used names for the primitive and/or derived rules given below are in accordance with the Łukasiewicz´s symbols of negation, conjunction, disjunction implication, and equivalence denoted by N, K, A, C, and E, respectively (introduced in the *parenthesis-free notation* called also *Polish notation*: Jan Łukasiewicz 1878 – 1956). Some commonly used symbols are given in parentheses. Other designations and/or abbreviations are the same as in (Słupecki J. and Borkowski L. 1967)*.

$\sim$	symbol of *negation*, called also *logical inversion* or *logical not* (called also e.g. '*quantum negation*', i.e. '*orthocomplement*' in quantum logic systems or '*linear negation*' in linear logic systems, etc.). Another used designations: $\neg$, ´ , ¯ , $\perp$ (e.g. $p^{\perp}$), *Not*, etc. The used symbol ' ´ ' may be also used as a citation of designations or formulae, e.g. ´~´ to denote the set *equinumerosity relation* or ´A $\cap$ B´ etc., depending on the context ;
$\wedge$	symbol of *conjunction* (logical multiplication, logical and: & , · , $\cap$, $\sqcap$, *And*) or *weak conjunction* (many-valued logics) ;
$\vee$	symbol of *disjunction* (logical sum, logical alternative, join: + , $\cup$, | , /, $\sqcup$, *Or*) or *weak disjunction* (many-valued logics) ;
&, $\veebar$, ⅋	the symbols of *strong conjunction* and *disjunction*, respectively: many-valued logics (& may also denote *additive and* in linear logic, depending on the context), ⅋ denotes *multiplicative or*, see: linear logic, subsection 2.4 ;
$\Rightarrow$	symbol of (the *material*) *implication* ($\rightarrow$, $\supset$, I) ;
$\Leftarrow$	$p \Leftarrow q \Leftrightarrow_{df} p \wedge \sim q$, the *co-implication* connective, known also as *subtraction* (or *difference*) *operator* ;
$\Leftrightarrow$	symbol of *equivalence* (*co-implication*, $\equiv$, iff , ~ , $\leftrightarrow$, E: '$\equiv$' may also denote the *congruence modulo m relation* on the set of integers or a linear logic equivalence, depending on the context (see linear logic, Subsection 2.4) ;
φ°	is an abbreviation for $\sim(\varphi \wedge \sim\varphi)$, i.e. $\varphi^{\circ} \Leftrightarrow_{df} \sim(\varphi \wedge \sim\varphi)$, see: *paraconsistent predicate logic* ;
$\varphi \Rightarrow\!> \psi$, $\varphi <\!\Leftrightarrow\!> \psi$	$\varphi \Rightarrow\!> \psi \Leftrightarrow_{df} \Box(\varphi \Rightarrow \psi)$ and $\varphi <\!\Leftrightarrow\!> \psi \Leftrightarrow_{df} \Box(\varphi \Leftrightarrow \psi)$, respectively (see temporal predicate calculus, Subsection 4.2) ;
$\Box$	(the modal) functor of *necessity* (other used designation: N): Provided there is no ambiguity by $\Box$ it is also denoted the temporal functor '*always*' (or '*henceforth*': see Subsection 2.3, Temporal logics) ;
$\Diamond$	(the modal) functor of *possibility* (other used designations: Δ or also P : Lewis C.I. 1883 - 1964): Provided there is no

* Słupecki Jerzy (1904 – 1987), Borkowski Ludwik (1914 – 1993).

	ambiguity by ◇ it is also denoted the temporal functor '*eventually*' (or '*sometimes*': see Subsection 2.3, Temporal logics) ;
⇒> , <⇔>	the modal logic symbols of *strict implication* (or *entailment*: ≺ , ∶ , —>> ,⇛,⇒: Lewis C.I.) and *strict equivalence*, respectively ;
- □	*rule of omitting the necessity modal functor* ;
+ ◇	*rule of joining the possibility modal functor* ;
N □	*rule of negating a necessity modal functor* ;
N◇	*rule of negating a possibility modal functor* ;
+ SI, + SE, – SI	*rules of joining a strict implication and a strict equivalence*, respectively, *rule* of *detachment for strict implication* (or *omitting a strict implication*) ;
+ OSE, – OSE	*ordinary rules of joining and omitting a strict equivalence*, respectively ;
□ /◇	*rule of changing a necessity modal functor into possibility modal functor ;*
□ K	*rule of exchanging a necessity of conjunction by conjunction of necessities ;*
□ C	*rule of necessity of implication ;*
□-Toll	*modal necessity tollens ;*
S-Toll	*strict implication tollens ;*
◇A	*rule of exchanging a possibility of disjunction by disjunction of possibilities ;*
A□	*rule of disjunction of necessities ;*
◇K	*rule of possibility of conjunction ;*
R □ , R◇	*rules of reduction of necessity and possibility*, respectively *;*
SDS, CSDS	*strict rule of Duns Scotus* and *contrapositive strict rule of Duns Scotus,* respectively ;
SMC, SAA	*strict laws of multiplication of consequents* and *addition of antecedents*, respectively ;
C◇	*rule of implication of possibilities* ;
◇-Toll	*modal possibility tollens* ;
N□K, N◇A	*modal De Morgan´s laws for necessity of conjunction* and *possibility of disjunction*, respectively ;
OBR	*obligation rule* ;
!	(the deontic) functor of *obligation* (O) or a connective in linear logic (pronounced as 'of course', see: linear logic, Subsection 2.4) ;
δ	(the deontic) functor of *permission* (P) ;
σ	(the deontic) functor of *forbiddance* (F, provided there is no ambiguity and depending on the context, by σ it is also denoted a binary relation or a program computation, i.e. a sequence of states, stack, i.e. an abstract data type: last in, first out, or also a map: see Subsection 2.3, Temporal logics, μ-calculus). Provided there is no ambiguity, the Greek letter μ

	also denotes a *path* (of a graph G): depending on the context ;
ζ	denotes a *formula* or the binary relation of *bisimulation*, depending on the context (see Subsection 2.3, Temporal logics,μ-calculus) ;
∗	deontic constant (i.e. 0-ary modal functor, denoted also by ´s´ or ´v´), standing for some *sanction* or related to some *violation* or may also denote a *belief revision action*, depending on the context ;
N !	*rule of negating a deontic functor of obligation* ;
N δ	*rule of negating a deontic functor of permission* ;
!K	*rule of exchanging an obligation of conjunction by conjunction of obligations* ;
δA	*rule of exchanging a permission of disjunction by disjunction of permissions* ;
A!	*rule of disjunction of obligations* ;
δK	*rule of permission of conjunction* ;
!/δ	*rule of changing an obligation deontic functor into permission deontic functor* ;
!C	*rule of obligation of implication* ;
Cδ	*rule of implication of permissions* ;
– !C	*deontic rule* of *detachment for implication* (or *omitting an implication*) ;
!-Toll	*deontic obligation tollens* ;
N!K	*deontic De Morgan´s law for obligation of conjunction* ;
NδA	*deontic De Morgan´s law for permission of disjunction* ;
!AA	*deontic law of addition of antecedents* ;
!MC	*deontic law of multiplication of consequents* ;
!MAC	*deontic rule of multiplication of the antecedents and consequents of two implications* ;
+!A	deontic rule of *joining a disjunction* ;
!TC	*deontic rule of transitivity for implication* ;
!(··· / ···), δ(··· / ···), σ(··· / ···)	*conditional* (or *relative*) *obligation*, *permission* and *forbiddance functors,* respectively, e.g. !(φ / ψ), where φ and ψ may be arbitrary classical logic formulae ;
○, ⊖, ⊖̃	the (temporal) functors *´next´*, *´previous´* and *'weak previous'* ;
U, W, S, B	the temporal logic functors *´until´*, *´unless* (*waiting-for*)´, *´since´* and *´back-to´* , respectively ;
⊡, ⟐	the temporal logic functors '*has-always-been*' and '*once*', respectively ;
Ꞡ	the temporal logic functor *´it will always be the case that...´* ;
Ꞙ	the primary tense logic functor *´it will be the case that...´* ;
Ħ	the temporal logic functor *´it has always been the case that...´* ;
Ᵽ	the primary tense logic functor *´it was the case that...´* ;
+ Ꞡ	*tense logic rule of joining* Ꞡ *functor* ;

+ Ħ	*tense logic rule of joining* Ħ *functor* ;
– ǤC, – ĦC	*tense logic* Ǥ and Ħ *rules* of *detachment for implication* (or *omitting an implication*) ;
Ǥ-Toll, Ħ-Toll	*tense logic rules* Ǥ and Ħ *tollens* ;
ꞘA, ᵽA	*distributive property rules of* Ꞙ and Ᵽ *functors* ;
NǤK, NĦK	*tense logic De Morgan´s laws for* Ǥ and Ħ *functors of conjunction* ;
NꞘA, NⱣA	*tense logic De Morgan´s laws for* Ꞙ and Ᵽ *functors of disjunction* ;
ǤTC, ĦTC	*tense logic rules of* Ǥ- and Ħ-*transitivity for implication* ;
$\mathscr{M} \models \varphi_\tau$, θ	*the formula* φ *in time instance* τ *is satisfied in model* $\mathscr{M}$, θ is a *time instance set* ($\mathscr{M}$ and τ may also denote a *transition system* and *transition label,* respectively: see Subsection 2.3, Temporal logics, μ-calculus: simulation logic) ;
l(u)	the *label associated with* u (an edge of a graph G) ;
N○	*rule of negating a temporal next functor* ;
○A, ○K	*distributive property rules for temporal next functor of disjunction* and *conjunction*, respectively ;
N○A, N○K	*De Morgan´s laws for temporal next functor of disjunction* and *conjunction*, respectively ;
$n^m(x)$, $a^m(x,y)$, $c^m(x,y)$	*Post's functions* corresponding to negation, disjunction and implication in m-valued systems (Post E.L. 1897 – 1954): the Post's functions corresponding to the classical disjunction, conjunction and negation, i. e. for m = 2, are denoted by a, k and n, respectively;
$=_{df}$	the metalogical symbol of *definitional equality* (:=, ≜ or also ≡) ;
≐	the *range equality* (set theory) :
$\Leftrightarrow_{df}$	symbol of *definitional equivalence ;*
– C	*rule of omitting an implication* (or *detachment for implication* RD called also *modus ponendo ponens* or briefly: *modus ponens* MP) ;
+ K (– K)	*rules of joining* (*omitting*) *a conjunction* (JC and OC , respectively) ;
+ A (– A)	*rules of joining* (*omitting*) *a disjunction* (JD and OD , respectively) ;
+ E (– E)	*rules of joining* (omitting*) an equivalence* (JE and OE, respectively) ;
+ C	*rule of joining an implication ;*
C^{-1}	*rule of conversion of implications ;*
DE	*rule of detachment for equivalence ;*
DS	*rule of Duns Scotus* (John Duns Scotus 1266 – 1308) ;
NC	*rule of negating an implication* ;
NA (NK)	*rules of negating a disjunction* (*conjunction*) ;
+ N (– N)	*rules of joining* (omitting*) double negation* ;

ER, SR	*extensionality rule, rule of substitution for equivalence* (called also: *substitution rule* or *rule of replacement for equivalence*) ;
RR	*rule of definitional replacement of one formula by another* (axiomatic system style) ;
– EA´,– EA´´	*rules of omitting an exclusive disjunction* ;
TC, TE, CR, CC, CE	the *transitivity for implication*, *transitivity for equivalence* and *implication rules*, the *law of transposition or contraposition of implication*n, *law of transposition* or *contraposition of equivalence* ;
$– N_a, – N_c, – K_a, – K_c, – A_a, – A_c, – C_a, – C_c, – E_a$, and $– E_c$	*rules of reduction of the basic logical functors* in the antecedent (consequent) *of a sequent (*subscripts ´a´ and ´c´ , respectively: here the Łukasiewicz´s symbols N,K,A,C, and E are used) ;
1 (0)	*truth* (*falsehood) of a sentence,* i.e. a sentence which is *always true* (*always false*, i.e. *inconsistent*) *proposition*, the logical constants (or equivalently: the *constant formulae*) *´true´* and *´false´* are also denoted by T and F, respectively or also by tt and ff: depending on the context: other possible designations: $\top$ and $\bot$, respectively,), $\top$ and $\bot$ may be also used to denote a *t-norm* and *t-conorm*, respectively ;
B	denotes '*both*', i.e. false and true: see paraconsistent logic (B may also denote e.g. a set: depending on the context);
Toll	*rule modus tollendo tollens* (or briefly: *modus tollens*);
AA, MC (AC)	*rules of addition of antecedents* and *multiplication* (*addition*) *of consequents* of two or more implications having the same consequent and antecedent, respectively;
AAC, MAC	*rules of addition and multiplication of the antecedents and consequents of two implications,* respectively;
iso, sim, $\sqsubseteq$, $\approx$, $\approx_\varphi$	binary relations of *machine isomorphism, simulation, inclusion*, *equivalence* (Pawlak Z. 1926 – 2006) and *isomorphism between two algebraic systems* (wrt some map φ) ;
GR	*Gödel's Rule* in modal logic systems (Gödel K. 1906 – 1978);
//	*symbol of replacement* ;
/	symbol of *alternative negation* (*Sheffer´s dash*: Sheffer H.M. 1882 – 1964) or also the (temporal logic) functor ´*first... then...*´: see Subsection 2.3, Temporal logics. This symbol may also denote '*such that*' or *arithmetical division* (depending on the context) ;
↓	symbol of *joint negation* (Peirce´s arrow: Peirce C.S. 1839 – 1914) ;
a	assumption(s) ;
a_i	the i^{th} axiom (i = 1, 2, ... ,k) ;
e	the *unit element* of an algebraic system (denoted also by u: provided there is no ambiguity by e it is also denoted an expression: see Subsection 2.3, Temporal logics) ;
aip	assumption(s) of indirect proof ;

ada, contr.	*additional assumption of a proof* , *contradiction* ;
BL, Ł-BL, G-BL, π-BL	the *basic fuzzy propositional logic* (called also: *basic many-valued logic*: Hájek P 1998),concerning the following three systems: *Łukasiewicz's BL*, Gödel's BL, and *product logic* BL, respectively (π may also denote *quantum program* or *proof*: depending on the context) ;
a.k.a., wrt, s.t.	means: ´*also known as*´, ´ *with respect to ...*´, and ´ *such that ...*´, respectively (denoted also by: /, $, or :): the 0-ary ´ $ ´ may also denote an ´intuitionistic absurd´, depending on the context ;
cnf(ϕ), dnf(ϕ)	the *conjunctive* (*disjunctive*) *normal form* of the formula ϕ ;
⊢	symbol of Gentzen´s *sequent* (→ , Gentzen G.K.E. 1909 – 1945) ;
⊥ , ⊤	the many-valued logic symbols of "*undefined*" (or "*undetermined*" also denoted by Kleene symbol "u": Kleene S.C. 1904 -1994) and "*overdefined*", respectively; the first symbol is also used as truth constant in strict fuzzy logic (denoting *falsity*, i.e. the truth degree 0 or equivalently: 0, similarly the second symbol denoting *truth*, i.e. the truth degree 1 or equivalently: 1). ⊥ and, ⊤ may also denote '*bottom*' and '*top*' in *linear logic* or also: '*machine*' and '*environment*' in *intuitionistic computational logic* (depending on the context, see Subsection 2.4):´⊥x ´ may denote 'the *decimal value* of a binary (in general m-ary) word´ ;
· , ⊽	*product logic*: *the usual arithmetic product* and *algebraic sum operations* (denoted also by '^{as}⊕'), where: a ⊽ b $=_{df}$ a + b – ab ;
min, max	the logical operations *minimum* and *maximum, respectively ;*
msc	the *main sequent connective* of a given formula φ , i.e. ´⊢ ´ in the sequent ´⊢ φ ´ ;
soa	'*state of affairs*' (wrt a database query) ;
⊨ , $⊨_t$	´ ⊨ ϕ ´ means ´ *ϕ is thesis*´ (⊢), ⊭ ϕ $⇔_{df}$ ~ (⊨ ϕ),´ $⊨_t$ ϕ ´ means ´ *ϕ is t- thesis*´ or also, depending on the context, '⊨ φ' requires φ to hold at the first state of any sequence of states (see Subsection 2.3, Temporal logics) ;
s ⊩ φ	denotes 'φ is *true* or *holds* at s' (another designation of the satisfaction relation: '⊩', used in μ-calculus: similarly for ⊯ and ⊮, see Subsection 2.3, Temporal logics) ;
$⇝_{(P,D)}$	the modal equivalence relation, denoted also by '⇝' (if P and D are known: see Subsection 2.3, Temporal logics, μ-calculus) ;
(σ,j) ⊨ φ	denotes 'φ holding at a position j , j ≥ 0, in a sequence σ' (similarly for ⊭: see Subsection 2.3, Temporal logics) ;
ϕ(j)	ϕ(j) $=_{df}$ '(σ,j) ⊨ φ' ;
⇎	symbol of *exclusive disjunction* (≢ , ⊻) ;

$\dfrac{\varphi}{\psi_1 \setminus \psi_2 \setminus \begin{matrix}\psi_1\\ \psi_2\end{matrix}}$	means ′ $\dfrac{\varphi}{\psi_1}$ or $\dfrac{\varphi}{\psi_2}$ or $\dfrac{\varphi}{\begin{matrix}\psi_1\\ \psi_2\end{matrix}}$ ′
p , q, r, s,...,p_1, p_2, ...	some *propositiona*l (called also *sentential*) *variables* (provided there is no ambiguity by p it is also denoted a property of a given program and s may also denote a state: see Subsection 2.3, Temporal logics) ;
I	symbol of an *interpretation* or *index set*, depending on the context: usually I coincides with the set of natural numbers: for convenience, instead of $\bigcup_{i \in I} X_i$, it is also used: $\bigcup_i X_i$ (assuming that I is known) ;
v(φ), v_I(φ), ψ ∈ Cn({φ_1, ... ,φ_n})	the *logical value of a propositional formula* φ , v(φ) for I , ψ is a *logical consequence wrt* φ_1, ... ,φ_n (*follows* from , is *deducible* by, equivalently: $\varphi_1, \dots ,\varphi_n \vDash \psi$) ;
$\varphi \vDash \psi$, $\varphi \vDash_t \psi$	ψ *follows from* (or is a *logical consequence wrt*) φ. Then $\vDash \varphi \Rightarrow \psi$ (and vice versa: similarly for $\varphi \vDash_t \psi$) ;
$\varphi \nvDash \psi$	~ ($\varphi \vDash \psi$) ;
χ(φ // ψ)	is obtained from χ by the replacement of its parts ψ by the formula φ ;
∈	′*is* (*are*) *element*(*s*) *of* ...′(may also denote an empty sequence ∈, depending on the context: see Subsection 2.3, Temporal logics: μ-calculus) ;
∅	empty (or null) set ;
ℕ , $ℕ_E$, $ℕ_O$, ℤ, ℚ, ℝ, $ℝ_+$, $ℝ_{++}$, ℂ , T, H	the sets of *natural numbers* ℕ $=_{df}$ {1, 2, ...}, the subsets of *even* and *odd natural numbers*, *integer numbers*, *rational numbers*, *real numbers*, *nonnegative real numbers* [0, ∞), *positive real numbers* (0, ∞), *complex numbers*, *transcendental numbers* and *quaternions*, respectively ;
IN_ω	*nonnegative integers* (i.e. *IN* $=_{df}$ ℕ ∪ {0}), extended by ‘ω’ (an *infinite number*) ;
$\aleph_0$, c	the cardinal numbers corresponding to the sets of *natural numbers* and *real numbers*, respectively ;
E, O	E, O ⊆ ℤ denote the subsets of *even* and *odd integer numbers* ;
{ a / Φ(a) }	the set of all ‘a’ satisfying condition Φ(a) , another designation: $\underset{a}{E}\,\Phi(a)$: ‘E’ from French word ‘*ensemble*’ ;
∣X∣	the *cardinality* of set X (denoted also by: $\overline{\overline{X}}$ (Cantor’s designation: G.F.L.P. Cantor 1845 – 1918) , card(X), nc(X), #(X) or #X ;
#(x,B)	the number of x’s in B (a *multiset*) ;
A_i, a_i	the i^{th} axiom A and its use a_i (i ∈ ℕ) ;
≤ , ⊴	denote ‘*is less than or equal to*’ (in general: *some partial ordering relation*) and ‘*is a subformula of*’, respectively ;
≽ , ≼ , ≻ , ≺	the *partial order relation*, the *opposite* (or *inverse*) *partial order relation* and their *strong* versions, respectively: denoted also by ≥ , ≤ , > and < , respectively ;

ρ^0	equivalently: 0_X or id it is denoted the set: $\{(x,x) \mid x \in X\}$;
$\mathscr{U}$	*universum* (the *universe set*, the *universe*, the *universal set*), other used designations: S, E, 1, etc., sometimes may be used another symbol (e.g. X: *fuzzy sets* or *intuitionistic fuzzy sets*) and this is depending on the context) ;
gcd(*a*,*b*)	the *greatest common divisor* (archaic: *greatest common factor*) of two integers *a* and *b* is the largest integer that divides them both. ;
(a) modulo m	denoted also by *rem*(a/m) $=_{df}$ b is the *remainder when a is divided by m*, where: $a = k \cdot m + b$ (a,m,k,,b – integers: $m > 0$ and $0 \le b < m$)* ;
$\subseteq$, $\subsetneq$, $\nsubseteq$	the *set inclusion* and the *proper set inclusion* (or *strict set inclusion*, denoted also by $\subset$†) binary relations. $X \nsubseteq Y \Leftrightarrow_{df} \sim(X \subseteq Y)$: if $X \subseteq Y$ then X is a *subset* of Y or equivalently, Y is a *superset* of X ;
$\subseteq_b$, $=_b$	the *multiset inclusion* and *equation*, respectively ;
inf{Y}, sup{Y}	*infimum* of Y (*supremum* of Y), denoted also by: inf Y (sup Y) or inf y / y $\in$ Y (sup y / y $\in$ Y),i.e. the greatest lower bound (the least upper bound) of $Y \subseteq X$, where X is a partial ordered set ;
supp(X)	*support of* X (a given set) ;
P, Φ	*non-empty* (equivalently: *non-void*) *sets of formulae*, e.g. *proposition letters* (i.e. *propositional variables* called also *atomic propositions* p, q, r, ... or also *free* or *bound variables* of a formula φ: x, y, z, etc., see Subsection 2.3, Temporal logic: μ-calculus; provided there is no ambiguity by P it is also denoted a given program or also process: see Subsection 2.3, Temporal logics): in set theory Φ and Ψ may also correspond to some conditions (depending on the context) ;
D	non-empty set of *atomic actions* (or *labels*: see Subsection 2.3, Temporal logics: μ-calculus) ;
X / ρ	the *quotient set* wrt the binary equivalence relation ρ (the used symbol ρ may also denote *metric* in a given *metric space*, denoted also by d, from the term '*distance*') ;
$\rho_{/B}$	the binary relation ρ on A, restricted to (the subset) $B \subseteq A$;
$[x]_\rho$	an *equivalence class* under ρ ;
$\underline{A}$ (X) , $\overline{A}$ (X), BR(X)	denote: *lower approchimation, upper approchimation* and *boundary region* of X wrt ρ (a binary relation, see: *rough sets*) ;
$\cup$, $\cap$, $-$, $\div$, $'$, $\circ$, *	the *set union* (*set sum*), *set intersection*, *set difference* and *symmetric* (or: symmetrical) *set difference* operations (another designation for set difference: ' \ ' , e.g. X \ Y, instead of $X - Y$), provided there is no ambiguity by X′ is denoted

* In general, the *congruence modulo m relation* on the set of integers '$\equiv$' is defined as follows: $a \equiv b \pmod m \Leftrightarrow_{df} m / a - b$, i.e. m divides the arithmetical difference $a - b$. Equivalently, $a \equiv b \pmod m$ iff there exists an integer $k \in \mathbb{Z}$ such that $a = k \cdot m + b$.

† This set operation may also denote usual set inclusion, e.g. see (Słupecki J. and Borkowski L. 1967).

	the *complement* of X (another designation: $\overline{X}$): the symbol ′∪′ may also denote a *union* (or *nondeterministic choice of events*, called also *regular expressions*), ∘ and ∗ are the corresponding algebraic operations of *concatenation* (called also *catenation*, *sequencing* or *composition*: another designation: ′ ; ′: ′∘′ may also denote *composition* or equivalently *superposition* of two or more, a finite number, binary relations) and the *Kleene star operator*, called *iteration*, Kleene S.C. (1904 – 1994: see Subsection 2.3, Temporal logics: dynamic logic) ;
$[a]$, $\langle a \rangle$	*dynamic logic functors* (called also *operators*, ′a′ denotes *action* or equivalently: *event*): $[a]\varphi$ and $\langle a \rangle \varphi$ denote the facts that *after performing* a i*t is necessarily* and *it is possible* the case that φ holds, respectively (for an arbitrary formula φ : see Subsection 2.3, Temporal logics: dynamic logic) ;
IND, LI, RTC, MGEN, MON	the *induction axiom*, the *loop invariance*, the *reflexive transitive closure*, the *modal generalization* and the *monotonicity* rules (see Subsection 2.3, Temporal logics: dynamic logic) ;
PML(D,P), μPML(D,P), FV(φ), BV(φ)	the *polymodal logic in D and P* , the *polymodal fixpoint logic in D and P*, the *sets of free* and *bound variables of a formula φ*, respectively (see Subsection 2.3, Temporal logics: μ-calculus) ;
K_c , B_c	the *epistemic* and *doxastic functors* (or: *operators,* denoted in short by *K* and *B*, if the *Agent* ′c′ is known). K_cp denotes "*Agent* c *knows* p" and B_cp denotes "*Agent* c *believes* p" ;
$\bigcup_i X_i$, $\bigcap_i X_i$	the *generalised set union* and *set intersection* operations, respectively ;
(x,y)	an *ordered pair* (*2-tuple* in short "*couple*": means "pair"), denoted also by <x,y>, Kuratowski K. (1896 – 1980) ;
× , X^n	*Cartesian product* of sets, $X^n =_{df} X \times X \times ... \times X$, n times, n ≥ 2 (René Descartes 1596 – 1650, Latinised: Renatus Cartesius) ;
ρ, ρ^{-1}, ρ^0	a *binary relation*: $\rho \subseteq A \times B$, where A and B are two sets (aρb $\Leftrightarrow_{df}$ (a,b) ∈ ρ); ρ^{-1} denotes the *transposed relation* or equivalently the *converse* of ρ, i.e. $a\rho^{-1}b \Leftrightarrow_{df} b\rho a$ (sometimes instead of the Greek ρ, σ, ... they are also used the latin letters R, S, ...) and the *identity relation* (denoted also by <u>id</u>): e.g. $\rho^0 =_{df} \{(x,x) / x \in X\}$;
dom(ρ), *cod*(ρ)	*domain* and *codomain* (called also: *range* or *image*) of a binary relation ρ ;
ρ^+, ρ^*, $\rho \circ \sigma$	*transitive closure* of ρ, *transitive and reflexive closure* of ρ, i.e. $\rho^* =_{df} \rho^+ \cup \rho^0$ and *composition* (called also: *superposition* or *relative product*) of two (or more, a finite number) binary relations: sometimes ′∘′ is omitted (′∘′ may also denote *concatenation*: language theory) ;
f : X → Y, Y^X	f is a *map* (*into* or *onto*) from X to Y, $Y^X =_{df} \{f / f : X \to Y\}$ (denoted also by Map(X,Y)) ;
S, $K_{D,P}S$, $[\|\varphi\|]^S$ ($\|\varphi\|^S_\varepsilon$)	a *(P,D)-(labelled) transition system* or *(P,D)-Kripke model*, $K_{D,P}S$ (in short K) denotes the *Kripke functor associated with D and P*, and and the corresponding *meaning* or *extension* of a formula φ in S (in S wrt the *environment* ε : see Subsection 2.3, Temporal logics: μ-calculus) ;

$\otimes , \oplus ; \hat{\otimes} , \hat{\oplus} ; \otimes_{if} , \oplus_{if} ; \hat{\otimes}_{if} , \hat{\oplus}_{if}$	the usual *triangular norm* and *conorm operators*, in short: *t-norm* and *t-conorm* (called also: *s-norm*), denoted also by $\top$ and $\bot$, respectively; the *generalised Łukasiewicz's fuzzy t-norm* and *t-conorm*; the *intuitionistic fuzzy t-norm* and *t-conorm*; *the generalised Łukasiewicz's intuitionistic fuzzy t-norm* and *t-conorm*, respectively. $\otimes$ and $\oplus$ may also denote *multiplicative and'* and *additive or*, depending on the context, see: linear logic, Subsection 2.4 ;
?, $\multimap$	is used in propositional dynamic logic, e.g. '?p' denotes: '*test* p *and proceed only if true*' (see Subsection 2.3) or also a connective in linear logic, pronounced as: '*why not*' and '$\multimap$ '
	denotes *linear implication*, e.g. $\varphi \multimap \psi$, see: linear logic, Subsection 2.4 ;
$\Rightarrow_{\alpha} , \Rightarrow_{fi} , \Rightarrow_{\alpha\text{-}fi}$	the *generalised Łukasiewicz's fuzzy implication*, the *fuzzy intuitionistic implication* and the *generalised Łukasiewicz's fuzzy intuitionistic implication*, respectively ;
$\circ$	denotes *logical, arithmetical operation* or another binary operation: depending on the context ;
$v_t(\varphi)$	the *logical value of a propositional formula* φ *wrt the continuous t-norm* $\otimes$: $v_t(\varphi) \in [0,1]$;
$\{ x / \varphi(x) \}$	the *set of all x such that* $\varphi(x)$ (or equivalently: $\underset{x}{E}\, \varphi(x)$) ;
$\mathbb{P}(X)$	the *power set* (or the *powerset*) of a set X, $\mathbb{P}(X) =_{df} \{ Y / Y \subseteq X \}$(also denoted by: $P(X)$, $\wp(X)$, $\wp X$, 2^X) ;
$\Pi(X)$	*partition of a set* X ;
BIPN	denotes the class of *Boolean interpreted Petri nets* ;
N	denotes a *Petri net* (provided there is no ambiguity, N denotes also *the Łukasiewicz´s symbols of negation*: depending on the context) ;
SM(N)	the *finite-state machine corresponding to* N (a Petri net) ;
b≈ , ≈	the *behaviour equivalence* and *state machine equivalence relations*, respectively ;
Coroll.	means: ´*Corollary*´;
Df. , df.	means: ´*Definition*´;
T-.-	e.g. ´T 1.2a´ means ´ *Thesis* or *Theorem 1.2a*´ (depending on the context) ;
TAU	the *state-tautology axiom* (see Subsection 2.3, Temporal logics) ;
GEN, SPEC, INST	the primitive rules of *generalisation*, *specialisation* and *instantiation* rules (see Subsection 2.3, Temporal logics) ;
TEMP, PAR, PR	the derived rules of *temporalisation*, *particularisation* and *propositional reasoning* (see Subsection 2.3, Temporal logics) ;
E(– C), ETC, EPR	the derived rules of *entailment omission of implication* (or: *entailment modus* ponens), *entailment transitivity for implication* (or: *entailment transitivity*) and *entailment propositional reasoning* (see Subsection 2.3, Temporal logics) ;
$\Sigma, \overline{\Sigma}$	an *alphabet* and the *set of all finite words* in Σ ;

$\forall$, $\underset{\varphi(x)}{\forall}\psi(x)$	the *universal quantifier* (called also: *general* or *big quantifier* and denoted also by: Π, A, ∧, etc.: Π or ∧ may also denote a *generalised conjunction*, ∧ may denote the *minimal element* of a given lattice: depending on the context) and the *bounded universal quantifier*: $\underset{\varphi(x)}{\forall}\psi(x) \Leftrightarrow_{df} \underset{x}{\forall}(\varphi(x) \Rightarrow \psi(x))$ (or equivalently: $\forall x\,(\varphi(x) \Rightarrow \psi(x))$ or $\forall x : \varphi(x) \Rightarrow \psi(x)$ or also $\forall x .\, \varphi(x) \Rightarrow \psi(x)$) ;
$\exists$, $\underset{\varphi(x)}{\exists}\psi(x)$	the *existential quantifier* (called also *existentional* or *little quantifier* and denoted also by: Σ, ∐, E, ∨, $\exists_{\geq 1}$, etc.: Σ or ∨ may also denote a *generalised disjunction*, Σ may also denote an *alphabet* and ∨ may denote the *maximal element* of a given lattice: depending on the context) and the *bounded existential quantifier*: $\underset{\varphi(x)}{\exists}\psi(x) \Leftrightarrow_{df} \underset{x}{\exists}(\varphi(x) \wedge \psi(x))$ (or equivalently: $\exists x\,(\varphi(x) \wedge \psi(x))$ or $\exists x : \varphi(x) \wedge \psi(x)$ or also $\exists x .\, \varphi(x) \wedge \psi(x)$) ;
Q	denotes a *quantifier*, i.e. $Q \in \{\forall, \exists\}$, a *set* or also a *predicate name* (depending on the context) ;
$\exists!$, $\exists*$	the unique (or ′exists unique′) *quantifier*, denoted also by $\exists_1$ or also by $\exists_{=1}$ and *the existential uniqueness quantifier*, denoted also by $\exists_{\leq 1}$, and so: $\underset{x}{\exists!}\varphi(x) \Leftrightarrow \underset{x}{\exists}\varphi(x) \wedge \underset{x}{\exists*}\varphi(x)$;
$\varphi(x/\xi)$	denotes an expression formed from φ by substituting (the *individual variable*) x for the *expression* ξ. ;
$\Phi(x)$	denotes a mathematical expression, describing some property associated with x ;
$N\forall, N\exists, -\forall, -\exists$ ($N\forall^*, N\exists^*, -\forall^*, -\exists^*$)	rules of *negating an universal quantifier*, *negating an existential quantifier*, *omitting an universal* and *an existential quantifiers* (the corresponding rules for *bounded quantifiers*) ;
$+\forall, +\exists$ ($+\forall^*, +\exists^*$), $\exists_*$	rules of *joining an universal* and *an existential quantifiers* (the corresponding rules for *bounded quantifiers*), and "*there exists at most one*" quantifier ;
EI, CER	the rule of *extensionality for identity*, *cut-elimination* rule ;
ι	′ ι ′ denotes a *descriptor's operator*, e.g. $\underset{x}{\iota}\varphi(x)$;
$-\exists!$	rule of *joining a descriptor's operator* (or *omitting an existential uniqueness quantifier*) ;
RES	the *resolvent rule* (see Subsection 3.6: resolution method) ;
fi	′*end-of-definition*′;
$=_{df}$	the symbol of definitional equality (or identity: *equals by definition*) ;
$\underline{x}$, $\underline{y}$, ...	some *vectors* (denoted also by **x**, **y**, ...) ;
λ	*empty formula* (*empty sequence*, *empty sentence, phrase or word, empty string*: denoted also by ε or ∈) or *real*

	parameter, depending on the context ;
ECSQ	*ex contradictione sequitur quodlibet* (latin), equivalently: ECQ ;
Cont, Weak, Perm	*Contraction, Weakening, Permutation*
i.e.	*id est* (latin): that is ;
LP	*logic of paradox* ;
PDL, LQA	*propositional dynamic logic*, *logic of quantum actions* ;
QL	(the traditional) *quantum logic* ;
℗$_h$, ©$^{e}_{h}$, ®$_h$	the *average*, *combinatorial* and *series* connectives (see Subsection 2.4, Fractal logic) ;
KB, DB, CWA	*knowledge base*, *database*, *closed world assumption* ;
$\mathcal{H}$	*Hilbert space* ;
π, κ	*quantum programs* or *proofs*: depending on the context ;
ite	*if ... then ... else*, e.g. $s = ite\{b / \alpha ; c\}$ denotes: '*if* α is true, *then* $s = b$, *else* $s = c$ (Subsection 2.4: fractal logic) ;
e	an *expression* ;
wrt, vs. , aka	*which respect to*, *versus*, *also known as* ;
IFN	*intuitionistic fuzzy number* ;
$\tilde{P}$	*Pytagorean fuzzy set*
LNC	*law of non-contradiction* ;
lt	´ lt ´ denotes ´*less than*´, i.e. ´< ´;
Z(X), Z(y)	´X *is a set*´ , ´y *is a set*´ ;
qr(φ)	the *quantifier rank* of φ ;
$\exists_{i,j}$	is an abbreviation: $\exists_{i,j} =_{df} \exists_i \exists_j$;
$\propto$	´*is proportional to*´ ;
□	´end of proof ´ (of an example, algorithm, or another formalised text) ;
'text'	citation of a text, e.g. 'a variety of problems that can be solved by … ' .
[boow], [boows]	denote (depending on the context): '*based on other work* ' and '*based on other works*', respectively.

Sets are denoted by capital letters, e.g. X, Y, etc. *Families of sets* are denoted by bold capital letters, e.g. **X**, **Y**, etc. Moreover, as in Słupecki J. and Borkowski L. (1967), below the lower case Greek letters ´ ϕ ´ , ´ φ ´ , ´ ψ´ , ´ χ ´ , etc., are used as metalanguage variables for which names of any formulae of the propositional calculus may be substituted, while for the variables ´ p ´ , ´ q ´ etc. , we may substitute any propositional formulae belonging to that calculus.

I. Propositions

The logical calculi considered here are based on a system of rules, which define the methods used in proofs from assumptions. The methodological problems in the deductive sciences are given only an elementary treatment (Słupecki J. and L. 1967). The last problems such as consistency, completeness, independence, etc. of a given system arise when a given axiomatic system is considered (in ways analogous to assumptional systems). Obviously, the most important of these properties is the consistency of a system. Some elements concerning direct reasoning and automated deduction methods are related to the sequent presentation. As it is shown, any Gentzen's rule can be represented as a corresponding thesis in the above-considered calculus. In the second section of this chapter some non-standard logic systems are considered, such as many-valued, fuzzy, modal, temporal and other non-classical systems (Zadeh L.A. 1974, Manna Z. and Pnueli A. 1992 and 1995, Carson J. 1995,Gotwald S. 1995, Suber P. 1997, Hájek P. 1998, Navara M. 2000, etc.). Here, the natural deduction methods are extended to some non-classical systems, e.g. such as many-valued, fuzzy, modal, temporal and some others. The Gentzen's sequent calculus is extended to Ł-BL systems. In particular, it is shown that any fuzzy propositional formula provable under Hájek's axioms of the logic BL is also provable under the above-proposed approach, i.e. using proofs from assumptions. This approach seems to be more attractive, more simpler and natural in practical use than the axiomatic one. Some new inference rules are also given in the case of modal and temporal systems. Some introductory notions related to the dynamic logic, modal μ-calculus and their applications are also presented (Pratt V.R. 1981, Kozen D.C. 1983, Venema Y. 2008, Gurov D. and Huisman M. 2013, etc.). As a first pass, some notions concerning the last calculus and related to the next two chapters may be omitted. There are also briefly presented (or commented) some other non-classical systems, such as: epistemic, game, quantum dynamic-epistemic, intuitionistic and fuzzy intuitionistic, linear, (intuitionistic) computability, paraconsistent, relevant and non-monotonic logic systems. Some comments concerning fractal logic are also given.

1. Classical propositional calculus

Logic is the science and art which so directs the mind in the process of reasoning and subsidiary processes as to enable it to attain clearness, consistency and validity in those processes (Turner W. 1999). The Greek word *logos*, meaning 'reason' is the origin of the term '*logic-logike*' (techen, pragmateia,or episteme,understood) as the name of a science or art., first occurs in the writings of the Stoics (The Stoic School: founded in 322 b.c. by Zeno of Cittium). The traditional mode of dividing logic into '*formal*' (called also '*symbolic*') and '*material*' is maintained in many modern treatises on the subject. The founder of logic, Aristoteles (born at Stagira, a Grecian colony in the Thracian peninsula Chalcidice,384 b.c. died at Chalcis,in Euboea,322 b.c.) in the six treatises*, which he devoted to the subject, examined and analysed the thinking processes for the purpose of formulating the laws of thought. In the wide sense, the discovery of logic should be associated with the classical Greek philosopher Socrates (469/70 – 399 b.c.) reckoned as the biggest wise man in Ancient Greece, one of the most famous of the whole history of philosophy, attracted many pupils and/or followers, such as: Euclides (430 – c.360 b.c.), Platon (c.424 – 347 b.c.), Xenophon (c.425 –c.386), Aristophanes (c.450 – c.386), Aristoteles (who was a student of Platon, whom Socrates taught), etc. The Socrates style of teaching was based on two different methods: *elenctic style* (elenkhos: critical thinking or cross-examing) and *maieutic style* (maieutics: constructive or creative style). It can be observed that the last two methods are related to the notions of indirect and direct proofs from assumptions, respectively. Some philosophy systems were studied early, e.g. in India, in the 5th century b.c.:'*nyāya'* meaning inference, logical argument or syllogism†. The *Nyāya school of logic*, based on Hindu philosophy, can be considered as a form of epistemology (i.e. theory of

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* The Categories, Interpretation, Prior Analytics, Posterior Analytics, Topics, and Sophisms.

† A *syllogism* (Greek: συλλογισμός , 'conclusion, inference') is a kind of logical argument that applies deductive reasoning to arrive at a conclusion based on two or more propositions that are asserted or assumed to be true (see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.).

knowledge) in addition to logic. In accordance with this school, there are exactly four sources of knowledge, i.e.'*paramanas*': perception, inference, comparison, and testimony (based on texts known as *Nyāya Sutras*, written by Aksapada Gautama from around the 2nd century b.c. (see T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*).

In the next considerations we shall concentrate our attention only to the formal logic (called in short also 'logic', provided there is no ambiguity). In the last case, logic can be divided into '*classical*'(known also as *Boolean*[*] or *bivalent*, i.e. the logical value of a propositional formula is either true or false) and '*non-standard'* (called also '*non-classica*l'). Typically, a logic consists of a formal or informal language together with a deductive system and/or a model-theoretic semantics. The language is, or corresponds to, a part of a natural language like English or Greek. The deductive system is to capture, codify, or simply record which *inferences* are correct for the given language, and the semantics is to capture, codify, or record the meanings, or truth-conditions, or possible truth conditions, for at least part of the language (Shapiro S. 1995).

Subsections 1.1 – 1.6 are a brief introduction to the methods of natural deduction according to the excellent work given by Słupecki J. and Borkowski L. (1967). The logical calculi considered here are based on a system of rules which define the methods used in proofs from assumptions. Such an approach was originally introduced during 1934 – 1935 by Jaśkowski S. and Gentzen G.K.E[†]. The last approach seems to be inspirated by the earlier presented deduction theorem (introduced independently by Tarski A. in 1923 and Herbrand J.[‡] in 1930: see Theorem 1.31 of Subsection 1.7 and related comments). Some elements concerning direct reasoning and automated deduction methods are related to the sequent presentation (Subsection 1.8).

1.1. Symbols and formulae

Consider the proposition: "Russell is a logicien or it is not true that Russell is a logicien". This proposition remains to be true if instead of the word "Russell", the word "Chaplin"[§] is used or also instead of the word "logicien" , e.g. the word "general" is used. However, the last property is not preserved if we consider only the first part of the above proposition: "Russell is a logicien" (Bertrand Russell: 1872 – 1970). So, it can be observed different texts may have different logical forms. In general, the *logical form* of a proposition is the formula obtained from this proposition by replacing *non-logical constants*[**] by variables, the same constants being replaced by the same variables and different constants by different variables. Unfortunately it is a very difficult task to be given some more general definition of the last notion. Some more fundamental investigations concerning the process of extraction of a logical form associated with a priori given text were presented by Barwise K. J. and Perry J. (1983) and also by Kamp H. and Reyle U. (1993)[††]. A more strict considerations of the notion of logical form are omitted here. According to the above example, the following formula can be obtained: $p \lor \sim p$, where the propositional variable p corresponds to " Russell is a logicien". The constants $\lor$ and $\sim$ correspond to the connectives 'or' and 'it is not true that', respectively. Hence, the *Aristotelian law of the excluded middle* (lat: "tertium non datur") was obtained.

In the *propositional calculus* any formula is constructed by using the following three kinds of symbols: (i) *propositional variables* (denoted by $p, q, r, s, \ldots, p_1, p_2, \ldots$), (ii) some *constants* of the calculus (called also *logical*

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[*] George Boole (1815 – 1864)

[†]Stanisław Jaśkowski (1906 – 1965), Gerhard Karl Erich Gentzen (1909 – 1945)

[‡] Alfred Tarski (1901 – 1983), Jacques Herbrand (1908 – 1931)

[§] Charlie Chaplin (1889 – 1977)

[**] Any logical expression consists of symbols (in the broad sense: constants and variables). In general, there exist two kinds of constants: *logical constants* (e.g. T, F, $\sim$, $\land$, $\lor$, $\Rightarrow$, $\Leftrightarrow$, $\Box$, $\Diamond$, ! , etc., having the same semantic content in any interpretation) and *non-logical* ones, depending on the used interpretation (and hence, the considered sentence may be true or false).

[††] A formal semantics approach including the Kamp's *discourse representation structures* having two critical components: a set of *discourse referents* representing entities which are under discussion and a set of *conditions* representing the obtained information about discourse referents (discourse representation theory: Johan Anthony Willem Kamp, born 1940)

functors or *connectives*, such as: $\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, etc.) and also (iii) *parentheses* (left: '(' and right: ')'). The *basic symbols* $\sim$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$ are called *negation*, *conjunction*, *disjunction* (below called also '*logical alternative*' or in short '*alternative*'), '(*material*) *implication*', and '*equivalence*', respectively. The expression '~ p' is read: '*not* p' (or: '*it is not true that* p'). In a similar way, the expressions 'p $\wedge$ q', 'p $\vee$ q', 'p $\Rightarrow$ q', and 'p $\Leftrightarrow$ q' are read as follows: 'p *and* q', 'p *or* q', '*if* p *then* q' (or: 'q *only if* p') and 'p *if and only if* q', respectively. The first and the second elements of an implication are called its *antecedent* (or *premise*) and its *consequent* (or *succedent*), respectively. As it was mentioned, an expression may be introduced without using any parentheses (in accordance with the *Łukasiewicz´s parenthesis-free notation*), e.g. instead of the expression '(p $\Rightarrow$ q) $\Rightarrow$ (~ q $\Rightarrow$ ~ p)' we can consider 'CCpqCNqNp' (this approach is omitted below).

To minimise the number of used parentheses in an expression, some priorities for logical connectives were introduced, e.g. (i) $\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ (i.e. the symbol of negation binds more strongly than the symbol of conjunction, the last binds more strongly than the symbol of disjunction, etc. Słupecki J. and Borkowski L. 1967) or (ii) $\sim$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\vee$ (Mostowski A.W and Pawlak Z. 1970) or also (iii) $\sim$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ (Chang C.-L. and Lee R.C.-T. 1973). The only difference concerns disjunction binding. The convention under Słupecki J. and Borkowski L. (1967) is used below. Moreover, in the next considerations we shall use the term '*proposition*' to denote only such expressions which are *declarative*, i.e. either *true* or *false* (so neither the interrogative nor the imperative sentences are propositions wrt the last sense).

The *set of propositional formulae* (called equivalently *propositional expressions*, in short: *expressions* or also *sentential formulae*) of the propositional calculus can be considered as the smallest set of formulae which includes propositional variables, and which is closed under the operations of forming the negation, conjunction, disjunction, implication and equivalence. Hence, any propositional variable can be considered as an expression and also the compound formulae formed from them by means of the corresponding logical functors. More formally, the following well-known inductive definition is used.

Definition 1.1

A *propositional formula* is:

1. Any propositional variable,
2. If φ and ψ are some propositional formulae, then such formulae are also: $\sim(\varphi)$, $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $(\varphi) \Rightarrow (\psi)$, and $(\varphi) \Leftrightarrow (\psi)$,
3. Every propositional formula in this propositional calculus either is a propositional variable or is formed from propositional variables by a single or multiple application of rule (2)*.

Let φ be a formula constructed under Definition 1.1. The *main purpose of this calculus* is to give an answer of the question whether this formula is a thesis or not.

1.2. Primitive rules

There exist two classical approaches in constructing of the propositional calculus: the *axiomatic approach* and the *approach from assumptions*. In general, a system based on assumptions is presented below. The proofs realised in such a system are very similar to the mathematical proofs or also to the reasoning in other disciplines. Hence, in accordance with the last property, the commonly used name for such systems is the name of *natural deduction*. As it was mentioned, the first systems based on assumptions were developed during 1934 – 1935 by Jaśkowski and Gentzen. The system considered below differs from them in some details (Borkowski L. and Słupecki J. 1958). The *proof* in the propositional calculus can be interpreted as a process of joining new lines by using some primitive or derived rules and/or other theses in accordance with the used assumptions. The following seven *primitive rules* are considered below.

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* Equivalently, the above collection of formulae, i.e. the language of this logic, can be recursively defined as follows: $\varphi =_{df} p \,/\, \sim\varphi \,/\, \varphi \wedge \psi \,/\, \varphi \vee \psi \,/\, \varphi \Rightarrow \psi \,/\, \varphi \Leftrightarrow \psi$. This definition corresponds to the *BNF notation*, i.e. the *Backus-Naur form*, a notation technique for context-free grammars: John Backus (1924 – 2007), Peter Naur (1928 – 2016).

(1) *Rule of detachment* (or *omitting an implication*, also known as „*modus ponens*” or „*modus ponendo ponens*” denoted below by ′– C′*):

The Stoic School founded in 322 b.c. by Zeno of Cittium (Zeno of Cittium b. 366; d. in 280 b.c.): the first system of propositional calculus

$$-C: \quad \frac{\begin{matrix}\varphi \Rightarrow \psi \\ \varphi\end{matrix}}{\psi}$$

(2) *Rule of joining a conjunction*:

The Stoic School

$$+K: \quad \frac{\begin{matrix}\varphi \\ \psi\end{matrix}}{\varphi \wedge \psi}$$

(3) *Rule of omitting a conjunction*:

Albert of Saxony (who was a student of Jean Buridan: 1320 – 1390)

$$-K: \quad \frac{\varphi \wedge \psi}{\varphi \setminus \psi \setminus \begin{matrix}\varphi \\ \psi\end{matrix}}$$

(4) *Rule of joining a disjunction*:

Robert Kildwarby: 1215 - 1279, Albert of Saxony, Jean Buridan (13^{th} and 14^{th} century, Jean Buridan 1295/1305 – 1358/61).

$$+A: \quad \frac{\varphi}{\varphi \vee \psi}$$

(5) *Rule of omitting a disjunction* (also known as „*modus tollendo ponens*” or also „*disjunctive syllogism*”):

The Stoic School: a rule formulated only for an exclusive disjunction.†

$$-A: \quad \frac{\begin{matrix}\varphi \vee \psi \\ \sim\varphi\end{matrix}}{\psi}$$

(6) *Rule of joining an equivalence*:

Boethius A.M.S.(c.e. 477/80 – 524): some laws for equivalence. However it seems that the rules ± E date from a much later time.

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*A schema having two parts, upper and lower, separated by a horizontal line will present any rule (corresponding to the sequential way of the human thinking and reasoning). Another possible convention, e.g. for the rule of detachment: $-C: \ \frac{\varphi \Rightarrow \psi, \varphi}{\psi}$, etc.

† In fact, the rule $\frac{\begin{matrix}\varphi \not\Leftrightarrow \psi \\ \sim\varphi\end{matrix}}{\psi}$, which can be considered as a particular case of the following rule of omitting an exclusive disjunction:

$-EA'$: $\frac{\begin{matrix}\varphi \not\Leftrightarrow \psi \\ \sim\varphi\end{matrix}}{\sim\varphi \setminus \psi \setminus \begin{matrix}\sim\varphi \\ \psi\end{matrix}}$ (in a similar way: $-EA''$: $\frac{\begin{matrix}\varphi \not\Leftrightarrow \psi \\ \varphi\end{matrix}}{\varphi \setminus \sim\psi \setminus \begin{matrix}\varphi \\ \sim\psi\end{matrix}}$).

$$+E: \quad \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}{\varphi \Leftrightarrow \psi}$$

(7) *Rule of omitting an equivalence*:

$$-E: \quad \frac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi \;\backslash\; \psi \Rightarrow \varphi \;\backslash\; \begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}$$

According to the last rule − E, we shall say $\varphi \Rightarrow \psi$ is a *direct* (or *if-*) *implication* and $\psi \Rightarrow \varphi$ is the corresponding *opposite* (*converse* or *only-if-*) *implication*. Sometimes $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ are also said to be *left-to-right* and *right-to-left implications*, respectively.

In general, the above primitive rules can be considered as a natural way of the human thinking and inference. As an example, assume that from our analysis it follows that some variable $x \geq 0$, i.e. $(x > 0) \vee (x = 0)$. But from the next steps of this analysis it follows that $x \neq 0$, i.e. $\sim (x = 0)$. And hence, in accordance with the rule of omitting a disjunction we can obtain: $x > 0$.

Next we shall concentrate our attention to the following two *rules for constructing a proof from assumptions*: direct proof from assumptions and indirect proof from assumptions. Let consider the following *generalised form of an expression*:

$$\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots)$$

In general, the *direct proof from assumptions* is realised as follows:

Proof:

(1) ϕ_1

(2) ϕ_2

(3) ϕ_3 $\{1,2,\ldots,n-1 \,/\, a\}$

...

(n − 1) ϕ_{n-1}

...

ϕ_n . □

In the first n − 1 lines the corresponding *primary assumptions* are given (called also: *assumptions of the direct proof* or *direct premises*). To the proof we may join: new proof lines in accordance with the existing primitive rules, some proved previously theorems and/or additional assumptions (i.e some secondary assumptions, if necessary). The proof is said to be *complete* if in its last line the formula ϕ_n appears (this line will not be numbered). If there exists some ϕ_k $(k \in \{1, \ldots, n-1\})$ which is a conjunction of the form $\varphi_1^{\,k} \wedge \varphi_2^{\,k} \wedge \ldots \wedge \varphi_{nk}^{\,k}$, according to − K , instead of assuming ϕ_k , n_k lines are obtained. In the case when the main symbol is not an implication the proof begins by writing down one or more theorems previously proved. This case is called an *ordinary direct proof.*

The rule for constructing a direct proof from assumptions is illustrated below.

Thesis 1.1 (*the law of multiplying implications by sides*)

$$\underbrace{(p \Rightarrow q) \wedge (r \Rightarrow s)}_{\phi_1} \Rightarrow (\underbrace{p \wedge r}_{\phi_2} \Rightarrow \underbrace{q \wedge s}_{\phi_3})$$

Proof:

(1) $p \Rightarrow q$

(2) $r \Rightarrow s$

(3) p $\{1,2,3,4 \,/\, a\}$

(4) r

(5) q $\{-C: \; 1,3\}$

(6) s {– C : 2,4}
$q \wedge s$. □ { + K : 5,6}

The *indirect proof from assumptions* is realised as follows:

Proof:

(1) ϕ_1
(2) ϕ_2
(3) ϕ_3 {1,2,...,n – 1 / a}
...
(n – 1) ϕ_{n-1}
(n) $\sim \phi_n$ {aip}
...
contr. □

In the first n – 1 lines the corresponding primary assumptions are given. In the n[th] line the formula $\sim \phi_n$ is placed as the *assumption of the indirect proof* (called also: *indirect premise*)[*]. To the proof we may join: new proof lines in accordance with the existing primitive rules, some proved previously theorems and/or additional assumptions (i.e some secondary assumptions, if necessary). The proof is said to be *complete* if two contradictory lines appear in it. If there exists some ϕ_k ($k \in \{1, \ldots, n-1\}$) which is a conjunction of the form $\varphi_1{}^k \wedge \varphi_2{}^k \wedge \ldots \wedge \varphi_{nk}{}^k$, according to – K , instead of assuming ϕ_k, n_k lines are obtained. The last approach will be used also for $\sim \phi_n$. In the case when the main symbol is not an implication the proof begins with the assumption of the indirect proof. This case is called an *ordinary indirect proof.*

The proper specification of the set of all primary assumptions is a very important process (see remark concerning T 1.15 and T 1.19, given in the next subsection). The rule for constructing an indirect proof from assumptions is illustrated below.

Thesis 1.2 (*law of reduction ad absurdum*)

$(\sim p \Rightarrow q \wedge \sim q) \Rightarrow p$

Proof:

(1) $\sim p \Rightarrow q \wedge \sim q$ {a}
(2) $\sim p$ {aip}
(3) $q \wedge \sim q$ {– C : 1,2}
(4) q {4,5 / – K : 3}
(5) $\sim q$
contr. □ {4,5}

An application of the law of multiplying implications by sides is given in the next example.

Example 1.1

Let ρ and σ be two *binary relations over* X ,i.e. $\rho, \sigma \subseteq X \times X$. Assume that ρ and σ are transitive[†]. Then the intersection $\rho \cap \sigma$ is also a transitive relation over X . So, the following implication have to be shown (for any $x,y,z \in X$).:

$$(x\rho y \wedge y\rho z \Rightarrow x\rho z) \wedge (x\sigma y \wedge y\sigma z \Rightarrow x\sigma z) \Rightarrow ((x\rho y \wedge x\sigma y) \wedge (y\rho z \wedge y\sigma z) \Rightarrow x\rho z \wedge x\sigma z)$$

Since $\wedge$ is a commutative and associative logical operation (see the commentary given after T 1.7a below), this implication is satisfied. □ {T 1.1}

[*] Lines (1 – n) can be considered as obtained by using (n – 1) times the rule NC of negating an implication (see T 1.19 below) wrt $\sim [\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots)]$.

[†] Any binary relation over X ρ is said to be *transitive* iff $x\rho y \wedge y\rho z \Rightarrow x\rho z$ (for any $x,y,z \in X$), where $x\rho y$ iff $(x,y) \in \rho$. Obviously, $x(\rho \cap \sigma)y$ iff $x\rho y \wedge x\sigma y$ (for any $x,y \in X$).

It can be observed the rule for a direct proof is a particular case of the rule of an indirect proof. However, direct proofs (if they exist) are usually more simpler than the indirect ones*. And finally, the only universal are the indirect (ordinary or ramified) proofs. And hence, if some formula φ is a thesis then such a proof always exists.

1.3. **Theses and derived rules**

The propositional variables (e.g. p, q, r, ...), appearing in the proof of any thesis given below, can be considered as *metavariables* and hence the obtained results, i.e. rules, can be generalised for arbitrary propositional formulae (e.g. φ, ψ, χ, ...). We shall first consider the laws of double negation.

Thesis 1.3a (*rule of omitting double negation*)

$\sim\sim p \;\Rightarrow\; p$

Proof:

(1)	$\sim\sim p$	{a}	In general: $-\mathrm{N}: \dfrac{\sim\sim\varphi}{\varphi}$
(2)	$\sim p$	{aip}	
	contr. □	{1,2}	

Thesis 1.3b (*rule of joining double negation*)

$p \;\Rightarrow\; \sim\sim p$

Proof:

(1)	p	{a}	In general: $+\mathrm{N}: \dfrac{\varphi}{\sim\sim\varphi}$
(2)	$\sim\sim\sim p$	{aip}	
(3)	$\sim p$	{T 1.3a}	
	contr. □	{1,3}	

Thesis 1.3

$\sim\sim p \;\Leftrightarrow\; p$. □ {+ E : T 1.3a , T 1.3b} The Stoic School

In accordance with the rule of detachment for implication and T 1.3a. , the following *law of reduction ad absurdum* can be also obtained (left to the reader).

Thesis 1.4

$(p \;\Rightarrow\; q \;\wedge \sim q) \;\Rightarrow\; \sim p$. □

In a (direct or indirect) proof from assumptions some additional assumptions can be also used. Let φ be an additional assumption of the corresponding proof and χ is obtained as a consequence of the previous lines and φ . Then the implication $\varphi \;\Rightarrow\; \chi$ may be jointed to the proof. We shall say the *rule of joining an implication* (denoted below by ′+ C′) has been used in the proof. This rule corresponds to the deduction theorem extended by

* For example, a typical indirect (ramified) proof of the following expression: $((p \vee q) \wedge r \Rightarrow s) \Rightarrow (p \wedge r \Rightarrow s) \wedge (q \wedge r \Rightarrow s)$, i.e. the rule '$-$ A_a' of removing a disjunction in the antecedent of a sequent (see Subsection 1.8) would require 20 lines. On the other hand, the corresponding direct proof would require only 3 lines. In fact, since $\wedge$ is distributive over $\vee$, by using the rule of addition of antecedents AA, SR and $-$ E the proof can be completed (the proof is left to the reader). But sometimes, obtained numbers of proof lines may be similar, e.g. see. the attached two proofs of the law of negating an implication T 1.19a given in the next subsection.

rules $\pm$ K, $\pm$ A, and $\pm$ E (see Theorem 1.31 and the comments at the end of Subsection 1.7: a more formal treatment is omitted here)*.

For simplicity an additional assumption will be preceded by some double number of the form ´i.j´ (to denote the j^{th} line of the i^{th} additional assumption; i,j = 1,2,...). It can be observed that any additional assumption, say φ, introduced as an "arbitrary formula" should be related to the considered proof. And so, we have some allowable degree of arbitrariness, e.g. it is not possible to accept ϕ_n as an additional assumption of the proof (i.e. the right side of the main implication) or also an expression φ which is in contradiction with some of the previous lines of this proof.

The rule + C is illustrated in the proof of the first implication of the next thesis called "law of multiplication of consequents" (the proof of the converse implication T 1.5b can be realised as a direct proof from assumptions similarly to T 1.1, so this is omitted).

Thesis 1.5 (*law of multiplication of consequents*: MC)

$p \Rightarrow q \wedge r \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow r)$

In accordance with the rule of omitting an equivalence, i.e. – E, the following two implications have to be proven.

Thesis 1.5a

$(p \Rightarrow q \wedge r) \Rightarrow (p \Rightarrow q) \wedge (p \Rightarrow r)$

Thesis 1.5b

$(p \Rightarrow q) \wedge (p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r)$

Proof T 1.5a:

(1)	$p \Rightarrow q \wedge r$	{a}
(1.1)	p	{ada}
(1.2)	$q \wedge r$	{– C : 1,1.1}
(1.3)	q	{1.3,1.4 / – K : 1.2}
(1.4)	r	
(2)	$p \Rightarrow q$	{+ C : 1.1⇒1.3}
(3)	$p \Rightarrow r$	{+ C : 1.1⇒1.4}
	$(p \Rightarrow q) \wedge (p \Rightarrow r)$. □	{+ K : 2,3}

Proof T 1.5a (a direct proof without using + C)*:*

(1)	$p \Rightarrow q \wedge r$	{a}
(2)	$\sim p \vee q \wedge r$	{CR : 1, i.e. according to the law of implication, see T 1.15 }
(3)	$(\sim p \vee q) \wedge (\sim p \vee r)$	{∨ is distributive over ∧}
	$(p \Rightarrow q) \wedge (p \Rightarrow r)$. □	{CR,SR : 3}

Here SR is the substitution rule (see: remarks associated with T 1.18). However, the corresponding indirect proof without using + C may be more complicated as it is illustrated below.

Proof T 1.5a (an indirect version)*:*

The used rules NK and NC (negating a conjunction or an implication, respectively) and the construction of ramified proofs with joined additional assumptions are described in the next considerations (see below: T 1.8 and T 1.19, respectively).

* Let p and q be two propositional variables corresponding to two, not necessarily adjacent, proof's lines. And so, by using '+ K' we have the conjunction $p \wedge q$. Since '$p \wedge q \Rightarrow (p \Rightarrow q)$' is a thesis, using '– C' the following implication can be obtained: $p \Rightarrow q$. It can be observed that this proof style do not corresponds to the above rule of joining an implication '+ C'.

(1)	$p \Rightarrow q \wedge r$	{a}
(2)	$\sim ((p \Rightarrow q) \wedge (p \Rightarrow r))$	{aip}
(3)	$\sim (p \Rightarrow q) \vee \sim (p \Rightarrow r)$	{NK : 2}
(1.1)	$\sim (p \Rightarrow q)$	{ada}
(1.2)	$p \wedge \sim q$	{NC : 1.1}
(1.3)	p	{1.3,1.4 / − K : 1.2}
(1.4)	$\sim q$	
(1.5)	$q \wedge r$	{− C : 1,1.3}
(1.6)	q	{1.6,1.7 / − K : 1.5}
(1.7)	r	
	contr.	{1.4,1.6}
(2.1)	$\sim (p \Rightarrow r)$	{ada}
(2.2)	$p \wedge \sim r$	{NC : 2.1}
(2.3)	p	{2.3,2.4 / − K : 2.2}
(2.4)	$\sim r$	
(2.5)	$q \wedge r$	{−C : 1,2.3}
(2.6)	q	{2.6,2.7 / − K : 2.5}
(2.7)	r	
	contr. □	{2.4,2.7}

It is possible to reduce the above proof wrt the lines (1.7) and (2.6) (see the rule of omitting a conjunction). The proof of T 1.5b is left to the reader. In fact, in accordance with the second above given direct proof version, all (algebraic) transformations, i.e. used derived rules are equivalencies. And hence, the opposite implication, i.e. T 1.5b is also satisfied. The following equivalence is also a thesis (*law of addition of consequents*, AC: the proof is left to the reader): $p \Rightarrow q \vee r \Leftrightarrow (p \Rightarrow q) \vee (p \Rightarrow r)$.

In accordance with T 1.5, the proof of any implication of the form $\varphi \Rightarrow \psi \wedge \chi$ can be replaced (and of course simplified) by equivalently proving the following corresponding two implications: $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow \chi$. In fact, assume that $\vDash \varphi \Rightarrow \psi$ and $\vDash \varphi \Rightarrow \chi$. Then according to the last two lines, the rule + K, and T 1.5, we have: $\vDash \varphi \Rightarrow \psi \wedge \chi$*. And so, this observation can be also extended in the case of proving implications on predicates, sets or non-standard formulae, e.g. the Hauber's law in Ł-, G -, and π - BL, in particular in G - and π - BL, where the classical law of negating an implication NC, i.e. T 1.19, is satisfied only in the case of Ł- BL: see Subsection 2.2). As an example, in the case of classical logic Hauber's law, a corresponding proof is given below: see the attached direct proof of T 1.13 using the law of multiplication of consequents T 1.5. It can be observed that a similar approach is possible in the case of T 1.9 (law of addition of antecedents: this is left to the reader).

According to T 1.3a and the rule of detachment for implication, i.e. − C , the following rule can be obtained.

Thesis 1.6 (*rule modus tollendo tollens*)

$(p \Rightarrow q) \wedge \sim q \Rightarrow \sim p$

The Stoic School, Boethius
A.M.S. etc.

Proof:

(1)	$p \Rightarrow q$	{1,2 / a}
(2)	$\sim q$	
(3)	$\sim \sim p$	{aip}
(4)	p	{− N : 3}
(5)	q	{− C : 1,4}
	contr. □	{2,5}

In general: $\text{Toll}: \dfrac{\varphi \Rightarrow \psi \quad \sim \psi}{\sim \varphi}$

* This symbol ´⊨´ concerns the whole expression on the right

It can be observed that the proof of T 1.6 follows directly from T 1.14a (according to the law of contraposition of implication), next using T 1.12b (the law of importation).

The following two very important theses, called *De Morgan´s laws* , can be obtained (Augustus De Morgan 1806 – 1871). In fact, they were already known in the middle ages (i.e. in the medieval period, e.g. W. Burleigh 1273 – 1357, William of Ockham 1285 – 1349, etc.). Only the proof of the first one is presented below (the proof of T 1.8 is left to the reader). More formally, the indirect proof of T 1.8a, in accordance with T 1.3, should require the use of substitution rule SR: introduced after T 1.18).

Thesis 1.7 (*the law of negating a disjunction*)

$\sim (p \lor q) \Leftrightarrow \sim p \land \sim q$

Thesis 1.8 (*the law of negating a conjunction*)

$\sim (p \land q) \Leftrightarrow \sim p \lor \sim q$

In accordance with the rule of omitting an equivalence, i.e. – E , the following two implications are obtained.

Thesis 1.7a

$\sim (p \lor q) \Rightarrow \sim p \land \sim q$

Thesis 1.7b

$\sim p \land \sim q \Rightarrow \sim (p \lor q)$

Let χ and $\sim\chi$ be two contradictory lines obtained for some additional assumption φ . Hence, by using rules + K and + C the following line can be obtained: $\varphi \Rightarrow \chi \land \sim\chi$. According to T 1.4 and the rule of detachment – C , as a next line of the proof, the formula $\sim \varphi$ can be used. This is illustrated in the proof of T 1.7a below.

Proof T 1.7a:

(1)	$\sim (p \lor q)$	{a}
(1.1)	p	{ada}
(1.2)	$p \lor q$	{+ A : 1.1}
(2)	$\sim p$	{1.1 $\Rightarrow$ contr.(1,1.2)}
(2.1)	q	{ada}
(2.2)	$p \lor q$	{+ A : 2.1}
(3)	$\sim q$	{2.1 $\Rightarrow$ contr.(1,2.2)}
	$\sim p \land \sim q$. □	{+ K : 2,3}

In general: $NA: \dfrac{\sim(\varphi \lor \psi)}{\sim\varphi \setminus \sim\psi \setminus {\sim\varphi \atop \sim\psi}}$

It is easily to show the $\lor$ and $\land$ logical operations satisfy the *commutative, associative, absorptive, idempotent, and distributive axioms.* In fact, any system $(P ; \lor , \land)$ can be considered as a Boolean finite lattice, where $P \neq \varnothing$[*] is an arbitrary set of formulae such that for any $p, q \in P$ we have: $p \lor q$, $p \land q \in P$ (we shall say the logical operations $\lor$ and $\land$ are *closed* in P , e.g. see Kerntopf P. 1967). Hence, for any $p, q, r \in P$ we have: $p \lor q \Leftrightarrow q \lor p$, $p \land q \Leftrightarrow q \land p$, $(p \lor q) \lor r \Leftrightarrow p \lor (q \lor r)$, $(p \land q) \land r \Leftrightarrow p \land (q \land r)$, $p \Leftrightarrow p \lor p \land q$, $p \Leftrightarrow p \land (p \lor q)$, $p \Leftrightarrow p \lor p$, $p \Leftrightarrow p \land p$, $p \land (q \lor r) \Leftrightarrow p \land q \lor p \land r$, and $p \lor q \land r \Leftrightarrow (p \lor q) \land (p \lor r)$ (e.g. the line (2.2) in the proof of T 1.7a is obtained in accordance with the commutative axiom for disjunction). The corresponding proofs are left to the reader. In general, the notion of a Boolean algebra is used (George Boole 1815 – 1864). So, let $P =_{df} \{0,1, p_1, p_2 , ..., \sim p_1 , ... , p_1 \land p_2 , ...\}$ be the set of all formulae containing 0,1, and the propositional variables p_1, p_2 , ..., closed wrt the logical operations $\lor$, $\land$,

[*] ´∅´ denotes "*empty set*". In general, the word "*empty*" may be used in other contexts, e.g. such as: empty formula, empty word, empty string, empty cover, empty graph, empty algebraic system, empty domain, empty music (*4'33"*: *four minutes, thirty-three second:* John Cage 1912 – 1992), etc.

and $\sim$ (it can be observed P is not finite). Hence, the corresponding algebraic system can be considered as a *Boolean algebra* (Mostowski and Pawlak 1970)[*].

Proof T 1.7b:

(1)	$\sim p$	{1,2 / a}
(2)	$\sim q$	
(3)	$\sim\sim (p \vee q)$	{aip}
(4)	$p \vee q$	{– N : 3}
(5)	q	{– A : 1,4}
	contr. □	{2,5}

For simplicity, in the next considerations we shall omit lines like (3) in the proof of T 1.7b assuming directly line (4) as an {aip}. According to T 1.8, the following rule of negating a conjunction can be obtained.

$$\text{NK}: \quad \frac{\sim(\varphi \wedge \psi)}{\sim\varphi \vee \sim\psi}$$

In accordance with the previous used convention , the above rules NA and NK can be considered as ′– NA′ and ′– NK′ in comparison with the rules + NA and + NK given below[†].

$$+\ \text{NA}: \quad \frac{\sim\varphi \wedge \sim\psi}{\sim(\varphi \vee \psi)}, \quad +\ \text{NK}: \quad \frac{\sim\varphi \vee \sim\psi}{\sim(\varphi \wedge \psi)}.$$

The following law is satisfied.

Thesis 1.9 (*law of addition of antecedents*: AA)

$p \vee q \Rightarrow r \;\Leftrightarrow\; (p \Rightarrow r) \wedge (q \Rightarrow r)$

Proof T 1.9a:

(1)	$p \vee q \Rightarrow r$	{a}
(1.1)	p	{ada}
(1.2)	$p \vee q$	{+ A : 1.1}
(1.3)	r	{– C : 1,1.2}
(2)	$p \Rightarrow r$	{+ C : 1.1⇒1.3}
(2.1)	q	{ada}
(2.2)	$p \vee q$	{+ A : 2.1}
(2.3)	r	{– C : 1,2.2}
(3)	$q \Rightarrow r$	{+ C : 2.1⇒2.3}
	$(p \Rightarrow r) \wedge (q \Rightarrow r)$. □	{+ K : 2,3}

Since $\wedge$ and $\vee$ are commutative and mutually distributive, the proof of T 1.9a (and hence T 1.9) can be simplified by using a similar approach as in T 1.5a (without + C) and this is left to the reader.

Proof T 1.9b:

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[*] It is assumed that any two equivalent formulae are identical. Hence, elements of this algebra are not single formulae ,but equivalence classes wrt the following binary relation $\rho \subseteq P \times P$: $\varphi_1 \rho \varphi_2 \Leftrightarrow_{df} (\varphi_1 \Leftrightarrow \varphi_2)$. Since ρ is an equivalence relation , the system $\mathscr{B} =_{df} (P/\rho\,;\, [0]_\rho, [1]_\rho\,;\, \vee', \wedge', \sim')$ is a *Boolean algebra*, where P/ρ is the quotient set wrt ρ and $\vee', \wedge', \sim'$ are the corresponding operations closed in P/ρ .

[†] Provided there is no ambiguity, a similar approach will be used for any derived rule having as a main connective in the corresponding thesis an equivalence, e.g. + CR wrt the law of implication CR, similarly + NC wrt the law of negating an implication NC (see T 1.15 and T 1.18 given below), etc.

(1)	$p \Rightarrow r$	
(2)	$q \Rightarrow r$	{1,2,3 / a}
(3)	$p \vee q$	
(4)	$\sim r$	{aip}
(5)	$\sim p$	{Toll : 1,4}
(6)	$\sim q$	{Toll : 2,4}
(7)	$\sim p \wedge \sim q$	{+ K : 5,6}
(8)	$\sim (p \vee q)$	{+ NA : 7}
	contr. □	{3,8}

It can be observed there may exist various proofs for a given thesis, e.g. an indirect version for T 1.9a (more complicated, like the indirect version of the proof of T 1.5a) or instead of using + K and + NA in the proof of T 1.9b, the use of the rule of omitting a disjunction ,i.e. – A wrt lines (3) and (5), etc. In the last case we can obtain some simplification of the proof (this is left to the reader). Since $\wedge$ and $\vee$ are associative logical operations, the above laws of multiplication of consequents and addition of antecedents (i.e. T 1.5 and T 1.9, respectively) can be generalised for a finite number (≥ 2) of propositional variables. The obtained laws are called *generalised laws of multiplication of consequents and addition of antecedents*.

Next we shall concentrate our attention on the construction *of ramified proofs with joined additional assumptions*. Consider the generalised form of a given expression. Let $\Phi =_{df} \{\varphi_1, \varphi_2, ..., \varphi_k\}$ be a finite set of additional assumptions. Also let $\varphi_1 \vee \varphi_2 \vee ... \vee \varphi_k$ be one of the proof lines. Then:

(i) a *ramified direct proof from assumptions* of this expression is recognised to be complete if the formula ϕ_n have been obtained as a consequence of each $\varphi \in \Phi$,

(ii) a *ramified indirect proof from assumptions* of this expression is recognised to be complete if a contradiction have been obtained as a consequence of each $\varphi \in \Phi$.

Let consider the *case (i).* Assume that ϕ_n have been obtained as a consequence of each $\varphi \in \Phi$. So, by using rule + C the k additional lines can be obtained: $\varphi \Rightarrow \phi_n$ (for each $\varphi \in \Phi$). According to the (generalised form of the) law of addition of antecedents we have: $\varphi_1 \vee \varphi_2 \vee ... \vee \varphi_k \Rightarrow \phi_n$. Finally, by rule of detachment – C wrt the last line and $\varphi_1 \vee \varphi_2 \vee ... \vee \varphi_k$, the formula ϕ_n can be obtained as the last proof line.

Let a contradiction have been obtained as a consequence of each $\varphi \in \Phi$. So, in accordance with T 1.4. to prove *case (ii)* the negation of each $\varphi \in \Phi$ can be joined, i.e. the following new proof lines can be obtained: $\sim \varphi_1$, $\sim \varphi_2$, ... , $\sim \varphi_k$. Next by using (k – 1) times the rule of omitting a disjunction – A two contradictory lines can be obtained: φ_k and $\sim \varphi_k$ (Słupecki J. and Borkowski L. 1967).

It can be observed that for $k \geq 4$ the proof of case (ii) can be simplified if instead of using (k – 1) times – A, the (generalised) rules + K and + NA are used (wrt the proof lines $\sim \varphi_1$, $\sim \varphi_2$, ... , $\sim \varphi_k$). Hence we can obtain the following two contradictory lines: $\varphi_1 \vee \varphi_2 \vee ... \vee \varphi_k$ and $\sim (\varphi_1 \vee \varphi_2 \vee ... \vee \varphi_k)$.

The following thesis (known in ancient times) is an illustration of the ramified direct proof from assumptions. An illustration of the ramified indirect proof from assumptions is the indirect version of the proof of T 1.5a.

Thesis 1.10 (*law of compound constructive dilemma*)

$(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \Rightarrow q \vee s$

Proof:

(1)	$p \Rightarrow q$	
(2)	$r \Rightarrow s$	{1,2,3 / a}
(3)	$p \vee r$	
(1.1)	p	{ada}
(1.2)	q	{– C : 1,1.1}

(1.3)	$q \vee s$	{+ A : 1.2}
(2.1)	r	{ada}
(2.2)	s	{– C : 2,2.1}
(2.3)	$q \vee s$	{+ A : 2.2}
	$q \vee s$. □	{1.3,2.3}

The last thesis can be also proven without using the ramified proof rule as it is shown below.

Proof T 1.10 (an indirect version)*:*

(1)	$p \Rightarrow q$	
(2)	$r \Rightarrow s$	{1,2,3 / a}
(3)	$p \vee r$	
(4)	$\sim (q \vee s)$	{aip}
(5)	$\sim q$	{5,6 / NA : 4}
(6)	$\sim s$	
(7)	$\sim p$	{Toll : 1,5}
(8)	$\sim r$	{Toll : 2,6}
(9)	$\sim p \wedge \sim r$	{+ K : 7,8}
(10)	$\sim (p \vee r)$	{+ NA : 9}
	contr. □	{3,10}

The proof of the next two theses is left to the reader. Is the law $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$ usable in the indirect proof of T 1.11? Give a direct proof of T 1.11 without using any additional assumptions. Similarly, is the law $(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow (p \vee r \Rightarrow q \vee s)$ usable in the direct proof of T 1.10?

Thesis 1.11 (*law of compound destructive dilemma*)

$(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge \sim (q \vee s) \Rightarrow \sim (p \vee r)$. □

Thesis 1.12 (*laws of exportation and importation*: implications (a) and (b), respectively)

$p \wedge q \Rightarrow r \Leftrightarrow p \Rightarrow (q \Rightarrow r)$. □

According to T 1.12, the following formula is an equivalent version of the law of compound constructive dilemma: $(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow (p \vee r \Rightarrow q \vee s)$ (in a similar way for T 1.11). The following two examples are an application of the ramified direct/indirect proofs with joined additional assumptions.

Example 1.2

Let ρ and σ be two *binary relations over* X (see Example 1.1). Assume that ρ and σ are symmetric*. Then the union $\rho \cup \sigma$ is also a symmetric relation over X . So, the following implication have to be shown (for any $x,y \in X$):

$(x\rho y \Rightarrow y\rho x) \wedge (x\sigma y \Rightarrow y\sigma x) \Rightarrow (x\rho y \vee x\sigma y \Rightarrow y\rho x \vee y\sigma x)$

This formula is an equivalent version of the law of compound constructive dilemma (T 1.10). The proof follows immediately from the law AAC (of addition of the antecedents and consequents of two implications: see T 1.20 given below). □

An application of the ramified indirect proof from assumptions is given in the next example (Tabakow 2001).

* Any binary relation over X ρ is said to be *symmetric* iff $x\rho y \Rightarrow y\rho x$ (for any $x,y \in X$), Obviously, $x(\rho \cup \sigma)y$ iff $x\rho y \vee x\sigma y$ (for any $x,y \in X$).

Example 1.3

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be an n-variable two-valued logic function. Assume that C is any circuit realisation of $f = f(\underline{x}) \in \{0.1\}$ where $\underline{x} =_{df} (x_1,...,x_n) \in \{0,1\}^n$ is the corresponding primary input vector. For C which realises the function $f(\underline{x})$, a *logical type fault* $\alpha \in \mathcal{F}$ changes the function realised to $f^{\alpha}(\underline{x})$ (a new n-variable two-valued logic function called a *faulty function*), where $\mathcal{F}$ is the set of all possible such faults. Next for any $\alpha \in \mathcal{F}$ by $T(\alpha) =_{df} \{ \underline{x} \in \{0,1\}^n / f(\underline{x}) \neq f^{\alpha}(\underline{x}) \}$ we shall denote the set of tests which *detect* α. Similarly, by $T(\alpha \not\approx \beta) =_{df} \{ \underline{x} \in \{0,1\}^n / f^{\alpha}(\underline{x}) \neq f^{\beta}(\underline{x}) \}$ we shall denote the set of tests that *distinguish* α and β (for any $\alpha, \beta \in F$, $\alpha \neq \beta$). The following implication is satisfied.

$\underline{x} \in T(\alpha) \cap T(\beta) \Rightarrow \underline{x} \notin T(\alpha \not\approx \beta)$ (for any $\underline{x} \in \{0,1\}^n$, $\alpha, \beta \in \mathcal{F}, \alpha \neq \beta$).

Proof:

(1)	$\underline{x} \in T(\alpha) \cap T(\beta)$	{a}
(2)	$\underline{x} \in T(\alpha \not\approx \beta)$	{aip}
(3)	$f(\underline{x}) = 0 \vee f(\underline{x}) = 1$	{df $f(\underline{x})$ }
(4)	$\underline{x} \in T(\alpha) \wedge \underline{x} \in T(\beta)$	{df ´∩´ : 1}
(5)	$\underline{x} \in T(\alpha)$	{5,6 / – K : 4}
(6)	$\underline{x} \in T(\beta)$	
(7)	$f(\underline{x}) \neq f^{\alpha}(\underline{x})$	{df T(α) : 5}
(8)	$f(\underline{x}) \neq f^{\beta}(\underline{x})$	{df T(β) : 6}
(9)	$f^{\alpha}(\underline{x}) \neq f^{\beta}(\underline{x})$	{df $T(\alpha \not\approx \beta)$: 2}
(1.1)	$f(\underline{x}) = 0$	{ada}
(1.2)	$f^{\alpha}(\underline{x}) = f^{\beta}(\underline{x})$	{3,7,8}
	contr.	{9,1.2}
(2.1)	$f(\underline{x}) = 1$	{ada}
(2.2)	$f^{\alpha}(\underline{x}) = f^{\beta}(\underline{x})$	{3,7,8}
	contr. □	{9,2.2}

It can be observed for an arbitrary *n-variable m-valued discrete logic function* $f : M^n \rightarrow M$ the above implication is not satisfied, where $M =_{df} \{0,1, ... ,m-1\}$ and $m \geq 2$. In fact, the above property is satisfied only for $m = 2$. □

The algorithm proving is another field of application of the logic rules. This is illustrated in the next example, where the well-known Donald Knuth´s version of the *Euclid´s algorithm* is presented (Euclides of Alexandria, a native of Megara: 430 b.c. – c.360 b.c.).

Example 1.4

Consider the following algorithm (without loss of generality, it is assumed below that a,b ∈ ℕ, the set of natural numbers: ´gcd´ denotes "*greatest common divisor*").

Input: a,b ∈ ℕ

Output: gcd(a,b) ∈ ℕ

(1) Let $p =_{df} a$ and $q =_{df} b$ (*initial step*) ; (2) Assume that $r =_{df}$ (p) modulo q ; (3) If $r = 0$ then $\gcd(a,b) =_{df} q$. End ; (4) Let $p =_{df} q$ and $q =_{df} r$. Go to (2) . □*

---

* The Euclid´s algorithm: (1) If $a > b$ then $a =_{df} a - b$. Go to (1) ; (2) If $a = b$ then $\gcd(a,b) =_{df} a$. End ; (3) $b =_{df} b - a$. Go to (1). □ It can be observed these two algorithms have the same behaviour. We shall say they are behavioural-*equivalent*, i.e. the outputs are the same for any input.

According to the above given algorithm it is sufficient to consider the case $a > b$.

Let:	$r_1 =_{df} a - c_1 b$	=	(a) modulo b ,	$r_1 = 0$	$\Rightarrow q = b$
	$r_2 =_{df} b - c_2 r_1$	=	(b) modulo r_1 ,	$r_2 = 0$	$\Rightarrow q = r_1$
	$r_3 =_{df} r_1 - c_3 r_2$	=	(r_1) modulo r_2 ,	$r_3 = 0$	$\Rightarrow q = r_2$
	...				
	$r_n =_{df} r_{n-2} - c_n r_{n-1}$	=	(r_{n-2}) modulo r_{n-1} .	$r_n = 0$	$\Rightarrow q = r_{n-1}$

For example, for $a = 1026$ and $b = 580$ we can obtain: $r_1 = 446, r_2 = 134, r_3 = 44, r_4 = 2$, and $r_5 = 0$. Hence gcd(1026,580) = 2. Similarly, assuming $a = 551$ and $b = 64$ we have: $r_1 = 39, r_2 = 25, r_3 = 14, r_4 = 11, r_5 = 3, r_6 = 2, r_7 = 1$, and $r_8 = 0$. So gcd(551,64) = 1, etc.

Provided there is no ambiguity, let $r_0 =_{df} b$. It can be observed $r_{k-1} > r_k$ (for any $k = 1,2, \ldots ,n$). It can be shown for any $a,b \in N$, there exists some step k such that $r_k = 0$ (i.e. the number of iterations required to reach a fixed point is finite and the algorithm converges: this is left to the reader).Moreover, the following property is satisfied.

Proposition (a): For any step $k = 1,2, \ldots ,n$:
$a = r_{k-1}\alpha(k) + r_k\gamma(k)$ and $b = r_{k-1}\beta(k) + r_k\delta(k)$,
where $\alpha(k)$, $\beta(k)$, $\gamma(k)$, and $\delta(k)$ are some sum-of-products forms wrt the c_i´s.

In fact, let $a = r_{s-1}\alpha(s) + r_s\gamma(s)$ and $b = r_{s-1}\beta(s) + r_s\delta(s)$. Since $r_n = r_{n-2} - c_n r_{n-1}$, for $n =_{df} s + 1$ we can obtain: $r_{s+1} = r_{s-1} - c_{s+1} r_s$. Hence $r_{s-1} = r_{s+1} + c_{s+1} r_s$ and so we have: $a = (r_{s+1} + c_{s+1} r_s)\alpha(s) + r_s\gamma(s)$ and $b = (r_{s+1} + c_{s+1} r_s)\beta(s) + r_s\delta(s)$. Then $a = r_s\alpha(s+1) + r_{s+1}\gamma(s+1)$ and $b = r_s\beta(s+1) + r_{s+1}\delta(s+1)$, where: $\alpha(s+1) =_{df} c_{s+1}\alpha(s) + \gamma(s)$, $\gamma(s+1) =_{df} \alpha(s)$, $\beta(s+1) =_{df} c_{s+1}\beta(s) + \delta(s)$, and $\delta(s+1) =_{df} \beta(s)$. □

For example:

k	$\alpha(k)$	$\beta(k)$	$\gamma(k)$	$\delta(k)$
1	c_1	1	1	0
2	$1 + c_1c_2$	c_2	c_1	1
3	$c_1 + c_3 + c_1c_2c_3$	$1 + c_2c_3$	$1 + c_1c_2$	c_2
etc.				

The following implication is satisfied.

Proposition (b): $r_n = 0 \Rightarrow \gcd(a,b) = q = r_{n-1}$ (for any $n \in \mathbb{N}$)

Proof:

The proof is inductive wrt n, e.g. for $n = 3$, assuming $r_3 = 0$ we can obtain $r_1 = c_3 r_2$. Hence $a = r_2 (c_1 + c_3 + c_1c_2c_3)$, $b = r_2 (1 + c_2c_3)$ and $q = r_2 = \gcd(a,b)$. In general the following implication have to be proven (the *inductive step* of the proof):

$$(r_k = 0 \Rightarrow \gcd(a,b) = r_{k-1}) \Rightarrow (r_{k+1} = 0 \Rightarrow \gcd(a,b) = r_k)$$

So we have:

(1)	$r_k = 0 \Rightarrow \gcd(a,b) = r_{k-1}$	
(2)	$r_{k+1} = 0$	{1,2 / a}
(3)	$\gcd(a,b) \neq r_{k-1} \Rightarrow r_k \neq 0$	{T 1.14: see below}
(1.1)	$\gcd(a,b) \neq r_{k-1}$	{ada}
(1.2)	$r_k \neq 0$	{– C : 3,1.1}

(4) $a = r_{k-1}\alpha(k) + r_k\gamma(k)$ and $b = r_{k-1}\beta(k) + r_k\delta(k)$ {Prop.(a)}
(5) $r_{k+1} = r_{k-1} - c_{k+1} r_k$ {df. r_n, $n =_{df} k+1$}
(6) $r_{k-1} = c_{k+1} r_k$ {2}
(7) $a = c_{k+1} r_k \alpha(k) + r_k\gamma(k)$ and $b = c_{k+1} r_k \beta(k) + r_k\delta(k)$ {4,6}
(8) $a = r_k \alpha(k+1)$ and $b = r_k \beta(k+1)$, where: {7}
$\alpha(k+1) =_{df} c_{k+1}\alpha(k) + \gamma(k)$ and $\beta(k+1) =_{df} c_{k+1}\beta(k) + \delta(k)$.
$\gcd(a,b) = r_k$. □ {8}

It can be observed that instead of the existing algorithm proving techniques, in general the last approaches may be nontrivial.

The following law of converting implications is satisfied (called also *law of a closed system of theorems* or *Hauber´s law*: Hauber E.D. 1695 - 1765).

Thesis 1.13 (*law of conversion of implications*)

$(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \wedge \sim(q \wedge s) \Rightarrow (q \Rightarrow p) \wedge (s \Rightarrow r)$

Proof (Słupecki J. and Borkowski L. 1967):

(1)	$p \Rightarrow q$	
(2)	$r \Rightarrow s$	{1,2,3,4 / a}
(3)	$p \vee r$	
(4)	$\sim(q \wedge s)$	
(5)	$\sim q \vee \sim s$	{NK : 4}
(1.1)	q	{ada}
(1.2)	$\sim s$	{– A : 5,1.1}
(1.3)	$\sim r$	{Toll : 2,1.2}
(1.4)	p	{– A : 3,1.3}
(6)	$q \Rightarrow p$	{+ C : 1.1⇒1.4}
(2.1)	s	{ada}
(2.2)	$\sim q$	{– A : 5,2.1}
(2.3)	$\sim p$	{Toll : 1,2.2}
(2.4)	r	{– A : 3,2.3}
(7)	$s \Rightarrow r$	{+ C : 2.1⇒2.4}
	$(q \Rightarrow p) \wedge (s \Rightarrow r)$. □	{+ K : 6,7}

In general: $C^{-1}: \dfrac{\varphi_1 \Rightarrow \psi_1 \quad \varphi_2 \Rightarrow \psi_2 \quad \varphi_1 \vee \varphi_2 \quad \sim(\psi_1 \wedge \psi_2)}{\psi_1 \Rightarrow \varphi_1 \quad \psi_2 \Rightarrow \varphi_2}$

The proof of the next five theses is left to the reader (T 1.17 and T 1.18 are some particular cases of the *laws of extensionality of equivalence*). According to T 1.14 and T 1.15, it can be also observed the implication is *transitive*, i.e. $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$. By using T 1.12, the following equivalent form of the last implication can be obtained, called *'first law of the hypothetical syllogism'* (or *conditional syllogism*: known in ancient times): $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

Thesis 1.14 (*law of transposition* or *contraposition of implication:* CC)

$p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$. □

By using CC we have:

$$\begin{aligned}(p \Leftrightarrow q) &\Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)\\ &\Leftrightarrow (\sim q \Rightarrow \sim p) \wedge (\sim p \Rightarrow \sim q)\\ &\Leftrightarrow (\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow \sim p)\\ &\Leftrightarrow (\sim p \Leftrightarrow \sim q). \;\square\end{aligned}$$

And so, the following *law of transposition* or *contraposition of equivalence* is obtained: CE : $\dfrac{\varphi \Leftrightarrow \psi}{\sim \varphi \Leftrightarrow \sim \psi}$.

Thesis 1.15 (*law of implication*)

$p \Rightarrow q \Leftrightarrow \sim p \vee q$. ▫

In general: $CR : \dfrac{\varphi \Rightarrow \psi}{\sim \varphi \vee \psi}$

The following law is also satisfied: $p \Rightarrow q \Leftrightarrow (p \wedge q \Leftrightarrow p)$. The proof is left to the reader.

By using T 1.15 another direct proof of T 1.13 can be obtained. This is illustrated below.

Proof T 1.13 (a direct proof)*:*

(1)	$p \Rightarrow q$	{1,2,3,4 / a}
(2)	$r \Rightarrow s$	
(3)	$p \vee r$	
(4)	$\sim (q \wedge s)$	
(5)	$\sim q \vee \sim s$	{NK : 4}
(1.1)	p	{ada}
(1.2)	q	{– C : 1,1.1}
(1.3)	$\sim s$	{– A : 5,1.2}
(1.4)	$\sim s \vee r$	{+ A : 1.3}
(1.5)	$p \vee \sim q$	{+ A : 1.1}
(1.6)	$s \Rightarrow r$	{CR : 1.4}
(1.7)	$q \Rightarrow p$	{CR : 1.5}
(1.8)	$(q \Rightarrow p) \wedge (s \Rightarrow r)$	{+ K : 1.6, 1.7}
(2.1)	r	{ada}
(2.2)	s	{– C : 2,2.1}
(2.3)	$\sim q$	{– A : 5,2.2}
(2.4)	$\sim q \vee p$	{+ A : 2.3}
(2.5)	$r \vee \sim s$	{+ A : 2.1}
(2.6)	$q \Rightarrow p$	{CR : 2.4}
(2.7)	$s \Rightarrow r$	{CR : 2.5}
(2.8)	$(q \Rightarrow p) \wedge (s \Rightarrow r)$	{+ K : 2.6, 2.7}
	$(q \Rightarrow p) \wedge (s \Rightarrow r)$. ▫	{1.8,2.8}

Proof T 1.13 (a direct proof usingToll)*:*

1)	$p \Rightarrow q$	{1,2,3,4 / a}
(2)	$r \Rightarrow s$	
(3)	$p \vee r$	
(4)	$\sim (q \wedge s)$	
(5)	$\sim q \vee \sim s$	{NK : 4}
(1.1)	$\sim q$	{ada}
(1.2)	$\sim q \vee p$	{+ A : 1.1}
(1.3)	$q \Rightarrow p$	{CR : 1.2}
(1.4)	$\sim p$	{Toll : 1,1.1}
(1.5)	r	{– A : 3,1.4}
(1.6)	$r \vee \sim s$	{+ A : 1.5}
(1.7)	$s \Rightarrow r$	{CR : 1.6}
(1.8)	$(q \Rightarrow p) \wedge (s \Rightarrow r)$	{+ K : 1.3,1.7}
(2.1)	$\sim s$	{ada}
(2.2)	$\sim s \vee r$	{+ A : 2.1}
(2.3)	$s \Rightarrow r$	{CR : 2.2}
(2.4)	$\sim r$	{Toll : 2,2.1}
(2.5)	p	{– A : 3,2.4}
(2.6)	$p \vee \sim q$	{+ A : 2.5}

(2.7) $q \Rightarrow p$ {CR : 2.6}
(2.8) $(q \Rightarrow p) \wedge (s \Rightarrow r)$ {+ K : 2.3,2.7}
$(q \Rightarrow p) \wedge (s \Rightarrow r)$. □ {1.8,2.8}

According to the notion of a ramified proof, it can be observed that sometimes we may have more than one possible alternative under consideration, e.g. lines (3) and (5) in the last proof, similarly in the indirect proof of the only-if-implication of the following thesis: $p \wedge (q \vee r) \Leftrightarrow p \wedge q \vee p \wedge r$ ($\wedge$ is distributive over $\vee$ and vice versa), etc. In general, any alternative may implicate a different proof effectiveness.

Thesis 1.16 (*detachment for equivalence*)

$(p \Leftrightarrow q) \wedge p \Rightarrow q$. □

In general: DE : $$\frac{\varphi \Leftrightarrow \psi \qquad \varphi}{\psi}$$

Thesis 1.17 (*extensionality of equivalence:* according to T 1.1 and T 1.14)

$(p \Leftrightarrow q) \Rightarrow (\sim p \Leftrightarrow \sim q)$. □

Thesis 1.18 (*extensionality of equivalence*, called below: *rule of extensionality*)

$(p \Leftrightarrow q) \wedge (r \Leftrightarrow s) \wedge (p \Leftrightarrow r) \Rightarrow (q \Leftrightarrow s)$. □

In general: ER : $$\frac{\varphi_1 \Leftrightarrow \psi_1 \qquad \varphi_2 \Leftrightarrow \psi_2 \qquad \varphi_1 \Leftrightarrow \varphi_2}{\psi_1 \Leftrightarrow \psi_2}$$

According to T 1.12, the above law of substitution for equivalence can be equivalently described as follows: $(p \Leftrightarrow q) \wedge (r \Leftrightarrow s) \Rightarrow ((p \Leftrightarrow r) \Rightarrow (q \Leftrightarrow s))$. It can be observed the converse implication of the right side is also satisfied, i.e. in general we have: $(p \Leftrightarrow q) \wedge (r \Leftrightarrow s) \Rightarrow ((p \Leftrightarrow r) \Leftrightarrow (q \Leftrightarrow s))$. Another form of the above *rule of extensionality* ER is the following *rule of substitution for equivalence* (*substitution rule* or *rule of replacement for equivalence*: Słupecki J. and Borkowski L. 1967).

SR : $$\frac{\varphi \Leftrightarrow \psi}{\chi \Leftrightarrow \chi(\varphi\ //\ \psi)}$$ (or equivalently: $$\frac{\varphi \Leftrightarrow \psi \qquad \chi(\psi)}{\chi(\varphi\ //\ \psi)}$$).

Here $\chi(\varphi\ //\ \psi)$ is obtained from χ by the replacement of its parts ψ by the formula φ, e.g. let χ corresponds to the law of compound constructive dilemma, i.e. $(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \Rightarrow q \vee s$, $r \Rightarrow s$ to ψ, and $\sim s \Rightarrow \sim r$ to φ. Then $(p \Rightarrow q) \wedge (\sim s \Rightarrow \sim r) \wedge (p \vee r) \Rightarrow q \vee s$ will correspond to $\chi(\varphi\ //\ \psi)$. In particular, the rule DE under T 1.16 can be considered, in a sense, as a special case of SR.

By using T 1.15, T 1.17, – N , and NA the following law can be proven.

Thesis 1.19 (*law of negating an implication*)

$\sim (p \Rightarrow q) \Leftrightarrow p \wedge \sim q$. □

In general: NC : $$\frac{\sim (\varphi \Rightarrow \psi)}{\varphi \setminus \sim \psi \setminus {\varphi \atop \sim \psi}}$$

Two kinds of proofs of T 1.19a are given below, i.e. direct and indirect. The first one uses the same technique as in the proof of T 1.7a (the proof of T 1.19b is left to the reader).

Proof T 1.19a (a direct proof):

(1)	$\sim (p \Rightarrow q)$	{a}
(1.1)	$\sim p$	{ada}
(1.2)	$\sim p \lor q$	{+ A : 1.1}
(1.3)	$p \Rightarrow q$	{CR : 1.2}
(2)	p	{1.1 $\Rightarrow$ contr.(1,1.3)}
(2.1)	q	{ada}
(2.2)	$q \lor \sim p$	{+ A : 2.1}
(2.3)	$p \Rightarrow q$	{CR : 2.2}
(3)	$\sim q$	{2.1 $\Rightarrow$ contr.(1,2.3)}
	$p \land \sim q$. □	{+ K : 2,3)

Proof T 1.19a (an indirect proof):

(1)	$\sim (p \Rightarrow q)$	{a}
(2)	$\sim (p \land \sim q)$	{aip}
(3)	$\sim p \lor q$	{NK, – N : 2}
(4)	$p \Rightarrow q$	{CR : 3}
	contr. □	{1,4}

Since $\vDash (\varphi \Leftrightarrow \psi) \Leftrightarrow (\sim \varphi \Leftrightarrow \sim \psi)$, according to DE (the rule of detachment for equivalence), any of the above two theses T 1.15 and T 1.19 can be proved by assuming that the another is satisfied. However, an independent proof of any of them would require an exact specification of the corresponding set of primary assumptions. For example, the following primary assumptions are associated with T 1.15b: $\phi_1 =_{df} \sim p \lor q$ and $\phi_2 =_{df} p$. In fact, if ϕ_2 have been omitted (using $\sim (p \Rightarrow q)$ as an assumption of indirect proof) then the proof of T 1.15b becomes depending on the proof of T 1.19a, but the last is depending on the proof of T 1.15b, etc. And hence, a cycle can be reached.

The rule NC was used in the indirect proof of T 1.5a. Another illustration may be the indirect proof of the law of conversion of implications (i.e. T 1.13, see below). The proofs of the laws *transitivity for implication*, TC (known also as *'second law of the hypothetical syllogism'*) : $(p \Rightarrow q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ *and transitivity for equivalence*, TE: $(p \Leftrightarrow q) \land (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ are left to the reader.

Proof T 1.13 (an indirect version)*:*

(1)	$p \Rightarrow q$	{1,2,3,4 / a}
(2)	$r \Rightarrow s$	
(3)	$p \lor r$	
(4)	$\sim (q \land s)$	
(5)	$\sim ((q \Rightarrow p) \land (s \Rightarrow r))$	{aip}
(6)	$\sim (q \Rightarrow p) \lor \sim (s \Rightarrow r)$	{NK : 5}
(7)	$q \land \sim p \lor s \land \sim r$	{NC, SR : 6}
(8)	$\sim q \lor \sim s$	{NK : 4}
(1.1)	q	{1.1,1.2 / ada}
(1.2)	$\sim p$	
(1.3)	$\sim s$	{– A : 8,1.1}
(1.4)	$\sim r$	{Toll : 2,1.3}
(1.5)	p	{– A : 3,1.4}
	contr.	{1.2,1.5}
(2.1)	s	{2.1,2.2 / ada}
(2.2)	$\sim r$	
(2.3)	$\sim q$	{– A : 8,2.1}
(2.4)	$\sim p$	{Toll : 1,2.3}
(2.5)	r	{– A : 3,2.4}
	contr. □	{2.2,2.5}

Proof T 1.13 (a direct proof using the law of multiplication of consequents T 1.5)*:*

The following two sub-theses should be proved: $(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \wedge \sim(q \wedge s) \Rightarrow (q \Rightarrow p)$ and $(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \wedge \sim(q \wedge s) \Rightarrow (s \Rightarrow r)$. The proof of the first one is illustrated below (the second proof is left to the reader).

(1)	$p \Rightarrow q$	
(2)	$r \Rightarrow s$	
(3)	$p \vee r$	{1,2,3,4,5 / a}
(4)	$\sim(q \wedge s)$	
(5)	q	
(6)	$\sim q \vee \sim s$	{NK : 4}
(7)	$\sim s$	{– A : 5,6}
(8)	$\sim r$	{Toll : 2,7}
	p. □	{– A : 3,8}

It accordance with the used approach, the set of all primary assumptions becomes redundant. In fact, the line (1), i.e. the implication '$p \Rightarrow q$' has not been used in the above proof (similarly '$r \Rightarrow s$' wrt the second sub-thesis). A ramified direct proof from assumptions can be obtained by using line (6): left to the reader.

The proof of the next theses is also left to the reader.

Thesis 1.20 (*addition of the antecedents and consequents of two implications*: AAC)

$(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow (p \vee r \Rightarrow q \vee s)$. □

In general: $\text{AAC}: \dfrac{\varphi_1 \Rightarrow \psi_1 \quad \varphi_2 \Rightarrow \psi_2}{\varphi_1 \vee \varphi_2 \Rightarrow \psi_1 \vee \psi_2}$

Is the law of addition of the antecedents and consequents of two implications usable in the indirect proof of T 1.13? Proof the following implication: $(p \vee r \Rightarrow q \vee s) \Rightarrow (p \Rightarrow q) \vee (r \Rightarrow s)$.

Thesis 1.21 (*multiplication of the antecedents and consequents of two implications*: MAC)

$(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow (p \wedge r \Rightarrow q \wedge s)$. □

In general: $\text{MAC}: \dfrac{\varphi_1 \Rightarrow \psi_1 \quad \varphi_2 \Rightarrow \psi_2}{\varphi_1 \wedge \varphi_2 \Rightarrow \psi_1 \wedge \psi_2}$

An illustration of the law of multiplication of the antecedents and consequents of two implications is given in the next example.

Example 1.5

Let M_1 and M_2 be two *isomorphic machines*, i.e. anyone can be simulated by the other. We have: $M_1 \text{ iso } M_2 \Leftrightarrow_{df} M_1 \text{ sim } M_2 \wedge M_2 \text{ sim } M_1$. We shall say that M_1 is a *submachine* wrt M_2, i.e. $M_1 \sqsubseteq M_2 \Leftrightarrow_{df} F_{M_1} \subseteq F_{M_2}$, where F_{M_i} is the set of functions realised by M_i $(i = 1,2)$. Also, we shall say that M_1 and M_2 are *equivalent*, i.e. $M_1 \approx M_2 \Leftrightarrow_{df} F_{M_1} = F_{M_2}$. The following property can be shown: $M_1 \text{ sim } M_2 \Rightarrow M_2 \sqsubseteq M_1$ (for any M_1 and M_2 . And so, in accordance with the law of multiplication of the antecedents and consequents of two implications (T 1.21) we can obtain: $M_1 \text{ iso } M_2 \Rightarrow M_1 \approx M_2$ (Pawlak Z. 1971). □

Thesis 1.22 (*law of contradiction, law of Duns Scotus*: John Duns Scotus 1266 – 1308)

$p \Rightarrow (\sim p \Rightarrow q)$*. □

In general: $\text{DS}: \dfrac{\varphi \quad \sim\varphi}{\psi}$

By using the law of importation and ′– C ′(see T 1.12) we can obtain: $p \wedge \sim p \Rightarrow q$ .Since $\vDash p \wedge \sim r \Rightarrow q \Leftrightarrow p \Rightarrow r \vee q$, for $r =_{df} p$ we have: $p \wedge \sim p \Rightarrow q \Leftrightarrow p \Rightarrow p \vee q$ (the proof is left to the reader: show that $(a^2 - 2ab = c^2 - 2bc) \Rightarrow (a = c) \vee (a + c = 2b)$ and $(a^2 - 2ab = c^2 - 2bc) \wedge (a \neq c) \Rightarrow (a + c = 2b)$ are equivalent, for any $a,b,c \in \mathbb{R}$). According to T 1.22, a contradictory proposition has any proposition as its consequence (i.e. everything follows from a contradiction). An illustration of the *rule* DS. is given in the next example.

Example 1.6

Let consider the following propositional formula:

$p \Rightarrow (q \Rightarrow (r \vee s \Rightarrow (\sim p \Rightarrow \sim q \wedge t)))$

Proof:

(1)	p	
(2)	q	{1,2,3,4 / a}
(3)	$r \vee s$	
(4)	$\sim p$	
	$\sim q \wedge t$. □	{DS.: 1,4}

Obviously, an indirect proof of this formula can be obtained immediately. In fact, assuming $\sim(\sim q \wedge t)$ we have a contradiction wrt (1) and (4).

Consider the following two truth functors ′/′ and ′↓′ called *symbol of alternative negation* and *symbol of joint negation*, respectively. The formula $p/q \Leftrightarrow_{df} \sim(p \wedge q)$ (is read ′*not both* p *and* q′: known as the *Sheffer′s dash*: Henry Maurice Sheffer 1882 - 1964) and the formula $p{\downarrow}q \Leftrightarrow_{df} \sim(p \vee q)$ (is read ′*neither* p *nor* q′: known as the *Peirce′s arrow*, Charles Sanders Peirce 1839 - 1914). It can be observed the last two functors differ from the other truth functors of two arguments in that either suffices to define all the remaining truth functors (this is left to the reader). We shall say these two functors are *functionally complete*. Functionally complete logical connectives are also negation and implication, since $p \wedge q \Leftrightarrow_{df} \sim(p \Rightarrow \sim q)$ and $p \vee q \Leftrightarrow_{df} \sim p \Rightarrow q$ (the *Łukasiewicz's system*: a more formal treatment is omitted here).

Definition 1.2

Let φ and ψ be two propositional formulae. Then φ and ψ are said to be *equivalent* iff $\vDash \varphi \Leftrightarrow \psi$.

Any propositional formula in the propositional calculus can be represented in an unique way by some *disjunctive normal form* or also equivalently by some *conjunctive normal form* (e.g. see Mostowski A.W and Pawlak Z. 1970). This is illustrated in the next example.

Example 1.7

Consider the following functor ′$\not\Leftrightarrow$′ called *exclusive disjunction*. The formula $p \not\Leftrightarrow q \Leftrightarrow_{df} \sim(p \Leftrightarrow q)$ (is read ′*either* p *or* q′: ′exclusive′ because the truth of one element excludes the truth of the other element). So, the following disjunctive and conjunctive normal forms can be obtained (a simple method for obtaining such forms is given in the next Subsection 1.4): $p \wedge \sim q \vee \sim p \wedge q$ and $(p \vee q) \wedge (\sim p \vee \sim q)$, respectively. According to Definition 1.2, the above two forms are equivalent, i.e. we have the following thesis:

$\vDash p \wedge \sim q \vee \sim p \wedge q \Leftrightarrow (p \vee q) \wedge (\sim p \vee \sim q)$

---

* By using the law of importation, i.e. T 1.12b, we can obtain: $p \wedge \sim p \Rightarrow q$. Among the *material implication paradoxes*, the following (theses) are also known: $p \Rightarrow (q \Rightarrow p)$, $(p \Rightarrow q) \vee (q \Rightarrow p)$ and $(p \Rightarrow q) \vee (q \Rightarrow r)$: the corresponding proofs are left to the reader.

Proof (a):

(1)	$p \wedge \sim q \ \vee \ \sim p \wedge q$	{a}
(1.1)	p	{1.1,1.2 / ada}
(1.2)	$\sim q$	
(1.3)	$p \vee q$	{+ A : 1.1}
(1.4)	$\sim p \vee \sim q$	{+ A : 1.2}
(1.5)	$(p \vee q) \wedge (\sim p \vee \sim q)$	{+ K : 1.3,1.4}
(2.1)	$\sim p$	{2.1,2.2 / ada}
(2.2)	q	
(2.3)	$p \vee q$	{+ A : 2.2}
(2.4)	$\sim p \vee \sim q$	{+ A : 2.1}
(2.5)	$(p \vee q) \wedge (\sim p \vee \sim q)$	{+ K : 2.3,2.4}
	$(p \vee q) \wedge (\sim p \vee \sim q)$. □	{1.5,2.5}

The proof of the converse implication (b) is left to the reader. The reader is also invited to verify the following implication: $p \not\Leftrightarrow q \ \Rightarrow \ p \vee q$.

Since $\vee$ is distributive over $\wedge$ (and the both logical operations are also commutative and associative), a simpler proof can be obtained. In fact, we have:

$$\begin{aligned} p \wedge \sim q \ \vee \ \sim p \wedge q \quad & \Leftrightarrow \quad (p \wedge \sim q \ \vee \ \sim p) \wedge (p \wedge \sim q \ \vee \ q) \\ & \Leftrightarrow \quad ((\sim p \vee p) \wedge (\sim p \vee \sim q)) \wedge ((q \vee p) \wedge (q \vee \sim q)) \\ & \Leftrightarrow \quad (p \vee q) \wedge (\sim p \vee \sim q)\,. \ \square \end{aligned}$$

In general, it can be shown any disjunctive normal form can be equivalently transformed to a conjunctive normal form and vice versa. It can be observed that a conjunctive normal form is a true formula if each disjunction includes at least one variable which in one case is negated and in another is not. Moreover, the *complexity* of the last two forms (obtained for a given formula, i.e. the number of propositional variables and the logical operations) may be different, e.g. the conjunctive normal form associated with $p \vee q$ coincides with the same formula, but the following disjunctive normal form can be obtained: $\sim p \wedge q \vee p \wedge \sim q \ \vee p \wedge q$. In fact, we have: $\vDash \ p \vee q \Leftrightarrow \ \sim p \wedge q \vee p \wedge \sim q \ \vee p \wedge q$. The following formula is also a thesis: $\vDash \ p \wedge \sim p \wedge q \vee r \Leftrightarrow r$ (the last two proofs are left to the reader).

1.4. Zero-one verification

The *zero-one verification* of a propositional formula $\varphi \in P$ is a functional method (called also *truth table method*) of determining in a finite number of steps whether φ is true (yes or no). And so, this method do not present the corresponding proof structure for a given thesis φ*. Moreover, in some computer science applications the method realisation may be difficult, but at the same time the assumptional system style is more natural and usable (see Example 1.3 in the previous subsection).

For simplicity, below any propositional variable is said to be an *atom*. The truth and the falsehood of a given formula φ will be symbolised by ´1´ and ´0´, respectively. Some introductory notions are first introduced (Chang and Lee 1973). Below we shall assume the *logical value of a propositional formula* φ, denoted by $v(\varphi) \in \{0,1\}$ (see Definition 1.1).

Definition 1.3

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* Let $\varphi = \varphi(p_1,p_2, \ldots ,p_n) \in P$ and $v(\underline{p}) =_{df} (v(p_1),v(p_2), \ldots ,v(p_n))$. Hence, the 0-1 verification of $\varphi(\underline{a})$ / i.e. $\varphi(\underline{p})$ for $v(\underline{p}) = \underline{a}$ is not a logical consequence of the 0-1 verification of $\varphi(\underline{b})$ / $v(\underline{p}) = \underline{b}$ (for any different $\underline{a}, \underline{b} \in \{0,1\}^n$).And so, we have not the structure of a typical assumptional proof (having a connected set of lines). In fact, the 0-1 verification method and the assumptional system style are very similar to the notions of 'black box' and of 'white box', respectively (well-known in automata theory).

Let $\varphi = \varphi(p_1,p_2, \ldots ,p_n) \in P$ be a given propositional formula where p_i are the corresponding atoms of φ $(i = 1,2, \ldots ,n)$. Then any $(v(p_1),v(p_2), \ldots ,v(p_n)) \in \{0,1\}^n$ is said to be an *interpretation* $\mathtt{I}$ of φ.

It can be observed the number of possible interpretations of a given formula φ having n different atoms is: $|\{0,1\}^n| = 2^n$. Let $v_{\mathtt{I}}(\varphi)$ be $v(\varphi)$ for $\mathtt{I}$.

Definition 1.4

A propositional formula φ *is true in* $\mathtt{I}$ iff $v_{\mathtt{I}}(\varphi) = 1$. Otherwise we shall say that φ is *false in* $\mathtt{I}$.

According to Definitions 1.2 and 1.4 it follows two propositional formulae φ and ψ are *equivalent* iff $v_{\mathtt{I}}(\varphi) = 1 \Leftrightarrow v_{\mathtt{I}}(\psi) = 1$ (for any $\mathtt{I}$). Moreover, the logical value of a propositional formula $\varphi \in P$ can be considered as a map $v : P \to \{0,1\}$ (for a given $\mathtt{I}$).

Definition 1.5

A formula $\varphi \in P$ is:

(i) *satisfied* (called also *thesis*, *true formula, valid formula* or *tautology**) iff φ is true in any $\mathtt{I}$,
(ii) *satisfiable* iff there exists some $\mathtt{I}$ such that φ is true in $\mathtt{I}$,
(iii) *contradictory* iff φ is false in any $\mathtt{I}$.

Let $\varphi \in P$ be a thesis. Then $\sim \varphi$ is contradictory. Hence, according to CC (contraposition of implication), if φ is contradictory, then $\sim \varphi$ is a thesis. For example, let φ be the law $p \vee \sim p$ (excluded middle), then $\sim (p \vee \sim p)$, i.e. $\sim p \wedge p$ is contradictory. Similarly, since $\models (p \wedge (q \Leftrightarrow r) \Rightarrow s) \Rightarrow (p \wedge q \wedge r \Rightarrow s) \wedge (p \Rightarrow q \vee r \vee s)$ then $\sim ((p \wedge (q \Leftrightarrow r) \Rightarrow s) \Rightarrow (p \wedge q \wedge r \Rightarrow s) \wedge (p \Rightarrow q \vee r \vee s))$ is contradictory, etc. In the case of the classical propositional calculus this follows immediately by the considered below zero-one verification method. A more formal proof is given in Subsection 2.2 (assuming basic fuzzy propositional logic systems: it can be observed that $\{0,1\} \subsetneq [0,1]$).

The zero-one verification requires the use of some tables, called *truth tables*, which show how the truth-value of the considered compound proposition is determined by the truth-values of the component propositions. The truth-values related to the basic two argument logical operations are shown in the table given below (according to Definition 1.1: provided there is no ambiguity, for simplicity instead of $v(\varphi)$ the same formula φ is used, e.g. p instead of $v(p)$, q instead of $v(q)$, $p \wedge q$ instead of $v(p \wedge q)$, etc.). Obviously, $v(\sim p) =_{df}$ if $v(p) = 1$ then 0 else 1 fi. The implication values follow immediately from T 1.15 (or also: T 1.19). It can be observed that the conjunction and disjunction connectives correspond to the logical operations minimum and maximum, respectively. Moreover, the logical value of the implication $p \Rightarrow q$ is equal to 1, i.e. $p \Rightarrow q = 1$ iff $p \le q$. And hence, the equivalence $p \Leftrightarrow q = 1$ iff $p \le q$ and $q \le p$, i.e. iff $p = q$ (see Subsection 2.1).

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$†	$p \Leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example 1.8

Consider the formula: $(p \Rightarrow q) \wedge p \Rightarrow q$ (corresponding to the rule of detachment $-$ C). So we can obtain the following table.

* From the Greek word ταυτολογία: in formal logic, a formula that is true in every possible interpretation.

† Filon of Megara, $4^{th} - 3^{th}$ century b.c.

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$(p \Rightarrow q) \wedge p \Rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

The same result can be obtained by using the *Hilbert´s abbreviated verification* (Hilbert D. 1862 – 1943). The last approach is most often used for formulae having the form of an implication. We can assume the antecedent (consequent) of the whole implication is true (false) and then examine whether its consequent (antecedent) can be false (true)[*].As an example, consider the above formula $(p \Rightarrow q) \wedge p \Rightarrow q$. Assume that $v(q) = 0$ (in short: $q = 0$). Since the antecedent of the whole implication is a conjunction, the logical value of p have to be equal to 1. Then the value of the first element of this conjunction, i.e. the implication $p \Rightarrow q$, is equal to 0 ($\neq 1$).So the verification is completed (see below: the index ´c´ denotes that the corresponding logical value is *critical*, i.e. there is no any other possibility).

$= 0 (\neq 1)$
$1^c \quad 0^c \quad 1^c$
$\underline{(p \Rightarrow q) \wedge p} \quad \Rightarrow \quad q$
$= 1? \qquad 0^c\,!$

A more complicated example may be the verification of the Hauber´s law (see T 1.13). So initially we can obtain:

$0^c \quad 1^c \quad x \quad x \quad 0^c \quad x \quad 1^c \quad x$		$1^c \quad 0^c \quad x \quad x$
$\underline{(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \wedge \sim(q \wedge s)}$	$\Rightarrow$	$\underline{(q \Rightarrow p) \wedge (s \Rightarrow r)}$
$= 1\,?$		$= 0\,!$

where $x \in \{0,1\}$.

Now it is necessary to verify if the antecedent of the main implication is equal to 1 (similarly for the rest two cases of the consequent of this implication: this is left to the reader). Since q is 1^c the value of $\sim(q \wedge s)$ will be ´1´ iff s is 0^c. In a similar manner, the value of $p \vee r$ will be ´1´ iff r is 1^c . Then the value of $r \Rightarrow s$ is uniquely specified from ´ $1^c \Rightarrow 0^c$ ´. Hence the obtained value of the antecedent of the main implication is 0 ($\neq 1$) and the verification process is completed. It can be observed the number of possible interpretations of this formula is equal to $2^4 = 16$ (we have four different atoms: p,q,r, and s). The reader is invited to verify the formula considered in Example 1.6. □

The reader is invited to verify the following implications:

1. $p \wedge q \Rightarrow (p \Rightarrow q)$
2. $((p \Rightarrow q) \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$
3. $((p \Rightarrow q) \Rightarrow r) \Rightarrow (p \wedge q \Rightarrow r)$
4. $(p \Rightarrow q \vee r) \wedge (s \wedge t \Rightarrow u) \Rightarrow ((r \Rightarrow s) \wedge p \wedge t \Rightarrow q \vee u)$: see Example 1.14 of Subsection 1.8.
5. $p \wedge \sim p \wedge q \vee r \Leftrightarrow r$

[*] One of these two approaches may not be applicable for some formulae, e.g. for the zero-one verification of the following formula: $p \vee q \Rightarrow (p \Rightarrow q) \vee p$. The logical value of the consequent $(p \Rightarrow q) \vee p$ is $\neq 0$ (for any $p, q \in \{0, 1\}$). And hence, the considered formula is a thesis.

The zero-one representation of a given formula φ may be also useful for obtaining the corresponding (disjunctive or conjunctive) normal form of φ . This is illustrated in the next example.

Example 1.9

Consider the truth table for exclusive disjunction:

$\not\Leftrightarrow$	0	1
0	0	1
1	1	0

The disjunctive normal form for $p \not\Leftrightarrow q$ can be easily obtained as a disjunction of all elementary conjunctions* corresponding to all 1´s of $p \not\Leftrightarrow q$. So we have two such conjunctions: $\sim p \wedge q$ and $p \wedge \sim q$ (for (v(p),v(q)) = (0,1) and (v(p),v(q)) = (1,0), respectively: see Example 1.7). In a similar way, the conjunctive normal form can be obtained as a conjunction of all elementary disjunctions corresponding to all 0´s of $p \not\Leftrightarrow q$ .We have two such disjunctions: $p \vee q$ and $\sim p \vee \sim q$ (for (v(p),v(q)) = (0,0) and (v(p),v(q)) = (1,1), respectively). The conjunctive normal form can be considered as a negation of the corresponding disjunctive normal form obtained for $\sim (p \not\Leftrightarrow q)$, i.e. for $p \Leftrightarrow q$ by using De Morgan´s laws (the proof is left to the reader). □

According to the last example, it can be shown any conjunctive normal form is a true formula iff each elementary disjunction includes at least one variable which is at the same time with and without negation.

Example 1.10

In the case of the law of Duns Scotus (see T 1.22),by using the law of implication T 1.15, the following conjunctive normal form can obtained: $p \Rightarrow (\sim p \Rightarrow q) \Leftrightarrow \sim p \vee p \vee q$. So we have only one elementary disjunction including at the same time the propositional variable p with and without negation. Similarly, in the case of the rule of detachment we can obtain:

$$\begin{aligned} p \wedge (p \Rightarrow q) \Rightarrow q \quad & \Leftrightarrow \sim (p \wedge (p \Rightarrow q)) \vee q \\ & \Leftrightarrow \sim p \vee p \wedge \sim q \vee q \\ & \Leftrightarrow (\sim p \vee p) \wedge (\sim p \vee \sim q) \vee q \\ & \Leftrightarrow (\sim p \vee p \vee q) \wedge (\sim p \vee \sim q \vee q). \end{aligned}$$

We have two elementary disjunctions including at the same time: $\sim p$ and p (in the first) and $\sim q$ and q (in the second). □

The reader is invited to verify and to proof the following formula: $(p \wedge q \Leftrightarrow p \wedge r) \wedge (p \vee q \Leftrightarrow p \vee r) \Rightarrow (q \Leftrightarrow r)$†.

Let $\varphi =_{df} \varphi(p_1,p_2, \ldots ,p_n)$ be an arbitrary propositional formula ($n \in \mathbb{N}$: the set of natural numbers). The following thesis is satisfied: $\models \varphi \Rightarrow F \Leftrightarrow \sim \varphi$. □ And hence: $\sim \varphi =_{df} \varphi \Rightarrow F$ (left to the reader).

In general, the Hilbert´s abbreviated verification can be considered as an *analysis* method, more effective than truth tables and usable in the case of verification if a priori given formula φ is a thesis or not. However in the case

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* In general, the notion of an *elementary conjunction* can be introduced as the following formula $p_1^{c1} \wedge p_2^{c2} \wedge \ldots \wedge p_n^{cn}$, where any $c_i \in \{0,1\}$ and $p_i^{ci} =_{df}$ if $c_i = 1$ then p_i else $\sim p_i$ fi (i = 1,2, ... ,n). The notion of an *elementary disjunction* can be introduced in a similar way by considering $\vee$ and $c_i' =_{df} 1 - c_i$ (see:Mostowski and Pawlak 1970). The Boolean constants c_i can be interpreted as some latin exponents.We shall say cnf(ϕ) is a *conjunctive normal form* of a formula ϕ iff it is equivalent to ϕ and is either an elementary disjunction or a conjunction of (two or more) elementary disjunctions. In general, cnf(ϕ) can be obtained from: ~ dnf(~ϕ) (similarly for dnf(ϕ) wrt the cnf(ϕ)).

† Use two times the absorptive, distributive axioms and the rule SR (starting with $q \Leftrightarrow \ldots$).

of *synthesis* of a formula φ having a priori required properties the only usable method are truth tables (e.g. see the proof of T 2.33 of Subsection 2.3). And hence, the last method is more universal.

1.5. Logical consequence

In the exact sciences and engineering, as in the usual life, often it is necessary to decide if a given assertion (i.e. thesis, sentence, declaration, etc.) follows from some others. The last involves the notion of ´*logical consequence*´ (or ´*entailment*´[*], i.e. a relation between set of sentences and a sentence). For example, the system fault diagnosis (or e.g. patient's disease diagnosis) should be a logical consequence of the obtained test outcomes (or consistency of the obtained measures, i.e. patient results). The single most important factor in general aviation flight safety is the decision of a pilot to begin or to continue with a flight in unsuitable weather, e.g. the landing of the plane at the airport should be a logical consequence wrt the corresponding set of conditions. The safety of nuclear central should be a logical consequence wrt the corresponding set of conditions related to the selected place of building, proposed type of project, the general concept of this project and quality of its realisation, and of course the human factor. The declaration of a candidate should be a logical consequence of the corresponding set of propositions used during the electioneering campaign. The judge's decision should be a logical consequence wrt the corresponding set of facts of the specific case. The specification validity of a reactive program should be a logical consequence wrt the a priori required properties associated with this program, etc.

The intuitive concept of consequence, the notion that one sentence follows logically from another, has driven the study of logic for more than two thousand years. But logic has moved forward dramatically in the past century, largely as a result of bringing mathematics to bear on the field (Etchemendy 1990). A generalised model-theoretic analysis of logical consequence was introduced by Tarski (Tarski A. 1902 – 1983). Here the following three simple axioms were assumed: $Cn(\varnothing) = \varnothing$, $A \subseteq B \Rightarrow Cn(A) \subseteq Cn(B)$, and $Cn(Cn(A)) = Cn(A)$, where ´Cn´ denotes ´consequence´, A and B are some finite sets of axioms. Some works concerning an examination of this model have been recently undertaken, e.g. the above cited work (a more formal treatment is omitted). Below only some introductory notions concerning the concept of a logical consequence are given (Chang and Lee 1973).

Definition 1.6

Let $\varphi_1,\varphi_2, \dots ,\varphi_n,\psi \in P$ be formulae. We shall say ψ is a *logical consequence* wrt $\varphi_1,\varphi_2, \dots , \varphi_n$ (or *follows logically from* $\varphi_1,\varphi_2, \dots ,\varphi_n$) iff $v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) = 1 \Rightarrow v(\psi) = 1$ (for any $\mathtt{I}$). We shall say $\varphi_1,\varphi_2, \dots ,\varphi_n$ are *axioms* for ψ .

Theorem 1.23

The formula ψ is a logical consequence wrt $\varphi_1,\varphi_2, \dots , \varphi_n \Leftrightarrow \models \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \Rightarrow \psi$[†] .

Proof:

(a) Assume that ψ is a logical consequence wrt $\varphi_1,\varphi_2, \dots , \varphi_n$. Let $\mathtt{I}$ is an interpretation such that $v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) = 1$. According to Definition 1.6 and the rule of detachment $-$ C, we have $v(\psi) = 1$. Hence $v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \Rightarrow \psi) = 1$ in $\mathtt{I}$. On the other hand, if $v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) = 0$ in $\mathtt{I}$ (i.e. there exists some φ_k having $v(\varphi_k) = 0$ in $\mathtt{I}$) then also $v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \Rightarrow \psi) = 1$ in $\mathtt{I}$. Hence $v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \Rightarrow \psi) = 1$ (for any $\mathtt{I}$). So according to Definition 1.5(i) we have: $\models \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \Rightarrow \psi$.

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[*] The notion of *entailment* is used in (at least) three meanings: *implication connective* (having some properties), the name of the *logical system* characterising this connective as well as the *area* in which this system is defined (see Subsection 2.4: Relevance logic).

[†] The generalised form of an expression can be transformed in a form similar to the right side of Theorem 1.23: using (n – 2) times T 1.12 (since logical equivalence is transitive).

(b) Let now $\models \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \Rightarrow \psi$. In accordance with the rule of detachment $-$ C, for any interpretation $\mathcal{I}$ we have: $v(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) = 1 \Rightarrow v(\psi) = 1$. According to Definition 1.6, ψ is a logical consequence wrt $\varphi_1, \varphi_2, \ldots, \varphi_n$. □

It can be observed ψ is not a logical consequence wrt $\varphi_1, \varphi_2, \ldots, \varphi_n \Leftrightarrow \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \wedge \sim \psi$ is a contradictory formula (by using the rule NC: see T 1.19). Below we shall denote $\psi \in Cn(\{\varphi_1, \varphi_2, \ldots, \varphi_n\}) \Leftrightarrow_{df} \psi$ is a logical consequence wrt $\varphi_1, \varphi_2, \ldots, \varphi_n$.

Consider the generalised form of an expression, i.e. $\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots)$. By T 1.12 it follows: $\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots) \Leftrightarrow \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \ldots \wedge \phi_{n-1} \Rightarrow \phi_n$ (the proof is left to the reader). Hence, according to T 1.23 we can obtain: $\phi_n \in Cn(\{\phi_1, \phi_2, \ldots, \phi_{n-1}\}) \Leftrightarrow \models \phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots)$.

Example 1.11

(a) Let $\varphi_1 \Leftrightarrow_{df} p \Rightarrow q$ and $\psi \Leftrightarrow_{df} q$. According to T 1.23, we can obtain: q is a logical consequence wrt $p \Rightarrow q \Leftrightarrow \models (p \Rightarrow q) \Rightarrow q$? The last requirement is not satisfied (for $v(p) = v(q) = 0$ we have $v(p \Rightarrow q) = 1$). So $q \notin Cn(\{p \Rightarrow q\})$.

(b) Assume now that $\varphi_1 \Leftrightarrow_{df} p \Rightarrow q$, $\varphi_2 \Leftrightarrow_{df} p$, and $\psi \Leftrightarrow_{df} q$. Since $\models (p \Rightarrow q) \wedge p \Rightarrow q$ then q is a logical consequence wrt $p \Rightarrow q$ and p (the rule of detachment). Hence $q \in Cn(\{p \Rightarrow q , p\})$.

(c) In a similar way for $\varphi_1 \Leftrightarrow_{df} p \Rightarrow q$, $\varphi_2 \Leftrightarrow_{df} p \vee r$, and $\psi \Leftrightarrow_{df} q \vee r$ we have: $q \vee r$ is a logical consequence wrt $p \Rightarrow q$ and $p \vee r$ (is equivalent to the ´law of a new element´: using the law of exportation, see T 1.12). Hence $q \vee r \in Cn(\{p \Rightarrow q , p \vee r\})$.

(d) Let consider the case (c) having ψ as one of the following three formulae: $q \vee \sim r$, $\sim q \vee r$, and $\sim q \vee \sim r$. It can be shown that : $q \vee \sim r , \sim q \vee r , \sim q \vee \sim r \notin Cn(\{p \Rightarrow q , p \vee r\})$. According to the rules + NK and ER , also $\sim (q \wedge r) \notin Cn(\{p \Rightarrow q , p \vee r\})$, e.g.

x $\;1^c\;$ x $\;1^c$		$1^c \;\; 1^c$
$\underline{(p \Rightarrow q) \wedge (p \vee r)}$	$\Rightarrow$	$\underline{\sim q \vee \sim r}$
= 1 ?		= 0 !

It can be observed for any $x \in \{0,1\}$ the antecedent of the last implication is always true. According to Definition 1.5, the last formula is satisfiable, but not satisfied. In fact, it is not satisfied for $\mathcal{I} = (v(p),v(q),v(r)) = (x,1,1)$. □

1.6. The consistency and completeness

Consider an arbitrary formula $\varphi \in P$ of the propositional calculus such that $v_{\mathcal{I}}(\varphi) = 1$ (for any interpretation $\mathcal{I}$). In accordance with Definition 1.5, φ is satisfied and hence a thesis of that calculus. The problem arises whether every thesis provable by the rules of the assumptional system of the propositional calculus is a true formula. In particular, a resolution of the last problem is related to the notion of order of a thesis, given in the next inductive definition (Słupecki J. and Borkowski L. 1967).

Definition 1.7

Let $\phi \in P$ is an arbitrary formula. Then:

(i) ϕ is a *thesis of order 1* iff there exists a (direct or indirect) proof from assumptions of ϕ by using only the primitive rules, i.e: $-$ C, $\pm$ K, $\pm$ A, and $\pm$ E ,

(ii) ϕ is a *thesis of order n* iff there exists a proof from assumptions in which only theses of order $\leq n - 1$ are used and ϕ is not a thesis of order $< n$,

(iii) ϕ is a *thesis* iff there exists some $n \in \mathrm{N}$ such that ϕ is a thesis of order n.

According to the last definition, it can be shown any thesis of the propositional calculus implicate proof of its corresponding metathesis (this is omitted). The proof of the next theorem is inductive wrt the order of the considered thesis.

Theorem 1.24

Every thesis of the assumptional system of the propositional calculus is a true formula.

Proof:

(1) Let $\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow ... \Rightarrow (\phi_{s-1} \Rightarrow \phi_s) ...)$ be a thesis of order 1 which is not true (aip). For simplicity, let also $\phi \Leftrightarrow_{df} (\phi_2 \Rightarrow (\phi_3 \Rightarrow ... \Rightarrow (\phi_{s-1} \Rightarrow \phi_s) ...)$. So there exists an interpretation $\mathcal{I}_0$ such that the logical value $v_{\mathcal{I}0}(\phi) = 0$. Hence we have: $v_{\mathcal{I}0}(\phi_k) = 1$ (for $k = 1, ... ,s - 1$) and $v_{\mathcal{I}0}(\phi_s) = 0$. Then $v_{\mathcal{I}0}(\sim \phi_s) = 1$. According to the assumption that $\vDash \phi$ of order 1, it follows there exists an indirect proof of ϕ such that the rules $-$ C, $\pm$ K, $\pm$ A, and $\pm$ E, applied to the assumptions ϕ_k $(k = 1, ... , s - 1)$ and $\sim \phi_s$ generate two contradictory lines, e.g. χ_1 and $\sim \chi_1$. Any primitive rule applied to formulae having value 1 for $\mathcal{I}_0$ will produce corresponding formulae having also 1 for $\mathcal{I}_0$. Since $v_{\mathcal{I}0}(\phi_k) = 1$ (for $k = 1, ... ,s - 1$) and $v_{\mathcal{I}0}(\sim \phi_s) = 1$ then all the formulae obtained from the former will have the logical value 1 for $\mathcal{I}_0$ .But this is a contradiction wrt the above formulae χ_1 and $\sim \chi_1$.

(2) Let now any thesis of order $\leq n - 1$ be a true formula (an inductive assumption) and $\vDash \phi$ of order n which is not true, where $n \geq 2$ (aip). As in the previous case (1) , there exists an indirect proof of ϕ . Starting with the assumptions $\phi_1, \phi_2, ... ,\phi_{s-1} ,\sim \phi_s$ and theses of orders lower than n , two contradictory lines, e.g. χ_n and $\sim \chi_n$, are generated.. Since $v_{\mathcal{I}0}(\phi_k) = 1$ (for $k = 1, ... ,s - 1$) , $v_{\mathcal{I}0}(\sim \phi_s) = 1$, and any thesis of order lower than n have the value 1 , then by the same reasons as in case (1) we shall obtain a contradiction wrt χ_n and $\sim \chi_n$. □

Definition 1.8

A system is said to be *consistent* iff its theses do not include two contradictory formulae.

According to T 1.24, every thesis of the assumptional system is true. So, it is no possible to meet two contradictory formulae being at the same time theses in this system. Hence, the following theorem is satisfied.

Theorem 1.25

The assumptional system of the propositional calculus is consistent. □

Theorem 1.26

Every true formula of the propositional calculus is a thesis of the assumptional system.

Proof:

Consider an arbitrary true formula ϕ . Let cnf(ϕ) be the corresponding equivalent conjunctive normal form, obtained in an unique way for ϕ (it can be shown for any φ and ψ : $\varphi \wedge \psi \Leftrightarrow$ cnf(φ) $\wedge$ cnf(ψ), and $\varphi \vee \psi \Leftrightarrow$ cnf(φ) $\vee$ cnf(ψ): the proof is left to the reader). According to Definition 1.2, we have: $\vDash \phi \Leftrightarrow$ cnf(ϕ). From T 1.24 it follows this equivalence is a true formula. Since ϕ is a true formula then a true formula is also cnf(ϕ). Moreover $\vDash$ cnf(ϕ). In accordance with the rule of detachment for equivalence DE (see T 1.16) wrt the theses $\phi \Leftrightarrow$ cnf(ϕ) and cnf(ϕ), it follows that $\vDash \phi$. □

Definition 1.9

A system is said to be *semantically complete* iff every true formula implemented in the language of that system is a thesis.

According to Definition 1.9, the above theorem T 1.24 cam be reformulated as follows.

Theorem 1.27

The assumptional system of the propositional calculus is semantically complete. □

Hence the process of verification of an arbitrary formula $\varphi \in P$ of the propositional calculus can be reduced to the process of checking whether cnf(φ) is a true formula.

Definition 1.10

A system is said to be *syntactically complete* iff every propositional formula $\varphi \in P$ implemented in the language of that system either is a thesis or (when joined to the theses of that system) results in a contradiction.

Theorem 1.28

The assumptional system of the propositional calculus is syntactically complete.

Proof:

Let $\varphi \in P$ and $\nvDash \varphi$. By T 1.24 and T 1.14 (the law of transposition of implication) it follows that φ is not a true formula. Hence there exists an interpretation I_0 such that $v_{I0}(\varphi) = 0$, e.g. for $v(p_i) = 1$ and $v(q_j) = 0$ ($i = 1,2, \ldots ,n_i$ and $j = 1,2, \ldots ,n_j$: according to Definition 1.3 the propositional variables p_i and q_j correspond to the set of atoms of φ and $n_i + n_j = n$). Starting with φ ,by using the substitution rule SR ,it can be obtained a new formula φ^* such that each p_i is substituted by a true formula and each q_j - by its negation, e.g. by ′$p \Rightarrow p$′ and ′$\sim (p \Rightarrow p)$′, respectively. Since $v_I(\varphi^*) = 0$ then $v_I(\sim \varphi^*) = 1$ (for any I). According to T 1.24, $\vDash (\sim \varphi^*)$. Hence the system of the propositional calculus extended by φ is a contradictory wrt φ^* and $\sim \varphi^*$. □

Definition 1.11

A system of rules is said to be *independent* iff none of these rules is secondary wrt the remaining rules.

According to the last definition, a system is independent if there is no a rule being a logical consequence of the remaining rules of this system.

Theorem 1.29

The system of primitive rules of the propositional calculus is an independent system. □

Let $\varphi = \varphi(p_1,p_2, \ldots ,p_n) \in P$ be a given propositional formula under Definition 1.1 and $v(\underline{p}) =_{df} (v(p_1),v(p_2), \ldots ,v(p_n)) \in \{0,1\}^n$. Since $\{0,1\}^n$ is finite then, in accordance with Definition 1.5, it is always possible to decide if φ is satisfied or not. In fact, the following theorem can be shown (e.g. see Grzegorczyk A. 1969).

Theorem 1.30 (*decidability of the classical propositional calculus*)

There exists an algorithm of determining in a finite number of steps whether a given $\varphi \in P$ is a true formula (i.e. thesis). □

Moreover, it can be shown the subset of all theses (say T) of the classical propositional calculus is a *theory*, i.e. $T = Cn(T)$ (see the last cited work).

There exists some closed relationship between the classical propositional calculus and the algebra of sets (i.e. the classical propositional logic has its corresponding set theory). So we have a possibility of obtaining directly from theses of the propositional calculus corresponding theorems of the algebra of sets consisting of two expressions, incorporating set operations, and connected by the symbol of equality or inclusion (Słupecki J. and Borkowski L. 1967). For example, from $p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$ (law of transposition of implication, see T 1.14) we can obtain: $P \subseteq Q \Leftrightarrow Q' \subseteq P'$ ('′' denotes *complement* of a set). From $(p \Rightarrow q) \Rightarrow (p \vee q \Leftrightarrow q) \vee (p \wedge q \Leftrightarrow p)$ we can obtain: $(P \subseteq Q) \Rightarrow (P \cup Q = Q) \vee (P \cap Q = P)$. Similarly, from $(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) \wedge \sim (q \wedge s) \Rightarrow (q \Rightarrow p) \wedge (s \Rightarrow r)$ (the Hauber′s law, see T 1.13) we have: $(P \subseteq Q) \wedge (R \subseteq S) \wedge (P \cup R = \mathscr{U}) \wedge (Q \cap S = \varnothing) \Rightarrow (Q \subseteq P) \wedge (S \subseteq R)$, where $\mathscr{U}$ is the universe, etc. (a more formal treatment is omitted, see Subsection 5.2).

1.7. The axiomatic approach

In general a scientific theory can be considered as an axiomatic system that obtains an empirical interpretation through appropriate statements called rules of correspondence, which establish a correlation between real objects (or real processes) and the abstract concepts of the theory. The language of a theory includes two kinds of terms: observational and theoretical. The statements of a theory are divided in two groups: analytic and synthetic. Observational terms denote objects or properties that can be directly observed or measured, while theoretical terms denote objects or properties we cannot observe or measure but we can only infer from direct observations. Analytic statements are a priori and their truth is based on the rules of the language; on the contrary, synthetic statements depend on experience, and their truth can be acknowledged only by means of the experience. This conception about the structure of scientific theories is perhaps the most durable philosophical principle of the logical positivism (Reichenbach H., Carnap R., etc.: see the *Internet Encyclopaedia of Philosophy* 2001).

The *axiomatic approach* as a rule involves a number of propositions that are not proved by means of the other propositions. Such propositions are said to be the *axioms* (or *postulates*) of the system. The next stage in the construction of the propositional calculus is related to the specification of the corresponding primitive rules of proving (inference) and hence the possibility of obtaining new theses. Usually the following two primitive rules of proving are introduced: the *rule of definitional replacement of one formula by another*, denoted below by RR and the *rule of detachment* (for implication: '– C'). The proof style used is always the direct style. For example, starting with the rule $(p \Rightarrow q) \wedge \sim q \Rightarrow \sim p$ (modus tollendo tollens, see T 1.6), by replacing each p with ′p ⇒ q′ a new thesis can be obtained: $((p \Rightarrow q) \Rightarrow q) \wedge \sim q \Rightarrow \sim (p \Rightarrow q)$ (the proof is left to the reader).

The rule of definitional replacement is introduced as follows (Słupecki J. and Borkowski L. 1967):

$$\text{RR}: \quad \frac{\begin{array}{c}\varphi =_{\text{df}} \psi \\ \chi(\varphi)\end{array}}{\chi(\psi\, //\, \varphi)},$$

where the Łukasiewicz's symbol of definitional equality '$=_{df}$' is not a logical connective, but it has a metalogical sense). As in SR, $\chi(\varphi // \psi)$ is obtained from χ by the replacement of its parts φ by the formula ψ. In fact, RR can be considered as a derived rule which is a generalisation of SR (since it is not necessarily usable only for equivalent formulae).

There exist very many different axiomatic systems*, e.g. including only one axiom or also having a relatively large set of axioms. A more enlarged axiomatic system seems to be preferable. In fact, we have a possibility of setting off some logical rules and also an easier way of deriving logical theses. However, independently of the transparent way of the obtained proof notations and in comparison with the assumptional style, the proofs in axiomatic systems are considerably difficult, complicated and hence, not so natural. Any axiomatic system is , in a sense, "dogmatic" (i.e. conservative, closed). In fact, there is not possibility of extending the actual set of axioms with new formulae derived by the same system. Moreover, any axiom can be considered as thesis and proved by using the assumptional style.

As an illustration, the following axiomatic system is presented below (Grzgorczyk A. 1969)†.

Positive axioms for implication

* The first such axiomatic system was introduced by Frege (Friedrich Ludwig Gottlob Frege 1848 – 1925).

† Andrzej Grzegorczyk (1922 – 2014).

(A1) $p \Rightarrow (q \Rightarrow p)$ {*law of simplification*}

(A2) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ {*Frege's law (or* syllogism): 1879}

Axioms describing the logical equivalence by means of implication

(A3) $(p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$

(A4) $(p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$

(A5) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow p) \Rightarrow (p \Leftrightarrow q))$

Axioms describing the logical conjunction and disjunction

(A6) $p \vee q \Rightarrow q \vee p$

(A7) $p \wedge q \Rightarrow q \wedge p$

(A8) $p \Rightarrow p \vee q$

(A9) $p \wedge q \Rightarrow p$

(A10) $p \Rightarrow (q \Rightarrow p \wedge q)$

(A11) $(p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$

Rules for negation

(A12) $(p \Rightarrow q \wedge \sim q) \Rightarrow \sim p$

(A13) $p \wedge \sim p \Rightarrow q$

The law of the excluded middle

(A14) $p \vee \sim p$

The set of axioms A1 – A11 corresponds to the so called *positive logic* (theses including negation are omitted). The set of axioms A1 – A13 is related to the so called *intuitionistic logic* (Brouwer L.E.J. 1881 – 1966, Heyting A. 1898 – 1980): see Subsection 2.4. Here the law of the excluded middle is omitted.

An example using of this axiomatic system is given below (provided there is no ambiguity, the use of Ai is denoted by a_i, for any i).

Example 1.12

The proof of the thesis $p \Rightarrow p$ can be realised as follows.

(1) $(p \Rightarrow (q \Rightarrow p)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow p))$ {RR : p // r wrt a_2}

(2) $(p \Rightarrow q) \Rightarrow (p \Rightarrow p)$ {– C : 1,a_1}

(3) $(p \Rightarrow (q \Rightarrow p)) \Rightarrow (p \Rightarrow p)$ {RR : $(q \Rightarrow p)$ // q wrt 2}

$p \Rightarrow p$. □ {– C : 3,a_1}

A more simple is the following *Łukasiewicz's implication-negation axiomatic system* (Słupecki J. and Borkowski L. 1967).

(A1) (the first law of the hypothetical syllogism): $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

(A2) (is one of the laws of reduction ad absurdum) $(\sim p \Rightarrow p) \Rightarrow p$

(A3) (*law of Duns Scotus*) $p \Rightarrow (\sim p \Rightarrow q)$

In accordance with the last system, the following proof of the above example thesis $p \Rightarrow p$ can be obtained.

(1) $(p \Rightarrow (\sim p \Rightarrow q)) \Rightarrow (((\sim p \Rightarrow q) \Rightarrow r) \Rightarrow (p \Rightarrow r))$ {RR : $(\sim p \Rightarrow q)$ // q wrt a_1}

(2) $((\sim p \Rightarrow q) \Rightarrow r) \Rightarrow (p \Rightarrow r)$ {– C : 1,a_3}

(3) $((\sim p \Rightarrow p) \Rightarrow p) \Rightarrow (p \Rightarrow p)$ {RR : p // q , p // r, 2}
$p \Rightarrow p$. □ {– C : 3,a2}

It is easily to express the remaining connectives in the Łukasiewicz's system (by means of implication and negation: left to the reader). Obviously, the proof of the above *law of identity for implication*, i.e. $p \Rightarrow p$ becomes very simple if an ordinary indirect proof is used. This is shown below.

(1) p {a}
(2) $\sim p$ {aip}
contr. □ {1,2}

To be proved the equivalence between two deduction systems, it is necessary to show that any axiom in the first system is an axiom or a thesis in the second one and also any primitive rule in the first system is a primitive or derived rule in the second one. In this way it is possible to show the equivalence between the assumptional system and any such axiomatic system (this is omitted). But any such proof is related to the well-known *deduction theorem* given below*. And so, the next considerations are a summary presentation under Słupecki J. and Borkowski L. (1967).

It can be observed that any axiomatic system is equivalent to a such one, where the only primitive proof rule is the rule of detachment, i.e. '– C' and also having as axioms all possible replacements related to the axioms of the original system. As an example, the above considered Łukasiewicz's implication-negation axiomatic system is equivalent to the following one.

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))$

(A2) $(\sim \varphi \Rightarrow \varphi) \Rightarrow \varphi$

(A3) $\varphi \Rightarrow (\sim \varphi \Rightarrow \psi)$

And so, in accordance with the above considered axiomatic systems, the letters p, q and r serve as metavariables for formulae.

Let '– C' be the only primitive proof rule in the considered propositional calculus. The notion of a proof is presented as follows.

Definition 1.12

Let $\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow ... \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) ...)$ be a generalised form of an expression. The finite sequence of formulae $\psi_1, \psi_2, ... , \psi_m$ is said to be a *proof* of ϕ_n wrt the formulae (antecedents) $\phi_1, \phi_2, ... , \phi_{n-1}$ iff $\psi_m = \phi_n$ and for any $i = 1, 2, ... ,m$: ψ_i is either an axiom or belongs to $\{\phi_1, \phi_2, ... , \phi_{n-1}\}$ or also there exist some $k, j < i$ such that $\psi_k = (\psi_j \Rightarrow \psi_i)$.

Definition 1.13

$\phi_1, \phi_2, ... , \phi_{n-1} \models \phi_n$ iff there exists a proof for ϕ_n wrt the formulae $\phi_1, \phi_2, ... , \phi_{n-1}$.

In accordance with the last definition, we shall say that ϕ_n is a *logical consequence* wrt (or *follows logically from*) the formulae $\phi_1, \phi_2, ... , \phi_{n-1}$ (see Definition 1.6, Subsection 1.5). And hence, $\phi_n \in Cn(\{\phi_1, \phi_2, ... , \phi_{n-1}\})$.

Let $\psi_1, \psi_2, ... , \psi_m$ be a proof of the implication $\phi \Rightarrow \psi$. From Definition 1.13 it follows that $\psi_m = (\phi \Rightarrow \psi)$. Then the sequence $\psi_1, \psi_2, ... , \psi_m, \psi_{m+1}, \psi_{m+2}$ is a proof of ψ, by assuming $\psi_{m+1} =_{df} \phi$ and next using '– C'. We can obtain: $\psi_{m+2} = \psi$. And hence, the following corollary is satisfied.

Corollary 1.1

* As a precursor of this theorem it is reckoned Bernard Bolzano (1837: 1781 – 1848) and the contemporary formulation of deduction theorem was given independently by Alfred Tarski (1923: 1901 – 1983) and Jacques Herbrand (1930: 1908 – 1931). The term itself was first used by David Hilbert (1862 – 1943) and Paul Isaac Bernays (1888 – 1977): *The little encyclopaedia of logic* (1988).

$\vDash (\phi \Rightarrow \psi) \Rightarrow (\phi \vDash \psi)$. □

Corollary 1.2

$(\phi_1, \phi_2, \ldots, \phi_n \vDash \psi) \wedge (\psi \vDash \chi) \Rightarrow (\phi_1, \phi_2, \ldots, \phi_n \vDash \chi)$

Proof:

Let $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ and $\psi \vDash \chi$. In accordance with Definition 1.13, there exists sequence $\psi_1, \psi_2, \ldots, \psi_m$ which is a proof of ψ wrt the formulae $\phi_1, \phi_2, \ldots, \phi_n$, where $\psi_m = \psi$. In a similar way, let $\chi_1, \chi_2, \ldots, \chi_r$ be the corresponding proof of χ wrt ψ, where $\chi_r = \chi$. And hence, according to Definition 1.12, the sequence $\psi_1, \psi_2, \ldots, \psi_m, \chi_1, \chi_2, \ldots, \chi_r$ is a proof of χ wrt $\phi_1, \phi_2, \ldots, \phi_n$. □

Corollary 1.3

$(\phi_1, \phi_2, \ldots, \phi_n \vDash (\psi \Rightarrow \chi)) \wedge (\phi_1, \phi_2, \ldots, \phi_n \vDash \psi) \Rightarrow (\phi_1, \phi_2, \ldots, \phi_n \vDash \chi)$

Proof: left to the reader. □

Consider now axiomatic systems in which the only primitive proof rule is the rule of detachment, i.e. '− C' and having as theses the following formulae (C follows from A and B: see Example 1.12):

(A) $\phi \Rightarrow (\psi \Rightarrow \phi)$ {law of simplification}
(B) $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$ {Frege's law}
(C) $\phi \Rightarrow \phi$ {law of identity for implication}

The following theorem is satisfied for any such system.

Theorem 1.31 (*deduction theorem*)

$(\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash \phi_n) \Rightarrow (\phi_1, \phi_2, \ldots, \phi_{n-2} \vDash (\phi_{n-1} \Rightarrow \phi_n))$*

Proof:

Assume that $\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash \phi_n$. Hence, in accordance with Definition 1.12, there exists a sequence $\psi_1, \psi_2, \ldots, \psi_m$ which is a proof of ϕ_n wrt the formulae $\phi_1, \phi_2, \ldots, \phi_{n-1}$.

Consider the following sequence of implications: $\phi_{n-1} \Rightarrow \psi_1, \ldots, \phi_{n-1} \Rightarrow \psi_m$, where any ψ_i is replaced by $\phi_{n-1} \Rightarrow \psi_i$ ($i = 1, \ldots, m$). Since $\psi_m = \phi_n$ the last implication in this sequence will be as follows: $\phi_{n-1} \Rightarrow \phi_n$. And so, after some completion, this way obtained sequence becomes a proof of the implication $\phi_{n-1} \Rightarrow \phi_n$ wrt $\phi_1, \phi_2, \ldots, \phi_{n-2}$. This is shown below.

One of the next three cases follow directly by Definition 1.12 ($i = 1, \ldots, m$).

1. ψ_i is an axiom
2. ψ_i belongs to $\{\phi_1, \phi_2, \ldots, \phi_{n-1}\}$
3. there exist some $k, j < i$ such that $\psi_k = (\psi_j \Rightarrow \psi_i)$

Let ψ_i be an axiom (*case 1*). By using '− C' wrt ψ_i and the law of simplification A, i.e. $\psi_i \Rightarrow (\phi_{n-1} \Rightarrow \psi_i)$, it follows that $\phi_{n-1} \Rightarrow \psi_i$ is a thesis.

Assume that $\psi_i \in \{\phi_1, \phi_2, \ldots, \phi_{n-1}\}$ (*case 2*). If $\psi_i =_{df} \phi_{n-1}$ then in accordance with the law of identity for implication C, it follows that $\phi_{n-1} \Rightarrow \phi_{n-1}$ is a thesis. Otherwise $\psi_i \in \{\phi_1, \phi_2, \ldots, \phi_{n-2}\}$ and hence by the law of simplification A, it follows that $\psi_i \Rightarrow (\phi_{n-1} \Rightarrow \psi_i)$ is a thesis. By using '− C' we can obtain $\phi_{n-1} \Rightarrow \psi_i$. According to Corollary 1.1, $\phi_{n-1} \Rightarrow \psi_i$ is a logical consequence wrt the formulae $\phi_1, \phi_2, \ldots, \phi_{n-2}$.

* The opposite implication in Corollary 1.1 becomes a particular case with $n = 2$ by assuming $\phi_1 =_{df} \phi$ and $\phi_2 =_{df} \psi$.

Consider now *case 3*. Since $(\phi_{n-1} \Rightarrow \psi_k) = (\phi_{n-1} \Rightarrow (\psi_j \Rightarrow \psi_i))$ and $\phi_{n-1} \Rightarrow \psi_j$ are earlier formulae than $\phi_{n-1} \Rightarrow \psi_i$ then in accordance with Frege's law B, $(\phi_{n-1} \Rightarrow (\psi_j \Rightarrow \psi_i)) \Rightarrow ((\phi_{n-1} \Rightarrow \psi_j) \Rightarrow (\phi_{n-1} \Rightarrow \psi_i))$ is a thesis. And hence, $\phi_{n-1} \Rightarrow \psi_i$ can be obtained by using two times '− C'.

According to the above considered three cases, any formula $\phi_{n-1} \Rightarrow \psi_i$ ($i = 1, \ldots, m$) of the sequence $\phi_{n-1} \Rightarrow \psi_1, \ldots, \phi_{n-1} \Rightarrow \psi_m$ is either a thesis or a logical consequence wrt the formulae $\phi_1, \phi_2, \ldots, \phi_{n-2}$ or also obtained from earlier formulae of this sequence by using '− C'. The last sequence of implications can be completed as follows: any thesis $\phi_{n-1} \Rightarrow \psi_i$ is completed by the corresponding proof preceding this thesis (wrt the axioms) and any formula $\phi_{n-1} \Rightarrow \psi_i$ being a logical consequence wrt the formulae $\phi_1, \phi_2, \ldots, \phi_{n-2}$ is completed by the corresponding proof of $\phi_{n-1} \Rightarrow \psi_i$ wrt $\phi_1, \phi_2, \ldots, \phi_{n-2}$. And this way completed sequence of implications is a proof of $\phi_{n-1} \Rightarrow \phi_n$ wrt the formulae $\phi_1, \phi_2, \ldots, \phi_{n-2}$. In fact, in accordance with Definition 1.12, any formula of the last obtained completed sequence is either an axiom or belongs to $\{\phi_1, \phi_2, \ldots, \phi_{n-2}\}$ or also obtained from preceding formulae using '− C' and the last formula of this sequence is $\phi_{n-1} \Rightarrow \phi_n$. □

It can be observed that, intuitively, the construction used in the above proof can be recognised as a ramified direct proof from assumptions (see Subsection 1.3). The above deduction theorem can be considered as a substantiation of the (earlier existing) natural deduction methods. In fact, the next two theorems hold.

The following theorem can be obtained by using multiple times Theorem 1.31.

Theorem 1.32

$(\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash \phi_n) \Rightarrow \vDash (\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots))$. □

The following formula is a thesis in the last two axiomatic systems.

(D) $(\sim \phi \Rightarrow \psi) \Rightarrow ((\sim \phi \Rightarrow \sim \psi) \Rightarrow \phi)$

Assume that Theorem 1.31 and thesis D are satisfied in our propositional calculus. Then the following theorem is also satisfied.

Theorem 1.33

$(\phi_1, \phi_2, \ldots, \phi_{n-1}, \sim \phi_n \vDash \psi) \wedge (\phi_1, \phi_2, \ldots, \phi_{n-1}, \sim \phi_n \vDash \sim \psi) \Rightarrow \vDash (\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots))$

Proof:

(1)	$\phi_1, \phi_2, \ldots, \phi_{n-1}, \sim \phi_n \vDash \psi$	{a}
(2)	$\phi_1, \phi_2, \ldots, \phi_{n-1}, \sim \phi_n \vDash \sim \psi$	
(3)	$\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash (\sim \phi_n \Rightarrow \psi)$	{T 1.31: 1}
(4)	$\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash (\sim \phi_n \Rightarrow \sim \psi)$	{T 1.31: 2}
(5)	$(\sim \phi_n \Rightarrow \psi) \vDash ((\sim \phi_n \Rightarrow \sim \psi) \Rightarrow \phi_n)$	{D, Coroll. 1.1}
(6)	$\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash ((\sim \phi_n \Rightarrow \sim \psi) \Rightarrow \phi_n)$	{Coroll. 1.2: 3,5}
(7)	$\phi_1, \phi_2, \ldots, \phi_{n-1} \vDash \phi_n$	{Coroll. 1.3: 4,6}
	$\vDash (\phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_3 \Rightarrow \ldots \Rightarrow (\phi_{n-1} \Rightarrow \phi_n) \ldots))$. □	{T 1.32: 7}

The notion of logical consequence introduced in Definition 1.12 can be extended by including beyond of '− C' also rules ± K, ± A, and ± E. The proof of the obtained extent of Theorem 1.31 would be similar. And this extent of Theorem 1.31 would correspond to the rule of joining an implication '+ C' used in the assumptional system. And hence, Theorems 1.32 and 1.33 would correspond to the rules for constructing a direct and indirect proof from assumptions, respectively. The above rules ± K, ± A, and ± E which are primitive in the assumptional system become derived in the considered axiomatic system. On the other hand, any axiom of the axiomatic system becomes either a thesis or primitive rule in the assumptional system. And then, the last two systems become equivalent.

In general, the deduction theorem can be considered as a formalisation of the common proof technique in which in proving some implication it is sufficiently to assume its antecedent and try to prove its consequent, using corresponding logical inference rules, e.g. in proving the implication $p \wedge q \Rightarrow p \vee q$ it is sufficiently to assume the antecedent $p \wedge q$ and try to prove the consequent $p \vee q$ using corresponding logical inference rules (left to the reader). And so, if $(p \wedge q) \models (p \vee q)$ is true then by deduction theorem and '– C' it follows that $\models (p \wedge q \Rightarrow p \vee q)$ is true. And this theorem provides rules for constructing proofs in the natural deduction systems. A more general version of the above theorem, known as a metatheorem of first-order predicate logic, can be presented as follows:

$$(\Phi, \varphi \models \psi) \Rightarrow (\Phi \models (\varphi \Rightarrow \psi)) \quad \text{or} \quad (\varphi \models \psi) \Rightarrow \models (\varphi \Rightarrow \psi), \text{ for } \Phi = \varnothing,$$

where Φ is a set of formulae in this theory and φ is a closed formula, i.e. a logic formula with no free variables (see Section 3 of Chapter II).

There is also satisfied the opposite implication of this theorem, called sometimes "*opposite deduction theorem*", which is presented as follows (*The little encyclopaedia of logic* 1988):

$$(\Phi \models (\varphi \Rightarrow \psi)) \Rightarrow (\Phi, \varphi \models \psi) \quad \text{or} \quad \models (\varphi \Rightarrow \psi) \Rightarrow (\varphi \models \psi), \text{ for } \Phi = \varnothing.$$

By using the opposite deduction theorem we have a possibility of introducing new inference rules wrt the already proved earlier formulae, e.g. having proved (or assumed as an axiom) the law $\sim\sim \varphi \Rightarrow \varphi$ we can accept the rule $\sim\sim \varphi \models \varphi$, i.e. it is allowed to deduce φ by assuming $\sim\sim \varphi$ (see T 1.3a: *rule of omitting double negation* '– N').

The deduction theorem is an important property of the well-known *Hilbert's style systems* where this theorem is used as a primitive rule of inference. But this approach is not generally followed. Moreover, there are first-order systems in which new inference rules are added for which the deduction theorem fails (Kohlenbach U. 2008). And finally, there exist systems in which this theorem is not satisfied, e.g. the classical deduction theorem does not hold in paraconsistent logic. Here, only the following "*two-way deduction theorem*" does hold in one form of this logic (Hewitt C. 2008): $\models (\varphi \Rightarrow \psi) \Rightarrow (\varphi \models \psi) \wedge (\sim \psi \models \sim \varphi)$, i.e. the *contrapositive inference* is also required (see T 1.14 of Subsection 1.3)[*].

In general, methodological problems such as consistency, completeness, independence, etc. arise when a given axiomatic system is considered (in ways analogous to assumptional systems). Obviously, the most important of these properties is the consistency of a system.

1.8. Sequent calculus

Reasoning is the ability to make inferences, and automated reasoning is concerned with the building of computing systems that automate this process. Although the overall goal is to mechanise different forms of reasoning, the term has largely been identified with valid deductive reasoning as practised in mathematics and formal logic. In this respect, automated reasoning is a kind of mechanical theorem proving (Portoraro F.D.2001).

The above-considered logical calculi were based on a system of rules, which define the methods used in proofs from assumptions. Some elements concerning direct reasoning and automated deduction methods are related to the Gentzen's *sequent calculus* (called also *sequent deduction* or *consecution calculus*) , originally denoted by LK, in contradistinction to the *natural deduction calculus*, denoted by NK[†]: Gentzen G.K.E. (1934, 1935)[‡]. A brief introduction to the notion of sequent is given below. Axioms and some used logical inference rules for a Gentzen's style presentation (called below in short: *reduction rules*) are also presented[§].

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[*] T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[†] The same designation is used as in the case of the corresponding rule of negating a conjunction.

[‡] Gerhard Karl Erich Gentzen (1909 – 1945).

[§] Another interesting approach having similar notation is the Lambek's calculus (concerning *formal grammars*: Joachim Lambek 1922 – 2014): a syntactic calculus that formalised the function type constructors along with various rules for the combination of functions. This approach

Let φ and ψ be some propositional formulae (see Definition 1.1). Next by Γ and Δ we shall denote some sequences of such formulae.

Definition 1.14

Any expression of the form $\Delta \vdash \Gamma$ is said to be a *sequent*. The first and the second elements of a sequent (here Δ and Γ) are called its *antecedent* and its *consequent*, respectively.

Any sequent of the form $\varphi_1,\varphi_2,...,\varphi_m \vdash \psi_1,\psi_2,...,\psi_n$ can be represented in an unique way by some propositional formula (known as a *generic interpretation*). In fact, the last sequent can be represented as follows:

$\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_m \Rightarrow \psi_1 \vee \psi_2 \vee ... \vee \psi_n$

An illustration of Definition 1.14 are the following sequents: $p \vdash p, q$ or $p \vee r, p \Rightarrow q \vdash q \vee r$ or also $\vdash \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$,etc.

Provided there is no ambiguity and to explain the use of the reduction rules given below, *before using a rule* wrt a given sequent, any missing element in this sequent is interpreted as an *empty* such one and denoted by the symbol λ. *After using this rule*, λ symbols are omitted in any new obtained sequent. And this process is continued as long as only elementary sequents are obtained (see below).

For example, the last of the above given sequents can be interpreted as follows: $\lambda \vdash \lambda, \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$. And hence, in accordance with the rule '$-\mathrm{E_c}$' and for convenience, the missing symbols Δ and Γ can be interpreted by λ And so, by using this rule the following two new sequents can be obtained: $\sim(p \vee q), \lambda \vdash \lambda, \sim p \wedge \sim q$ and $\sim p \wedge \sim q, \lambda \vdash \lambda, \sim(p \vee q)$. In the next step any such empty symbol is omitted. Then we have: $\sim(p \vee q) \vdash \sim p \wedge \sim q$ and $\sim p \wedge \sim q \vdash \sim(p \vee q)$. Similarly as in the case of using '$-\mathrm{E}$', i.e. the classical rule of omitting an equivalence. In the same way, the following simplified rule of removing an implication in the consequent of a sequent can be obtained: $-\mathrm{C}_c^*: \dfrac{\vdash A \Rightarrow B}{A \vdash B}$ or e.g. the rule of removing an equivalence in the consequent of a sequent: $-\mathrm{E}_c^*: \dfrac{\vdash A \Leftrightarrow B}{A \vdash B \quad A \vdash B}$ (a more formal treatment is omitted: see rules $-\mathrm{C_c}$ and $-\mathrm{E_c}$ given below).

The considered system includes only one *axiom**. This is a sequent having at least one *elementary formula* (i.e. at least one tiny latin letter such as: p, q, r, s, ...) in common in its antecedent and its consequent. So, according to the above example sequents, only the first of them $p \vdash p, q$ is an axiom. Any sequent $\Delta \vdash \Gamma$ having Δ and Γ as elementary sequences, i.e. sequences of elementary formulae without any logical connectives, is said to be an *elementary sequent* or *atom*. So any axiom is an atom, but not vice versa, e.g. the atom $p \vdash r, q$ is not an axiom.

In general, the Gentzen's sequent calculus can be used wrt some sequent transformations. This is done by introducing some (finite) set of rules. The used set of rules may be different depending on the following two cases:

(i) Generation of some true formula (i.e. thesis) for a priori given axiom by using *introduction rules*† (e.g. Glushkov V.M. 1964: 1921 - 1982), or also

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is an effort to capture mathematical aspects of natural language syntax in logical form (a very influential work in *computational linguistics*): *The Free Encyclopaedia, The Wikimedia Foundation, Inc*. The logical system called *full Lambek's calculus* is obtained by removing the following three structural rules: *exchange*, *contraction* and *weakening*. On the other hand, by adding all or some of the last three structural rules we can obtain various *intuitionistic substructural logics* (Kawaguchi M.F. et al. 2005). See Subsection 2.4 of this book: linear logic.

* In fact, any Gentzen's system uses only one axiom: $A \vdash A$, known as '*identity*' and denoted by Id. Moreover, introduction rules become an important proof technique (e.g. see Subsection 2.4: *Relevance logic*).

† In particular, such rules are used in *relevance logic*, e.g. $+\mathrm{C_a}$ or $+\mathrm{C_c}$ (known also as 'arrow on the left' and 'arrow on the right', respectively: see Subsection 2.4).

(ii) Validation if a priori given formula is a thesis by using *reduction* (called also: *elimination*) *rules* (e.g. Pawlak Z. 1965, Huzar Z. 2002, etc.).

Example 1.13

(i) $+C_a$: $$\frac{\Delta \vdash \Lambda, A \quad B, \Gamma \vdash \Theta}{A \Rightarrow B, \Delta, \Gamma \vdash \Lambda, \Theta}$$

is an example rule, shortly denoted by $'+C_a'$, *of adding an implication to the antecedent of a sequent* (composed wrt the previously given two sequents $\Delta \vdash \Lambda, A$ and $B, \Gamma \vdash \Theta$, where Δ, Λ, Γ and Θ denote some sequences of propositional formulae).

(ii) $-C_a$: $$\frac{\Delta, A \Rightarrow B \vdash \Gamma}{\Delta, B \vdash \Gamma \quad \Delta \vdash \Gamma, A}$$

is an example rule, shortly denoted by $'-C_a'$, of *removing an implication in the antecedent of a sequent*. In consequence, two new sequents are obtained. ▫

Any rule of the considered Gentzen's sequent calculus can be represented in an equivalent way by some propositional formula, which is a thesis. Some proofs of the above given two rules $+C_a$ and $-C_a$ are given in the next example.

Example 1.14

Below any sequence, say Δ , of propositional formulae $\varphi_1, \varphi_2, ..., \varphi_k$ is replaced by some conjunction $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_k$ (if Δ belongs to the antecedent of the considered sequent) or also by some disjunction $\psi_1 \vee \psi_2 \vee ... \vee \psi_k$ (if Δ belongs to the consequent of this sequent, in accordance with the above given sequent interpretation). By Ψ_Δ we shall denote the obtained propositional formula associated with Δ (similarly: Ψ_Γ and Ψ_Θ, associated with Γ and Θ, respectively).

(i) $+C_a$: $\models (\Psi_\Delta \Rightarrow \Psi_\Lambda \vee A) \wedge (B \wedge \Psi_\Gamma \Rightarrow \Psi_\Theta) \Rightarrow ((A \Rightarrow B) \wedge \Psi_\Delta \wedge \Psi_\Gamma \Rightarrow \Psi_\Lambda \vee \Psi_\Theta)$

Proof:

(1)	$\Psi_\Delta \Rightarrow \Psi_\Lambda \vee A$	
(2)	$B \wedge \Psi_\Gamma \Rightarrow \Psi_\Theta$	
(3)	$A \Rightarrow B$	{1,2,3,4,5 / a}
(4)	Ψ_Δ	
(5)	Ψ_Γ	
(6)	$\sim (\Psi_\Lambda \vee \Psi_\Theta)$	{aip}
(7)	$\sim \Psi_\Lambda$	{NA : 6}
(8)	$\sim \Psi_\Theta$	
(9)	$\Psi_\Lambda \vee A$	{− C : 1,4}
(10)	A	{− A : 7,9}
(11)	B	{− C : 3,10}
(12)	$B \wedge \Psi_\Gamma$	{ + K : 5,11}
(13)	Ψ_Θ	{− C : 2,12}
	contr. ▫	{8,13}

Unfortunately, the opposite implication is not satisfied (the reader is invited to use the Hilbert′s abbreviated verification).

(ii) $-C_a$: $\models (\Psi_\Delta \wedge (A \Rightarrow B) \Rightarrow \Psi_\Gamma) \Rightarrow (\Psi_\Delta \wedge B \Rightarrow \Psi_\Gamma) \wedge (\Psi_\Delta \Rightarrow \Psi_\Gamma \vee A)$

Proof:

(1)	$\Psi_\Delta \wedge (A \Rightarrow B) \Rightarrow \Psi_\Gamma$	{a}
(2)	$\sim((\Psi_\Delta \wedge B \Rightarrow \Psi_\Gamma) \wedge (\Psi_\Delta \Rightarrow \Psi_\Gamma \vee A))$	{aip}
(3)	$\Psi_\Delta \wedge B \wedge \sim\Psi_\Gamma \vee \Psi_\Delta \wedge \sim\Psi_\Gamma \wedge \sim A$	{NK,NC,NA: 2}
(1.1)	Ψ_Δ	
(1.2)	B	{1.1,1.2,1.3 / ada}
(1.3)	$\sim\Psi_\Gamma$	
(1.4)	$\sim(\Psi_\Delta \wedge (A \Rightarrow B))$	{Toll : 1,1.3}
(1.5)	$\sim\Psi_\Delta \vee \sim(A \Rightarrow B)$	{NK: 1.4}
(1.6)	$\sim(A \Rightarrow B)$	{– A : 1.1,1.5}
(1.7)	A	{NC: 1.6}
(1.8)	$\sim B$	
	contr.	{1.2,1.8}
(2.1)	Ψ_Δ	
(2.2)	$\sim\Psi_\Gamma$	{2.1,2.2,2.3 / ada}
(2.3)	$\sim A$	
(2.4)	$\sim(\Psi_\Delta \wedge (A \Rightarrow B))$	{Toll : 1,2.2}
(2.5)	$\sim\Psi_\Delta \vee \sim(A \Rightarrow B)$	{NK: 2.4}
(2.6)	$\sim(A \Rightarrow B)$	{– A : 2.1,2.5}
(2.7)	A	{NC: 2.6}
(2.8)	$\sim B$	
	contr. □	{2.3,2.7}

Without loss of generality and for simplicity, instead of Ψ_Δ, Ψ_Λ, Ψ_Γ, Ψ_Θ, A, B, etc. we can use some propositional variables, e.g. such as: p, q, r, s ...p_1, p_2, etc. For example, in the case of proving (ii), i.e. the above rule $-C_a$, we can obtain the following propositional formula: $(p \wedge (q \Rightarrow r) \Rightarrow s) \Rightarrow (p \wedge r \Rightarrow s) \wedge (p \Rightarrow s \vee q)$. And hence, it is sufficiently to prove that this formula is a thesis.

According to case (ii), the opposite implication, i.e. $(\Psi_\Delta \wedge B \Rightarrow \Psi_\Gamma) \wedge (\Psi_\Delta \Rightarrow \Psi_\Gamma \vee A) \Rightarrow (\Psi_\Delta \wedge (A \Rightarrow B) \Rightarrow \Psi_\Gamma)$, is also satisfied.

Proof:

(1)	$\Psi_\Delta \wedge B \Rightarrow \Psi_\Gamma$	
(2)	$\Psi_\Delta \Rightarrow \Psi_\Gamma \vee A$	{1,2,3,4 / a}
(3)	Ψ_Δ	
(4)	$A \Rightarrow B$	
(5)	$\sim\Psi_\Gamma$	{aip}
(6)	$\Psi_\Gamma \vee A$	{– C : 2,3}
(7)	A	{– A : 5,6}
(8)	B	{– C : 4,7}
(9)	$\Psi_\Delta \wedge B$	{ + K : 3,8}
(10)	Ψ_Γ	{– C : 1,9}
	contr. □	{5,10}

And so the following rule can be generated:

$$\frac{\Delta, B \vdash \Gamma \quad \Delta \vdash \Gamma, A}{\Delta, A \Rightarrow B \vdash \Gamma}.$$

The last rule can be obtained from $+C_a$ by interchanging Γ with Δ and next Λ and Θ with Γ. □

Below we shall restrict our attention only to the problem of validation of propositional formulae. For any of the logical functors (i.e. the basic symbols $\sim$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$) two reduction rules are given, depending on the fact if this logical functor belongs to the antecedent or also to the consequent of the considered sequent. So the following ten *reduction rules* can be used*.

(1) *Rule of removing a negation in the antecedent of a sequent* $\quad -\mathrm{N_a}: \dfrac{\sim A, \Delta \vdash \Gamma}{\Delta \vdash \Gamma, A}$

(2) *Rule of removing a negation in the consequent of a sequent* $\quad -\mathrm{N_c}: \dfrac{\Delta \vdash \Gamma, \sim A}{A, \Delta \vdash \Gamma}$

(3) *Rule of removing a conjunction in the antecedent of a sequent* $\quad -\mathrm{K_a}: \dfrac{A \wedge B, \Delta \vdash \Gamma}{A, B, \Delta \vdash \Gamma}$

(4) *Rule of removing a conjunction in the consequent of a sequent* $\quad -\mathrm{K_c}: \dfrac{\Delta \vdash \Gamma, A \wedge B}{\Delta \vdash \Gamma, A \quad \Delta \vdash \Gamma, B}$

(5) *Rule of removing a disjunction in the antecedent of a sequent* $\quad -\mathrm{A_a}: \dfrac{A \vee B, \Delta \vdash \Gamma}{A, \Delta \vdash \Gamma \quad B, \Delta \vdash \Gamma}$

(6) *Rule of removing a disjunction in the consequent of a sequent* $\quad -\mathrm{A_c}: \dfrac{\Delta \vdash \Gamma, A \vee B}{\Delta \vdash \Gamma, A, B}$

(7) *Rule of removing an implication in the antecedent of a sequent* $\quad -\mathrm{C_a}: \dfrac{\Delta, A \Rightarrow B \vdash \Gamma}{\Delta, B \vdash \Gamma \quad \Delta \vdash \Gamma, A}$

(8) *Rule of removing an implication in the consequent of a sequent* $\quad -\mathrm{C_c}: \dfrac{\Delta \vdash \Gamma, A \Rightarrow B}{A, \Delta \vdash \Gamma, B}$

(9) *Rule of removing an equivalence in the antecedent of a sequent* $\quad -\mathrm{E_a}: \dfrac{\Delta, A \Leftrightarrow B \vdash \Gamma}{A, B, \Delta \vdash \Gamma \quad \Delta \vdash \Gamma, A, B}$

(10) *Rule of removing an equivalence in the consequent of a sequent* $\quad -\mathrm{E_c}: \dfrac{\Delta \vdash \Gamma, A \Leftrightarrow B}{A, \Delta \vdash \Gamma, B \quad B, \Delta \vdash \Gamma, A}$

As an example, since $\wedge$ is distributive over $\vee$, the rule of removing a disjunction $-\mathrm{A_a}$ can be proved directly by using the law of addition of antecedents of two implications having the same consequent. Also, the following two different indirect proof versions for $-\mathrm{E_a}$, i.e. $(p \wedge (q \Leftrightarrow r) \Rightarrow s) \Rightarrow (p \wedge q \wedge r \Rightarrow s) \wedge (p \Rightarrow q \vee r \vee s)$, can be obtained.

Proof of $-\mathrm{E_a}$ (*the first version is ramified*):

(1)	$p \wedge (q \Leftrightarrow r) \Rightarrow s$	{a}
(2)	$\sim((p \wedge q \wedge r \Rightarrow s) \wedge (p \Rightarrow q \vee r \vee s))$	{aip}
(3)	$p \wedge q \wedge r \wedge \sim s \vee p \wedge \sim q \wedge \sim r \wedge \sim s$	{NK,NC,NA,SR : 2}

* Since the rules corresponding to the first three connectives are very simple, only these rules can be considered as sufficient. But the obtained proofs may be longer than these ones related to the use of complete set of rules, e.g. the obtained proofs of T 1.10 (the law of compound constructive dilemma) without using $'-\mathrm{C_a}'$ and with using $'-\mathrm{C_a}'$ have different complexity (left to the reader). We observe that commas placed in the antecedents or consequents are "commutative" (since $\wedge$ and $\vee$ are commutative). In general, any of the above inference rules (called also '*cut rules*' can be considered as a generalisation of the classical modus ponens, i.e. $'-\mathrm{C}'$. It was shown (the Gentzen's cut-elimination theorem, 1934) that any judgement that possesses a proof in the sequent calculus making use of the cut rule also possesses a cut-free proof (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.).

(1.1)	p	
(1.2)	q	{1.1,1.2,1.3,1.4 / ada}
(1.3)	r	
(1.4)	$\sim s$	
(1.5)	$\sim p \lor \sim (q \Leftrightarrow r)$	{Toll,NK,SR : 1,1.4}
(1.6)	$\sim (q \Leftrightarrow r)$	{$-$ A : 1.1,1.5}
(1.7)	$q \land \sim r \lor \sim q \land r$	{SR : 1.6}
(1.8)	$q \land r \land (q \land \sim r \lor \sim q \land r)$	{+K : 1.2,1.3,1.7}
	contr.	{1.8}
(2.1)	p	
(2.2)	$\sim q$	{2.1,2.2,2.3,2.4 / ada}
(2.3)	$\sim r$	
(2.4)	$\sim s$	
(2.5)	$\sim p \lor \sim (q \Leftrightarrow r)$	{Toll,NK,SR : 1,2.4}
(2.6)	$\sim (q \Leftrightarrow r)$	{$-$ A : 2.1,2.5}
(2.7)	$q \land \sim r \lor \sim q \land r$	{SR : 2.6}
(2.8)	$\sim q \land \sim r \land (q \land \sim r \lor \sim q \land r)$	{+K : 2.2,2.3,2.7}
	contr. □	{2.8}

Since $q \not\Leftrightarrow r \Leftrightarrow_{df} \sim (q \Leftrightarrow r)$ the above-ramified proof can be simplified. In fact, by using the (corresponding short forms of the derived) rules of omitting an exclusive disjunction $-$ EA′′ (wrt lines (1.2) and (1.6) we can obtain a contradiction: r and $\sim r$) and $-$ EA′ (wrt lines (2.2) and (2.6): the same two contradictory lines can be obtained, see remark to ′$-$ A′, Subsection 1.2). Another proof technique is given below.

Proof of $-$ E_a (*the second version*):

(1)	$p \land (q \Leftrightarrow r) \Rightarrow s$	{a}
(2)	$\sim ((p \land q \land r \Rightarrow s) \land (p \Rightarrow q \lor r \lor s))$	{aip}
(3)	$p \land q \land r \land \sim s \lor p \land \sim q \land \sim r \land \sim s$	{NK,NC,NA,SR : 2}
(4)	$p \land \sim s \land (q \land r \lor \sim q \land \sim r)$	{$\land$ is distributive over $\lor$}
(5)	$p \land (q \Leftrightarrow r)$	{$-$ K, $\land$ is commutative and associative, SR : 4}
(6)	$\sim s$	
(7)	s	{$-$ C : 1,5}
	contr. □	{6,7}

Another version can be obtained by using Toll wrt the lines (1,6) and then $-$ A (this is omitted). Instead of two contradictory lines, in the first version two contradictory formulae are obtained (in accordance with the considered cases). Similarly, the last rule of removing an equivalence $-$ E_c can be proved directly by using the following two theses (this is left to the reader): $\models (p \Rightarrow q \lor (r \Leftrightarrow s)) \Rightarrow (p \Rightarrow q \lor (r \Rightarrow s))$ and $\models (p \Rightarrow q \lor (r \Leftrightarrow s)) \Rightarrow (p \Rightarrow q \lor (s \Rightarrow r))$, etc. The reader is invited to prove the rule $-$ E_c by using T 1.5.

In general, any Gentzen's rule is provable using indirect proof techniques.

Any propositional formula φ can be considered as a sequent of type ′$\vdash \varphi$′, where the symbol ′$\vdash$′ is said to be the *main sequent connective* for φ. The validation of φ and so the process of reduction of the logical functors associated with φ is started always wrt the main sequent connective. This process is repeated as long as the whole set of logical functors is reduced and so only a set of atoms is obtained. Hence, φ is a thesis iff any such atom is an axiom.

Any of the above presented rules (1 $-$ 10) can be considered as a derived (or secondary) wrt the natural deduction approach. The corresponding proofs are similar to these ones given in Example 1.14 (the reader is invited to present proofs for the remaining rules). Moreover, as in the axiomatic approach, the proof style used is always the direct style. The methodological aspects of the above calculus are omitted here. The proposed system is exact wrt any propositional formula φ.

An illustration of the above-considered approach is given in the next example (the proof style used below is related to the so called ‘*turnstile*’).

Example 1.15

(a) Let consider the proof of the law of implication (see T 1.15).It is assumed below the first line of the proof always is corresponding to the *main sequent connective* (shortly: *msc*)*.

(1)	$\vdash p \Rightarrow q \Leftrightarrow \sim p \vee q$	{msc}	
(2)	$p \Rightarrow q \;\vdash\; \sim p \vee q$	$\{2,3 \,/\, - E_c : 1\}$	
(3)	$\sim p \vee q \;\vdash\; p \Rightarrow q$		
(4)	$q \;\vdash\; \sim p \vee q$	$\{4,5 \,/\, - C_a : 2\}$	
(5)	$\vdash\; \sim p \vee q, p$		
(6)	$p, \sim p \vee q \;\vdash\; q$	$\{- C_c : 3\}$	
(7)	$q \;\vdash\; \sim p, q$	$\{- A_c : 4\}$	
(8)	$\vdash\; \sim p, q, p$	$\{- A_c : 5\}$	
(9)	$p, \sim p \;\vdash\; q$	$\{9,10 \,/\, - A_a : 6\}$	
(10)	$p, q \;\vdash\; q$		{axiom}
(11)	$p, q \;\vdash\; q$	$\{- N_c : 7\}$	{axiom}
(12)	$p \;\vdash\; q, p$	$\{- N_c : 8\}$	{axiom}
(13)	$p \;\vdash\; q, p.$ □	$\{- N_a : 9\}$	{axiom}

(b) The following proof of the law of negating a disjunction (see T 1.7) can be obtained.

(1)	$\vdash\; \sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$	{msc}	
(2)	$\sim (p \vee q) \;\vdash\; \sim p \wedge \sim q$	$\{2,3 \,/\, - E_c : 1\}$	
(3)	$\sim p \wedge \sim q \;\vdash\; \sim (p \vee q)$		
(4)	$\vdash\; \sim p \wedge \sim q, p \vee q$	$\{- N_a : 2\}$	
(5)	$\sim p, \sim q \;\vdash\; \sim (p \vee q)$	$\{- K_a : 3\}$	
(6)	$\vdash\; \sim p, p \vee q$	$\{6,7 \,/\, - K_c : 4\}$	
(7)	$\vdash\; \sim q, p \vee q$		
(8)	$\sim q \;\vdash\; \sim (p \vee q), p$	$\{- N_a : 5\}$	
(9)	$p \vdash p \vee q$	$\{- N_c : 6\}$	
(10)	$q \vdash p \vee q$	$\{- N_c : 7\}$	
(11)	$\vdash\; \sim (p \vee q), p, q$	$\{- N_a : 8\}$	
(12)	$p \;\vdash\; p, q$	$\{- A_c : 9\}$	{axiom}
(13)	$q \;\vdash\; p, q$	$\{- A_c : 10\}$	{axiom}
(14)	$p \vee q \;\vdash\; p, q$	$\{- N_c : 11\}$	
(15)	$p \;\vdash\; p, q$	$\{15,16 \,/\, - A_a : 14\}$	{axiom}
(16)	$q \;\vdash\; p, q\,.$ □		{axiom}

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* In general, in accordance CR and SR, the sequent $\vdash \varphi \Rightarrow \psi$ can be equivalently represented as: $\vdash \sim \varphi \vee \psi$. Next by using $- A_c$ and then $-N_c$ we can obtain: $\varphi \vdash \psi$. In a similar way, by using $- E$ and SR, the sequent $\vdash \varphi \Leftrightarrow \psi$ can be equivalently represented as: $\vdash (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$. According to $- K_c$, the following two sequents can be obtained: $\vdash \varphi \Rightarrow \psi$ and $\vdash \psi \Rightarrow \varphi$. And finally, we have: $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

The reader is invited to show the law of addition an arbitrary proposition to the antecedent and consequent of a given implication (see Example 2.7 of Subsection 2.2) and also the following two theses:

$$(p \Rightarrow q) \wedge p \Leftrightarrow p \wedge q$$

and

$$(p \Rightarrow q \vee r) \wedge (s \wedge t \Rightarrow u) \Rightarrow ((r \Rightarrow s) \wedge p \wedge t \Rightarrow q \vee u).$$

It can be observed that the size of the generated proof for a given φ will depend on the number of possible branchings (i.e. the size of the obtained branch-out tree, in each step of the reduction process for φ). For example, the direct proof of the converse implication T 1.5b (of the law MC of multiplication of consequents) requires two times ´– C´ and then ´+ K´. At the same time, the corresponding sequent calculus will require eight possible branchings. The reader is invited to prove the following laws: addition of antecedents, compound constructive and compound destructive dilemmas, and the law of conversion of implications (i.e. T 1.9, T 1.10, T 1.11, and T 1.13, respectively).

An extension of the above given set of reduction rules (1 – 10) is omitted here. It will be given in the next chapter, where the (first-order) predicate logic is presented. And hence, four new reduction rules should be added wrt the *universal* ($\forall$) and *existential* ($\exists$) *quantifiers* (depending on the fact if any such quantifier is in the antecedent or also in the consequent of the sequent associated with the considered predicate formula: see Subsection 3.4).

The sequent calculus becomes elusive for some non-classical logic systems. A generalisation of Gentzen-style sequent systems (known as '*hypersequent calculus*') is considered in Subsection 4.1.

A brief survey of the commonly studied non-standard logics, e.g. such as: many-valued, fuzzy, modal, temporal, and some others is given in the next section.

2. Non-standard logics

Non-standard logics such as many-valued, fuzzy, modal, temporal, etc. have been increasingly gaining the attention mainly wrt the following two reasons. First, the natural desire to extend the existing classical deduction techniques to new domains of logic. And second - the need of providing a suitable foundation for artificial intelligence. Any such system is constructed on the basis of classical logic, e.g. by deleting or also introducing some axioms (e.g. intuitionistic logic or paraconsistent logic), by introducing some new connectives (called also functors, e.g. such as: □, ◇, =>>, <⇔> , ! , δ , ○ , / , U , etc. in the modal and temporal logics), by introducing more than two (but a finite number of) logical values or also an extension to a non-countable set of values in the closed interval, i.e. based on the *truth-values set* [0,1] (e.g. *many-valued* or *fuzzy logics*) and so on. The first non-classical systems were introduced to the middle of the 20th century, e.g. such as: the 3-valued Łukasiewicz's system (Łukasiewicz J. 1917), system of strict implication (Lewis C.I. 1918), intuitionistic logic (Heyting A. 1930, Gentzen G.K.E. 1935 and Kleene S.C. 1952)*.

The Łukasiewicz's intention in the construction of the 3-valued system was to use a third additional truth value for "possible", and to model in this way the modalities "it is necessary that" and "it is possible that"(considered as special or alethic such ones). This intended application to modal logic was not materialised. The outcome of these investigations are, however, the Łukasiewicz's systems, and a series of theoretical results concerning these systems, e.g. such as: the basic idea of additional truth degrees wrt problems of the representability of functions (Post E.L. 1921), reasoning the intuitionistic logic in terms of many truth degrees (Gödel K. 1932), an infinite valued characteristic matrix for intuitionistic logic (Jaś S. 1936), a philosophical application of 3-valued logic to the discussion of paradoxes (Bochvar D.A. 1938), and a mathematical one to partial functions and relations (Kleene S.C.

* See Subsection 2.4: *Intuitionistic and fuzzy intuitionistic logics*.

1938), the determination of the fixed points in the revision theory of truth (Kripke S. 1975), and so on (see Gottwald S. 2000).

The construction of the systems of many-valued logic can be considered as one of the most important logical discoveries in the past century (Słupecki J. and Borkowski L. 1967). Below we shall first concentrate our attention to some such systems.

2.1. Systems of many-valued logic

Many-valued logics are non-classical logics. But they are similar to classical logic because they accept the *principle of truth-functionality*, i.e. the truth of a compound proposition is determined by the truth-values of its component propositions (and so remains unaffected when one of its component propositions is replaced by another having the same truth-value). Here a fundamental fact is that the above logics do not restrict the number of truth-values to only two, e.g. in the grey-scale images the assertion that some colour is not white is not equivalent to the assertion that this colour is the black itself and vice versa. And this is because the set of all possible colours in the grey level scale gradation $\{0,1, \ldots, 255\}$ includes more than two elements (i.e. the black colour '0' and the white one '255').

There are two kinds of semantics for systems of many-valued logic: standard logical matrices and algebraic semantics. The main types of logical calculi are all available for these systems (such as: the natural deduction approach, Gentzen's type sequent calculi or tableau calculi). The main systems of many-valued logic often come as families, which comprise uniformly defined finite-valued as well as infinite-valued systems such as: Łukasiewicz's logics, Gödel's logics, t-norm related systems, 3-valued systems, product systems, Dunn/Belnap's 4-valued system, etc. (Gottwald S. 2000).

Łukasiewicz's logics (Jan Łukasiewicz 1878 – 1956)

The systems $Ł_m$ and $Ł_\infty$ are defined by means of the following two truth degree sets:

$W_m =_{df} \left\{ \frac{k}{m-1} / k \in \{0,1,\ldots,m-1\}, m \geq 2 \right\}$ and $W_\infty =_{df} [0,1] \subseteq \mathbb{R}_+$ (the set of all nonnegative real numbers), respectively*.

The degree 1 is the only designated truth degree. The main *conjunction*s of these systems are the *strong* and *weak conjunctions*, denoted by & and $\wedge$, respectively. The following two truth degree functions are used:

$p \;\&\; q =_{df} \max\{0, p + q - 1\}$† and

$p \wedge q =_{df} \min\{p,q\}$.

It can be observed that the logical value $p \;\&\; q = p \wedge q = 0$ for $p = 0$ or $q = 0$. Let now $p, q > 0$. Then we have: $p \;\&\; q = p \wedge q$, for $p + q > 1$ and $\max\{p,q\} = 1$. Similarly $p \;\&\; q \neq p \wedge q$, for $p + q \leq 1$. For example, $p \;\&\; q = p \wedge q = \frac{1}{2}$, for $p = \frac{1}{2}$ and $q = 1$. In a similar way, $p \;\&\; q = 0 \neq \frac{1}{3} = p \wedge q$, for $p = \frac{1}{2}$ and $q = \frac{1}{3}$. But the above two truth degree functions coincide for $m = 2$.

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* The first Łukasiewicz's system (a ternary logic system with $W_3 =_{df} \{0, 1/2, 1\}$) was given in 1918: usually 0, 1/2 and 1 are denoted by the logical constants F, U (unknown) and T, respectively.

† Below we shall assume the *logical value of a propositional formula* φ, denoted by $v(\varphi) \in W_m$ or also $v(\varphi) \in W_\infty$, depending on the used system. Provided there is no ambiguity, for simplicity instead of $v(\varphi)$ the same formula φ is used, e.g. p instead of v(p), q instead of v(q), $p \wedge q$ instead of $v(p \wedge q)$, etc. Moreover, to minimise the number of used parentheses we shall assume the logical functors bind more strongly than the sign of equality. Hence, e.g. $(p \Rightarrow q) \wedge p \Rightarrow q = 1$ denotes $((p \Rightarrow q) \wedge p \Rightarrow q) = 1$ or equivalently: $v((p \Rightarrow q) \wedge p \Rightarrow q) = 1$.

In accordance with the above two conjunctions, the following two (*strong* and *weak*) *disjunction* connectives can be introduced:

$p \veebar q =_{df} \min\{1, p + q\}$ and

$p \vee q =_{df} \max\{p, q\}$.

The last two disjunctions also coincide for $m = 2$ (a similar verification of the relations '=' and '≠' between the logical values of $p \veebar q$ and $p \vee q$ is left to the reader).

The *Łukasiewicz's negation and* i*mplication connectives* are defined as follows:

$\sim p =_{df} 1 - p$ and

$p \Rightarrow q =_{df} \min\{1, 1 - p + q\}$ (or equivalently: $p \Rightarrow q =_{df}$ if $p \le q$ then 1 else $1 - p + q$).

There were given also another many-valued logic systems, e.g. *Kleene's ternary logic* (1952: Stephen Cole Kleene 1904 – 1994): *Kleene's ternary implication* differs in its definition in that 'U implies U' is 'U' (instead of 'T', as in the case of Łukasiewicz's one): In accordance with the law of implication (T 1.15 , Subsection 1.3), the logical value of this implication, $p \Rightarrow q =_{df} \sim p \vee q =_{df} \max\{1 - p, q\}$. The corresponding definitions for negation, conjunction and disjunction connectives are the same as in Łukasiewicz's system: concerning the weak conjunction and disjunction connectives. Instead of $W_3 =_{df} \{0, 1/2, 1\}$, there was used the set {F, U, T}, where 0, 1/2 and 1 correspond to F, U and T, respectively: Kleene's symbol 'U' denotes '*unknown*'. Kleene's ternary implication was used as a natural way of generating a paraconsistent logic (see Subsection 2.4: left to the reader).

Proposition 2.1

$(p \Rightarrow q) \wedge (q \Rightarrow p) = (p \Rightarrow q) \,\&\, (q \Rightarrow p)$

Proof:

Let $L =_{df} (p \Rightarrow q) \wedge (q \Rightarrow p) = \min\{\min\{1, 1 - p + q\}, \min\{1, 1 - q + p\}\}$. Also assume that $R =_{df} (p \Rightarrow q) \,\&\, (q \Rightarrow p) = \max\{0, \min\{1, 1 - p + q\} + \min\{1, 1 - q + p\} - 1\}$. Obviously, for $p = q$ we have: $L_{/p=q} = 1 = R_{/p=q}$.

Assume now that $p < q$. Hence $\min\{1, 1 - p + q\} = \min\{1, 1 + (q - p)\} = 1$ and $\min\{1, 1 - q + p\} = \min\{1, 1 - (q - p)\} = 1 - q + p$. And so we can obtain: $L_{/p<q} = \min\{1, 1 - (q - p)\} = 1 - q + p = \max\{0, 1 + (1 - (q - p)) - 1\} = R_{/p<q}$.

In a similar way for $p > q$ we can obtain: $L_{/p>q} = 1 + q - p = R_{/p>q}$. □

It can be observed that $v(\varphi) = v(\psi)$ will implicate $\varphi \Leftrightarrow \psi$. An example equivalence connective defined for $Ł_4$ is shown in the next table (independent on the used type of conjunction: strong or weak). This table is a generalisation of the classical corresponding to $Ł_2$.

⇔	0	1/3	2/3	1
0	1	2/3	1/3	0
1/3	2/3	1	2/3	1/3
2/3	1/3	2/3	1	2/3
1	0	1/3	2/3	1

Next we shall assume that the symbol of negation binds more strongly than the remaining symbols. An algebraic interpretation of *De Morgan's laws* is given below.

(i) $\sim(p \,\&\, q) = \sim p \veebar \sim q$,

(ii) $\sim(p \veebar q) = \sim p \,\&\, \sim q$,

(iii) $\sim(p \wedge q) = \sim p \vee \sim q$, and

(iv) $\sim(p \vee q) = \sim p \wedge \sim q$.

Proof (i):

The following equality has to be shown:

$$1 - \max\{0, p + q - 1) \overset{?}{=} \min\{1, (1-p) + (1-q)\}$$

We have three cases for consideration depending on the value of the sum $p + q$ ($= 1$ or < 1 or > 1). For example, assuming $p + q > 1$ the left side $L =_{df} 1 - \max\{0, p + q - 1\} = 2 - (p + q)$. Since $p + q > 1$ then $2 - (p + q) < 1$. Hence the right side $R =_{df} \min\{1, 2 - (p + q)\} = 2 - (p + q) = L$ (similarly for the rest two cases). □

It is easily to show the weak operations $\wedge$ and $\vee$ satisfy the commutative, associative, absorptive, idempotent, and distributive axioms. Unfortunately the strong conjunction and disjunction are only commutative and associative. For example, the absorptive axioms are not satisfied for $p = \frac{1}{2}$ and $q = 1$ (or 0, depending of the considered axiom), the distributive axioms are not satisfied for $p = \frac{1}{3}$ and $q = r = \frac{1}{2}$, etc. In fact, the following properties are satisfied:

(v) $p \,\&\, q = q \,\&\, p$,

(vi) $p \veebar q = q \veebar p$,

(vii) $(p \,\&\, q) \,\&\, r = p \,\&\, (q \,\&\, r)$, and

(viii) $(p \veebar q) \veebar r = p \veebar (q \veebar r)$.

According to the associative axiom, the above strong and weak operations can be generalised for a finite number of more than two arguments. As an example, the proof of (vii) is illustrated below.

Proof (vii):

The following equality has to be shown:

$$\max\{0, \max\{0, p + q - 1\} + r - 1\} \overset{?}{=} \max\{0, p + \max\{0, q + r - 1\} - 1\}$$

Hence the following $3^2 = 9$ cases have to be shown depending on the values of $p + q - 1$ and $q + r - 1$ (if they are $= 0$ or < 0 or also > 0). For example, assuming $p + q - 1 < 0$ and $q + r - 1 > 0$ the left side $L =_{df} \max\{0, r - 1\}$ and the right side $R =_{df} \max\{0, p + q + r - 2\}$. Since $0 \leq r \leq 1$ we have: $L = 0$. On the other hand: $p + q + r = (p + q) + r < 1 + r \leq 2$. Hence $R = 0$ and so $L = R$ (the rest cases can be analysed in an similar way). □

The following *monotonic property* is satisfied

$p \leq q \Rightarrow p \,\&\, r \leq q \,\&\, r$ and

$p \leq q \Rightarrow p \wedge r \leq q \wedge r$.

Post's logics (Emil Leon Post 1897 – 1954)

Let $i, m \in \mathbb{N}$ (the set of natural numbers) be some parameters such that $i < m$ and $m \geq 2$. Consider the following family of many-valued logic systems $P_i^m =_{df} (A, B, n^m, a^m)$, where the sets $A =_{df} \{0,1,2, \dots, m-i-1, m-i, m-i+1, \dots, m-1\}$, $B =_{df} \{m-i, m-i+1, \dots, m-1\} \subseteq A$ and the two functions n^m and a^m are defined as follows (see: *Formal logic. Encyclopedical outline with applications to informatics and linguistics* 1987):

$n^m(x)$ $=_{df}$ if $x = 0$ then $m - 1$ else $x - 1$ fi and

$a^m(x,y)$ $=_{df}$ $\max\{x,y\}$ (for any $x, y \in A$).

The above two functions are functionally complete. It can be observed that n^2 and a^2 in P_l^2 correspond to the classical negation and disjunction (i.e. logical alternative), respectively.

Example 2.1

Consider the example system $P_l^3 = (\{0,1,2\}, \{2\}, n^3, a^3)$. In accordance with the above definitions, the following two tables for n^3 and a^3 are shown below.

x	$n^3(x)$
0	2
1	0
2	1

$a^3(x,y)$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

Since n^3 and a^3 are functionally complete in P_l^3, any other function can be expressed by them. □

Example 2.2

It can be observed that the variables x and y in the above given definitions can be considered as logical values of some propositional variables, say p and q, respectively. This is in accordance with our previous style, and so $n^m(x)$ can be denoted by $\sim p$ and $a^m(x,y)$ by $p \vee q$, and vice versa. So, e.g. assuming $m = 3$, the Łukasiewicz's negation and implication connectives can be defined as follows: $n_Ł^3(x)$ $=_{df}$ $2 - x$ and $c_Ł^3(x,y)$ $=_{df}$ $\min\{2, 2 - x + y\}$ (multiplying each element in W_3 by 2). And hence, e.g. in the Łukasiewicz's implication can be expressed as follows: $c_Ł^3(x,y) = a^3(a^3(n_Ł^3(x), y), n_Ł^3(a^3(n^3(x), n^3(y))))$. The last obtained expression is equivalent to the original one, i.e. $\min\{2, 2 - x + y\}$. In fact, the columns in the corresponding two truth tables will be identical (for any $(x,y) \in \{0,1,2\}^2$). The reader is invited to find the expression related to $n_Ł^3(x)$.

The logical values 0, 1, and 2 can be expressed as follows: $n^3(n^3(a^3(a^3(x, n^3(x)), n^3(n^3(x)))))$, $n^3(a^3(a^3(x, n^3(x)), n^3(n^3(x))))$ (is the famous Słupecki's t function), and $a^3(a^3(x, n^3(x)), n^3(n^3(x)))$, respectively. In accordance with the previously used style, e.g. the last expression corresponds to the following propositional formula: $p \vee \sim p \vee \sim\sim p$. □

It can be observed that the involutivity property is not satisfied in the case of Post's negation, i.e. $n^m(n^m(x)) \neq x$ (in contradistinction with the Łukasiewicz's negation, see below: t-norm related systems*)*. Obviously, different expressions may correspond to the same function, e.g. $g = h$ (i.e. $g(x) = h(x)$, for any $x \in \{0,1,2)$), where $g(x)$ $=_{df}$ $n^3(n^3(a^3(n^3(a^3(x, n^3(x))), a^3(x, n^3(x)))))$ and $h(x)$ $=_{df}$ $n^3(n^3(a^3(x, n^3(x))))$ (the corresponding propositional formulae are equivalent and hence we can obtain a thesis).

The following interesting interpretation was proposed by Post. So, any element of the *universe* A can be interpreted as a binary sequence of length $(m - 1)$. Moreover, if some element of a given sequence is equal to 1, then all rest elements of this sequence are also equal to 1. And hence, the logical value $m - 1 \in A$ is interpreted as a sequence of 1's of length $(m - 1)$. The value $m - 2 \in A$ is interpreted as a binary sequence having the first element equal to 0 and the rest (m − 2) elements equal to 1. Similarly, the element $1 \in A$ is interpreted as a binary sequence of length (m − 1) having all elements 0's except the last one, equal to 1. And finally, the logical value $0 \in A$ is interpreted as a sequence of length (m − 1) having all elements equal to 0. In accordance with the above given interpretation, the *distinguished elements* of $B \subseteq A$ are considered as binary sequences of length (m − 1) having less than i 0's. Moreover, it can be observed that the logical values in A for different systems P_i^m will be also different. And hence, the elements of A (and $B \subseteq A$) will depend on the parameter m. And so, more formally the above sets A and B can be introduced as follows: $A =_{df} \{0^m, 1^m, 2^m, \ldots, m-i-1^m, m-i^m, m-i+1^m, \ldots, m-1^m\}$ and $B =_{df} \{m-i^m, m-i+1^m, \ldots, m-1^m\}$, respectively.

Example 2.3

Let $A =_{df} \{0,1,2,3,4,5\}$. Consider the system $P_3^6 =_{df} (\{0,1,2,3,4,5\}, \{3,4,5\}, n^6, a^6)$. The following interpretation of the elements of the universe A can be obtained:

0^6	=	(0,0,0,0,0)	the strongest false value
1^6	=	(0,0,0,0,1)	
2^6	=	(0,0,0,1,1)	the weakest false value
3^6	=	(0,0,1,1,1)	the weakest true value
4^6	=	(0,1,1,1,1)	
5^6	=	(1,1,1,1,1)	the strongest true value

. □

It was shown by Post that any many-valued formula can be presented by a composition of classical logic formulae. In fact, according to the above interpretation, any $x, y \in A$ can be considered as binary vectors $\underline{x}, \underline{y} \in \{0,1\}^{m-1}$, where $\underline{x} =_{df} (x_1, x_2, \ldots, x_{m-1})$ and $\underline{y} =_{df} (y_1, y_2, \ldots, y_{m-1})$. Let a, k and n be the functions corresponding to the classical disjunction, conjunction and negation, respectively. Also assume that $z : \{0,1\}^{m-1} \to \{0,1\}$ be a map such that $z(\underline{x}) = z(x_1, x_2, \ldots, x_{m-1}) =_{df} a(a(\ldots(a(a(x_1,x_2), x_3)\ldots), x_{m-2}), x_{m-1})$. Hence, the following interpretation can be obtained for n^m and a^m.

$n^m(x) = (n(z), a(n(z), k(x_1, x_2)), a(n(z), k(x_2, x_3)), \ldots, a(n(z), k(x_{m-2}, x_{m-1})))$ and

$a^m(x,y) = (a(x_1, y_1), a(x_2, y_2), \ldots, a(x_{m-1}, y_{m-1}))$.

Example 2.4

Let consider the system P_1^3 of Example 2.1. We have: $A = \{0^3, 1^3, 2^3\}$. Since $z = a(x_1,x_2)$, the following interpretation can be obtained for n^3 and a^3.

$n^3(x) = (n(a(x_1,x_2)), a(n(a(x_1,x_2)), k(x_1, x_2)))$ and

$a^3(x,y) = (a(x_1, y_1), a(x_2, y_2))$.

The integers 0^3, 1^3 and 2^3 (i.e 0, 1 and 2 with m = 3) are interpreted as follows: $\underline{0}^3 = (0,0)$, $\underline{1}^3 = (0,1)$ and $\underline{2}^3 = (1,1)$. Directly by the above definitions we can obtain, e.g. $n^3(0) = 2$ and also $a^3(1,2) = 2$. By using the last interpretation we can obtain for n^3 and a^3 binary vectors corresponding to the same values. In fact, we have: $n^3(0) =_{df} (n(a(0,0)), a(n(a(0,0)), k(0,0))) = (n(0), a(n(0), 0)) = (1, \max\{1,0\}) = (1,1) = \underline{2}^3$. Since $x = (0,1)$ and $y = (1,1)$ then in a similar way we can obtain: $a^3(1,2) =_{df} (a(0,1), a(1,1)) = (1,1) = \underline{2}^3$. The reader is invited to find the corresponding interpretation for the functions n^6 and a^6 of Example 2.3. □

From historical point of view the philosophical hopes related to Łukasiewicz's logics have not been completely satisfied. Doubtless, the Łukasiewicz's three-valued system is now recognised as one of the most important developments of the past century*. On the other hand, the Post's interpretations initiated the today's use of many-valued logics in such areas as: discrete mathematics, in particular combinatorics and (the chain-based) Post algebras and their technical applications in multiple-valued switching theory (e.g synthesis of multiple-valued or also quantum logic circuits). And so, all these applications are a confirmation of their own increasing importance.

Some other many-valued systems were also introduced, e.g. such as: Sobociński's logics (using two connectives related to the Łukasiewicz's negation and implication), Słupecki's logics, etc (see: *The little encyclopaedia of logic* 1988). In particular, the last system can be considered as the largest finite such one. A brief presentation of this system is given below.

Słupecki logics (Jerzy Słupecki 1904 – 1987)

Consider the system $S_i^m =_{df} (A, B, C, R, S)$, where $i < m$ and $m \geq 2$ $(i, m \in \mathbb{N})$. The *universe* $A =_{df} \{1,2, \dots , m\}$ and $B =_{df} \{1,2, \dots , i\}$ is the *subset of distinguished elements*. The logical connectives C, R and S are defined as follows (for any $p, q \in A$):

1. if $1 \leq p \leq i$ then $Cpq = q$ else $Cpq = 1$,
2. if $1 \leq p \leq m - 1$ then $Rp = p + 1$ else $Rp = 1$ and
3. if $p = 1\ (2)$ then $Sp = 2\ (1)$ else $Sp = p$.

The used designation C corresponds to the Łukasiewicz´s symbol of implication. It was shown that any S_i^m is a semantically and syntactically complete system (see Definitions 1.9 and 1.10 of Subsection 1.6). Moreover, with any such system a corresponding set of axioms was associated (a more formal treatment is omitted).

Gödel's logics (Kurt Gödel 1906 – 1978)

The systems G_m and G_∞ are defined by means of the above two truth degree sets: W_m and W_∞. The degree 1 is the only designated truth degree.

The basic connectives of these systems are a conjunction and a disjunction, denoted by $\wedge$ and $\vee$, respectively. They are defined as follows:

$p \wedge q =_{df} \min\{p,q\}$ and

$p \vee q =_{df} \max\{p,q\}$.

The *negation and* the implication *connectives* are defined as follows:

$\sim p =_{df}$ if $p = 0$ then 1 else 0 fi and

$p \Rightarrow q =_{df}$ if $p \leq q$ then 1 else q fi.

The Łukasiewicz´s and Gödel's systems $Ł_\infty$ and G_∞ are directly related to fuzzy logic (see the next Subsection 2.2). These two systems together with product logic are defined as basic (Hájek P. 1998). Another two many-valued systems are summarised below.

Three-valued systems (Stephen Cole Kleene 1904 – 1994)

Multiple-valued logics have been introduced for many reasons: philosophical, as with Łukasiewicz, or purely mathematical, as with Post. Three-valued systems seem to be particularly simple cases, which offer intuitive

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* Early ideas for construction of systems having more than two logical values were presented of the turn *of* 19th and 20th centuries, e.g. by the following forerunners of multi-valued logics: Hugh MacColl (1837 – 1909): "*logic of three dimensions*", Charles S. Peirce (1839 – 1914): "*trichotomic mathematics*" based on "*triadic logic*" and Nikolai Alexandrovich Vasiliev (1880 – 1940): "*imaginary non-Aristotelian logics*". Vasiliev is reckoned also as the forerunner of *paraconsistent logic*: see Subsection 2.4.

interpretations of the truth degrees. Few have been as useful or as natural as the three-valued logics of Kleene (1952), introduced for computer science purposes (or at least they would have been if computer science had existed at the time).Kleene thought of the third truth value as *undefined* or *undetermined*, rather than as contingent or of probability $\frac{1}{2}$. This third truth degree for "undefined" was introduced in the context of partial recursive functions. So this reading suggests a natural condition: the behaviour of the third truth value should be compatible with any increase in information. That is, if the value of some propositional letter, p say, is changed from undefined to either true or false, the value of any formula φ with p as a component should never change from true to false or from false to true, though a change from undefined to one of false or true is allowed (Fitting M. 1990). The above condition, originally referred as *regularity* can be considered in terms of *monotonicity* in an ordering that places undefined below both false and true. It was observed (Kleene S.C. 1952) among three-valued logics satisfying the regularity condition there is a weakest and a strongest logic and also there are several intermediate ones. An illustration of Kleene 3-valued systems is given below (Fitting M. 1990).

Most implementations of Pascal require that (classical) truth values of all components of a Boolean expression be available before the value of the expression itself is calculated. Consider the propositions p and q. In particular, $p \wedge q$ must be given the value "undefined", i.e. $\perp$, if either p or q has the value $\perp$. Otherwise $p \wedge q$ behaves classically. Similarly for $\vee$ and $\sim$. This is the *weak Kleeene logic*. The third value $\perp$ can be interpreted, e.g. as a nontermination of a procedure call calculating some Boolean expression (Fitting M. 1990) or also as an "indeterminate" value of a signal line (Breuer M.A. and Friedman A.D. 1977).

On the other hand, one can imagine a language allowing a degree of parallelism, in which the calculation of a value for $p \wedge q$ proceeds as follows. Values for p and q are calculated in parallel, if either p or q turns out *false*, work on the other is halted and $p \wedge q$ is assigned the value *false* on the grounds that the value of the other won't matter. If one of p or q turns out true (say p), work must continue on the other component (q) because its value now is critical. And so, the value of $p \wedge q$ is whatever the value of q turns out to be. In such a system $p \wedge q$ is *true* if both components are true, $p \wedge q$ is *false* if one component is *false* and $p \wedge q$ is $\perp$ otherwise. This means $p \wedge q$ is $\perp$ if one of p or q is *true* but the other is $\perp$, or if both are $\perp$. Similarly for $\vee$ and $\sim$. This is the *strong Kleeene logic*.

There could also be a sequential evaluation of $p \wedge q$, say from left to right, so that p is evaluated first. If p evaluates to *false*, work stops and $p \wedge q$ is assigned *false*. If p evaluates to true then q is evaluated, and its value becomes the value assigned to $p \wedge q$. This is an asymmetric logic, e.g. if p is *false* but q is $\perp$, $p \wedge q$ evaluates to *false*, but $q \wedge p$ evaluates to $\perp$. This logic is also regular in Kleene's sense, and its connectives correspond to the AND, OR and NOT of *Lisp* (also the logic of *Prolog* with its left-right, top-down evaluation).

The importance of many-valued logics in computer science is generally recognised. They have been used in artificial intelligence (Ginsberg M.I. 1988), logic programming (Fitting M. 1985), algebraic specification of data types (Pigozzi D. 1990), epistemic structures (to represent the generally many-valued truth values of propositions about the external world: Muravitsky A.Y. 1994), digital circuits (e.g. Breuer M.A. and Friedman A.D. 1977, Lu H. and Lee S.C. 1985, Muzio J.C and Wesselkamper T.C. 1986), etc. As an illustration, an application of three-valued (strong Kleeene) logic to hazard detection is given below.

Hazard detection using three-valued algebra

One of the most common causes of circuit malfunction is due to circuit delays. Delay is an inherent property of all circuit elements and interconnections. In combinational circuits one form of malfunction caused by delays associated with elements is referred to as a *hazard*. A simple model for gate delays is shown in Figure 2.1 below, where only pure delay elements are considered, i.e. elements whose output value at time t is equal to its input value at time $(t - d)$ and gate G_i have delay d_i (here G_i' is assumed to have 0-delay, $i = 1,2,3$).

Suppose that the inputs at some time t_1 are $x_1 = x_2 = x_3 = 1$ and at time $t_2 > t_1$, x_1 changes to 0. Ideally, the output of the circuit should be 1 both before and after the change. However, if $d_1 < d_2$, even by a very small amount ε, the output may contain a 0-pulse. Hence, due to delays a combinational circuit may produce a transient error or spike. This is called a hazard (strictly: *static 1-hazard*, i.e. 0-pulse in a stable 1-signal). Such an error, if applied to the input of a flip-flop, may result in a permanent incorrect state. In synchronous circuits this possibility is eliminated by ensuring that no clock pulse occurs until the combinational circuit stabilises. In asynchronous circuits hazards cannot be masked in this way and they must be eliminated by proper design, or the circuit must be designed to operate properly in spite of the presence of transients caused by hazards (Breuer M.A. and Friedman A.D. 1977).

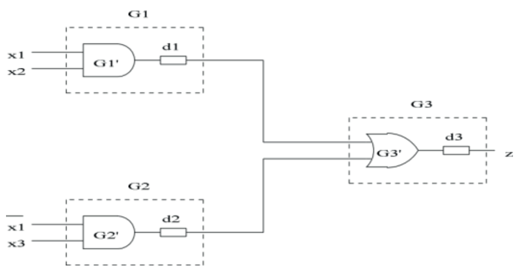

Figure 2.1 A gate delay model

One of the first hazard detection procedures was given by Eichelberger E.B. (1965). It was observed that when a signal line changes values, it goes through a transition period where its value may be interpreted by each of its loads, independently, as either a 0 or 1. To denote this undefined value the symbol ‘u’ was used by Breuer M.A. and Friedman A.D. (1977), to represent the fact that the value is unknown (Eichelberger used the symbol ‘ $\frac{1}{2}$ ’ rather than ‘u’: here ‘$\perp$’ is denoted by u). And so, it was developed a ternary algebra ($\{0,u,1\};0,1; +, \cdot, -$), where $+, \cdot$ correspond to $\vee, \wedge$ in the strong Kleene logic (the same as the weak disjunction and the weak conjunction in the Łukasiewicz’s system) and $\bar{u} =_{df} u$ (it can be observed that any multi-valued logic involves corresponding algebraic system).

Consider a combinational circuit C having input signals $\underline{x} =_{df} (x_1, x_2, ..., x_n)$. Let the current input vector be $\underline{x}(t) =_{df} (a_1, a_2, ..., a_n)$ and the next input vector be $\underline{x}(t+1) =_{df} (c_1, c_2, ..., c_n)$, where $a_i, c_i \in \{0,1\}$. Assume now that $\underline{x}(t+) =_{df} (b_1, b_2, ..., b_n)$ is a pseudo input vector such that $b_i =_{df}$ if $a_i = c_i$ then a_i else u (for any i). Let the response on line z in C to the input sequence $\underline{x}(t)\underline{x}(t+)\underline{x}(t+1)$ be $\underline{z}(t)\underline{z}(t+)\underline{z}(t+1)$. The following necessary and sufficient condition was shown.

Proposition 2.2

Let $X =_{df} \underline{x}(t)\underline{x}(t+)\underline{x}(t+1)$ be an input sequence for C and $Z =_{df} \underline{z}(t)\underline{z}(t+)\underline{z}(t+1)$ be the corresponding response to X. Then a static hazard on line z exists iff $Z \in \{0u0, 1u1\}$. □

Example 2.5

Consider the circuit C of Figure 2.1. Assume that $\underline{x}(t) =_{df} (1,1,1)$ and $\underline{x}(t+1) =_{df} (0,1,1)$. Hence $\underline{x}(t+) = (u,1,1)$ and $X = (1,1,1)\,(u,1,1)\,(0,1,1)$. Then $Z = (1)(u)(1)$ or for simplicity 1u1 (here $u + \bar{u} = u$).We have a static 1-hazard. □

In accordance with the last example, assuming two OR gates (G_1, G_2) and one AND gate (G_3), a static 0-hazard can be obtained for $\underline{x}(t) =_{df} (0,0,0)$ and $\underline{x}(t+1) =_{df} (1,0,0)$. Then $Z = 0u0$. Moreover, the above-considered model can be extended in the case of asynchronous sequential circuits (by using an appropriate defined iterative array model: this is omitted).

The above proposition was used in the logic level simulation domain[*]. Systems of many-valued logic were also used in circuit design and test generation for m-logic circuits, e.g. (Lu H. and Lee S.C. 1985, Tabakow I.G.

[*] *Logic simulation* is the process of building and exercising a model of a digital circuit on a digital computer. By exercising we mean the evaluation of signal values in the modelled circuit as a function of time for some applied input sequence. There are two main applications for a logic simulator: *design oriented simulation*, i.e. the evaluation of a new design, and *fault analysis simulation*. In the first case the logic designer is interested in testing for logical correctness, as well as timing and signal propagation characteristics. He may desire information related to race, hazard and oscillatory circuit conditions. In the second case the test engineer or logic designer may desire information related to what faults are detected by a proposed test sequence, what is the operational characteristic of the circuit under specific fault conditions, or

1993). Some interesting and practical uses of many-valued logic have been done, e.g. such as: Intel's flash memory with more than two levels of logic, applications to standard binary logic synthesis and optimisation, highly integrated devices, programmable gate arrays, resonant tunneling diodes, vlsi chips, biocomputing, variable-valued logic, Łukasiewicz's machines to control mini-robots, etc.

Dunn / Belnap's four-valued system*

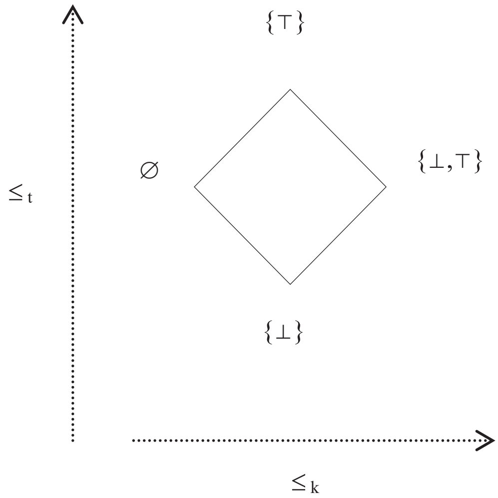

Figure 2.2 A double Hasse diagram for the four-valued system

Relevance logics are non-classical logics (called "*relevance logics*" in North America and "*relevant logics*" in Britain and Australasia) these systems have been developed as attempts to avoid the paradoxes of material and strict implication (e.g. such as: $p \Rightarrow (q \Rightarrow p)$, $p \Rightarrow (\sim p \Rightarrow q)$, $(p \Rightarrow q) \vee (q \Rightarrow p)$, $(p \Rightarrow q) \vee (q \Rightarrow r)$, $p \wedge \sim p \Rightarrow> q$, $p \Rightarrow> (q \vee \sim q)$, , $p \Rightarrow> (q \Rightarrow> q)$, etc.)†. Relevant logicians claim that what is unsettling about these so-called paradoxes is that in each of them the antecedent seems irrelevant to the consequent. In addition, relevant logicians have had qualms about certain inferences that classical logic makes valid. They have attempted to construct logics that reject theses and arguments that commit "fallacies of relevance" (Mares E.D. 1998). Hence, any such system assumes a semantical relationship between the antecedent and consequent of the considered true implication. A shift in emphasis from programming languages to databases makes it natural to move from a three to a four-valued logic. This particularly interesting many-valued system was the result of research on relevant logic but it also has significance for computer science applications (Fitting M. 1990, Gottwald S. 2000). The following truth degree set is assumed (Gottwald S. 2000): $\{\varnothing, \{\bot\},\{\top\},\{\bot,\top\}\}$. The

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what degree of fault resolution is obtainable with a given test sequence? However, for many simulation applications two logic values (0,1) are not sufficient, e.g. in ambiguity delay processing, circuit race or hazard detection, etc.(Breuer M.A. and Friedman A.D. 1977).

* Nuel D. Belnap, Jr., born 1930 and J.Michael Dunn, born 1941.

† In general, these logics can be considered as a kind of *paraconsistent logic* systems: for a more formal treatment: see Subsection 2.4: *Relevance logic*. A more formal definition of strict implication is given in Subsection 2.3 (see: Modal logic).

truth degrees can be interpreted as indicating (e.g. wrt a database query for some state of affairs, in short: *soa*) that there is: no information concerning this *soa*, information saying that the *soa* fails, information saying that the *soa* obtains, and conflicting information saying that the *soa* fails as well as obtains.

In accordance with the above given double Hasse* diagram (see: Figure 2.2), the set of truth degrees has two natural (lattice) orderings: a *truth ordering* (in short: $\leq_t$) which has $\{\top\}$ on top of the incomparable degrees $\varnothing$, $\{\bot,\top\}$, and has $\{\bot\}$ at the bottom and also an *information ordering* (or: *knowledge ordering*, denoted by $\leq_k$) which has $\{\bot,\top\}$ on top of the incomparable degrees $\{\bot\}$,$\{\top\}$, and has $\varnothing$ at the bottom. The truth degree functions for the conjunction and disjunction connectives can be defined in an unique way by using the corresponding inf- and sup-operations under $\leq_t$. A negation is determined by a truth degree function which exchanges the degrees $\{\bot\}$ and $\{\top\}$, and which leaves the degrees $\{\bot,\top\}$ and $\varnothing$ fixed. The definition of implication connective and the choice of the designated truth degrees is depending on the used application (computer science or relevant logic). This is still an open research topic.

Direct-product systems

The truth degrees can be interpreted as different aspects in the evaluation of a given proposition. By assuming e.g. k different aspects, the truth degrees may be chosen as k-tuples of values which evaluate the single aspects. The compound truth degree function over such k-tuples can be defined "componentwise" from truth degree (or: truth-value) functions for the values of the single components. In this manner, k logical systems may be combined into one many-valued product system. In this way, the truth degrees of Dunn/Belnap's 4-valued system can be considered as evaluating two aspects of *soa* related to a database: whether there is positive information about the truth of this *soa* or not, and whether there is positive information about the falsity of this *soa* or not. Both aspects can use standard truth-values for this evaluation. In this case, the conjunction, disjunction, and negation of Dunn/Belnap's 4-valued system are componentwise definable by conjunction, disjunction, and negation of classical logic, respectively. And so, this 4-valued system is a direct-product of two copies of classical two-valued logic (Gottwald S. 2000). A more formal treatment of topics such as: algebraic systems, in particular lattices, Cartesian (or: direct-) products of such systems, logical bilattices†, etc., is omitted in this part of study (will be presented in Part II of this book, e.g. see MacLane S. and Birkhoff G. 1967, Kerntopf P. 1967, Ginsberg M.L. 1988, Bronstein I.N. et al. 2001, etc.).

2.2. Fuzzy logic

Consider a set of propositions. To any element of this set may be assigned some *degree of truth*, which may be "*absolutely true*", "*absolutely false*" or some *intermediate truth degree*: a proposition may be more true than another one. And so, any proposition can be represented with some degree of truthfulness and falsehood. In the analogy to various definitions of operations on fuzzy sets (such as: intersection, union, complement, etc.) one may ask how propositions can be combined by *logical connectives* (such as: conjunction, disjunction, negation, etc.). And also, if the truth degree of a composed proposition (or equivalently: *propositional formula*) is determined by the truth degrees of its components, i.e. if the connectives have their corresponding *truth functions* (like truth tables of classical logic). Saying "yes" (which is the mainstream of fuzzy logic) one accepts the *truth-functionality principle*. And this makes fuzzy logic to something distinctly different from probability theory since the latter is not truth-functional. In fact, the probability of conjunction of two propositions is not determined by the probabilities of those propositions (Hájek P. 2002).

The following two main directions in fuzzy logic can be distinguished: *fuzzy logic in the broad sense* and *fuzzy logic in the narrow sense*, i.e. in senso stricto. In the broad sense (older, better known, heavily applied but

* Helmut Hasse (1898 – 1979)

† A *bilattice* is a system $\mathscr{B} =_{df} (B, \leq_t, \leq_k, \sim)$ such that B is a non-empty set containing at least two elements, $(B,\leq_t)$, $(B,\leq_k)$ are complete lattices, and $\sim$ is a unary operation on B that has the following properties: $a \leq_t b \Rightarrow \sim a \geq_t \sim b$, $a \leq_k b \Rightarrow \sim a \leq_k \sim b$, and $\sim\sim a = a$ (for any $a,b \in B$: see Ginsberg M.L. 1988).

not asking deep logical questions) this logic serves mainly as apparatus for fuzzy control*, analysis of vagueness in natural language and several other application domains. It is one of the techniques of *soft computing*, i.e. computational methods tolerant to suboptimality and impreciseness (vagueness) and giving quick, simple and sufficiently good solutions.

Basic fuzzy propositional logic (called also: *basic many-valued logic*, in short: BL) is symbolic logic with a comparative notion of truth developed fully in the spirit of classical logic (syntax, semantics, axiomatisation, truth-preserving deduction, completeness, etc. both propositional and predicate logic). It is a branch of may-valued logic based on the paradigm of *inference under vagueness*. BL is a strict fuzzy logic system using the logic of continuous triangular norms.

Triangular norms (in short: *t-norms*) are a generalisation of the classical two-valued conjunction. They were originally introduced by Menger K. (1942)† in the framework of the probabilistic (statistical) metric spaces as a generalisation of the classical triangle inequality for ordinary metric spaces.This paper was followed almost immediately by a paper of Wald A. (1943). The next investigations (Schweizer B. and Sklar A. 1960, 1963) were related with axiomatic of these norms. A more detailed treatment was given in (Schweizer B. and Sklar A. 1983, 2005). For infinite valued systems with truth degree set $W_\infty =_{df} [0,1]$, the influence of fuzzy set theory (Zadeh L. A. 1965, 1974) quite recently initiated the study of a whole class of such systems of many-valued logic. In fuzzy logic systems, the basic aggregation operations are performed by the logical connectives AND and OR which provide point wise implementations of the intersection and union operations. It has been well established in the literature that the appropriate characterisations of these operations in the multi-valued logic environment are the triangular norm operators (Hájek P. 1998). In general, the most of the studies concerning Hájek's system BL focus attention on methodological problems such as compactness, consistency, decidability or satisfiability of t-tautologies, various proving techniques or also introducing some new t-norms, e.g. (Hájek P. and Godo L. 1997, Klement, E.P. and Navara M. 1999, Navara M. 2000, Cintula P. and Navara M. 2004, etc.).

The b*asic fuzzy propositional logic* is a relatively young discipline, both serving as a foundation for the fuzzy logic in a broad sense and of independent logical interest, since it turns out that strictly logical investigation of this kind of logical calculus can go rather far (Hájek P. 1998). It is broadly accepted that t-norms (dually, t-conorms) are possible truth functions of *fuzzy conjunction* (of *fuzzy disjunction*). The best-known candidate for *fuzzy negation* is the *Łukasiewicz's negation* $x' =_{df} 1 - x$. However, some other notions, e.g. such as *Sugeno's fuzzy negation* or *Yager's fuzzy negation* are also applicable (Bronstein I.N. et al. 2001: see below). The *fuzzy implication* connective is sometimes disregarded but is of fundamental importance for fuzzy logic in the narrow sense. A straightforward but logically less interesting possibility is to define implication from disjunction and negation or conjunction and negation using the corresponding theses of classical logic T 1.15 and T 1.19, respectively (see Subsection 1.3). Such implications are called *S-implications*‡. In fact, more useful and interesting are the so-called *R-implications* (any such implication is specified as a residuum with respect to the used t-norm).

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* Traditional control systems are based on mathematical models in which the control system is described using one or more differential equations that define the system response to its inputs. Such systems are often implemented as "proportional-integral-derivative (PID)" controllers. They are the products of decades of development and theoretical analysis, and are highly effective. If PID and other traditional control systems are so well-developed, why bother with fuzzy control? It has some advantages. In many cases, the mathematical model of the control process may not exist, or may be too "expensive" in terms of computer processing power and memory, and a system based on empirical rules may be more effective. Furthermore, fuzzy logic is well suited to low-cost implementations based on cheap sensors, low-resolution analogue-to-digital converters, and 4-bit or 8-bit one-chip microcontroller chips. Such systems can be easily upgraded by adding new rules to improve performance or add new features. In many cases, fuzzy control can be used to improve existing traditional controller systems by adding an extra layer of intelligence to the current control method. Hence, the *broad sense fuzzy logic* has become a common buzzword in machine control. However, the term itself inspires certain scepticism, sounding equivalent to "half-baked logic" or "bogus logic". Some other nomenclature might have been preferable, but it's too late now, and this fuzzy logic is actually very straightforward. This logic is a way of interfacing inherently analogue processes, that move through a continuous range of values, to a digital computer, that likes to see things as well-defined discrete numeric values (*G*oebel G. 2003).

† Karl Menger (1902 – 1985)

‡ In accordance with T 1.19 and CE (the law of transposition or contraposition of equivalence) we have: $p \Rightarrow q \Leftrightarrow \sim(p \wedge \sim q)$. Let e.g. $x' =_{df} 1 - x$ be the Łukasiewicz's fuzzy negation and $x \otimes y =_{df} \min\{x,y\}$ be the Zadeh's t-norm. Hence, the following S-implication can be obtained: $x \Rightarrow y =_{df} 1 - \min\{x, 1 - y\} = \max\{1 - x, y\}$ (the logical value of this implication: the proof of the last equality is left to the reader). Obviously, it is possible also the use of other fuzzy negations and / or t-norms. It can be observed that sometimes the above two S- and R-implications may coincide, e.g. in $Ł_\alpha$-BL (and hence in Ł-BL, assuming Yager's fuzzy negation: see Corollary 2.4 of this subsection).

There exist two classical approaches in constructing of the propositional calculus: the *axiomatic approach* and the *approach from assumptions* (Słupecki J. and Borkowski L. 1967). In general, the actual research methodology and extensions have been related to the Hájek's axiomatic approach in constructing of the fuzzy propositional calculus.

The subject of this subsection is fuzzy propositional calculus. The proposed approach is related to the infinite valued Łukasiewicz's system, recognised as one of the most important basic fuzzy propositional logics (in common with Gödel's and product logic systems, in short: Ł-BL, G-BL and π-BL). There are first introduced some basic notions and definitions concerning t- norms and t-conorms. A proof method for fuzzy propositional logics based on natural deduction is next presented (Tabakow I.G. 2006). This approach seems to be more attractive, more simpler and natural in practical use than the axiomatic one. Next, a new t-norm and t-conorm are introduced and then it is defined a generalised Łukasiewicz's system, denoted by $Ł_\alpha$-BL ($\alpha > 0$), where the previous one becomes a particular case with $\alpha = 1$. The system $Ł_\alpha$-BL is presented in two versions, called first and second order $Ł_\alpha$-BL, depending on the used continuous fuzzy negations (Łukasiewicz's one or the more general Yager's one, respectively). And hence, in accordance with the generalised De Morgan's laws, two possible t-conorms are obtained. The used fuzzy implication is specified as a residuum of the above t-norm (which is continuous) and hence, the last implication is unique. This subsection is an extension of the previously study (Tabakow I.G. 2006). As an illustration, various assumptional proofs and corresponding derived rules are given.

Triangular norms and conorms: basic notions and definitions

The *t-norm operator* (called also: *fuzzy t-norm*) provides the characterisation of the AND operator. It is a binary operation $\otimes : [0,1]^2 \rightarrow [0,1]$ with the following properties (for any $x,y,u,v \in [0,1]$: Hájek P. 1998, Klement, E.P. and Navara M. 1999, Bronstein I.N. et al. 2001, etc.):

$x \otimes y = y \otimes x$	*commutative*
$x \otimes y \geq u \otimes v$ for $x \geq u$ and $y \geq v$	*monotonic*
$x \otimes (y \otimes z) = (x \otimes y) \otimes z$	*associative*
$x \otimes 1 = 1 \otimes x = x$	has 1 as *unit element*

The dual *t-conorm operator* (called also: *fuzzy t-conorm* or *fuzzy s-norm*), characterises the OR operator. It is a binary operation $\oplus : [0,1]^2 \rightarrow [0,1]$ having properties as follows (for any $x,y,u,v \in [0,1]$):

$x \oplus y = y \oplus x$	*commutative*
$x \oplus y \geq u \oplus v$ for $x \geq u$ and $y \geq v$	*monotonic*
$x \oplus (y \oplus z) = (x \oplus y) \oplus z$	*associative*
$x \oplus 0 = 0 \oplus x = x$	has 0 as *unit element*

In accordance with the monotonic property, any t-norm is non-decreasing in both arguments having 1 as unit element and 0 as *zero* (or *null*) element, i.e. $x \otimes 0 = 0$ (for any $x \in [0,1]$, similarly for any t-conorm with respect to 1 as a zero element). In fact, the system $\mathscr{A} =_{df} ([0,1] ; 1 , 0 ; \otimes)$ is an Abelian* algebraic system with the above monotonic property, similarly for $\mathscr{B} =_{df} ([0,1] ; 0 , 1 ; \oplus)$, where 1, 0 and 0, 1 are the *constants* of these two algebraic systems. Obviously, the systems $\mathscr{A}$ and $\mathscr{B}$ can be considered as *Abelian monoids*† with respect to the constants 1 and 0, respectively.

In general, the notion of (continuous) *fuzzy negation* can be introduced as a function $f : [0,1] \rightarrow [0,1]$ with the following properties (for any $x, y \in [0,1]$, Bronstein I.N. et al. 2001):

$f(0) = 1$ and $f(1) = 0$	*the terminal point values*
$x < y \Rightarrow f(x) \geq f(y)$	*monotonicity*

* Any algebraic system $\mathscr{A} =_{df} (A ; \circ)$ is said to be *groupoid*, where ′∘′ is a *binary operation*, i.e. $\circ : A^2 \rightarrow A$. The system $\mathscr{A}$ is *Abelian* if ∘ is commutative (Niels Henrik Abel: 1802 – 1829).

†Let $\mathscr{A} =_{df} (A ; e ; \circ)$ be an algebraic system having $e \in A$ as unit element and ′∘′ as a binary operation. If ′∘′ is associative then $\mathscr{A}$ is said to be *monoid*.

f(f(x)) = x *involutivity*
f(x) is a continuous function *continuity* (for any $x \in [0,1]$)

It can be observed that a very simple function satisfying the above properties is the classical Łukasiewicz's negation $f_Ł(x) =_{df} 1 - x$. Some generalisations were also introduced, e.g. such as: *Sugeno's fuzzy negation*[*] $f_S(x) =_{df} \frac{1-x}{1+\lambda x}$, where $\lambda \in (-1, \infty)$ or also *Yager's fuzzy negation*[†] $f_Y(x) =_{df} (1 - x^{\alpha})^{1/\alpha}$, where $\alpha \in \mathbb{R}_{++} =_{df} (0, \infty)$. For example, the *involutivity property* for $f_\lambda(x)$ is presented as follows:

Let $y =_{df} f_S(x)$. Then $f_S(y) = \frac{1-y}{1+\lambda y} = \frac{1-\frac{1-x}{1+\lambda x}}{1+\lambda\cdot\frac{1-x}{1+\lambda x}} = \frac{x\cdot(\lambda+1)}{\lambda+1} = x$. Similarly for the Yager's fuzzy negation $f_Y(x)$ (this is omitted).

It can be observed that the continuity property is not satisfied for the following fuzzy negation, used in G-BL and π-BL (Hájek P. 2002): $x' =_{df}$ if $x = 0$ then 1 else 0 fi.

Let $x' =_{df} f(x)$ be a continuous fuzzy negation. So any t-conorm is dual to the corresponding t-norm under the order-reversing operation which assigns x' to x on [0,1]. And hence, for a given t-norm the complementary conorm is defined as follows (a generalisation of *De Morgan's laws*): $x \oplus y =_{df} (x' \otimes y')'$.

As a continuous fuzzy negation may be selected any one. But the most important representatives are as follows: the *Łukasiewicz's fuzzy negation* $x' =_{df} 1 - x$ (assumed in Ł-BL, i.e $Ł_\alpha$-BL with $\alpha = 1$) and *Yager's fuzzy negation* $x' =_{df} (1 - x^{\alpha})^{1/\alpha}$ (which is a generalisation of the previous one with $\alpha = 1$), where $\alpha > 0$. Since t-conorms are "negation oriented", we have the possibility of constructing two different versions of the generalised Łukasiewicz's BL system. The last two versions are presented below, where traditionally the Łukasiewicz's fuzzy negation is first assumed.

We shall say that $\otimes$ is *continuous t-norm* if it is continuous as a function, i.e. $\forall \varepsilon > 0\ \forall x_1, x_2, y_1, y_2 \in [0,1]\ \exists \delta > 0\ (\ |x_1 - x_2| < \delta\ \wedge\ |y_1 - y_2| < \delta\ \Rightarrow |\ x_1 \otimes y_1 - x_2 \otimes y_2\ | < \varepsilon\ )$. The left- and right-continuity can be introduced in a similar way (this is omitted). And $\otimes$ is said to be an *Archimedean t-norm* if it has the *Archimedean property*[‡], i.e. $\forall x, y \in (0,1)\ \exists n \in \mathbb{N}\ (x^n \le y)$. Any x is *nilpotent* if $\exists n \in \mathbb{N}\ (x^n = 0)$, where x^n denotes $x \otimes x \otimes ... \otimes x$ (*n times*) and $\mathbb{N}$ is the *set of natural numbers*. An element x is *idempotent* if $x \otimes x = x$. Any continuous t-norm is Archimedean if it has no idempotents between 0 and 1, i.e. there is no any idempotent $x \in (0,1)$[§]. And this t-norm is *strict* if 0 is its only nilpotent element, i.e. if $x \otimes x > 0$ for all $x > 0$. Continuous Archimedean t-norms which are not strict are called *nilpotent*. For example, the *product t-norm*, i.e. $x \otimes y =_{df} xy$ is strict and the Łukasiewicz's t-norm $x \otimes y =_{df} \max\{0, x + y - 1\}$ is nilpotent. All nilpotent t-norms are isomorphic with the Łukasiewicz's t-norm being their prototypical representative. Similarly, all strict t-norms are isomorphic with the product t-norm[**]. The *partial ordering* of t-norms can be

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[*] Michio Sugeno, born 1940.

[†] Ronald R. Yager, born 1941.

[‡] Named after Archimedes (287 b.c. – 212 b.c. axiom v of Archimedes) and first used in some algebraic systems by Otto Stolz (1842 – 1905). In general, this property is related to the impossibility of having infinitely large or infinitely small elements.

[§] More generally, each continuous Archimedean t-norm can be obtained by using an increasing (decreasing) bijection called a *multiplicative* (*additive*) *generator*, which is not uniquely determined (*public domain*).

[**] The set of idempotents of each continuous t-norm is a closed subset of [0,1] and its complement is a union of countable many non overlapping open intervals. The restriction of the t-norm to any such interval (including its endpoints) is Archimedean. And hence, the obtained restriction is isomorphic either to the Łukasiewicz's t-norm or to the product t-norm. For such x, y that do not fall into the same open interval, the t-norm evaluates to the min{x,y}. These conditions are known as the Mostert-Shields theorem. And hence any continuous t-norm is decomposable (*public domain*). Another characterisation theorem was given by Schweizer and Sklar (this is omitted here: see Larsen H.L. 1998).

introduced in an usual manner, i.e. $\otimes_1 \le \otimes_2 \Leftrightarrow_{df} \forall x, y \in [0,1]\ (x \otimes_1 y \le x \otimes_2 y)$. The point wise larger t-norms are sometimes called *stronger*. It was observed that $\min\{x,y\} \ge x \otimes y$ and $\max\{x,y\} \le x \oplus y$ (for any $x, y \in [0,1]$ and t-norm / conorms $\otimes$ and $\oplus$). It can be observed the linear convex combination of two different t-norms is not necessary t-norm, e.g. for $x \otimes_1 y =_{df} \min\{x,y\}$ and $x \otimes_2 y =_{df} xy$ the obtained linear convex combination $\lambda(x \otimes_1 y) + (1 - \lambda)(x \otimes_2 y)$ is not associative ($\lambda \in [0,1]$). It is desired sometimes to be introduced some operators which are interpolating between t-norms and t-conorms. Any such operator is said to be *compensating* (Bronstein I.N. et al. 2001), e.g. the *λ-operator* $x \bigcirc_\lambda y =_{df} \lambda xy + (1 - \lambda)(x + y - xy)$ and the *γ-operator*: $x \bigcirc_\gamma y =_{df} (xy)^{1-\gamma} \cdot (x + y - xy)^{\gamma}$ ($\lambda, \gamma \in [0,1]$). In general, these operators are also not associative (a more formal treatment is omitted).

Some example t-norms with the corresponding dual t-conorms are given in the next table (the parameter $\alpha \in \mathbb{R}$: Bronstein I.N. et al. 2001)*.The aggregations modelled by Zadeh's t-norm and t-conorm are sometimes referred as the *pure AND*, and the *pure OR*, respectively.

Author	*t-norm*	*t-conorm (s-norm)*
Zadeh	$x \otimes y =_{df} \min\{x,y\}$	$x \oplus y =_{df} \max\{x,y\}$
Łukasiewicz	$x \otimes y =_{df} \max\{0, x + y - 1\}$	$x \oplus y =_{df} \min\{1, x + y\}$
algebraic product and sum	$x \otimes y =_{df} xy$	$x \oplus y =_{df} x + y - xy$
drastic product and sum	$x \otimes y =_{df}$ if $(x = 1)$ or $(y = 1)$ then $\min\{x,y\}$ else 0 fi	$x \oplus y =_{df}$ if $(x = 0)$ or $(y = 0)$ then $\max\{x,y\}$ else 1 fi
Hamacher ($\alpha \ge 0$)	$x \otimes y =_{df} \dfrac{xy}{\alpha + (1-\alpha)(x + y - xy)}$	$x \oplus y =_{df} \dfrac{x + y - xy - (1-\alpha)xy}{1 - (1-\alpha)xy}$
Einstein	$x \otimes y =_{df} \dfrac{xy}{1 + (1-x)(1-y)}$	$x \oplus y =_{df} \dfrac{x + y}{1 + xy}$
Frank ($\alpha > 0, \alpha \ne 1$)	$x \otimes y =_{df} \log_\alpha\left[1 + \dfrac{(\alpha^x - 1)(\alpha^y - 1)}{\alpha - 1}\right]$	$x \oplus y =_{df} 1 - \log_\alpha\left[1 + \dfrac{(\alpha^{1-x} - 1)(\alpha^{1-y} - 1)}{\alpha - 1}\right]$
Yager ($\alpha > 0$)	$x \otimes y =_{df} 1 - \min\{1, ((1-x)^\alpha + (1-y)^\alpha)^{1/\alpha}\}$	$x \oplus y =_{df} \min\{1, (x^\alpha + y^\alpha)^{1/\alpha}\}$
Schweizer ($\alpha > 0$)	$x \otimes y =_{df} \max\{0, (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha}\}$	$x \oplus y =_{df}$ $1 - \max\{0, ((1-x)^{-\alpha} + (1-y)^{-\alpha} - 1)^{-1/\alpha}\}$
Dombi ($\alpha > 0$)	$x \otimes y =_{df} \left\{1 + \left[\left(\dfrac{1-x}{x}\right)^\alpha + \left(\dfrac{1-y}{y}\right)^\alpha\right]^{1/\alpha}\right\}^{-1}$	$x \oplus y =_{df} \left\{1 + \left[\left(\dfrac{x}{1-x}\right)^\alpha + \left(\dfrac{y}{1-y}\right)^\alpha\right]^{1/\alpha}\right\}^{-1}$
Weber ($\alpha \ge -1$)	$x \otimes y =_{df} \max\{0, (1+\alpha)(x + y - 1)\}$	$x \oplus y =_{df} \min\{1, x + y + \alpha xy\}$

* Sometimes there are required additional *interpolating operators* between t-norms and t-conorms: see Subsection 7.1.

Dubois $(0 \le \alpha \le 1)$	$x \otimes y =_{df} \dfrac{xy}{\max\{x, y, \alpha\}}$	$x \oplus y =_{df} \dfrac{x + y - xy - \min\{x, y, (1-\alpha)\}}{\max\{(1-x), (1-y), \alpha\}}$

A proof method for fuzzy propositional BL-system based on natural deduction

Next we shall say that $\otimes$ and $\oplus$ are *t- conjunction* and *t- disjunction*, respectively. The *fuzzy implication* connective is sometimes disregarded but it is of fundamental importance for fuzzy logic in the narrow sense. Next only continuous t-norms are considered as good candidates for *truth functions* of a conjunction. Each t-norm determines uniquely its corresponding implication $\Rightarrow$ (not necessarily continuous) satisfying for any $x,y,z \in [0,1]$ the following *crucial adjointness condition*: $z \le (x \Rightarrow y)$ iff $x \otimes z \le y$. For all those t-norms which have the *sup-preservation property*, i.e. $x \otimes \sup_i y_i = \sup_i (x \otimes y_i)$* there is a standard way to introduce the related *implication connective* with a truth degree function. In fact, any t-norm based fuzzy implication can be interpreted as a binary operation over [0,1] and specified as a residuum of the corresponding t-norm (called also *R-implication*). It was shown that this residuum is unique if the considered t-norm is at least left-continuous. In general, the logical value of any R-implication can be defined as follows: $x \Rightarrow y =_{df} \sup\{z \in [0,1] / x \otimes z \le y\}$ (for any $x, y \in [0,1]$ and any left-continuous $\otimes$: Hájek P. and Godo L. 1997, Hájek P. 1998, Navara M. 2000, Gottwald S. 2000, etc.). Also, the *implication* and *negation connectives* are assumed under

Let $\otimes$ be an arbitrary left-continuous t-norm and $x \le y$. In accordance with the monotonic property, we can obtain: $x \otimes z \le y \otimes z$ and $y \otimes z \le y \otimes 1 = y$. Then $x \otimes z \le y$ (for any $x, y, z \in [0,1]$ and $x \le y$). In general, the following well-known property of the logical value of any R-implication is satisfied: $x \Rightarrow y = 1$ iff $x \le y$.

Below we shall use the same names for the primitive and/or derived rules as in the case of classical propositional calculus (see Section 1). Some additional inference rules of the first-order predicate logic calculus are also used, e.g. such as: the rule of *negating an universal quantifier* (denoted by N∀), the rules of *omitting an universal* and *an existential quantifiers* (denoted by: −∀ and −∃, respectively: see Subsection 3.3 of the next Chapter II).

In the fuzzy propositional calculus any formula is constructed by using the following three kinds of symbols: (i) propositional variables (denoted by p, q, r, s,...,p_1,p_2,...), (ii) some fuzzy connectives (depending on the used system), e.g. such as: the *Łukasiewicz's* (*strong*) *fuzzy conjunction*, *fuzzy disjunction*, *implication*, *logical equivalence*, and *negation*, denoted by: &, $\veebar$, $\Rightarrow$, $\Leftrightarrow$, and ~, respectively or the *Gödel's* (*weak*) *fuzzy conjunction*, *fuzzy disjunction*, *implication*, *logical equivalence*, and *negation*, denoted by: $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and ~, respectively or also the *product logic's fuzzy conjunction*, *fuzzy disjunction*, *implication*, *logical equivalence*, and *negation*, denoted by: $\cdot$, $\triangledown$, $\Rightarrow$, $\Leftrightarrow$, and ~, respectively and also (iii) parentheses (left: '(' and right: ')'). The truth and falsity constants, corresponding to 'T' and 'F' in the classical case, are denoted by $\underline{1}$ and $\underline{0}$ (or also by $\top$ and $\bot$ respectively). To minimise the number of used parentheses in an expression, some priorities for logical connectives can be introduced. The following convention is assumed below (Słupecki J. and Borkowski L. 1967): ~, $\otimes$, $\oplus$, $\Rightarrow$, $\Leftrightarrow$ (i.e. the symbol of negation binds more strongly than the symbol of t-conjunction, the last binds more strongly than the symbol of t-disjunction, etc.), where $\otimes \in \{\&, \wedge, \cdot\}$ and $\oplus \in \{\veebar, \vee, \triangledown\}$ are depending on the used system (Ł-, G- or π-BL). It can be observed that the fuzzy connectives related to Ł- and G-BL are the same as in the case of many-valued logic systems (see Subsection 2.1) but extended to the whole interval [0,1]. The negation connective in π-BL is the same as in G-BL. The fuzzy conjunction and disjunction

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* Let $(X, \ge)$ be a *partially ordered set*. An element $a \in X$ is said to be an *upper bound* of $Y \subseteq X$ iff $a \ge y$ (for any $y \in Y$). The upper bound a of Y is said to be a *least upper bound* (i.e. *supremum*) of Y iff for any upper bound a′ of Y : $a' \ge a$. The least upper bound of Y is denoted by sup(Y) (also supY or $\sup_{y \in Y} y$) . The notions of a *lower bound* of $Y \subseteq X$, a *greatest lower bound* (i.e. *infimum*) of Y and inf(Y) can be introduced in a similar way, e.g. for $Y =_{df} \{-1, +1/2, -1/3, +1/4, \ldots, (-1)^n/n, \ldots\}$ we can obtain: $\inf(Y) = -1$ and $\sup(Y) = 1/2$. The used term 'sup-preservation' is related to the well-known distributive property wrt an algebraic system. If Y has a supremum, then the supremum is unique. Let s_1 and s_2 are two different suprema of Y (aip).We have: $s_1 \le s_2$ and $s_2 \le s_1$. Since $\le$ is weak antisymmetric it follows that $s_1 = s_2$.

connectives in π-BL are defined as follows: x·y (the usual *arithmetic product*) and $x \mathbin{\overline{\vee}} y =_{df} x + y - x \cdot y$ (called *algebraic* or "*probabilistic*" *sum*). The following proposition is satisfied.

Proposition 2.3

The sup-preservation property is satisfied for any $a \in [0,1]$ and $\otimes \in \{\wedge, \&, \cdot\}$.

Proof:

(i) Let $\otimes =_{df} \wedge$. We have: $a \otimes \sup_{x \in [0,1]} x = \min\{a, \sup_{x \in [0,1]} x\} = \min\{a, \sup[0,1]\} = \min\{a,1\} = a$. On the other hand $\sup_{x \in [0,1]} (a \otimes x) = \sup\{x \in [0,1] / \min\{a,x\}\} = \sup\{\min\{a,0\}, \dots , \min\{a,1\}\} = \sup[0,a] = a$.

(ii) For $\otimes =_{df} \&$ we can obtain: $a \otimes \sup_{x \in [0,1]} x = \max\{0, a + \sup_{x \in [0,1]} x - 1\} = \max\{0, a + 1 - 1\} = \max\{0,a\} = a$. The right side $\sup_{x \in [0,1]} (a \otimes x) = \sup\{x \in [0,1] / \max\{0, a + x - 1\}\} = \sup\{ x \in [0,1] / \max\{0, a - (1 - x)\}\} = \sup\{a - 1, \dots , a\} = \sup[a - 1, a] = a$ (for $0 \le x \le 1$ we have $a - 1 \le a - (1 - x) \le a$).

(iii) Assuming $\otimes =_{df} \cdot$ We have: $a \otimes \sup_{x \in [0,1]} x = a \cdot \sup_{x \in [0,1]} x = a \cdot 1 = a$. In a similar way the right side $\sup_{x \in [0,1]} (a \otimes x) = \sup\{x \in [0,1] / a \cdot x\} = \sup\{a \cdot 0, \dots , a \cdot 1\} = \sup[0,a] = a$. □

The next example is an illustration of the notion of R-implication introduced in the Hájek's system BL.

Example 2.6 (*Ł-BL, G-BL and π-BL implications*)

(i) Let $x \otimes y =_{df} x \,\&\, y = \max\{0,x + y - 1\}$. So the implication in the *Łukasiewicz's logic* system with the strong conjunction ´&´ can be defined as follows: $x \Rightarrow y =_{df}$ if $x \le y$ then 1 else $1 - x + y$ fi (or equivalently: $x \Rightarrow y =_{df} \min\{1,1 - x + y\}$).

(ii) Let $x \otimes y =_{df} x \wedge y = \min\{x,y\}$. In the case of the *Gödel's logic system* with the weak conjunction ´∧´ we have: $x \Rightarrow y =_{df}$ if $x \le y$ then 1 else y.

(iii) Let now $x \otimes y =_{df} x \cdot y = xy$ (the usual arithmetic product). So, for the *product logic* (in short: *π-logic*) we can define: $x \Rightarrow y =_{df}$ if $x \le y$ then 1 else y / x (is the usual arithmetic division: since $a > b$ then $a \ne 0$). □

The logical value of the fuzzy equivalence $(a \Rightarrow b) \otimes (b \Rightarrow a)$ is equal to 1 for $a = b$ (Ł-, G-, π-BL) and equal to: $1 - \max\{a,b\} + \min\{a,b\} = 1 - |a - b|$ or $\min\{a,b\}$, or also $\dfrac{\min\{a,b\}}{\max\{a,b\}}$, for $a \ne b$ (depending on the considered system: Ł-, G- or π-BL, respectively). The absolute value $|a - b|$ corresponds to the value of the Ł-BL *fuzzy difference* $a \not\Leftrightarrow b$ defined as follows: $\sim (a \Leftrightarrow b)$. And this fuzzy difference coincides with the notion of *M-difference* given by Lu H. and Lee S.C. (1984).

In accordance with the associativity axiom, any t-norm (s-norm) can be extended to more than two arguments. And hence, the logical values of the obtained generalised connectives are given below (the corresponding proofs are inductive wrt n and they are left to the reader).

$\mathop{\&}\limits_{i=1}^{n} p_i = \max\{0, \sum\limits_{i=1}^{n} p_i - n + 1\}$, $\mathop{\underline{\vee}}\limits_{i=1}^{n} p_i = \min\{1, \sum\limits_{i=1}^{n} p_i\}$ and $\mathop{\overline{\vee}}\limits_{i=1}^{n} p_i = \sum\limits_{i=1}^{n} p_i - \sum\limits_{i_1, i_2 = 1 / i_1 < i_2}^{n} p_{i_1} \cdot p_{i_2} + \dots + (-1)^{n+1} \cdot \prod\limits_{i=1}^{n} p_i$ (for any $p_i \in [0,1]$ ($i = 1, 2, \dots , n$; $n \ge 2$). The algebraic sum connective is very similar to the well-known Poincaré's formula concerning the probability $P(\bigcup\limits_{i=1}^{n} A_i)$. The above sum can be also equivalently represented as follows: $\mathop{\overline{\vee}}\limits_{i=1}^{n} p_i = 1 - \prod\limits_{i=1}^{n} (1 - p_i)$. Obviously, for any $\otimes \in \{\wedge, \&, \cdot\}$: $\bigotimes\limits_{i=1}^{n} p_i = 1$ iff $p_i = 1$ (for any $i = 1, \dots , n$).

It can be shown that min and max are definable from $\otimes$ and $\Rightarrow$, i.e. the following identities are satisfied (for any continuous t-norm $\otimes$: Hájek P. and Godo L. 1997): (i) $\min\{x,y\} = x \otimes (x \Rightarrow y)$ and (ii) $\max\{x,y\} = \min\{(x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x\}$.

In general, the class of all t-norms, even of those, which have the sup-preservation property, is very large. And so, the infinite valued Łukasiewicz's and Gödel's systems, also the product logic can be considered as some particular cases.

The set of *fuzzy propositional formulae* (called equivalently *fuzzy propositional expressions*, in short: *expressions* or also *sentential formulae*) of this propositional calculus can be considered as the smallest set which includes propositional variables, and which is closed under the operations of forming the negation, conjunction, disjunction, implication and equivalence. Hence, any propositional variable can be considered as an expression and also the compound formulae formed from them by means of the corresponding logical functors. More formally, the following inductive definition can be used (a generalisation of the classical one: see Definition 1.1).

Definition 2.1

A *fuzzy propositional formula* is:

1. Any propositional variable,
2. If φ and ψ are some fuzzy propositional formulae, then such formulae are also: $\sim(\varphi)$, $(\varphi) \otimes (\psi)$, $(\varphi) \oplus (\psi)$, $(\varphi) \Rightarrow (\psi)$, and $(\varphi) \Leftrightarrow (\psi)$,
3. Every fuzzy propositional formula in this propositional calculus either is a propositional variable or is formed from propositional variables by a single or multiple application of rule (2). And this should be in accordance with the used definitions of fuzzy connectives, depending on the considered system, where $\otimes \in \{\&, \wedge, \cdot\}$ and $\oplus \in \{\veebar, \vee, \bigtriangledown\}$.

The main purpose of this calculus is the same as in the classical case. Here, '$\Rightarrow$' and '$\Leftrightarrow$' denote R-implication and R-equivalence, respectively.

Any evaluation of fuzzy propositional variables can be considered as a map v assigning to each fuzzy propositional variable p its truth-value in [0,1]. This extends to each fuzzy propositional formula φ as an evaluation of propositional variables in φ by truth degrees in [0,1] (Hájek P. and Godo L. 1997, Hájek P. 1998). Below by $v_\otimes(\varphi) \in [0,1]$ (in short: $\varphi \in [0,1]$, e.g. $\varphi = a \in [0,1]$) we shall denote the *logical value of the fuzzy propositional formula* φ with respect to $\otimes$. In a similar way, e.g. by $\varphi \le \psi$ we shall denote: $\varphi = a$, $\psi = b$, and $a \le b$ $(a,b \in [0,1])$.

Definition 2.2 (Hájek P. 2002)

Let $\otimes$ be a given continuous t-norm and $v_\otimes(\varphi) \in [0,1]$ be the *logical value of the fuzzy propositional formula* φ *wrt* $\otimes$. So, we shall say φ is *t-tautology*, *t-thesis* or also *standard BL-tautology* of that calculus if $v_\otimes(\varphi) = 1$ for each evaluation of propositional variables in φ by truth degrees in [0,1] and each continuous t-norm.

The following *t*-tautologies are taken as *axioms of the logic* Ł-BL (the used letters p, q and r serve as metavariables for formulae).

A1. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

A2. $p \,\&\, q \Rightarrow p$

A3. $p \,\&\, q \Rightarrow q \,\&\, p$

A4. $p \,\&\, (p \Rightarrow q) \Rightarrow q \,\&\, (q \Rightarrow p)$

A5a. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \,\&\, q \Rightarrow r)$

A5b. $(p \,\&\, q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$

A6. $((p \Rightarrow q) \Rightarrow r) \Rightarrow (((q \Rightarrow p) \Rightarrow r) \Rightarrow r)$

A7. $0 \Rightarrow p$

A8. $\sim\sim p \Rightarrow p$

In the G- and π-BL systems, instead of A8 , the following two axioms are used: $p \Rightarrow p \wedge p$ and $\sim\sim p \Rightarrow ((p \Rightarrow p \cdot q) \Rightarrow q \cdot \sim\sim q)$, respectively. It was shown that any formula φ is a t-thesis iff it is provable under BL (Cignoli, R. et al. 2000). But, the main problem here is the time and space effectiveness of the corresponding proofs using only ´– C´ and ´RR´. Another approach is proposed below.

Let now φ be a fuzzy propositional formula obtained under Definition 2.1. Hence, as in the classical case, the main task is to verify if φ is a t-thesis. Unfortunately, the usefulness of Definition 2.2 seems to be limited considering arbitrary t-norms. Next we shall assume only t-norms related to the basic fuzzy propositional logics. Any such t-thesis is said to be a *strong t-thesis* (or equivalently: *strong t-tautology*, *strong standard BL-tautology*). The last definition can be modified assuming "t-norm dependence", i.e. the following definition can be introduced.

Definition 2.3

Let ⊗ be a given continuous t-norm and $v_{\otimes}(\varphi) \in [0,1]$, or $v(\varphi)$ if ⊗ is understood, be the *logical value of the fuzzy propositional formula* φ *wrt* ⊗. We shall say φ a *weak t-thesis* if $v_{\otimes}(\varphi) = 1$ for each evaluation of propositional variables in φ by truth degrees in [0,1].

The *proof* in the fuzzy propositional calculus can be interpreted as a process of joining new lines by using some primitive or derived rules and/or other theses in accordance with the used assumptions. The proposed approach is an extension of the classical one (Słupecki J. and Borkowski L. 1967). An illustration is given in the next example.

Example 2.7

Consider the following well-known classical law (of *addition an arbitrary proposition to the antecedent and consequent of a given implication*):

$(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

This law can be proved both using a direct or also an indirect proof. In general, the indirect proof is a more universal approach, but corresponding to more proof lines than the direct one (if it exists). The following indirect proof can be obtained (a thesis is also $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$: left to the reader).

Proof:

(1)	$p \Rightarrow q$	{1,2 / a}
(2)	$p \vee r$	
(3)	$\sim(q \vee r)$	{aip}
(4)	$\sim q$	{4,5 / NA: 3}
(5)	$\sim r$	
(6)	p	{– A : 2,5}
(7)	q	{– C : 1,6}
	contr. □	{4,7}

Since any $\varphi \in \{a\} \cup \{aip\}$ is assumed to be a true formula (i.e. true in any interpretation), the following proof technique can be equivalently introduced*.

(1)	$\forall\, p, q \in \{0,1\}\,(p \Rightarrow q = 1)$	{1,2 / a}
(2)	$\forall\, p, r \in \{0,1\}\,(p \vee r = 1)$	
(3)	$\sim\forall\, q, r \in \{0,1\}\,(q \vee r = 1)$	{aip}
(4)	$\exists\, q, r \in \{0,1\}\,(q \vee r \neq 1)$	{N∀: 3}
(5)	$q_0 \vee r_0 = 0$	{−∃: 4}
(6)	$(q_0 = 0) \wedge (r_0 = 0)$	{df.'∨': 5}
(7)	$q_0 = 0$	{7,8 / – K: 6}
(8)	$r_0 = 0$	
(9)	$p_0 \Rightarrow q_0 = 1$	{−∀: 1}
(10)	$p_0 \vee r_0 = 1$	{−∀: 2}
(11)	$p_0 \leq q_0$	{df.'⇒': 9}
(12)	$(p_0 = 1) \vee (r_0 = 1)$	{df.'∨': 10}
(13)	$p_0 = 0$	{7,11}
(14)	$r_0 = 1$	{– A: 12,13}
	contr. □	{8,14}

The above proof technique can be easily extended to the whole interval [0,1]. Hence, the following implication is satisfied.

Thesis 2.1 (*law of addition an arbitrary fuzzy proposition to the antecedent and consequent of a given implication*)

$$(p \Rightarrow q) \Rightarrow (p \oplus r \Rightarrow q \oplus r)$$

Proof (e.g. Ł-BL: $\oplus =_{df} \veebar$ *):*

(1)	$\forall\, p, q \in [0,1]\,(p \Rightarrow q = 1)$	{1,2 / a}
(2)	$\forall\, p, r \in [0,1]\,(p \veebar r = 1)$	
(3)	$\sim\forall\, q, r \in [0,1]\,(q \veebar r = 1)$	{aip}
(4)	$\exists\, q, r \in [0,1]\,(q \veebar r \neq 1)$	{N∀: 3}
(5)	$q_0 \veebar r_0 \neq 1$	{−∃: 4}
(6)	$q_0 + r_0 < 1$	{df.' ⊻': 5}
(7)	$p_0 \Rightarrow q_0 = 1$	{−∀: 1}
(8)	$p_0 \veebar r_0 = 1$	{−∀: 2}
(9)	$p_0 \leq q_0$	{df.'Ł-⇒': 7}
(10)	$p_0 + r_0 \geq 1$	{df.' ⊻': 8}
(11)	$p_0 + r_0 \leq q_0 + r_0$	{+ r_0: 9}
(12)	$q_0 + r_0 \geq 1$	{10,11}
	contr. □	{6,12}

---

* Since the logical value of any fuzzy propositional formula $\varphi \in [0,1]$ and provided there is no ambiguity, in the next considerations of this subsection any quantifier restricted to [0,1] is interpreted in standard predicate logic style (the subset over which ranges any such quantifier coincides with the universe of discourse, i.e. [0,1]). And hence, the used designations have only an auxiliary sense. A more formal treatment concerning the notion of a restricted quantifier is presented in Chapter II.

In accordance with our considerations, T1 is a strong t-thesis. Also, the following example strong t-theses are satisfied (the corresponding proofs are omitted here).

Thesis 2. 2 (*law of compound constructive dilemma*)

$(p \Rightarrow q) \otimes (r \Rightarrow s) \otimes (p \oplus r) \Rightarrow q \oplus s.$ □

Thesis 2.3 (*law of compound destructive dilemma*)

$(p \Rightarrow q) \otimes (r \Rightarrow s) \otimes \sim (q \oplus s) \Rightarrow \sim (p \oplus r).$ □

Thesis 2.4 (De Morgan's law of *negating a t-disjunction*)

$\sim (p \oplus q) \Leftrightarrow \sim p \otimes \sim q.$ □

Thesis 2.5 (De Morgan's law of *negating a t-conjunction*)

$\sim (p \otimes q) \Leftrightarrow \sim p \oplus \sim q.$ □

Thesis 2.6 (*rule modus tollendo tollens*)

$(p \Rightarrow q) \otimes \sim q \Rightarrow \sim p.$ □

Thesis 2.7 (*law of transitivity for implication*)

$(p \Rightarrow q) \otimes (q \Rightarrow r) \Rightarrow (p \Rightarrow r).$ □

Thesis 2.8 (*laws of exportation and importation*)

$p \otimes q \Rightarrow r \Leftrightarrow p \Rightarrow (q \Rightarrow r).$ □

Thesis 2.9 (*law of reduction ad absurdum*)

$(p \Rightarrow q \otimes \sim q) \Rightarrow \sim p.$ □

Thesis 2.10 (*law of transposition* or *contraposition of implication*)

$p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p.$ □

Thesis 2.11 (*law of the hypothetical, called also conditional, syllogism*)

$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)).$ □ {T2.7, T2.8}

T 2.11 corresponds to the first axiom A1 of the logic BL (Hájek P. and Godo L. 1997, Hájek P. 1998, Hájek P. 2002). It is easily to show any of the remaining axioms (this is omitted). An example proof of A4 is given below.

Thesis 2.12 (the A4 axiom)

$p \otimes (p \Rightarrow q) \Rightarrow q \otimes (q \Rightarrow p).$ □

Proof (Ł-BL: $\otimes =_{df}$ &, the proof is the same for G- and π-BL):

(1)	$\forall p \in [0,1] (p = 1)$	{1,2 / a}
(2)	$\forall p, q \in [0,1] (p \Rightarrow q = 1)$	
(3)	$\sim\forall q, p \in [0,1] (q \ \& \ (q \Rightarrow p) = 1)$	{aip}
(4)	$\exists q, p \in [0,1] (q \ \& \ (q \Rightarrow p) \neq 1)$	{N∀: 3}
(5)	$q_0 \ \& \ (q_0 \Rightarrow p_0) \neq 1$	{−∃: 4}

(6)	$(q_0 \neq 1) \vee (q_0 \Rightarrow p_0 \neq 1)$	{df.'&': 5}
(7)	$p_0 = 1$	{−∀: 1}
(8)	$p_0 \Rightarrow q_0 = 1$	{−∀: 2}
(9)	$p_0 \leq q_0$	{df.'Ł -⇒': 8}
(1.1)	$q_0 \neq 1$	{ada}
(1.2)	$q_0 \geq 1$	{7,9}
(1.3)	$q_0 \leq 1$	{$q_0 \in [0,1]$}
(1.4)	$q_0 = 1$	{1.2, 1.3}
	contr.	{1.1, 1.4}
(2.1)	$q_0 \Rightarrow p_0 \neq 1$	{ada}
(2.2)	$q_0 > p_0$	{df.'Ł -⇒': 2.1}
(2.3)	$q_0 > 1$	{2.2, 7}
(2.4)	$q_0 \leq 1$	{$q_0 \in [0,1]$}
	contr. ▫	{2.3, 2.4}

The proof of T 2.12 is a ramified indirect one with joined additional assumptions (see proof lines (1.1) and (2.1)). An illustration of this technique is also the proof of the next thesis T 2.13: see below).

Thesis 2.13 (*law of converting implications, called also law of a closed system of theorems* or *Hauber´s law*)

$(p \Rightarrow q) \otimes (r \Rightarrow s) \otimes (p \oplus r) \otimes \sim (q \otimes s) \Rightarrow (q \Rightarrow p) \otimes (s \Rightarrow r)$. ▫

Thesis 2.14 (*law of multiplication of consequents*)

$(p \Rightarrow q) \otimes (p \Rightarrow r) \Leftrightarrow p \Rightarrow q \otimes r$. ▫

As an example, the used in T 2.1 assumptional system style is also illustrated in the proof of T 2.13 below, where Ł-, G- or π-BL are considered..

Proof (Ł-BL: $\otimes =_{df}$ &, $\oplus =_{df} \veebar$):

(1)	$\forall p, q \in [0,1] (p \Rightarrow q = 1)$	{1,2,3,4 / a}
(2)	$\forall r, s \in [0,1] (r \Rightarrow s = 1)$	
(3)	$\forall p, r \in [0,1] (p \veebar r = 1)$	
(4)	$\forall q, s \in [0,1] (\sim (q \,\&\, s) = 1)$	
(5)	$\sim \forall p, q, r, s \in [0,1] ((q \Rightarrow p) \,\&\, (s \Rightarrow r) = 1)$	{aip}
(6)	$\exists p, q, r, s \in [0,1] ((q \Rightarrow p) \,\&\, (s \Rightarrow r) \neq 1)$	{ N∀: 5}
(7)	$(q_0 \Rightarrow p_0) \,\&\, (s_0 \Rightarrow r_0) \neq 1$	{−∃: 6}
(8)	$(q_0 \Rightarrow p_0 \neq 1) \vee (s_0 \Rightarrow r_0 \neq 1)$	{df.'&': 7}
(9)	$(q_0 > p_0) \vee (s_0 > r_0)$	{df.'Ł -⇒',SR: 8}
(10)	$p_0 \Rightarrow q_0 = 1$	{−∀: 1}
(11)	$p_0 \leq q_0$	{df.'Ł -⇒',SR: 10}
(12)	$r_0 \Rightarrow s_0 = 1$	{−∀: 2}
(13)	$r_0 \leq s_0$	{df.'Ł -⇒',SR: 12}
(14)	$p_0 \veebar r_0 = 1$	{−∀: 3}
(15)	$p_0 + r_0 \geq 1$	{df.'⊻': 14}
(16)	$\sim (q_0 \,\&\, s_0) = 1$	{−∀: 4}
(17)	$q_0 \,\&\, s_0 = 0$	{df.'Ł - ~',SR: 16}
(18)	$q_0 + s_0 \leq 1$	{df.'&': 17}
(19)	$p_0 + r_0 \leq q_0 + s_0$	{11 + 13}

(20)	$p_0 + r_0 = 1$	{20,21 / 15,18,19}
(21)	$q_0 + s_0 = 1$	
(1.1)	$q_0 > p_0$	{ada}
(1.2)	$r_0 = 1 - p_0$	{20}
(1.3)	$s_0 = 1 - q_0$	{21}
(1.4)	$1 - p_0 \leq 1 - q_0$	{13}
(1.5)	$p_0 \geq q_0$	{1.4}
	contr.	{1.1, 1.5}
(2.1)	$s_0 > r_0$	{ada}
(2.2)	$p_0 = 1 - r_0$	{20}
(2.3)	$q_0 = 1 - s_0$	{21}
(2.4)	$1 - r_0 \leq 1 - s_0$	{11}
(2.5)	$r_0 \geq s_0$	{2.4}
	contr. □	{2.1, 2.5}

Proof (G-BL: $\otimes =_{df} \wedge$, $\oplus =_{df} \vee$):

(1)	$\forall\, p, q \in [0,1]\, (p \Rightarrow q = 1)$	{1,2,3,4 / a}
(2)	$\forall\, r, s \in [0,1]\, (r \Rightarrow s = 1)$	
(3)	$\forall\, p, r \in [0,1]\, (p \vee r = 1)$	
(4)	$\forall\, q, s \in [0,1]\, (\sim (q \wedge s) = 1)$	
(5)	$\sim \forall\, p, q, r, s \in [0,1]\, ((q \Rightarrow p) \wedge (s \Rightarrow r) = 1)$	{aip}
(6)	$\exists\, p, q, r, s \in [0,1]\, ((q \Rightarrow p) \wedge (s \Rightarrow r) \neq 1)$	{ N∀: 5}
(7)	$(q_0 \Rightarrow p_0) \wedge (s_0 \Rightarrow r_0) \neq 1$	{–∃: 6}
(8)	$(q_0 \Rightarrow p_0 \neq 1) \vee (s_0 \Rightarrow r_0 \neq 1)$	{df.' ∧': 7}
(9)	$(q_0 > p_0) \vee (s_0 > r_0)$	{df.'G -⇒',SR: 8}
(10)	$p_0 \Rightarrow q_0 = 1$	{–∀: 1}
(11)	$p_0 \leq q_0$	{df.'G -⇒',SR: 10}
(12)	$r_0 \Rightarrow s_0 = 1$	{–∀: 2}
(13)	$r_0 \leq s_0$	{df.'G -⇒',SR: 12}
(14)	$p_0 \vee r_0 = 1$	{–∀: 3}
(15)	$(p_0 = 1) \vee (r_0 = 1)$	{df.' ∨': 14}
(16)	$\sim (q_0 \wedge s_0) = 1$	{–∀: 4}
(17)	$q_0 \wedge s_0 = 0$	{df.'G - ~',SR: 16}
(18)	$(q_0 = 0) \vee (s_0 = 0)$	{df.' ∧': 17}
(19)	$(q_0 \leq p_0) \Rightarrow (s_0 > r_0)$	{+ N, CR, SR : 9}
(20)	$(s_0 \leq r_0) \Rightarrow (q_0 > p_0)$	{CC : 19}
(1.1)	$p_0 = 1$	{ada}
(1.2)	$q_0 = 1$	{1.1,11: $q_0 \in [0,1]$}
(1.3)	$s_0 = 0$	{– A : 18,1.2}
(1.4)	$r_0 = 0$	{1.3,13: $r_0 \in [0,1]$}
(1.5)	$q_0 = p_0$	{1.1,1.2}
(1.6)	$s_0 > r_0$	{– C : 19,1.5}
(1.7)	$s_0 = r_0$	{1.3,1.4}
	contr.	{1.6,1.7}
(2.1)	$r_0 = 1$	{ada}
(2.2)	$s_0 = 1$	{2.1,13: $s_0 \in [0,1]$}
(2.3)	$q_0 = 0$	{– A : 18,2.2}
(2.4)	$p_0 = 0$	{2.3,11: $p_0 \in [0,1]$}
(2.5)	$s_0 = r_0$	{2.1,2.2}
(2.6)	$q_0 > p_0$	{– C : 20,2.5}
(2.7)	$q_0 = p_0$	{2.3,2.4}
	contr. □	{2.6,2.7}

Proof (π-BL: $\otimes =_{df} \cdot$, $\oplus =_{df} \triangledown$):

(1)	$\forall p, q \in [0,1]\ (p \Rightarrow q = 1)$	
(2)	$\forall r, s \in [0,1]\ (r \Rightarrow s = 1)$	{1,2,3,4 / a}
(3)	$\forall p, r \in [0,1]\ (p \triangledown r = 1)$	
(4)	$\forall q, s \in [0,1]\ (\sim (q \cdot s) = 1)$	
(5)	$\sim \forall p, q, r, s \in [0,1]\ ((q \Rightarrow p) \cdot (s \Rightarrow r) = 1)$	{aip}
(6)	$\exists p, q, r, s \in [0,1]\ ((q \Rightarrow p) \cdot (s \Rightarrow r) \neq 1)$	{ N$\forall$: 5}
(7)	$(q_0 \Rightarrow p_0) \cdot (s_0 \Rightarrow r_0) \neq 1$	{$-\exists$: 6}
(8)	$(q_0 \Rightarrow p_0 \neq 1) \vee (s_0 \Rightarrow r_0 \neq 1)$	{df.'$\cdot$': 7}
(9)	$(q_0 > p_0) \vee (s_0 > r_0)$	{df.'π -$\Rightarrow$',SR: 8}
(10)	$p_0 \Rightarrow q_0 = 1$	{$-\forall$: 1}
(11)	$p_0 \leq q_0$	{df.'π -$\Rightarrow$',SR: 10}
(12)	$r_0 \Rightarrow s_0 = 1$	{$-\forall$: 2}
(13)	$r_0 \leq s_0$	{df.'π -$\Rightarrow$',SR: 12}
(14)	$p_0 \triangledown r_0 = 1$	{$-\forall$: 3}
(15)	$(p_0 = 1) \vee (r_0 = 1)$	{df.'$\triangledown$': 14}
(16)	$\sim (q_0 \cdot s_0) = 1$	{$-\forall$: 4}
(17)	$q_0 \cdot s_0 = 0$	{df.'π -$\Rightarrow$',SR: 16}
(18)	$(q_0 = 0) \vee (s_0 = 0)$	{df.'$\cdot$': 17}
(19)	$p_0 + r_0 \leq q_0 + s_0$	{11 + 13}
(20)	$p_0 + r_0 \geq 1$	{15: $p_0, r_0 \in [0,1]$}
(21)	$q_0 + s_0 \geq 1$	{19,20}
(22)	$\sim (q_0 + s_0 > 1)$	{18}
(23)	$q_0 + s_0 = 1$	{– A : 21,22}
(24)	$p_0 + r_0 = 1$	{19,20,23}
(1.1)	$q_0 > p_0$	{ada}
(1.2)	$r_0 = 1 - p_0$	{24}
(1.3)	$s_0 = 1 - q_0$	{23}
(1.4)	$1 - p_0 \leq 1 - q_0$	{13,1.2,1.3}
(1.5)	$p_0 \geq q_0$	{1.4}
	contr.	{1.1,1.5}
(2.1)	$s_0 > r_0$	{ada}
(2.2)	$p_0 = 1 - r_0$	{24}
(2.3)	$q_0 = 1 - s_0$	{23}
(2.4)	$1 - r_0 \leq 1 - s_0$	{11}
(2.5)	$r_0 \geq s_0$	{2.4}
	contr. □	{2.1, 2.5}

The following simple property is also a strong t-thesis: $p \otimes q \Rightarrow p \oplus q$ (the corresponding proofs in Ł-, G- and π-BL are left to the leader.

The following example weak t-theses are satisfied (Ł-BL only): $\sim \sim p \Leftrightarrow p$ (the law of double negation, and hence the rules ± N), $p \Rightarrow q \Leftrightarrow \sim p \veebar q$ (the law of implication, i.e. the rule CR), $\sim (p \Rightarrow q) \Leftrightarrow p \,\&\, \sim q$ (the law of negating an implication, i.e. NC), $p \Rightarrow p \wedge p$ (idempotence of t-conjunction: G-BL only, the opposite implication is strong t-thesis related to – K), the following axiom (Hájek P. and Godo L. 1997): $\sim\sim p \Rightarrow ((p \Rightarrow p \otimes q) \Rightarrow q \otimes \sim\sim q)$ is not satisfied for G-BL, the well-known absorptive and distributive axioms are satisfied only in G-BL, the law of addition of antecedents $(p \Rightarrow r) \otimes (q \Rightarrow r) \Leftrightarrow p \oplus q \Rightarrow r$ is satisfied only in G- and π-BL, etc. The corresponding proofs are omitted.

In general, any strong t-thesis can be considered as a new derived rule. The following strong t-theses are considered as a generalisation or extension of the classical primitive and derived rules.

$$-C: \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \varphi\end{array}}{\psi}, \quad +K: \frac{\begin{array}{c}\varphi \\ \psi\end{array}}{\varphi \otimes \psi}, \quad -K: \frac{\varphi \otimes \psi}{\varphi \setminus \psi \setminus \begin{array}{c}\varphi \\ \psi\end{array}}, \quad +A: \frac{\varphi}{\varphi \oplus \psi}, \quad -A: \frac{\begin{array}{c}\varphi \oplus \psi \\ \sim \varphi\end{array}}{\psi}, \quad +E: \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}{\varphi \Leftrightarrow \psi},$$

$$-E: \frac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi \setminus \psi \Rightarrow \varphi \setminus \begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}, \quad \text{Toll}: \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \sim \psi\end{array}}{\sim \varphi}, \quad CC: \frac{\varphi \Rightarrow \psi}{\sim \psi \Rightarrow \sim \varphi}, \quad NA: \frac{\sim (\varphi \oplus \psi)}{\sim \varphi \setminus \sim \psi \setminus \begin{array}{c}\sim \varphi \\ \sim \psi\end{array}},$$

$$NK: \frac{\sim (\varphi \otimes \psi)}{\sim \varphi \oplus \sim \psi}, \quad SR: \frac{\varphi \Leftrightarrow \psi}{\chi \Leftrightarrow \chi(\varphi // \psi)}, \quad TC: \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \chi\end{array}}{\varphi \Rightarrow \chi}, \quad MC: \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \varphi \Rightarrow \chi\end{array}}{\varphi \Rightarrow \psi \otimes \chi}.$$

The following example proofs are an illustration of using the above rules (for simplicity only Ł-BL proofs are presented below).

Example 2.8

Consider the proof of T2.1 under the above-introduced primitive rules. This proof can be realised as follows.

(1)	$p \Rightarrow q$	{1,2 / a}
(2)	$p \veebar r$	
(3)	$\sim (q \veebar r)$	{aip}
(4)	$\sim q$	{4,5 / NA, NK: 3}
(5)	$\sim r$	
(6)	p	{− A : 2,5}
(7)	q	{− C : 1,6}
	contr. □	{4,7}

A more extended (ramified indirect) proof is given in the next example.

Example 2.9

Let consider the proof of the Hauber´s law T2.13. Since NC is a Ł-BL weak t-thesis, the following proof can be obtained.

(1)	$p \Rightarrow q$	
(2)	$r \Rightarrow s$	{1,2,3,4 / a}
(3)	$p \veebar r$	
(4)	$\sim (q \;\&\; s)$	
(5)	$\sim ((q \Rightarrow p) \;\&\; (s \Rightarrow r))$	{aip}
(6)	$\sim (q \Rightarrow p) \veebar \sim (s \Rightarrow r)$	{NK: 5}
(7)	$q \;\&\; \sim p \veebar s \;\&\; \sim r$	{NC, SR: 6}
(1.1)	q	{1.1,1.2 / ada}
(1.2)	$\sim p$	
(1.3)	r	{− A: 3,1.2}
(1.4)	s	{− C: 2,1.3}
(1.5)	$q \;\&\; s$	{+ K: 1.1,1.4}
	contr.	{4,1.5}
(2.1)	s	{2.1,2.2 / ada}
(2.2)	$\sim r$	
(2.3)	p	{− A: 3,2.2}

(2.4)	q	{– C: 1,2.3}
(2.5)	q & s	{+ K: 2.1,2.4}
	contr. □	{4,2.5}

The next definition is an extension of the corresponding classical one.

Definition 2.4

Let $\varphi = \varphi(p_1, p_2, \dots, p_n)$ be a fuzzy propositional formula under Definition 2.1. We shall say that φ is a *contradictory formula* iff $\forall\, p_1, p_2, \dots, p_n \in [0,1]\ (\varphi(p_1, p_2, \dots, p_n) = 0)$.

Proposition 2.4

Let φ be a fuzzy propositional formula under Definition 2.1. Then $\varphi \otimes \sim \varphi$ is a contradictory formula.

Proof:

(1) Ł-BL: $\varphi \,\&\, \sim \varphi = \max\{0, \varphi + (1 - \varphi) - 1\} = 0$,

(2) G-BL: $\varphi \wedge \sim \varphi = \min\{\varphi, \sim \varphi) = 0$, and

(3) π-BL: $\varphi \cdot \sim \varphi = 0$. □ {Df.2.4, df.'Ł-,G -,and π- ~'}

In accordance with Proposition 2.4 and + K, any two contradictory proof lines φ and $\sim \varphi$ will correspond to a contradictory formula $\varphi \otimes \sim \varphi$. A contradictory formula has any proposition as its consequence (the law of Duns Scotus). A more general form of a contradictory formula, e.g. may be the following: $\bigoplus_{i=1}^{n}(\varphi_i \otimes \sim \varphi_i \otimes \psi_i)$, where φ_i and ψ_i are arbitrary fuzzy propositional formulae (i = 1, 2, ... , n).

The following proposition is satisfied.

Proposition 2.5

Let φ be a t-thesis (either strong or weak). Then $\sim \varphi$ is a contradictory formula.

In fact, this property follows immediately by the definitions of the notions of fuzzy negation, t-thesis and a contradictory formula. A more formal proof is given below.

Proof (Ł-BL):

Let $\varphi = \varphi(p_1, p_2, \dots, p_n)$ be a t-thesis. So we have:

(1)	$\forall\, p_1, \dots, p_n \in [0,1]\ (\varphi(p_1, p_2, \dots, p_n) = 1)$	{a}
(2)	$\sim \forall\, p_1, \dots, p_n \in [0,1]\ (\sim \varphi(p_1, p_2, \dots, p_n) = 0)$	{aip}
(3)	$\exists\, p_1, \dots, p_n \in [0,1]\ (\sim \varphi(p_1, p_2, \dots, p_n) \neq 0)$	{ N∀: 2}
(4)	$\sim \varphi(a_1, a_2, \dots, a_n) \neq 0$	{–∃: 3}
(5)	$1 - \varphi(a_1, a_2, \dots, a_n) \neq 0$	{df.'Ł-~': 4}
(6)	$\varphi(a_1, a_2, \dots, a_n) \neq 1$	{5}
(7)	$\varphi(a_1, a_2, \dots, a_n) = 1$	{–∀: 1}
	contr. □	{6,7}

In the case of G- or π-BL the proof is similar and hence it is omitted. A more general form of a contradictory formula is used in the proof of the next example.

Example 2.10

Consider the following rule of the well-known Gentzen's sequent calculus called 'rule of removing an equivalence in the antecedent of a sequent' (in short: $- E_a$, the sequent symbol is denoted by '$\vdash$'):

$$-E_a: \quad \frac{\theta, A \Leftrightarrow B \vdash \Delta}{A, B, \theta \vdash \Delta \quad \theta \vdash \Delta, A, B}.$$

Since any sequent can be interpreted in an unique way by some propositional formula, the following fuzzy propositional formula can be obtained (without loss of generality and for simplicity, the used sequences of formulae θ, A, B and Δ are interpreted as propositional variables, e.g. p, q, r and s, respectively). And this formula is a Ł-BL weak t-thesis as it is shown below.

$(p \,\&\, (q \Leftrightarrow r) \Rightarrow s) \Rightarrow (p \,\&\, q \,\&\, r \Rightarrow s) \,\&\, (p \Rightarrow q \veebar r \veebar s)$

Proof:

(1)	$p \,\&\, (q \Leftrightarrow r) \Rightarrow s$	{a}
(2)	$\sim((p \,\&\, q \,\&\, r \Rightarrow s) \,\&\, (p \Rightarrow q \veebar r \veebar s))$	{aip}
(3)	$\sim(p \,\&\, q \,\&\, r \Rightarrow s) \veebar \sim(p \Rightarrow q \veebar r \veebar s)$	{NK: 2}
(4)	$p \,\&\, q \,\&\, r \,\&\, {\sim}s \veebar p \,\&\, {\sim}q \,\&\, {\sim}r \,\&\, {\sim}s$	{NC, NA, SR: 3}
(1.1)	p	
(1.2)	q	{1.1,1.2,1.3,1.4 / ada}
(1.3)	r	
(1.4)	${\sim}s$	
(1.5)	$\sim(p \,\&\, (q \Leftrightarrow r))$	{Toll: 1,1.4}
(1.6)	$\sim p \veebar \sim(q \Leftrightarrow r)$	{NK: 1.5}
(1.7)	$\sim(q \Leftrightarrow r)$	{– A: 1.1,1.6}
(1.8)	$\sim((q \Rightarrow r) \,\&\, (r \Rightarrow q))$	{– E, SR: 1.7}
(1.9)	$\sim(q \Rightarrow r) \veebar \sim(r \Rightarrow q)$	{NK: 1.8}
(1.10)	$q \,\&\, {\sim}r \veebar r \,\&\, {\sim}q$	{NC, SR: 1.9}
(1.11)	$(q \,\&\, {\sim}r \veebar r \,\&\, {\sim}q) \,\&\, q \,\&\, r$	{+ K: 1.2,1.3,1.10}
	contr.	{1.11}
(2.1)	p	
(2.2)	${\sim}q$	{2.1,2.2,2.3,2.4 / ada}
(2.3)	${\sim}r$	
(2.4)	${\sim}s$	
(2.5)	$\sim(p \,\&\, (q \Leftrightarrow r))$	{Toll: 1,2.4}
(2.6)	$\sim p \veebar \sim(q \Leftrightarrow r)$	{NK: 2.5}
(2.7)	$\sim(q \Leftrightarrow r)$	{– A: 2.1,2.6}
(2.8)	$\sim((q \Rightarrow r) \,\&\, (r \Rightarrow q))$	{– E, SR: 2.7}
(2.9)	$\sim(q \Rightarrow r) \veebar \sim(r \Rightarrow q)$	{NK: 2.8}
(2.10)	$q \,\&\, {\sim}r \veebar r \,\&\, {\sim}q$	{NC, SR: 2.9}
(2.11)	$(q \,\&\, {\sim}r \veebar r \,\&\, {\sim}q) \,\&\, {\sim}q \,\&\, {\sim}r$	{+ K: 2.2,2.3,2.10}
	contr. □	{2.11}

The considered formula is also a weak t-thesis in π-BL, but it is not satisfied in G-BL, e.g. for p, q, r, and s equal to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{3}$, respectively (the value of the main implication is equal to $\frac{1}{3} \neq 1$).

Let ϕ be a classical / many-valued logic formula (see: Subsection 2.1) and $t(\phi)$ be the corresponding fuzzy propositional formula obtained under Definition 2.1 (by interchanging the classical two-valued / many-valued connectives and the corresponding t- and s-norms ones). Since $\{0,1\} \subseteq \mathbb{W}_m \subsetneq [0,1]$, the following proposition is satisfied.

Proposition 2.6

If $\vDash_t t(\phi)$ then $\vDash \phi$. □

And hence, the condition that a formula φ is a thesis is not sufficient for $t(\varphi)$ to be a t-thesis.

Definition 2.5

Let $\phi_1, ..., \phi_n, \varphi$ be formulae under Definition 2.1. We shall say that φ is a (*fuzzy*) *t-consequence* of $\phi_1, ..., \phi_n$ (in short: $\varphi \in Cn_t(\{\phi_1, ..., \phi_n\})$) iff $\vDash_t \bigotimes_{i=1}^{n} \phi_i \Rightarrow \varphi$.

The above-proposed approach preserves the BL-provability property, i.e. the following proposition is satisfied.

Proposition 2.7 (*BL-provability*)

Any t-thesis provable under the Hájek's axiomatic approach of the logic BL is also provable under the above-proposed approach from assumptions.

Proof:

Let $A =_{df} \{a_1, a_2, ..., a_n\}$ be a finite non-empty set of BL-axioms and $\varphi \in Cn_t(A)$. Since $\varphi \in Cn_t(A)$ then $\vDash_t \bigotimes_{i=1}^{n} a_i \Rightarrow \varphi$. Since any $a_i \in A$ is a t-thesis (e.g. T2.12 is a proof of A4, i.e. a_4) then any fuzzy propositional formula $\varphi \in Cn_t(A)$ can be considered as a consequence of the above-proposed approach from assumptions. A more formal treatment is given below.

Let $R_i \neq \varnothing$ be the subset of (primitive and/or derived) inference rules of the first-order predicate logic calculus related to the proof of $a_i \in A$ (including the rules for constructing an indirect proof from assumptions) and $R =_{df} \bigcup_{i=1}^{n} R_i$ be the whole set of rules covering A. For simplicity, let $R =_{df} \{r_1, r_2, ..., r_m\}$.

In general, any inference rule can be interpreted as some axiom. Assume that φ is a fuzzy propositional formula under Definition 2.1 such that φ is provable by A. Since any $a_i \in A$ is a t-thesis in our system then $a_i \in Cn(R_i)$. On the other hand $R_i \subseteq R$. Then $Cn(R_i) \subseteq Cn(R)$ and $a_i \in Cn(R)$. Hence, the following implication is necessary to be shown.

$\forall a_i \in A\ (a_i \in Cn_t(R)) \otimes (\varphi \in Cn_t(A)) \Rightarrow (\varphi \in Cn_t(R))$.

Hence we have:

(1)	$\forall a_i \in A\ (a_i \in Cn_t(R))$	{1,2 / a}
(2)	$\varphi \in Cn_t(A)$	
(3)	$\forall a_i \in A\ (\bigotimes_{k=1}^{m} r_k \Rightarrow a_i)$	{Df.2.5,SR: 1}
(4)	$\bigotimes_{k=1}^{m} r_k \Rightarrow a_1$	
(5)	$\bigotimes_{k=1}^{m} r_k \Rightarrow a_2$	{4,5,...,n+3 / SR,– K: 3}
…	…	
(n+3)	$\bigotimes_{k=1}^{m} r_k \Rightarrow a_n$	
(n+4)	$\bigotimes_{k=1}^{m} r_k \Rightarrow \bigotimes_{i=1}^{n} a_i$	{MC: 4,5,...,n+3}
(n+5)	$\bigotimes_{i=1}^{n} a_i \Rightarrow \varphi$	{Df.2.5,SR: 2}
(n+6)	$\bigotimes_{k=1}^{m} r_k \Rightarrow \varphi$	{TC: n+4,n+5}
	$\varphi \in Cn_t(R)$. □	{Df.2.5,SR: 7}

Since any φ is a t-thesis iff it is provable under BL (Cignoli, R. et al. 2000) then the above presented assumptional approach and the axiomatic approach are equivalent. A more formal treatment of this equivalence seems to be realised in a similar way as in Subsection 1.7, i.e. by extending the notion of *proof*, i.e. *t-proof*, including beyond of '$-$ C' also the above obtained rules (or their representative subset). And next, by introducing a (*fuzzy*) *deduction theorem* similarly as in the classical case, i.e. Theorem 1.31. But any such approach seems to be difficult and it is omitted here.

Below it is introduced a new t-norm and then it is defined a generalised Łukasiewicz's system, denoted by $Ł_\alpha$-BL ($\alpha > 0$), where the previous one becomes a particular case with $\alpha = 1$ (Tabakow I.G. 2010).

A generalisation of the Łukasiewicz's BL system

Let now consider the following two *similar* Abelian systems, i.e. of the same type (0,0,2): $\mathscr{A}_1 =_{df} ([0,1] ; 1, 0 ; \hat{\otimes})$ and $\mathscr{A}_2 =_{df} ([0,1] ; 1, 0 ; \otimes)$, where $\hat{\otimes}$ and $\otimes$ are two t-norms. We shall assume that $\otimes$ is a priori given t-norm called *source t-norm* (or *prototypical representative*). Assume that $f : [0,1] \to [0,1]$ is a given increasing bijection* and $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic with respect to f. Hence, the following two conditions should be satisfied (for any $x, y \in [0,1]$):

1. $f(1) = 1, f(0) = 0$ (the *algebraic constants preservation*)
2. $f(x \hat{\otimes} y) = f(x) \otimes f(y)$ (the *algebraic operations preservation*).

Since f is bijection and in accordance with the above assumptions, there exists an inverse function f^{-1} (having the same properties as the original function f) such that $f^{-1} (f(x \hat{\otimes} y)) = f^{-1} (f(x) \otimes f(y))$. Therefore, the new $\hat{\otimes}$ can be obtained in an unique way by the following well-known equality: $x \hat{\otimes} y =_{df} f^{-1} (f(x) \otimes f(y))$.

Let now consider the increasing bijection $y = f(x) =_{df} x^\alpha$ defined in [0,1]. The inverse function $y = f^{-1} (x) =_{df} \sqrt[\alpha]{x}$, where $x \geq 0$ and $\alpha > 0$. It is selected as a source t-norm the Łukasiewicz's one, i.e. $x \otimes y =_{df} \max\{0, x + y - 1\}$. And hence, the following generalised t-norm can be obtained.

$$x \hat{\otimes} y =_{df} (\max\{0, x^\alpha + y^\alpha - 1\})^{1/\alpha} = \max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha} .$$

By assuming Łukasiewicz's fuzzy negation the following t-conorm is obtained as below.

$$x \hat{\oplus} y =_{df} 1 - \max\{0, (1 - x)^\alpha + (1 - y)^\alpha - 1\}^{1/\alpha} .$$

Assume now Yager's fuzzy negation. For simplicity, let $z =_{df} x' \hat{\otimes} y'$. By definition we have:

$$\begin{aligned} z &= \max\{0, (x')^\alpha + (y')^\alpha - 1 \}^{1/\alpha} \\ &= \max\{0, 1 - x^\alpha - y^\alpha\}^{1/\alpha} . \end{aligned}$$

Hence we can obtain:

$$\begin{aligned} x \hat{\oplus} y &=_{df} (1 - z^\alpha)^{1/\alpha} \\ &= (1 - \max\{0, 1 - x^\alpha - y^\alpha\})^{1/\alpha} . \end{aligned}$$

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* A *bijection* (*bijective function*: in general *bijective map* or *one-to-one correspondence*) is a map $f : X \to Y$ which is at the same time *one-to-one* (*injective*, i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for any $x_1,x_2 \in X$) and *onto* (i.e. *surjective*).

Since $(1 - \max\{0, 1 - x^\alpha - y^\alpha\}) = \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$ (The left side and the right side coincide for $x^\alpha + y^\alpha \geq 1$ and $x^\alpha + y^\alpha < 1$). And so, the following *generalised Łukasiewicz's t-conorm* is obtained: $x \hat{\oplus} y =_{df} \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$.* □

Proposition 2.8

The above systems $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic with respect to $y = x^\alpha$ defined in [0,1].

Proof:

We have: $f(1) = 1^\alpha = 1$, $f(0) = 0^\alpha = 0$ and $f(x \hat{\oplus} y) = f(\max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha}) = \max\{0, x^\alpha + y^\alpha - 1\} = \max\{0, f(x) + f(y) - 1) = f(x) \otimes f(y)$†. □

Moreover, the above two norms are well-defined and the corresponding axioms are satisfied (this is omitted)‡.

Corollary 2.1

Let $\hat{\otimes}$ and $\hat{\oplus}$ be the new obtained t-norm and t–conorm. Assume that $\otimes$ and $\oplus$ are the corresponding source Łukasiewicz's ones. Hence, if $\alpha = 1$ then $\hat{\otimes} = \otimes$ and $\hat{\oplus} = \oplus$.

Proof:

Since $\hat{\otimes} = \otimes$ is obvious, we shall show that $\hat{\oplus} = \oplus$. And so, let $\alpha = 1$ and $x, y \in [0,1]$. Then we can obtain: $x \hat{\oplus} y = 1 - \max\{0, 1 - x + 1 - y - 1\} = 1 - \max\{0, 1 - (x + y)\} = \min\{1, x + y\}$ (since $1 - \max\{0, 1 - a\} = \min\{1, a\}$: this is omitted). □

It can be observed that the graph of $\hat{\otimes}$ (i.e. of the two-argument function $z = x \hat{\otimes} y$) and this one associated with Łukasiewicz's t-norm are different. In fact, assuming $z = 0$, all points of plane XOY corresponding to the Yager's negation $y = (1 - x^\alpha)^{1/\alpha}$ will be located on the left side and the right side of the line $y = 1 - x$ (the Łukasiewicz's negation), depending on the used values for α ($\alpha < 1$ or $\alpha > 1$, respectively). And the last two functions will coincide with $\alpha = 1$. In accordance with Proposition 2.8, $\hat{\otimes}$ is a continuous t-norm (it is a superposition of continuous functions). Moreover, there is no any idempotent $x \in (0,1)$ and hence $\hat{\otimes}$ is Archimedean t-norm. In fact, by assuming $x \hat{\otimes} x = x$, for some $x \in (0,1)$, we can obtain: $\max\{0, 2x^\alpha - 1\} = x^\alpha$. And so, if $x^\alpha = 0$ then $x = 0$ (a contradiction, since $x \neq 0$). If $x^\alpha = 2x^\alpha - 1$ then $x^\alpha = 1$ and hence $x = 1$ (a contradiction, since $x \neq 1$).

Proposition 2.9

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* The generalised Łukasiewicz's t-norm / conforms can be generalised for more than two (but a finite number of arguments), e.g. $\hat{\bigoplus}_{i=1}^{n} x_i = \min\{1, \sum_{i=1}^{n} x_i^\alpha\}^{1/\alpha}$ (the corresponding profs are inductive wrt n : left to the reader)

† Obviously, the systems $\mathscr{A}_2$ and $\mathscr{A}_1$ are isomorphic with respect to the inverse function $y = x^{1/\alpha}$. In fact, since $0^{1/\alpha} = 0$ and $1^{1/\alpha} = 1$, it is necessary to show that: $f^{-1}(x \otimes y) = f^{-1}(x) \hat{\otimes} f^{-1}(y)$. This is left to the reader (as in Prop.2.8, starting with the right side of the last equality). In general, any such isomorphism can be considered as a *binary relation of equivalence*, i.e. *reflexive*, *symmetric* and *transitive*, denoted below by ´≈´ and defined on the set of all *similar* algebraic systems (a more formal treatment will be presented in the next part of this work) . And hence, the use of the inverse function corresponds to the *symmetric* property of this relation. More formally: $\mathscr{A}_1 \approx_f \mathscr{A}_2 \Rightarrow \mathscr{A}_2 \approx_{f^{-1}} \mathscr{A}_1$.

‡ It can be observed that In the case of using as a source t-norm the usual arithmetic product xy, the new obtained t-norm would be identical with the source one (since $\sqrt[\alpha]{x^\alpha y^\alpha} = xy$).

The binary operation $\hat{\otimes}$ is a nilpotent Archimedean t-norm.

Proof:

Let $x \in (0,1)$ be a nilpotent element in $\mathscr{A}_2$. So, there exists $n \in \mathbb{N}$ such that $x \otimes x \otimes ... \otimes x = 0$. Assume that this is satisfied for $n = n_0$. Then $x^{n_0} = 0$. In accordance with Proposition 2.8, the two algebraic systems $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic and hence we have: $(x \hat{\otimes} x \hat{\otimes} ... \hat{\otimes} x)^\alpha = x^\alpha \otimes x^\alpha \otimes ... \otimes x^\alpha$ (n_0 times) $= (x^\alpha)^{n_0} = (x^{n_0})^\alpha = 0^\alpha = 0$. Hence $x \hat{\otimes} x \hat{\otimes} ... \hat{\otimes} x = 0$ and x is nilpotent in $\mathscr{A}_1$. And so, the only idempotents are 0 and 1. Since $\hat{\otimes}$ is continuous then $\hat{\otimes}$ is a nilpotent Archimedean t-norm. □

Consider the following R-implication: $x \Rightarrow_\alpha y =_{df} \sup\{z \in [0,1] / x \hat{\otimes} z \le y\}$, where $x \hat{\otimes} z =_{df} \max\{0, x^\alpha + z^\alpha - 1\}^{1/\alpha}$ and $\alpha > 0$. According to the last considerations, $x \Rightarrow_\alpha y = 1$ iff $x \le y$. In fact, assume that $z = 0$. Since $x^\alpha \le 1$ then $x^\alpha - 1 \le 0$ and $\max\{0, x^\alpha - 1\}^{1/\alpha} = \sqrt[\alpha]{0} = 0 \le y$ is satisfied. Let now $z = 1$. Then $\max\{0, x^\alpha + 1 - 1\}^{1/\alpha} \le y$ iff $\sqrt[\alpha]{x^\alpha} = x \le y$. And hence, $x \Rightarrow_\alpha y = 1$.

Assume now that $x > y$. The case $z = 0$ is the same as in the previous considerations and hence it is omitted. Since $z = 1$ implies $x \le y$ then $x > y$ implies $z \ne 1$, i.e. $z < 1$ (contraposition of implication). Consequently, assuming $x > y$ the above supremum will be associated with some subinterval $[0,z] \subsetneq [0,1]$. The least upper bound of $[0,z]$ can be obtained as follows: $\max\{0, x^\alpha + z^\alpha - 1\}^{1/\alpha} \le y$ iff $\max\{0, x^\alpha + z^\alpha - 1\} \le y^\alpha$. Let $x^\alpha + z^\alpha - 1 > 0$. Then we have: $z \le (1 - x^\alpha + y^\alpha)^{1/\alpha}$, where $1 - x^\alpha + y^\alpha \ge 0$ (since $1 - x^\alpha \ge 0$ and $y^\alpha \ge 0$). And hence, the following proposition is satisfied.

Proposition 2.10

Let $\hat{\otimes}$ be the above introduced t-norm. The fuzzy implication $x \Rightarrow_\alpha y$ having logical value as follows: $x \Rightarrow_\alpha y =_{df}$ if $x \le y$ then 1 else $(1 - x^\alpha + y^\alpha)^{1/\alpha}$ is a well-defined and unique R-implication, where $\alpha > 0$. □

Since x^α is increasing in $[0,1]$ then $x^\alpha \le y^\alpha$ if $x \le y$. And hence, $y^\alpha - x^\alpha \ge 0$. Then $1 + y^\alpha - x^\alpha \ge 1$ and $(1 - x^\alpha + y^\alpha)^{1/\alpha} \ge 1$. In a similar way, assuming $x > y$ we can obtain $(1 - x^\alpha + y^\alpha)^{1/\alpha} < 1$. And so, the following corollary is satisfied.

Corollary 2.2

$x \Rightarrow_\alpha y = \min\{1, 1 - x^\alpha + y^\alpha\}^{1/\alpha}$. □

Corollary 2.3

The Łukasiewicz's implication is a particular case with $\alpha = 1$. □

Corollary 2.4

The S- and R-implications coincide in Ł$_\alpha$-BL if Yager's fuzzy negation is assumed.

Proof:

$$
\begin{aligned}
x \Rightarrow_\alpha y &=_{df} (x \hat{\otimes} y')' \\
&= (1 - (\max\{0, x^\alpha + ((1 - y^\alpha)^{1/\alpha})^\alpha - 1\}^{1/\alpha})^\alpha)^{1/\alpha} \\
&= (1 - \max\{0, x^\alpha - y^\alpha\})^{1/\alpha} \\
&= \min\{1, 1 - x^\alpha + y^\alpha\}^{1/\alpha}. \ \square
\end{aligned}
$$

In accordance with Proposition 2.8, the considered two algebraic systems $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic all original properties are preserved. In particular, the following proposition is satisfied.

Proposition 2.11

The sup-preservation property is satisfied for $\hat{\otimes}$ and any $a \in [0,1]$.

Proof:

We can obtain: $a \,\hat{\otimes}\, \sup_{x \in [0,1]} x = \max\{0, a^\alpha + (\sup_{x \in [0,1]} x)^\alpha - 1\}^{1/\alpha} = \max\{0, a^\alpha + 1^\alpha - 1\}^{1/\alpha} = a$. On the other hand: $\sup_{x \in [0,1]} (a \,\hat{\otimes}\, x). = \sup\{x \in [0,1] \,/\, \max\{0, a^\alpha + x^\alpha - 1\}^{1/\alpha}\} = \sup\{\max\{0, a^\alpha - 1\}^{1/\alpha}, \ldots, \max\{0, a^\alpha\}^{1/\alpha}\} = \sup[0,a] = a$ (since $a^\alpha - 1 \le 0$). □

The relations between the Zadeh's t-norm and t-conorm and the presented ones are given in the next proposition.

Proposition 2.12

$\min\{x,y\} \ge \max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha}$ and $\max\{x,y\} \le \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$

Proof:

Assume that $x \le y$. Then $\min\{x,y\} = x$ and $\max\{x,y\} = y$ ($x, y \in [0,1]$). Since $x \ge 0$ and $y \le 1$ it is sufficient to show that $x \ge (x^\alpha + y^\alpha - 1)^{1/\alpha}$ and $y \le (x^\alpha + y^\alpha)^{1/\alpha}$. We have: $x \ge (x^\alpha + y^\alpha - 1)^{1/\alpha}$ iff $x^\alpha \ge x^\alpha + y^\alpha - 1$ iff $y^\alpha \le 1$. On the other hand, $y \le (x^\alpha + y^\alpha)^{1/\alpha}$ iff $y^\alpha \le x^\alpha + y^\alpha$ iff $x^\alpha \ge 0$ (the proof for $x > y$ is omitted). □

Therefore, $\max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha} \le \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$. The following fuzzy propositional system, called *generalised Łukasiewicz's system* and denoted by Ł$_\alpha$-BL is presented below (in short: *first-order Ł$_\alpha$-BL*). The *propositional variables* $p, q \in [0,1]$* and $\alpha > 0$.

1. *t- conjunction*: $p \,\&_\alpha\, q =_{df} \max\{0, p^\alpha + q^\alpha - 1\}^{1/\alpha}$,

2. *t- disjunction*: $p \,\veebar_\alpha\, q =_{df} 1 - \max\{0, (1-p)^\alpha + (1-q)^\alpha - 1\}^{1/\alpha}$,

3. *t-implication*: $p \Rightarrow_\alpha q =_{df}$ if $p \le q$ then 1 else $(1 - p^\alpha + q^\alpha)^{1/\alpha}$ and

4. *fuzzy negation*: $\sim p =_{df} 1 - p$.

Another system, called second order Ł$_\alpha$-BL can be obtained by using the Yager's fuzzy negation, where the logical value of the obtained t- disjunction is specified as below.

Proposition 2.13

---

* As in Subsection 1.4 and for simplicity, instead of $v(\varphi)$ the same formula φ will be used. Hence, e.g. such notions as: '$\varphi = \ldots$' or '$\varphi \in [0,1]$' or also '$\varphi \le \psi$', etc. should be interpreted as: '$v(\varphi) = \ldots$', '$v(\varphi) \in [0,1]$' and '$v(\varphi) \le v(\psi)$', respectively. Moreover, as in the previous section and to minimise the number of used parentheses we shall assume the logical functors bind more strongly than the sign of equality. Hence, e.g. $(p \Rightarrow q) \wedge p \Rightarrow q = 1$ denotes $((p \Rightarrow q) \wedge p \Rightarrow q) = 1$ or equivalently: $v((p \Rightarrow q) \wedge p \Rightarrow q) = 1$.

Let $\sim p =_{df} (1 - p^{\alpha})^{1/\alpha}$ be the Yager's fuzzy negation. Then the logical value of $p \veebar_{\alpha} q = \min\{1, p^{\alpha} + q^{\alpha}\}^{1/\alpha}$.

Proof:

Let $\sim p =_{df} (1 - p^{\alpha})^{1/\alpha}$ $(p \in [0,1])$. In accordance with generalised De Morgan's laws, the logical value $p \veebar_{\alpha} q =_{df} \sim(\sim p \,\&_{\alpha} \sim q)$. Since $\sim p \,\&_{\alpha} \sim q = (1 - p^{\alpha})^{1/\alpha} \&_{\alpha} (1 - p^{\alpha})^{1/\alpha} = \max\{0, 1 - (p^{\alpha} + q^{\alpha})\}^{1/\alpha}$. Hence, $p \veebar_{\alpha} q = (1 - \max\{0, 1 - (p^{\alpha} + q^{\alpha})\})^{1/\alpha}$. Since $1 - \max\{0, 1 - (p^{\alpha} + q^{\alpha})\} = \min\{1, p^{\alpha} + q^{\alpha}\}$ then $p \veebar_{\alpha} q = \min\{1, p^{\alpha} + q^{\alpha}\}^{1/\alpha}$. □

The *second order generalised Łukasiewicz's system* (in short: *second order Ł$_{\alpha}$-BL*) is defined as follows.

1. *t- conjunction*: $p \,\&_{\alpha} q =_{df} \max\{0, p^{\alpha} + q^{\alpha} - 1\}^{1/\alpha}$,
2. *t- disjunction*: $p \veebar_{\alpha} q =_{df} \min\{1, p^{\alpha} + q^{\alpha}\}^{1/\alpha}$,
3. *t-implication*: $p \Rightarrow_{\alpha} q =_{df}$ if $p \leq q$ then 1 else $(1 - p^{\alpha} + q^{\alpha})^{1/\alpha}$ and
4. *fuzzy negation*: $\sim p =_{df} (1 - p^{\alpha})^{1/\alpha}$.

Proposition 2.14

For any order Ł$_{\alpha}$-BL the logical value of fuzzy equivalence $p \Leftrightarrow_{\alpha} q$ is equal to $(1 - |p^{\alpha} - q^{\alpha}|)^{1/\alpha}$. In the case of the second order Ł$_{\alpha}$-BL the logical value of fuzzy difference $p \not\Leftrightarrow_{\alpha} q$ is equal to $|p^{\alpha} - q^{\alpha}|^{1/\alpha}$ (for any $p, q \in [0,1]$).

Proof:

By definition the logical value of $p \Leftrightarrow_{\alpha} q$ is equal to the logical value of the following t- conjunction: $(p \Rightarrow_{\alpha} q) \,\&_{\alpha} (q \Rightarrow_{\alpha} p)$. And hence, in accordance with Corollary 2.2 we can obtain: $(p \Rightarrow_{\alpha} q) \,\&_{\alpha} (q \Rightarrow_{\alpha} p) =_{df} \max\{0, \min\{1, (1 - p^{\alpha} + q^{\alpha})^{1/\alpha}\}^{\alpha} + \min\{1, (1 - q^{\alpha} + p^{\alpha})^{1/\alpha}\}^{\alpha} - 1\}^{1/\alpha}$. Then the logical value of $p \Leftrightarrow_{\alpha} q$ is equal to 1 if $p = q$. Assume that $p < q$. Hence, the logical value of $p \Leftrightarrow_{\alpha} q$ is equal to $(1 - q^{\alpha} + p^{\alpha})^{1/\alpha}$. In a similar way, for $p > q$ this logical value is equal to $(1 - p^{\alpha} + q^{\alpha})^{1/\alpha}$. And finally we have: $(p \Rightarrow_{\alpha} q) \,\&_{\alpha} (q \Rightarrow_{\alpha} p) = (1 - |p^{\alpha} - q^{\alpha}|)^{1/\alpha}$.

According to the Yager's negation, the logical value of $p \not\Leftrightarrow_{\alpha} q$ is equal to $(1 - (p \Leftrightarrow_{\alpha} q)^{\alpha})^{1/\alpha}$ and hence it is equal to $|p^{\alpha} - q^{\alpha}|^{1/\alpha}$. □

Since $\&_{\alpha}$ is associative it can be extended to more than two, i.e. a finite number of arguments. And so, the following proposition is satisfied.

Proposition 2.15

The logical value of the *generalised t- conjunction* $\mathop{\&_{\alpha}}_{i=1}^{n} p_i = \max\{0, \sum_{i=1}^{n} p_i^{\alpha} - n + 1\}^{1/\alpha}$.

Proof:

The proof is inductive with respect to n. In consequence, finally it is necessary to show that $\max\{0, \beta + p_{k+1}^{\alpha} - 1\} = \max\{0, \max\{0, \beta\} + p_{k+1}^{\alpha} - 1\}$, where $\beta =_{df} \sum_{i=1}^{k} p_i^{\alpha} - k + 1$. But this equation is satisfied for any $\beta \leq 0$ or $\beta > 0$ (a more formal treatment is omitted). □

Similarly, the logical value of the generalised t- disjunction in the second order Ł$_{\alpha}$-BL (the corresponding proof is left to the reader): $\underline{\vee}_{\alpha\, i=1}^{n} p_i = \min\{1, \sum_{i=1}^{n} p_i^{\alpha}\}^{1/\alpha}$.

Provided there is no ambiguity, and to simplify the corresponding proofs, the logical value $v_t(\varphi) \in [0,1]$ of any fuzzy propositional formula φ will be also denoted by φ (depending on the context and used t-norm). Moreover, since any R-implication is associated with some t-norm, for convenience instead of 'R-implication' the term '*t-implication*' is equivalently used. As an illustration, in the next some proofs in (first or second order) Ł$_{\alpha}$-BL are presented and new derived rules are given, where the assumptional proof system is used (Tabakow I.G. 2006).

Assumptional proofs in Ł$_{\alpha}$-BL and derived rules

As in the previous section, the used names for the primitive and/or derived rules given below are in accordance with the Łukasiewicz´s symbols of negation, conjunction, disjunction, implication, and equivalence denoted by N, K, A, C, and E, respectively. The following (generalised *primitive) rules* are considered below: − C (*rule of detachment for t-implication* or *omitting a t-implication*), ± K (*rules of joining / omitting a t-conjunction*), ± A (*rules of joining / omitting a t-disjunction*), and ± E (*rules of joining / omitting a t-equivalence*). The *rule of substitution* is denoted by SR. Some additional rules are also used, such as: ± N (rules of *joining / omitting double negation*), CR (*implication rule*), CC (the *law of transposition or contraposition of implication*), NA (rule of *negating a t-disjunction*), NK (rule of *negating a t-conjunction*), Toll (*rule modus tollendo tollens*), TC (the law of *transitivity for implication*), MC (the law of *multiplication of consequents* of two or more implications having the same antecedents). Some additional inference rules of the first-order predicate logic calculus are also used below, such as: the rule of *negating an universal quantifier* (denoted by N∀), the rules of *omitting an universal* and *an existential quantifiers* (denoted by: −∀ and −∃, respectively: see Subsection 3.3 of Chapter II). The used construction of any formula in Ł$_{\alpha}$-BL (first or second order) is the same as in Definition 2.1 and the same priorities for logical connectives are assumed. Provided there is no ambiguity, the notion of t-thesis is interpreted under Definition 2.3.

As in the classical case, the main task of this calculus is to verify if φ is a t-thesis. If yes then there exists a proof. Any such proof in the fuzzy propositional calculus can be interpreted as a process of joining new lines by using some primitive or derived rules and/or other theses in accordance with the used assumptions. The proposed here approach is an extension of the previous one (Tabakow I.G. 2006) to Ł$_{\alpha}$-BL. Some proofs and new derived rules are given below.

The next thesis is an illustration of using the same proof technique as in Ł-BL (i.e. using first-order predicate logic calculus, see: Example 2.7 and Thesis 2.1).

Thesis 2.15 (*law of addition an arbitrary fuzzy proposition to the antecedent and consequent of a given implication*)

$(p \Rightarrow_{\alpha} q) \Rightarrow_{\alpha} (p \,\underline{\vee}_{\alpha}\, r \Rightarrow_{\alpha} q \,\underline{\vee}_{\alpha}\, r)$

Proof / first-order Ł$_{\alpha}$-BL:

(1)	$\forall\, p, q \in [0,1]\, (p \Rightarrow_{\alpha} q = 1)$	{1,2 / a}
(2)	$\forall\, p, r \in [0,1]\, (p \,\underline{\vee}_{\alpha}\, r = 1)$	
(3)	$\sim\forall\, q, r \in [0,1]\, (q \,\underline{\vee}_{\alpha}\, r = 1)$	{aip}
(4)	$\exists\, q, r \in [0,1]\, (q \,\underline{\vee}_{\alpha}\, r \neq 1)$	{N∀: 3}

(5)	$q_0 \veebar_\alpha r_0 \neq 1$	{−∃: 4}
(6)	$p_0 \Rightarrow_\alpha q_0 = 1$	{−∀: 1}
(7)	$p_0 \veebar_\alpha r_0 = 1$	{−∀: 2}
(8)	$p_0 \leq q_0$	{df.'Ł-$\Rightarrow_\alpha$': 6}
(9)	$(1 - q_0)^\alpha + (1 - r_0)^\alpha > 1$	{df. '$\veebar_\alpha$': 5}
(10)	$(1 - p_0)^\alpha + (1 - r_0)^\alpha \leq 1$	{df. '$\veebar_\alpha$': 7}
(11)	$(1 - q_0)^\alpha > 1 - (1 - r_0)^\alpha$	{9}
(12)	$(1 - q_0)^\alpha \leq 1 - (1 - r_0)^\alpha$	{Since $q_0 \geq p_0$ then $-q_0 \leq -p_0$ and $1 - q_0 \leq 1 - p_0$. Hence $(1 - q_0)^\alpha \leq (1 - p_0)^\alpha \leq 1 - (1 - r_0)^\alpha$: 8,10}
	contr. □	{11,12}

Proof / second order Ł$_\alpha$-BL:

(1)	$\forall\, p, q \in [0,1]\,(p \Rightarrow_\alpha q = 1)$	{1,2 / a}
(2)	$\forall\, p, r \in [0,1]\,(p \veebar_\alpha r = 1)$	
(3)	$\sim\forall\, q, r \in [0,1]\,(q \veebar_\alpha r = 1)$	{aip}
(4)	$\exists\, q, r \in [0,1]\,(q \veebar_\alpha r \neq 1)$	{N∀: 3}
(5)	$q_0 \veebar_\alpha r_0 \neq 1$	{−∃: 4}
(6)	$p_0 \Rightarrow_\alpha q_0 = 1$	{−∀: 1}
(7)	$p_0 \veebar_\alpha r_0 = 1$	{−∀: 2}
(8)	$p_0 \leq q_0$	{df.'Ł-$\Rightarrow_\alpha$': 6}
(9)	$q_0^\alpha + r_0^\alpha < 1$	{df. '$\veebar_\alpha$': 5}
(10)	$p_0^\alpha + r_0^\alpha \geq 1$	{df. '$\veebar_\alpha$': 7}
(11)	$q_0^\alpha + r_0^\alpha \geq 1$	{Since $q_0 \geq p_0$ then $q_0^\alpha \geq p_0^\alpha$ and $-q_0^\alpha \leq -p_0^\alpha$. Hence, $1 - q_0^\alpha \leq 1 - p_0^\alpha \leq r_0^\alpha$: 8,10}
	contr. □	{9,11}

The above two proofs of T2.15 are very similar, in fact the first eight proof lines are identical. In accordance with our considerations, as an example, the following derived rules have been obtained (the corresponding proofs are omitted: the used assumptional system style is very similar to this one shown in the proof of T2.15):

$$-\mathrm{C}: \frac{\begin{matrix}\varphi \Rightarrow_\alpha \psi\\ \varphi\end{matrix}}{\psi},\quad +\mathrm{K}: \frac{\begin{matrix}\varphi\\ \psi\end{matrix}}{\varphi \,\&_\alpha\, \psi},\quad -\mathrm{K}: \frac{\varphi \,\&_\alpha\, \psi}{\varphi \backslash \psi \backslash \begin{matrix}\varphi\\ \psi\end{matrix}},\quad +\mathrm{A}: \frac{\varphi}{\varphi \veebar_\alpha \psi},\quad -\mathrm{A}: \frac{\begin{matrix}\varphi \veebar_\alpha \psi\\ \sim\varphi\end{matrix}}{\psi},\quad +\mathrm{E}: \frac{\begin{matrix}\varphi \Rightarrow_\alpha \psi\\ \psi \Rightarrow_\alpha \varphi\end{matrix}}{\varphi \Leftrightarrow_\alpha \psi},$$

$$-\mathrm{E}: \frac{\varphi \Leftrightarrow_\alpha \psi}{\varphi \Rightarrow_\alpha \psi \backslash \psi \Rightarrow_\alpha \varphi \backslash \begin{matrix}\varphi \Rightarrow_\alpha \psi\\ \psi \Rightarrow_\alpha \varphi\end{matrix}},\quad \mathrm{Toll}: \frac{\begin{matrix}\varphi \Rightarrow_\alpha \psi\\ \sim\psi\end{matrix}}{\sim\varphi},\quad \mathrm{CC}: \frac{\varphi \Rightarrow_\alpha \psi}{\sim\psi \Rightarrow_\alpha \sim\varphi},\quad \mathrm{NA}: \frac{\sim(\varphi \veebar_\alpha \psi)}{\sim\varphi \backslash \sim\psi \backslash \begin{matrix}\sim\varphi\\ \sim\psi\end{matrix}},$$

$$\text{NK}: \frac{\sim(\varphi \;\&_\alpha\; \psi)}{\sim\varphi \;\veebar_\alpha \sim\psi}, \qquad \text{TC}: \frac{\begin{matrix}\varphi \Rightarrow_\alpha \psi \\ \psi \Rightarrow_\alpha \chi\end{matrix}}{\varphi \Rightarrow_\alpha \chi}, \qquad \text{MC}: \frac{\begin{matrix}\varphi \Rightarrow_\alpha \psi \\ \varphi \Rightarrow_\alpha \chi\end{matrix}}{\varphi \Rightarrow_\alpha \psi \;\&_\alpha\; \chi}, \qquad -\text{N}_\alpha: \frac{\sim\sim\varphi}{\varphi}, \qquad +\text{N}_\alpha: \frac{\varphi}{\sim\sim\varphi},$$

$$\text{SR}: \frac{\varphi \Leftrightarrow_\alpha \psi}{\chi \Leftrightarrow_\alpha \chi(\varphi // \psi)},$$

where $\chi(\varphi // \psi)$ is obtained from χ by the replacement of its parts ψ by the formula φ.

The rules NA, NK and ± N are direct consequence of the involutivity property, which is by definition satisfied for any continuous fuzzy negation (Bronstein I.N. et al. 2001). The above derived rules are satisfied as well as in first and in second order Ł$_\alpha$-BL. An illustration of using these rules are the next proofs. The proof of Thesis 2.15 is first presented. It is of the same complexity as in the classical case.

Proof of T2.15:

(1)	$p \Rightarrow_\alpha q$	{1,2 / a}
(2)	$p \veebar_\alpha r$	
(3)	$\sim(q \veebar_\alpha r)$	{aip}
(4)	$\sim q$	{4,5 / NA: 3}
(5)	$\sim r$	
(6)	p	{– A : 2,5}
(7)	q	{– C : 1,6}
	contr. □	{4,7}

Thesis 2.16 (*law of multiplication of consequents of two or more implications having the same antecedents: MC*)

$(p \Rightarrow_\alpha q) \;\&_\alpha\; (p \Rightarrow_\alpha r) \Leftrightarrow_\alpha (p \Rightarrow_\alpha q \;\&_\alpha\; r)$

In accordance with the rule of omitting t-equivalence, i.e. – E, the following two implications have to be proven.

Thesis 2.16a (*if-implication*)

$(p \Rightarrow_\alpha q) \;\&_\alpha\; (p \Rightarrow_\alpha r) \Rightarrow_\alpha (p \Rightarrow_\alpha q \;\&_\alpha\; r)$

Thesis 2.16b (*only-if-implication*)

$(p \Rightarrow_\alpha q \;\&_\alpha\; r) \Rightarrow_\alpha (p \Rightarrow_\alpha q) \;\&_\alpha\; (p \Rightarrow_\alpha r)$

Proof of T2.16a:

(1)	$p \Rightarrow_\alpha q$	
(2)	$p \Rightarrow_\alpha r$	{1,2 / a}
(3)	p	
(4)	$\sim(q \;\&_\alpha\; r)$	{aip}
(5)	q	{– C : 1,3}
(6)	r	{– C : 2,3}
(7)	$q \;\&_\alpha\; r$	{+ K : 5,6}
	contr. □	{4,7}

The proof of T2.16b is left to the reader. It can be observed that some rules satisfied in the classical Ł-BL may be or not satisfied in the generalised $Ł_\alpha$-BL depending on the used system version. For example, CR and NC which are satisfied in Ł-BL, are not satisfied in the first-order $Ł_\alpha$-BL, but they are satisfied in the second order $Ł_\alpha$-BL, e.g. the left side and the right side of the formula $\sim (p \Rightarrow_\alpha q) \Leftrightarrow_\alpha p \,\&_\alpha \sim q$ are different in the first-order $Ł_\alpha$-BL for $p = 1/2$, $q = 1/3$, and $\alpha = 2$: $1 - \sqrt{31}/6$ and 0, respectively.

An illustration of a more complete use of the above introduced proof rules is given in the proof of the next thesis. Here, a construction of ramified indirect proof with joined additional assumptions is presented (Słupecki J. and Borkowski L. 1967).

Thesis 2.17 (*law of converting implications, called also law of a closed system of theorems* or *Hauber´s law*)

$(p \Rightarrow_\alpha q) \,\&_\alpha (r \Rightarrow_\alpha s) \,\&_\alpha (p \veebar_\alpha r) \,\&_\alpha \sim (q \,\&_\alpha s) \Rightarrow_\alpha (q \Rightarrow_\alpha p) \,\&_\alpha (s \Rightarrow_\alpha r)$

Proof / second order $Ł_\alpha$-BL:

(1)	$p \Rightarrow_\alpha q$	
(2)	$r \Rightarrow_\alpha s$	
(3)	$p \veebar_\alpha r$	{1,2,3,4,5 / a}
(4)	$\sim (q \,\&_\alpha s)$	
(5)	$\sim ((q \Rightarrow_\alpha p) \,\&_\alpha (s \Rightarrow_\alpha r))$	{aip}
(6)	$q \,\&_\alpha \sim p \;\veebar_\alpha\; s \,\&_\alpha \sim r$	{NK, NC, SR : 5}
(1.1)	q	{ada}
(1.2)	$\sim p$	
(1.3)	r	{– A : 3,1.2}
(1.4)	s	{– C : 2,1.3}
(1.5)	$q \,\&_\alpha s$	{+ K : 1.1,1.4}
	contr.	{4,1.5}
(2.1)	s	{ada}
(2.2)	$\sim r$	
(2.3)	p	{– A : 3,2.2}
(2.4)	q	{– C : 1,2.3}
(2.5)	$q \,\&_\alpha s$	{+ K : 2.1,2.4}
	contr. □	{4,2.5}

It can be observed that in accordance with MC, it is sufficient to prove the following two implications.

$(p \Rightarrow_\alpha q) \,\&_\alpha (r \Rightarrow_\alpha s) \,\&_\alpha (p \veebar_\alpha r) \,\&_\alpha \sim (q \,\&_\alpha s) \Rightarrow_\alpha (q \Rightarrow_\alpha p)$

$(p \Rightarrow_\alpha q) \,\&_\alpha (r \Rightarrow_\alpha s) \,\&_\alpha (p \veebar_\alpha r) \,\&_\alpha \sim (q \,\&_\alpha s) \Rightarrow_\alpha (s \Rightarrow_\alpha r)$

The proof of the first implication is given below. The proof of the second one is very similar and hence it is omitted.

Proof:

(1)	$p \Rightarrow_\alpha q$	
(2)	$r \Rightarrow_\alpha s$	
(3)	$p \veebar_\alpha r$	{1,2,3,4,5 / a}

(4) $\sim (q \,\&_\alpha\, s)$
(5) q
(6) $\sim p$ {aip}
(7) $\sim q \veebar_\alpha \sim s$ {NK : 4}
(8) $\sim s$ {– A : 5,7}
(9) $\sim r$ {Toll : 2,8}
(10) p {– A : 3,9}
contr. □ {6,10}

In accordance with Proposition 2.7, any formula which is satisfied in (the first or second order) Ł$_\alpha$-BL is also satisfied in Ł-BL and hence provable by using the Hájek's axiomatic approach, but not vice versa.

The above-introduced approach from assumptions seems to be more attractive, more simpler and natural in practical use than the axiomatic one and it is related to the infinite valued Łukasiewicz's system Ł-BL, recognised as one of the most important basic fuzzy propositional logics (in common with Gödel's and product logic systems). The use of Yager's continuous fuzzy negation seems to be the most natural extension of Ł-BL. In application the obtained t-connectives have very similar properties as the classical ones. And hence, we have a new possibility of extending the classical notion of logical consequence in terms of the obtained generalised Łukasiewicz's system Ł$_\alpha$-BL. In fact, with any fuzzy propositional formula in Ł$_\alpha$-BL, which is a logical consequence of a finite set of another such formulae, can be associated some "crisp" formula by using a threshold value (e.g. 0.5, as in fuzzy sets). And so, we have the possibility of introducing a new generation control systems using fuzzy logic and corresponding fuzzy rules in senso stricto. Moreover, the obtained Ł$_\alpha$-BL is directly related to binary operations in multiple-valued algebras. Any methodological aspects, e.g. such as compactness, consistency, decidability or satisfiability of t-tautologies have been omitted in this research. Obviously, the most important of these properties is the consistency of Ł$_\alpha$-BL. Several areas for future investigations of Ł$_\alpha$-BL may be also related to approximative reasoning and automated theorem proving, and also extension of the Hájek's axiomatic Ł-BL system to Ł$_\alpha$-BL. In particular, Gentzen's sequents can be easily introduced as derived rules in our system.

The notion of *fuzzy flip-flop* was first introduced in (Hirota K. and Ozawa K. 1989): in particular, there was also considered a group of *bounded fuzzy operations*. The last operations were also used in (Diamond J. et al. 1994). As an example, it is shown below an application of the above generalised Łukasiewicz's fuzzy t-norm / -conorm in the area of modelling and synthesis of control systems using fuzzy interpreted Petri nets (Gniewek L. and Kluska J. 1998, 2004), (Gniewek L. 2012).The presented here transformation procedure (net into logic circuit) is based on fuzzy flip-flops using bounded fuzzy operations: there were introduced four such JK flip-flops, of type SA, AA, AB and SB.

Example 2.22 (*fuzzy JK flip-flop of type SA*)*

Let $\hat{\otimes}$, $\hat{\oplus}$ and $'$ be the generalised Łukasiewicz's fuzzy t-norm, t-conorm and Yager's fuzzy negation, where: $x \hat{\otimes} y =_{df} \max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha}$, $x \hat{\oplus} y =_{df} \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$ and $x' =_{df} (1 - x^\alpha)^{1/\alpha}$ (for any $x, y \in [0,1]$ and $\alpha \in (0, \infty)$). The following generalisation of the first version SA of the fuzzy JK flip-flop can be obtained: $Y = [(J \hat{\otimes} y') \hat{\oplus} y] \hat{\otimes} (K' \hat{\oplus} y')$, where the *boundary values* of the primary inputs of this flip-flop are as follows: $Y = \max\{J^\alpha, y^\alpha\}$, for $K = 0$ and $Y = \min\{(K')^\alpha, y^\alpha\}$, for $J = 0$. A proof is given below.

For simplicity, let $a =_{df} J \hat{\otimes} y'$, $b =_{df} a \hat{\oplus} y$ and $c =_{df} K' \hat{\oplus} y'$ (and hence: $Y = b \hat{\otimes} c$). And so, we can obtain:

$$a = \max\{0, J^\alpha + (y')^\alpha - 1\}^{1/\alpha}$$

* Instead of Q_{n+1} and Q, the next and present states are here denoted by Y and y, respectively.

$$= \quad \max\{0, J^\alpha - y^\alpha\}^{1/\alpha}.$$

$$b = a \hat{\oplus} y$$
$$= \min\{1, a^\alpha + y^\alpha\}^{1/\alpha}$$
$$= \min\{1, \max\{0, J^\alpha - y^\alpha\} + y^\alpha\}^{1/\alpha}.$$

$$c = K' \hat{\oplus} y'$$
$$= \min\{1, (K')^\alpha + (y')^\alpha\}^{1/\alpha}$$
$$= \min\{1, 2 - K^\alpha - y^\alpha\}^{1/\alpha}.$$

$$Y = b \hat{\otimes} c$$
$$= \max\{0, b^\alpha + c^\alpha - 1\}^{1/\alpha}$$
$$= \max\{0, \min\{1, \max\{0, J^\alpha - y^\alpha\} + y^\alpha\} + \min\{1, 2 - K^\alpha - y^\alpha\} - 1\}^{1/\alpha}.$$

Assume that $K = 0$. Since $Y/_{K=0} = \max\{0, \min\{1, \max\{0, J^\alpha - y^\alpha\} + y^\alpha\} + \min\{1, 2 - y^\alpha\} - 1\}$, the following equality should be shown. In fact, The left side and the right side coincide for: $J = y$, $J < y$, and $J > y$.

$$\max\{0, \min\{1, \max\{0, J^\alpha - y^\alpha\} + y^\alpha\} + \min\{1, 2 - y^\alpha\} - 1\} = \max\{J^\alpha, y^\alpha\}$$

Let $J = y$. Then (the left side) $L = \max\{0, \min\{1, y^\alpha\} + \min\{1, 2 - y^\alpha\} - 1\}$. Since $y \le 1$ then $y^\alpha \le 1^\alpha = 1$ and $1 - y^\alpha \ge 0$. Hence $2 - y^\alpha \ge 1$. So $\min\{1, 2 - y^\alpha\} = 1$ and $L = \max\{0, y^\alpha + 0\} = y^\alpha$. Since (the right side) $R = y^\alpha$ then $L = R$.

If $J < y$ then $J < y$ then $J^\alpha < y^\alpha$ and hence $R = y^\alpha$. Since $J^\alpha - y^\alpha < 0$ and $2 - y^\alpha \ge 1$, we can obtain: $L = \max\{0, y^\alpha + 1 - 1\} = y^\alpha = R$.

Let now $J > y$. We have: $J^\alpha - y^\alpha > 0$, $2 - y^\alpha \ge 1$ and $R = J^\alpha$. Hence: $L = \max\{0, \min\{1, J^\alpha - y^\alpha + y^\alpha\} + 1 - 1\} = J^\alpha = R$.

Assume now $J = 0$. We have: $Y/_{J=0} = \max\{0, \min\{1, y^\alpha\} + \min\{1, 2 - K^\alpha - y^\alpha\} - 1\}^{1/\alpha} = \max\{0, y^\alpha + \min\{1, 2 - K^\alpha - y^\alpha\} - 1\}^{1/\alpha}$. And hence, the following equality should be shown.

$$\max\{0, y^\alpha + \min\{1, 2 - K^\alpha - y^\alpha\} - 1\} = \min\{(K')^\alpha, y^\alpha\}$$

By definition, $(K')^\alpha =_{df} ((1 - K^\alpha)^{1/\alpha})^\alpha = 1 - K^\alpha$. And hence, It is easily to show the left side and the right side coincide for: $1 - K^\alpha = y^\alpha$, $1 - K^\alpha < y^\alpha$, and $1 - K^\alpha > y^\alpha$.

Let $1 - K^\alpha = y^\alpha$. Then $1 - K^\alpha - y^\alpha = 0$ and $L = \max\{0, y^\alpha + \min\{1, 1 + 1 - K^\alpha - y^\alpha\} - 1\} = \max\{0, y^\alpha + 1 - 1\} = y^\alpha = R$.

If $1 - K^\alpha < y^\alpha$ then $R = 1 - K^\alpha$. Since $1 - K^\alpha - y^\alpha < 0$ then $1 + (1 - K^\alpha - y^\alpha) < 1$ and $\min\{1, 1 + (1 - K^\alpha - y^\alpha)\} = 2 - K^\alpha - y^\alpha$. Hence $L = \max\{0, y^\alpha + 2 - K^\alpha - y^\alpha\} - 1\} = 1 - K^\alpha = R$.

Assume now $1 - K^\alpha > y^\alpha$. We have: $1 - K^\alpha - y^\alpha > 0$ and $2 - K^\alpha - y^\alpha > 1$. Hence: $L = \max\{0, y^\alpha + \min\{1, 2 - K^\alpha - y^\alpha\} - 1\} = \max\{0, y^\alpha + 1 - 1\} = y^\alpha = R$.

The above cited fuzzy JK flip-flop of type SA becomes a particular case with $\alpha = 1$. And so, there exists a possibility of using more *flexible* such control systems (i.e. easily changed to suit new conditions: Oxford dictionary). □

In general, among the most important examples of t-norms is the Łuasiewicz's one. It have been introduced a new generalised such t-norm which implies some new applications, e.g. such as: fuzzy sets, fuzzy rough sets, intuitionistic fuzzy sets and intuitionistic fuzzy rough sets, fuzzy similarity and fuzzy equivalence relations, specification of new probabilistic metric space (by using as a definition of the corresponding triangle function the above introduced t-norm), and so on. Several areas for future investigations may be related to the introduction of new t-norm based measures or also computations related to the probability of fuzzy events, specification of new commutative and associative copulas, new possibilities to combine criteria in multicriteria decision making (for evaluation the truth degrees of compound formulae), new kind of fuzzy t-equivalence relation and hence new possibilities of using this relation in image processing (fuzzy segmentation and clusterisation), artificial intelligence (approximative reasoning and automated theorem proving using Gentzen's sequents, fuzzy consequence), and so on. Another area of application seems to be the fault isolation in discrete event systems with partially degradable components or fuzzy control in the narrow sense, in particular *interval type-2 fuzzy sets and systems* (Mendel J.M. 2001)*: e.g. instead of the *algebraic product t-norm*, using the generalised Łuasiewicz's one (concerning the notion of *firing interval*), etc. And so, essentially most remains to be done.

2.3. Modal, deontic and temporal logics

A *modal* is an expression (like ´necessarily´ or ´possibly´) that is used to qualify the truth of a judgement. Modal logic is ,strictly speaking, the study of the deductive behaviour of the expressions ´it is necessary that´ and ´it is possible that´. However, the term '*modal logic*' may be used more broadly for a family of related systems. These include logics for belief, for tense and other temporal expressions, for the deontic (moral) expressions such as ' it is obligatory ' and ' it is permitted ' , etc. *Temporal logic* (sometimes used to refer to *tense logic*) is a deduction system in which the times at which propositions bear certain truth-values can be indicated, in which the ´tense´ of the assertion can be indicated, and in which truth-values can be affected by the passage of time (*public domain*). A brief introduction to the modal, deontic and temporal logic systems is given below†.

Modal logic‡

Some preliminary study concerning modal logic have been observed in ancient times (e.g. Aristotelian syllogistic, Stoic School or also Megarian School of philosophy: Diodorus Cronus' ideas related to "strict" or "strong" implication, died c.284 b.c., etc.) or also in the middle ages (e.g. John Duns Scotus 1266 – 1308, William of Ockham 1288 – 1348, etc.). In antiquity and in the middle ages modality was understood as the truth value of a proposition: it can be necessarily, actually, or possibly true, e.g. (Sulkunen P. and Törrönen J. 1997): in nowadays called "*alethic*" (from the Greek word *alētheia*: refers to the various modalities of truth, such as necessity, possibility or impossibility) or sometimes also "*special*" modality (from the Latin "species"). However, contemporary theory of modal logic systems is related to the notion of strict implication introduced by Lewis C.I. (1883 – 1964) and inspired by the earlier work of McCool H. (1837 – 1909). The above strict implication system was extended in the next work of Lewis C.I. and Langford C.H. (1932).

In general, the main approach in constructing of modal logic systems is the axiomatic one. In fact, there were proposed several different systems, e.g. such as: S1 – S5 systems§ (Lewis C.I. and Langford C.H., Becker O.),

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* Jerry M. Mendel, born 1938.

† For a more information see: *The little encyclopaedia of logic* (1988) or *Formal logic. Encyclopedical outline with applications to informatics and linguistics* (1987) or also T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

‡ The sequent calculus becomes elusive for such logics as: *modal logics*, *intermediate* (i.e. *consistent superintuitionistic*) *logics* and *substructural logics* (logics lacking one of the usual structured rules such as: *weakening, contraction*, *exchange* or *associativity*, e.g. two of the more significant such logics are *relevance logic* and *linear logic*), see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*. To treat such systems, a *hypersequent calculus* should be used. In particular, hypersequents can be also used to obtain analytic calculi for various *fuzzy logic systems* As an illustration, see Subsection 4.1.

§ „S": an abbreviation from „Strict".

T - (called also M -) system (Feys R., Wright G.H. von.), K. Gödel's system, McKinsey J.C.C. and Tarski A., Kripke S., and so on. In particular, the set of axioms in S1 is a proper subset wrt S2, S4 and S5. As an example, a version of S5 is shown below (Lewis C.I. 1918, new edition 1960: Clarence Irving Lewis 1883 – 1964). A8 was introduced by Becker O. (1930: Oskar Becker 1889 – 1964) and this axiom corresponds to the Gödel's axiom G3 (see the next considerations). Here ´ $\Rightarrow>$ ´ and ´ $\Diamond$ ´ denote strict implication and the modal functor of possibility, respectively.

(A1) $p \wedge q \Rightarrow> q \wedge p$
(A2) $p \wedge q \Rightarrow> p$
(A3) $p \Rightarrow> p \wedge p$
(A4) $(p \wedge q) \wedge r \Rightarrow> p \wedge (q \wedge r)$
(A5) $p \Rightarrow> \sim(\sim p)$
(A6) $(p \Rightarrow> q) \wedge (q \Rightarrow> r) \Rightarrow> (p \Rightarrow> r)$
(A7) $p \wedge (p \Rightarrow> q) \Rightarrow> q$
(A8) $\sim \Diamond \sim p \Rightarrow> \sim \Diamond \sim(\sim \Diamond \sim p)$

A new approach to S4 was introduced in *Gödel's system*, i.e. to any classical propositional calculus there are added the following three axioms* (for any modal propositional formulae φ and ψ, see Definition 2.6 given below) :

(G1) $\Box \varphi \Rightarrow \varphi$,

(G2) $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box \varphi \Rightarrow \Box \psi)$ and

(G3) $\Box \varphi \Rightarrow \Box \Box \varphi$.

Moreover, the following *Gödel's rule* was assumed (*necessitation*):

(GR) $\models \varphi \Rightarrow \models \Box \varphi$.

As in the classical logic, the above axiomatic approach uses rules – C and RR, but extended also with GR. Here, the Gödel's axiom (G2) represents the familiar *distribution axiom* schema. A more formal treatment is omitted (see *Encyclopedical outline with applications to informatics and linguistics* 1987).

The set of constants of the classical propositional calculus can be extended by introducing the following two new connectives related to the above considered two expressions: $\Box$ (*box*) and $\Diamond$ (*diamond*) , i.e. the *modal functors* (called also "*operators*") *of necessity* and *possibility* (sometimes also known as: *potentiality*), respectively: $\Diamond$ can be considered as a *dual modal functor* wrt $\Box$. It can be observed that any of the last two connectives can be expressed by the another one, i.e. $\Box \varphi \Leftrightarrow_{df} \sim \Diamond \sim \varphi$ and $\Diamond\varphi \Leftrightarrow_{df} \sim \Box \sim \varphi$. The *strict implication* and *strict equivalence* can be introduced as follows: $p \Rightarrow> q \Leftrightarrow_{df} \Box(p \Rightarrow q)$ and $p <\Leftrightarrow> q \Leftrightarrow_{df} \Box(p \Leftrightarrow q)$. The next inductive definition is a generalization of the classical one (see Definition 1.1).

Definition 2.6

A *modal propositional formula* is:

1. Any propositional variable,

* Axioms (G1) and (G2), sometimes denoted by ´ T ´ and ´ K ´ respectively, are sufisient for introduction of the so called *standard modal logic system*. For convenience, this rule we shall also denote by ´- $\Box$´.

2. If φ and ψ are some modal propositional formulae, then such formulae are also: $\sim(\varphi)$, $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $(\varphi) \Rightarrow (\psi)$, $(\varphi) \Leftrightarrow (\psi)$, $\Box(\varphi)$, $\Diamond(\varphi)$, $(\varphi) \Rightarrow> (\psi)$, and $(\varphi) <\Leftrightarrow> (\psi)$,
3. Every modal propositional formula in this propositional calculus either is a propositional variable or is formed from propositional variables by a single or multiple application of rule (2).

The main purpose of this calculus ic the same as in the classical case.

Let φ be the implication $\varphi_1 \Rightarrow \varphi_2$. According to GR, we have: $\vDash (\varphi_1 \Rightarrow \varphi_2) \Rightarrow \vDash \Box(\varphi_1 \Rightarrow \varphi_2)$. Next, by using the above introduced notion of strict implication, SR and TC (transitivity for implication) we can obtain: $\vDash \varphi_1 \Rightarrow> \varphi_2$. And hence, the following corollary is satisfied.

Corollary 2.5

$\vDash (\varphi_1 \Rightarrow \varphi_2) \Rightarrow \vDash (\varphi_1 \Rightarrow> \varphi_2)$. □

Corollary 2.6

$\vDash (\varphi_1 \Leftrightarrow \varphi_2) \Rightarrow \vDash (\varphi_1 <\Leftrightarrow> \varphi_2)$

Proof:

It can be observed that any event requiring the necessity of a conjunction 'p $\wedge$ q' can be considered as an equivalent to the event requiring the necessities of each of its arguments p and q. In fact, next we shall assume that: $\Box(p \wedge q) \Leftrightarrow \Box p \wedge \Box q$ (see T 2.23 given below). And this observation can be generalised for any two formulae φ and ψ.

According to the notion of strict equivalence, we can obtain: $p <\Leftrightarrow> q \Leftrightarrow_{df} \Box(p \Leftrightarrow q)$. Next, by using two times SR and the corresponding definitions of strict implication and strict equivalence we have: $\Box(p \Leftrightarrow q) \Leftrightarrow \Box((p \Rightarrow q) \wedge (q \Rightarrow p))$. And hence: $\Box(p \Leftrightarrow q) \Leftrightarrow \Box(p \Rightarrow q) \wedge \Box(q \Rightarrow p)$. Then: $p <\Leftrightarrow> q \Leftrightarrow (p \Rightarrow> q) \wedge (q \Rightarrow> p)$.

Assume now that $\vDash (\varphi_1 \Rightarrow \varphi_2) \Rightarrow \vDash (\varphi_1 \Rightarrow> \varphi_2)$ and $\vDash (\varphi_2 \Rightarrow \varphi_1) \Rightarrow \vDash (\varphi_2 \Rightarrow> \varphi_1)$ are two implications under Corollary 2.5. Then, in accordance with the law of multiplication of the antecedents and consequents of two implications (MAC: see T 1.21, Subsection 1.3), SR and the corresponding definitions for equivalence and strict equivalence connectives we can obtain: $\vDash (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_2 \Rightarrow \varphi_1) \Rightarrow \vDash (\varphi_1 \Rightarrow> \varphi_2) \wedge (\varphi_2 \Rightarrow> \varphi_1)$. And hence, we have this corollary. □

Let now φ be arbitrary. By G1 it follows that $\Box \sim \varphi \Rightarrow \sim \varphi$. And hence, in accordance with the law of transposition or contraposition of implication CC (i.e. T 1.14, Subsection 1.3), SR and the definition of the modal functor of possibility we can obtain: $\varphi \Rightarrow \sim \Box \sim \varphi \Leftrightarrow \varphi \Rightarrow \Diamond\varphi$. By Corollary 2.5 it follows that $\varphi \Rightarrow> \Diamond\varphi$ is a thesis. Moreover, by definition of the above two modal functors, the law of transposition or contraposition of equivalence CE, – N and SR we have: $\sim \Box \varphi \Leftrightarrow_{df} \Diamond \sim \varphi$ and $\sim \Diamond\varphi \Leftrightarrow_{df} \Box \sim \varphi$.

In particular, the following rules are used below.

(1) *Rule of omitting a necessity modal functor**
(denoted below by ´- □´):

$$- \Box : \frac{\Box \varphi}{\varphi}$$

* In accordance with *Gödel's* axiom G1.

(2) *Rule of joining a possibility modal functor*
(denoted below by ´+ ◇´):

$$+\Diamond : \quad \frac{\varphi}{\Diamond\varphi}$$

(3) *Rule of negating a necessity modal functor*
(denoted below by ´ N □´):

$$N\Box : \quad \frac{\sim \Box\, \varphi}{\Diamond \sim \varphi}$$

(4) *Rule of negating a possibility modal functor*
(denoted below by ´ N◇´):

$$N\Diamond : \quad \frac{\sim \Diamond\, \varphi}{\Box \sim \varphi}$$

(5) *Rule of joining a strict implication*
(denoted below by ´+ SI´):

$$+ SI : \quad \frac{\vDash (\varphi_1 \Rightarrow \varphi_2)}{\vDash (\varphi_1 \Rightarrow\!> \varphi_2)}$$

(6) *Rule of joining a strict equivalence*
(denoted below by ´+ SE´):

$$+ SE : \quad \frac{\vDash (\varphi_1 \Leftrightarrow \varphi_2)}{\vDash (\varphi_1 <\!\Leftrightarrow\!> \varphi_2)}$$

It can be observed that the above two rules + SI and + SE are directly related to the Gödel’s rule GR. Another way of joining and/or omitting a strict equivalence are the following two rules (see the above Corollary 2.6). The obtained rules are very similar to the classical ones and hence they are said to be *ordinary*.

(7) *Ordinary rule of joining a strict equivalence*
(denoted below by ´+ OSE´):

$$+ OSE : \quad \frac{\begin{array}{c}\varphi \Rightarrow\!> \psi \\ \psi \Rightarrow\!> \varphi\end{array}}{\varphi <\!\Leftrightarrow\!> \psi}$$

(8) *Ordinary rule of omitting a strict equivalence*
(denoted below by ´– OSE´):

$$- OSE : \quad \frac{\varphi <\!\Leftrightarrow\!> \psi}{\varphi \Rightarrow\!> \psi \,/\, \psi \Rightarrow\!> \varphi \,/\, \begin{array}{c}\varphi \Rightarrow\!> \psi \\ \psi \Rightarrow\!> \varphi\end{array}}$$

It is illustrated below the use of the assumptional system style. As in the classical propositional calculus, the obtained theses related to some propositional variables, e.g. p, q, r, etc. are generalised for arbitrary modal propositional formulae, e.g. φ, ψ, χ, etc., respectively. Moreover, by rules + SI and + SE it follows that it is sufficiently to show the corresponding proofs for the usual, i.e. non-strict connectives. And such an approach is realised below, e.g. see T 2.22.

The following simple property is satisfied.

Thesis 2.18

$\Box p \Rightarrow \Diamond p$

Proof:

(1)	$\Box p$	{a}
(2)	p	$\{- \Box :\ 1\}$
	$\Diamond p$. □	$\{+ \Diamond :\ 2\}$

Thesis 2.19

$(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Diamond q)$

Proof:

(1)	$p \Rightarrow q$	{1,2 / a}
(2)	$\Box p$	
(3)	p	$\{- \Box :\ 2\}$
(4)	q	$\{- C : 1,3\}$
	$\Diamond q$. □	$\{+ \Diamond :\ 4\}$

Assume that $\models \varphi \Rightarrow \psi$, where $\varphi \Leftrightarrow_{df} \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$. In accordance with the Gödel's rule GR , we can obtain $\models \Box (\varphi \Rightarrow \psi)$. And hence, by G2 and − C it follows that $\models \Box \varphi \Rightarrow \Box \psi$. Since $\Box (\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \Leftrightarrow \Box \varphi_1 \wedge \Box \varphi_2 \wedge \ldots \wedge \Box \varphi_n$ then by SR it follows that $\models \Box \varphi_1 \wedge \Box \varphi_2 \wedge \ldots \wedge \Box \varphi_n \Rightarrow \Box \psi$. And hence by using + SI we have: $\models \Box \varphi_1 \wedge \Box \varphi_2 \wedge \ldots \wedge \Box \varphi_n \Rightarrow\!> \Box \psi$. And so, as in the classical case (T 1.23, Subsection 1.5), the notion of modal logical consequence may be introduced, i.e. ψ may be considered as a *modal logical consequence* wrt $\varphi_1, \varphi_2, \ldots, \varphi_n$. And then, any classical logical consequence involves some modal logical consequence (a more formal treatment is omitted).

As an illustration of the assumptional system style, some example theses and corresponding proofs related to the T–system* are presented below. Obviously, any thesis of the classical propositional calculus belongs to this system. And so, the following formulae are theses in this system (see: *The little encyclopaedia of logic* 1988).

Thesis 2.20

$\Box p \Rightarrow\!> p$. □

Thesis 2.21

$p \Rightarrow\!> \Diamond p$. □

Thesis 2.22

$\Box p \Rightarrow\!> \Diamond p$. □ {T 2.18, + SI}

Thesis 2.23

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* Robert Feys (1889 – 1961)

$\Box(p \wedge q) <\Leftrightarrow> \Box p \wedge \Box q$. ▫

Thesis 2.24

$\Diamond(p \vee q) <\Leftrightarrow> \Diamond p \vee \Diamond q$. ▫

Since $\wedge$ and $\vee$ are associative the last two theses can be generalised for more than two, but a finite number, arguments. Moreover, in accordance with T 2.23 and T 2.24, the modal functors $\Box$ and $\Diamond$ are *distributive* wrt the logical connectives $\wedge$ and $\vee$, respectively. An example proof of T 2.24 by using T 2.23 is given below. In a similar way, the proof of T 2.23 can be realised by using T 2.24. In accordance with the duality property one of these two theses can be obtained from the another by using the rule CE (contraposition of equivalence: left to the reader). In particular, e.g. the formulae: ' $\Box(p \wedge q) \Leftrightarrow \Box p \wedge \Box q$ ' and ' $\Diamond(p \vee q) \Leftrightarrow \Diamond p \vee \Diamond q$ ' are also theses in Feys' T-system *.

Proof T 2.24a:

(1)	$\Diamond(p \vee q)$	{a}
(2)	$\sim(\Diamond p \vee \Diamond q)$	{aip}
(3)	$\Box \sim p \wedge \Box \sim q$	{NA, N$\Diamond$, SR : 2}
(4)	$\Box(\sim p \wedge \sim q)$	{T 2.23 : 3}
(5)	$\Box \sim(p \vee q)$	{NA, SR : 4}
(6)	$\sim \Diamond(p \vee q)$	{ N$\Diamond$: 5}
	contr. ▫	{1,6, + SI}

Proof T 2.24b:

(1)	$\Diamond p \vee \Diamond q$	{a}
(2)	$\sim \Diamond(p \vee q)$	{aip}
(3)	$\Box(\sim p \wedge \sim q)$	{ N$\Diamond$, NA, SR : 2}
(4)	$\Box \sim p \wedge \Box \sim q$	{T 2.23 : 3}
(5)	$\sim \Diamond p$	{ N$\Diamond$, SR, – K : 4}
(6)	$\sim \Diamond q$	
(7)	$\Diamond q$	{– A : 1,5}
	contr. ▫	{6,7, + SI, + SE}

Thesis 2.25

$\Box p \vee \Box q \Rightarrow> \Box(p \vee q)$. ▫

It is obvious that the possibility of a conjunction of two or more (a finite number) arguments implicates the conjunction of the possibilities of its arguments, i.e. by CC, NK and SR it follows that a conjunction '$p \wedge q$' is not possible if it is not possible at least one of its arguments. In fact, the following necessary condition holds.

* Provided there is no ambiguity and for simplicity, it is assumed that any rule corresponding to a formula having as a main connective logical equivalence is *two-sided binding* (i.e. from top to down and from bottom to up, e.g. NA, NK, N $\Box$, N$\Diamond$, etc. Another possibility was considered in Subsection 1.3 where the "from bottom to up" case was interpreted by the prefix '+', e.g. + NA, + NK, + N $\Box$, + N$\Diamond$, etc.). Moreover, the used priorities for logical connectives are similar as in the classical case, see Subsection 1.1, e.g. the symbol of disjunction binds more strongly than the symbol of strict implication, the last binds more strongly than the symbol of strict equivalence, etc.

Thesis 2.26

$\Diamond(p \wedge q) \Rightarrow> \Diamond p \wedge \Diamond q.$ □

T 2.25 can be used as a derived rule in the proof of T 2.26, and vice versa. An example proof of T 2.25 using T 2.26 is given below (the proof of T 2.26 by using T 2.25 is left to the reader).

Proof T 2.25:

(1)	$\Box p \vee \Box q$	{a}
(2)	$\sim \Box (p \vee q)$	{aip}
(3)	$\Diamond (\sim p \wedge \sim q)$	{ N □, NA, SR : 2}
(4)	$\Diamond \sim p \wedge \Diamond \sim q$	{– C : T 2.26, 3}
(5)	$\Diamond \sim p$	{– K : 4}
(6)	$\Diamond \sim q$	
(7)	$\sim \Box p$	{N □ : 5}
(8)	$\sim \Box q$	{N □ : 6}
(9)	$\Box q$	{– A : 1,7}
	contr. □	{8,9, + SI}

It is obvious that any thesis gives a new derived rule, e.g. the following derived rules follow from T 2.22 - T 2.26.

(9) *Rule of changing a necessity modal functor into possibility modal functor*
(denoted below by ´ □ /◇´):

$$\Box/\Diamond: \quad \frac{\Box \varphi}{\Diamond \varphi}$$

(10) *Rule of exchanging a necessity of conjunction by conjunction of necessities*
(denoted below by ´ □ K´):

$$\Box K: \quad \frac{\Box(\varphi \wedge \psi)}{\Box \varphi \,/\, \Box \psi \,/\, \begin{matrix}\Box \varphi \\ \Box \psi\end{matrix}}$$

(11) *Rule of exchanging a possibility of disjunction by disjunction of possibilities*
(denoted below by ´ ◇A´):

$$\Diamond A: \quad \frac{\Diamond(\varphi \vee \psi)}{\Diamond \varphi \vee \Diamond \psi}$$

(12) *Rule of disjunction of necessities*
(denoted below by ´ A□´):

$$A\Box: \quad \frac{\Box \varphi \vee \Box \psi}{\Box(\varphi \vee \psi)}$$

(13) *Rule of possibility of conjunction*
(denoted below by ´ ◇K´):

$$\Diamond K: \quad \frac{\Diamond(\varphi \wedge \psi)}{\Diamond\varphi \;/\; \Diamond\psi \;/\; \Diamond\varphi \atop \Diamond\psi}$$

By the *law of idempotence*: p ∨ p ⇔ p. Hence, according to CR , ' – N ' and SR we have: ~ p ⇒ p ⇔ p. And so, by using SR and ' + SE ' the following thesis can be obtained.

Thesis 2.27

□ p <⇔> □ (~ p ⇒ p). ▫

Thesis 2.28

(p ⇒> q) ∧ p ⇒> q

Proof:

It is sufficient to show the following implication: (p ⇒> q) ∧ p ⇒ q.

(1)	p ⇒> q	{a}
(2)	p	
(3)	□ (p ⇒ q)	{df. '⇒>': 1}
(4)	p ⇒ q	{– □ : 3}
	q. ▫	{– C : 2,4; + SI}

The next two theses (T 2.30 and T 2.31) belonging to the Feys' T–system are related to the axiom G2 introduced in Gödel's system. And hence, the proof of this axiom is first given below.

Thesis 2.29 (G2)

□ (p ⇒ q) ⇒ (□ p ⇒ □ q)

Proof:

(1)	□ (p ⇒ q)	{a}
(2)	□ p	
(3)	~ □ q	{aip}
(4)	□ (p ⇒ q) ∧ □ p	{+ K : 1,2}
(5)	□ ((p ⇒ q) ∧ p)	{□ K : 4}
(6)	□ (p ∧ q}	{⊨ ((p ⇒ q) ∧ p ⇔ p ∧ q) , SR : 5}
(7)	□ q	{□ K : 6}
	contr. ▫	{3,7}

Thesis 2.30

(p ⇒> q) ∧ □ p ⇒> □ q

Proof:

(1)	p ⇒> q	{a}
(2)	□ p	
(3)	□ (p ⇒ q)	{df. '⇒>': 1}
(4)	□ p ⇒ □ q	{– C : T 2.29, 3}

$\Box$ q. $\square$ {– C : 2,4; + SI}

Thesis 2.31

$(p \Rightarrow\!> q) \wedge \Diamond \sim q \Rightarrow\!> \Diamond \sim p$

Proof:

(1)	$p \Rightarrow\!> q$	{a}
(2)	$\Diamond \sim q$	
(3)	$\sim \Diamond \sim p$	{aip}
(4)	$\Box (p \Rightarrow q)$	{df. '⇒>': 1}
(5)	$\Box p \Rightarrow \Box q$	{– C : T 2.29, 4}
(6)	$\Box p$	{ N◇ , – N, SR : 3}
(7)	$\Box q$	{– C : 5,6}
(8)	$\sim \Box q$	{ N□ : 2}
	contr. $\square$	{7,8; + SI}

The above theses T 2.30 and T 2.31 can be considered as equivalent (wrt the laws of exportation and importation T 1.12, the law of transposition or contraposition of implication T 1.14 and SR (see Subsection 1.3). In fact, T 2.31 can be transformed as follows: $(p \Rightarrow\!> q) \wedge \sim \Box q \Rightarrow\!> \sim \Box p$, left to the reader).

Thesis 2.32

$(p \Rightarrow\!> q) \wedge \sim q \Rightarrow\!> \sim p$

Proof:

(1)	$p \Rightarrow\!> q$	{a}
(2)	$\sim q$	
(3)	p	{aip}
(4)	$\Box (p \Rightarrow q)$	{df. '⇒>': 1}
(5)	$p \Rightarrow q$	{– □ : 4}
(6)	q	{– C : 3,5}
	contr. $\square$	{2,6; + SI}

The following derived rules can be obtained (according to T 2.28, T 2.29, T 2.31 and T 2.32).

(14) Rule of *detachment for strict implication* (or *omitting a strict implication*) (denoted below by ´– SI´):

$$- SI : \quad \frac{\varphi \Rightarrow\!> \psi \qquad \varphi}{\psi}$$

(15) *Rule of necessity of implication* (denoted below by ´□ C´):

$$\Box C : \quad \frac{\Box (\varphi \Rightarrow \psi)}{\Box \varphi \Rightarrow \Box \psi}$$

(16) *Modal necessity tollens* (denoted below by ´ □-Toll ´):

$$\Box\text{-Toll}: \quad \frac{\varphi \Rightarrow> \psi \qquad \sim \Box\, \psi}{\sim \Box\, \varphi}$$

(17) *Strict implication tollens* (denoted below by ´ S-Toll ´):

$$\text{S-Toll}: \quad \frac{\varphi \Rightarrow> \psi \qquad \sim \psi}{\sim \varphi}$$

Thesis 2.33

$(p \Rightarrow> q) \wedge (q \Rightarrow> r) \Rightarrow> (p \Rightarrow> r)$

Proof:

(1)	$p \Rightarrow> q$	{a}
(2)	$q \Rightarrow> r$	
(3)	$\Box(p \Rightarrow q)$	{df. '⇒>': 1}
(4)	$\Box(q \Rightarrow r)$	{df. '⇒>': 2}
(5)	$\Box((p \Rightarrow q) \wedge (q \Rightarrow r))$	{□ K : 3,4}
(6)	$\Box((p \Rightarrow r) \wedge \zeta)$	{SR : 5, i.e. it is assumed that there exists some formula ζ such that the formulae $(p \Rightarrow q) \wedge (q \Rightarrow r)$ and $(p \Rightarrow r) \wedge \zeta$ are equivalent, see below}
(7)	$\Box(p \Rightarrow r)$	{□ K : 6}
	$p \Rightarrow> r$. □	{df. '⇒>': 7, + SI}

The construction of ζ is related to the notion of interpretation (see: Definitions 1.3 and 1.4 of Subsection 1.4). Let now $\varphi =_{df} (p \Rightarrow q) \wedge (q \Rightarrow r)$ and $\psi =_{df} (p \Rightarrow r) \wedge \zeta$. According to T 2.33, the new introduced formula ζ should satisfy the following conditions: (1) ζ should not be equivalent to the previous ones, i.e φ and '$p \Rightarrow r$', (2) ζ should depend on the same propositional variables, i.e. p, q and r, and (3) φ and ψ should be equivalent. In fact, by using the Karnaugh's map method* different such formulae can be obtained, e.g. $\zeta_1 =_{df} (p \Leftrightarrow q) \vee q \wedge r$ or $\zeta_2 =_{df} (q \Leftrightarrow r) \vee \sim p \wedge r$ or also $\zeta_3 =_{df} \sim p \wedge r \vee p \wedge q \vee \sim(q \vee r)$, etc. (left to the reader).

Thesis 2.34

$(p \wedge q \Rightarrow> r) \Rightarrow> (p \Rightarrow> (q \Rightarrow r))$

Proof:

(1)	$p \wedge q \Rightarrow> r$	{a}
(2)	$\Box(p \wedge q \Rightarrow r)$	{df. '⇒>': 1}
(3)	$\Box(p \Rightarrow (q \Rightarrow r))$	{T 1.12 : 2, i.e. laws of exportation and importation: see Subsection 1.3 }
	$p \Rightarrow> (q \Rightarrow r)$. □	{df. '⇒>': 3, + SI}

Thesis 2.35

* Maurice Karnaugh: 1953, a refinement of Veitch's diagram: Edward W. Veitch: 1952 (e.g. see Breuer M.A. and Friedman A.D. 1977).

$\Box p \Rightarrow> (q \Rightarrow> p)$

Proof:

(1)	$\Box p$	{a}
(2)	$\sim (q \Rightarrow> p)$	{aip}
(3)	$\sim \Box (q \Rightarrow p)$	{df. '⇒>', SR : 2}
(4)	$\Diamond(q \wedge \sim p)$	{ N □ , NC, SR : 3}
(5)	$\Diamond \sim p$	{◇K : 4}
(6)	$\sim \Box p$	{N □ : 5}
	contr. □	{1,6; + SI}

Thesis 2.36

$\Box \sim p \Rightarrow> (p \Rightarrow> q)$

Proof:

(1)	$\Box \sim p$	{a}
(2)	$\sim (p \Rightarrow> q)$	{aip}
(3)	$\sim \Box (p \Rightarrow q)$	{df. '⇒>', SR : 2}
(4)	$\Diamond(p \wedge \sim q)$	{ N □ , NC, SR : 3}
(5)	$\Diamond p$	{◇K : 4}
(6)	$\sim \Diamond p$	{ N◇ : 1}
	contr. □	{5,6; + SI}

The following (Gödel's system G3) axiom can be introduced to Feys' T–system *: $\Box p \Rightarrow \Box \Box p$. And hence, the following *thesis of reduction* holds: $\Box \Box p \Leftrightarrow \Box p$ (left to the reader). In accordance with CE (i.e. law of transposition or contraposition of equivalence: see T 1.14 and the next text, Subsection 1.3), we can equivalently obtain: $\sim \Box \Box p \Leftrightarrow \sim \Box p$. Then: $\Diamond \Diamond \sim p \Leftrightarrow \Diamond \sim p$. Since p is an arbitrary proposition we have: $\Box \Box \varphi \Leftrightarrow \Box \varphi$ and $\Diamond \Diamond \sim \varphi \Leftrightarrow \Diamond \sim \varphi$.

And hence, from T 2.18 it follows that the following two formulae are theses: $\Box \Diamond p \Rightarrow \Diamond \Diamond p$ and $\Box \Box \sim p \Rightarrow \Diamond \Diamond \sim p$ (left to the reader). In fact, the following two rules can be introduced.

(18) *Rule of reduction of necessity* (denoted below by ´R □´):

$$R \Box : \quad \frac{\Box \Box \varphi}{\Box \varphi}$$

(19) *Rule of reduction of possibility* (denoted below by ´R◇´):

$$R \Diamond : \quad \frac{\Diamond \Diamond \sim \varphi}{\Diamond \sim \varphi}$$

* *The little encyclopaedia of logic* (1988)

The following thesis is satisfied.

Thesis 2.37

$\Box\,(p \Rightarrow> p)$

Proof:

(1)	$\sim \Box\,(p \Rightarrow> p)$	{aip}
(2)	$\Diamond \sim (p \Rightarrow> p)$	{ N$\Box$: 1}
(3)	$\Diamond \sim \Box\,(p \Rightarrow p)$	{df. '$\Rightarrow>$', SR : 2}
(4)	$\Diamond\Diamond \sim (p \Rightarrow p)$	{ N$\Box$, SR : 3}
(5)	$\Diamond \sim (p \Rightarrow p)$	{ R$\Diamond$: 4}
(6)	$\sim \Box\,(p \Rightarrow p)$	{ N$\Box$: 5}
(7)	$\Box\,(p \Rightarrow p)$	{GR, – C : since $\models (p \Rightarrow p)$, i.e. law of identity for implication}
	contr. □	{6,7}

The proofs of the next three theses are left to the reader: Theses 2.38 and 2.39 are known as *strict implication paradoxes*. Another one is also the following: $p \Rightarrow> (q \Rightarrow> q)$*. In fact, since $p \Rightarrow (q \Rightarrow q)$ is a thesis, in accordance with ´+ SI´, we have: $p \Rightarrow> (q \Rightarrow q)$. The remaining part of this proof is left to the reader.

Thesis 2.38

$p \wedge \sim p \Rightarrow> q$. □ {+ SI}

Thesis 2.39

$p \Rightarrow> q \vee \sim q$. □ {+ SI}

Thesis 2.40

$(p \Rightarrow> q) \Rightarrow> (\Box\, p \Rightarrow \Box\, q)$. □ {T 2.29, + SI}

Some additional theses directly related to the above-considered Feys' T–system are given below.

Thesis 2.41

$(p \Rightarrow> q) \wedge (p \Rightarrow> r) <\Leftrightarrow> p \Rightarrow> q \wedge r$

Proof T 2.41a:

(1)	$p \Rightarrow> q$	{a}
(2)	$p \Rightarrow> r$	
(3)	$\Box\,(p \Rightarrow q)$	{df. '$\Rightarrow>$': 1}
(4)	$\Box\,(p \Rightarrow r)$	{df. '$\Rightarrow>$': 2}
(5)	$\Box\,((p \Rightarrow q) \wedge (p \Rightarrow r))$	{$\Box$K : 3,4}
(6)	$\Box\,(p \Rightarrow q \wedge r)$	{MC, SR : 5, i.e. *law of multiplication of consequents* T 1.5: see Subsection 1.3}
	$p \Rightarrow> q \wedge r$. □	{df. '$\Rightarrow>$': 6, + SI, + SE}

* It can be observed that there are no propositional variables in common on the left and on the right sides of the corresponding main implications.

Proof T 2.41b:

(1)	$p \Rightarrow> q \wedge r$	{a}
(2)	$\Box (p \Rightarrow q \wedge r)$	{df. '$\Rightarrow>$': 1}
(3)	$\Box ((p \Rightarrow q) \wedge (p \Rightarrow r))$	{MC, SR : 2, i.e. *law of multiplication of consequents* T 1.5: see Subsection 1.3}
(4)	$\Box (p \Rightarrow q)$	{$\Box$ K : 3}
(5)	$\Box (p \Rightarrow r)$	
(6)	$p \Rightarrow> q$	{df. '$\Rightarrow>$': 4}
(7)	$p \Rightarrow> r$	{df. '$\Rightarrow>$': 5}
	$(p \Rightarrow> q) \wedge (p \Rightarrow> r)$. □	{+ K : 6,7; + SI}

The proof of the next thesis is very similar to the previous one. And hence, it is left to the reader.

Thesis 2.42

$(p \Rightarrow> r) \wedge (q \Rightarrow> r)$ <⇔> $p \vee q \Rightarrow> r$. □

Thesis 2.43

$(p \Rightarrow> \Box q) \wedge (q \Rightarrow> \Box r) \Rightarrow> (p \Rightarrow \Box r)$

Proof:

(1)	$p \Rightarrow> \Box q$	
(2)	$q \Rightarrow> \Box r$	{a}
(3)	p	
(4)	$\Box (p \Rightarrow \Box q)$	{df. '$\Rightarrow>$': 1}
(5)	$\Box (q \Rightarrow \Box r)$	{df. '$\Rightarrow>$': 2}
(6)	$p \Rightarrow \Box q$	{− □ : 4}
(7)	$q \Rightarrow \Box r$	{− □ : 5}
(8)	$\Box q$	{− C : 3,6}
(9)	q	{− □ : 8}
	$\Box r$. □	{− C : 7,9; + SI}

Thesis 2.44

$(\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond (p \Rightarrow q)$

Proof:

(1)	$\Diamond p \Rightarrow \Diamond q$	{a}
(2)	$\sim \Diamond (p \Rightarrow q)$	{aip}
(3)	$\Box (p \wedge \sim q)$	{ N◇ , NC, SR : 2}
(4)	$\Box p$	{□ K : 3}
(5)	$\Box \sim q$	
(6)	$\Diamond p$	{□ /◇: 4}
(7)	$\Diamond q$	{− C : 1,6}
(8)	$\sim \Diamond q$	{ N◇ : 5}
	contr. □	{7,8}

Thesis 2.45

$(p \Rightarrow> \Diamond q) \wedge \sim \Diamond q \Rightarrow> \Diamond \sim p$

Proof:

(1)	$p \Rightarrow> \Diamond q$	{a}
(2)	$\sim \Diamond q$	
(3)	$\sim \Diamond \sim p$	{aip}
(4)	$\Box (p \Rightarrow \Diamond q)$	{df. '⇒>': 1}
(5)	$p \Rightarrow \Diamond q$	{– □ : 4}
(6)	$\Box p$	{N◇, – N, SR : 3}
(7)	p	{– □ : 6}
(8)	$\Diamond q$	{– C : 5,7}
	contr. □	{2,8; + SI}

The following *De Morgan's laws* are satisfied.

Thesis 2.46

$\sim \Box (p \wedge q) <\Leftrightarrow> \sim \Box p \vee \sim \Box q$

Proof:

$\sim \Box (p \wedge q)$	$\Leftrightarrow$	$\Diamond (\sim p \vee \sim q)$	{N □ , NK, SR}
	$\Leftrightarrow$	$\Diamond \sim p \vee \Diamond \sim q$	{◇A}
	$\Leftrightarrow$	$\sim \Box p \vee \sim \Box q$. □	{N□ , SR, + SE}

Thesis 2.47

$\sim \Diamond (p \vee q) <\Leftrightarrow> \sim \Diamond p \wedge \sim \Diamond q$. □

The proof of T 2.47 is similar to the proof of the previous thesis and it is left to the reader. The next rules follow directly from the last considered theses (T 2.38, T 2.39, T 2.41, T 2.42, T 2.44 – T 2.47).

(20) *Strict rule of Duns Scotus* (denoted below by ´SDS´):

$$\text{SDS} : \frac{\varphi \quad \sim \varphi}{\psi}$$

(21) *Contrapositive strict rule of Duns Scotus* (denoted below by ´CSDS´):

$$\text{CSDS} : \frac{\varphi}{\psi \vee \sim \psi}$$

(22) *Strict law of multiplication of consequents* (denoted below by ´SMC´):

$$\text{SMC} : \frac{\varphi \Rightarrow> \psi \quad \varphi \Rightarrow> \chi}{\varphi \Rightarrow> \psi \wedge \chi}$$

(23) *Strict law of law of addition of antecedents* (denoted below by ´SAA´):

$$\text{SAA}: \quad \frac{\varphi \Rightarrow> \chi \qquad \psi \Rightarrow> \chi}{\varphi \vee \psi \Rightarrow> \chi}$$

(24) *Rule of implication of possibilities* (denoted below by ´C◇´):

$$\text{C}\Diamond: \quad \frac{\Diamond\varphi \Rightarrow \Diamond\psi}{\Diamond(\varphi \Rightarrow \psi)}$$

(25) *Modal possibility tollens* (denoted below by ´◇-Toll´):

$$\Diamond\text{-Toll}: \quad \frac{\varphi \Rightarrow> \Diamond\psi \qquad \sim\Diamond\psi}{\Diamond\sim\varphi}$$

(26) *Modal De Morgan´s law for necessity of conjunction* (denoted below by ´N□K´):

$$\text{N}\Box\text{K}: \quad \frac{\sim\Box(\varphi \wedge \psi)}{\sim\Box\varphi \vee \sim\Box\psi}$$

(27) *Modal De Morgan´s law for possibility of disjunction* (denoted below by ´N◇A´):

$$\text{N}\Diamond\text{A}: \quad \frac{\sim\Diamond(\varphi \vee \psi)}{\sim\Diamond\varphi \,/\, \sim\Diamond\psi \,/\, \sim\Diamond\varphi \quad \sim\Diamond\psi}$$

There exist two approaches concerning semantics of modal logic systems: *algebraic semantics* and *Kripke's semantics* (called also *relational* or *frame semantics*: see Kripke S. 1959, 1963, 1965)*. Since the most of research is related to the second approach, a short review is given below.

Let consider the following system $\mathscr{K} =_{df} (K ; g ; \rho)$, where $K \neq \varnothing$ is the set of all possible "*worlds*", $g \in K$ is said to be "*real world*", and the binary relation $\rho \subseteq K \times K$ is said to be an *accessibility relation*, where $h_1 \rho h_2$ iff the world h_2 is possible wrt h_1 (i.e. any proposition which is satisfied in h_2 should be possible in h_1, for any $h_1, h_2 \in K$). In accordance with the law of contraposition of implication, any proposition which is not possible in h_1 should not be satisfied in h_2.

Let $h \in K$ and φ be a formula which is satisfied in h. Then φ is possible in h. And hence, it is assumed that the above relation ρ is *reflexive* in K, i.e. $h \rho h$ (for any $h \in K$). Next we shall assume that K is partitioned into two disjoint and non-empty subsets of *normal* and *non-normal* (or *quarrel*) *worlds*, denoted by N and Q, respectively. Any such system $\mathscr{K}$ is said to be a *Kripke's frame* (or a *modal frame*). It can be observed that contradictions are excluded in normal worlds, but in the case of non-normal worlds all becomes be possible. The need of introducing non-normal worlds follows from the analysis of some modal logic systems where the Gödel's rule GR fails.

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* The models introduced by S. Kripke were oriented to some modal propositional calculi (Saul Aaron Kripke, born 1940). An earlier similar constructions concerning deontic logics was presented by Kanger S. (1957).

Let P be the *set of modal propositional formulae* obtained under Definition 2.6. The *Kripke's model* is introduced as a pair $(\mathcal{K}, f)$, where $f : P \times K \rightarrow \{0,1\}$ is an arbitrary map. We shall say that f is a *modeling function for* $\varphi \in P$ *wrt* $\mathcal{K}$ and that φ is *satisfied in* $h \in K$ iff $f(\varphi, h) = 1$. A formula φ is a *thesis in Kripke's model* $(\mathcal{K}, f)$ iff $f(\varphi, h) = 1$ (for any $h \in K$). And this formula is a *thesis in Kripke's frame* $\mathcal{K}$ iff it is a thesis in $(\mathcal{K}, f)$ (for all possible choices of f). In a natural manner the last two notions can be extended for classes of such frames or models by assuming that φ is a thesis in any member of the corresponding class. The *contents of the frame* $\mathcal{K}$ is introduced as the set of all formulae satisfied in any $h \in K$ and for all possible choices of f.

The values of f for compound formulae can be defined by an inductive way, e.g. $f(\sim \varphi, h) = 1$ iff $f(\varphi, h) = 0$, $f(\varphi \Rightarrow \psi, h) = 1$ iff $f(\varphi, h) = 0 \vee f(\psi, h) = 1$, $f(\varphi \wedge \psi, h) = 1$ iff $f(\varphi, h) = 1 \wedge f(\psi, h) = 1$, etc.

Next by $h_1{\bullet} =_{df} \{ h_2 \in K \ / \ h_1 \rho h_2 \}$ we shall denote *the subset of all worlds out-incident to* $h_1 \in K$. We shall say that $\varphi \in P$ *is necessary in the world* h_1, i.e. $f(\Box \varphi, h_1) = 1$ iff $\forall h_2 \in h_1{\bullet} (f(\varphi, h_2) = 1)$. In a similar way, $\varphi \in P$ *is possible in the world* h_1, i.e. $f(\Diamond \varphi, h_1) = 1$ iff $\exists h_2 \in h_1{\bullet} (f(\varphi, h_2) = 1)$.

There exists a possibility of obtaining some different Kripke's frames depending on the required properties for ρ, e.g. by assuming a restriction of ρ to N and $g \in N$ we can obtain the *S2 - Kripke's frame* (related to the Lewis' system S2) or also *S4 -* and *S5 - Kripke's frames*, assuming that the accessibility relation ρ is transitive and equivalence, respectively, etc.

The above presented notion of thesis, related to a given modal logic system, allows for introducing the *completeness problem* associated with any such system (this is omitted here).

The above presented semantics can be generalised for *fuzzy modal systems* by assuming as a codomain of f the whole closed interval [0,1]. The last may be topic for a further research.

There exist very many applications in modal logic field, e.g. such as: agentive and situational applications, terminological logic, topological applications and complex systems, model constructions in set theory (such as: forcing and non-wellfounded set theory), unravelling, etc. One of them is related to the notion of *grounding in communications*[*], first introduced by Clark H.H. and Brennan S.E. (1991). And this notion was used in the case of grounding modal language in communications of artificial cognitive agent systems (Katarzyniak R.P. 2007). Here, the analysis of grounding requirements was given in the case of logic equivalences extended with modal functors of possibility, belief and knowledge (a more formal treatment is omitted).

The two basic modal functors of necessity and possibility are used below in the case of constructing deontic logic systems.

Deontic logic

Deontic logic is a deduction system in which theses appear logical constants such as: '*it is obligatory that*' (or '*it ought to be the case that*'), '*it is permitted* (or *permissible*) *that*' and '*it is forbidden* (or *prohibited*) *that*'. The term '*deontic'* is derived from ancient Greek "δεον", roughly meaning "*that which is binding or proper*".

This deduction system is a formal study of the *normative concepts* of *obligation*, *permission* and *prohibition* represented by the above logical constants. These concepts and their logical relationships to one another are distinguished from *value concepts* such as: goodness and badness (or evil) as well as from such *agent-based concepts* as: act, choice, decision, freedom, and will. And hence, deontic logic is not itself an *ethical theory* that tells us what in fact is permitted, obligatory or forbidden, but it is or should be a part of such a theory. A complete ethical or moral theory would encompass the logic of all these different concepts and not just the normative ones. And so, with any ethical theory should be associated some deontic logic system. The study of different deontic logics is a part of *metaethics*. Moreover, the normative concepts of obligation and permission are similar in many

[*] Grounding in communication involves mutual knowledge, mutual beliefs, and mutual assumptions, which are necessary for communication between two people: "They cannot even begin to coordinate on content without assuming a vast amount of shared information or common ground – that is, mutual knowledge, mutual beliefs, and mutual assumptions. And to coordinate on process, they need to update their common ground moment by moment".

respects to the modal concepts of necessity and possibility. In fact, the above normative concepts are also modal ones (Cocchiarella N.B., in the late 1960s).

Some logical relations based on deontic concepts have been observed in ancient times, e.g. *Indian Mimamsa school* as a part of Hindu philosophy (' Mīmāṃsā ' derived from the Sanskrit word* ' investigation ' and having as a central aim the elucidation of the nature of '*dharma*', understood as a set ritual obligations and prerogatives to be performed properly and hence this nature is not accessible to reason or observation…), similar constructions have been observed in Ancient Greece, also in the late middle ages (a comparison of deontic concepts with the alethic ones, e.g. the logical relations between the notions *licitum*, *illicitum*, *debitum*, and *indifferens* wrt the notions *possible impossible*, *necessarium*, and *contingens*, respectively: Gottfried Wilhelm Leibniz 1646 – 1716), etc. However, the first modern formal system of deontic logic was proposed by Mally E. (1926: Ernst Mally 1879 – 1944). In the Mally's deontic system were introduced two logical constants, one unary and two binary connectives. But this system led to some counterintuitive results. (Sart F. 2009). In fact, such earlier attempts were fragmentary. And so, the first viable system of deontic logic was introduced by von Wright (1951: Georg Henrik von Wright 1916 – 2003). The formula validation in the last system bears many resemblances with Wittgenstein's truth table method (Wittgenstein L. 1922: Ludwig Wittgenstein 1889 – 1951). As in modal logic systems, the general approach in constructing deontic logic systems is the axiomatic one. An example deontic system is considered below.

The notion of deontic propositional formula can be introduced in a similar way as in the case of modal logic. We have the following definition. Provided there is no ambiguity, below the deontic *functors of obligation* (O), *permission* (P) and *forbiddance* (F) are denoted by (*the Mally's unary connective*) $!$, δ and σ, respectively.

Definition 2.7

A *deontic propositional formula* is:

1. Any propositional variable,
2. If φ and ψ are some propositional formulae, then such formulae are also: $\sim(\varphi)$, $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $(\varphi) \Rightarrow (\psi)$, and $(\varphi) \Leftrightarrow (\psi)$, $!(\varphi)$, $\delta(\varphi)$, and $\sigma(\varphi)$,
3. Every propositional formula in this propositional calculus either is a propositional variable or is formed from propositional variables by a single or multiple application of rule (2).

The main purpose of this calculus is the same as in the classical case.

In the next considerations we shall use some deontic constant (i.e. 0-ary modal operator), standing for some *sanction* or related to some *violation* (e.g. bad thing, prohibition, conflict situation, etc.) and denoted by asterisk ($\ast$). By using this constant we have a possibility of defining the above unary functor $!$ in terms of the alethic modal functor $\Box$ (Anderson A.R. 1958)†:

$$!\varphi \Leftrightarrow_{df} \Box(\sim \varphi \Rightarrow \ast),$$

for any deontic propositional formula φ.

The following two definitions are also used below.

$$\delta\varphi \Leftrightarrow_{df} \Diamond(\varphi \wedge \sim \ast) \quad \text{and}$$

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* *Sanskrit* (*saṃskṛtā vāk*: "refined speech"), is a historical Indo-Aryan language and the primary liturgical language of Hinduism and Buddhism as laid out in the grammar of *Pānini,* around the 4th century b.c. Its position in the cultures of the Greater India is similar to that of Latin and Greek in Europe and it has significantly influenced most modern languages of the Indian subcontinent, particularly in India, Pakistan and Nepal. Today, it is listed as one of the 22 scheduled languages of India and is an official language of the state of Uttarakhand (The *Free Encyclopaedia, The Wikimedia Foundation, Inc*.).

† Alan Ross Anderson (1925 – 1973)

$\sigma\varphi \Leftrightarrow_{df} \sim \delta\varphi$.

There are a number of possible choices that one can make in regard to what deontic logic to adopt. Much will depend on various metaethical considerations. The axiomatic approach in any deontic logic system is restricted to the use of the following three inference rules: RR, - C and also the following implication (*obligation rule*).

(OBR) $\vDash \varphi \Rightarrow \vDash !\varphi$,

i.e. that what is provable in deontic logic is obligatory (Cocchiarella N.B., in the late 1960s)*.

Let $\varphi =_{df} \psi \Rightarrow \chi$. According to OBR, the following derived rules can be obtained.

(D1) $\vDash (\psi \Rightarrow \chi) \Rightarrow \vDash (!\psi \Rightarrow !\chi)$,

(D2) $\vDash (\psi \Rightarrow \chi) \Rightarrow \vDash (\delta\psi \Rightarrow \delta\chi)$,

(D3) $\vDash (\psi \Rightarrow \chi) \Rightarrow \vDash (\sigma\chi \Rightarrow \sigma\psi)$ and

(D4) $\vDash (\psi \Leftrightarrow \chi) \Rightarrow \vDash (!\psi \Leftrightarrow !\chi)$.

The first two derived rules are directly related to theses T 2.55 and T 2.59 given below. The next rule is related to SR and the law of contraposition of implication CC.

Let $\psi \Leftrightarrow \chi$ be a thesis. Hence, by '– E' it follows that $\vDash (\psi \Rightarrow \chi)$ and $\vDash (\chi \Rightarrow \psi)$. According to D1, we have: $\vDash (\psi \Rightarrow \chi) \Rightarrow \vDash (!\psi \Rightarrow !\chi)$ and $\vDash (\chi \Rightarrow \psi) \Rightarrow \vDash (!\chi \Rightarrow !\,\psi)$. And then, by using MAC (rule of multiplication of the antecedents and consequents of two implications), '+ E' and SR we have D4 (a more formal treatment is omitted).

The assumptional system style is used in the next considerations. Some example theses and their corresponding proofs are given. Moreover, the following formula is accepted, said to be a *basic axiom* (*of negating the twofold standards*).

$\sim !(\varphi \wedge \sim \varphi)$.

And so, contradictory obligations are not allowed. Any iteration of deontic constants, e.g. such as !!p, !σp, !(!p ⇒ p), etc. is omitted below. In fact, such iterations may lead to difficult interpretation problems. The following two theses are satisfied (! and δ are *mutually dual* and any of these two connectives can be expressed by the another one).

Thesis 2.48

$\sim !p \Leftrightarrow \delta \sim p$

Proof:

$\sim !p$	$\Leftrightarrow$	$\sim \Box (\sim p \Rightarrow *)$	{df. '!', SR}
	$\Leftrightarrow$	$\Diamond \sim (\sim p \Rightarrow *)$	{N □}
	$\Leftrightarrow$	$\Diamond (\sim p \wedge \sim *)$	{NC, SR}
	$\Leftrightarrow$	$\delta \sim p$. □	{df. 'δ'}

Thesis 2.49

* But not vice versa, in fact it can be observed the opposite implication may involve some speculative inference.

$\sim \delta p \Leftrightarrow\ ! \sim p$

Proof:

$$
\begin{aligned}
\sim \delta p &\Leftrightarrow \sim \Diamond (p \wedge \sim *) && \{\text{df. `}\delta\text{', SR}\} \\
&\Leftrightarrow \Box (\sim p \vee *) && \{\text{N}\Diamond\text{, NK, – N, SR}\} \\
&\Leftrightarrow \Box (p \Rightarrow *) && \{\text{CR, SR}\} \\
&\Leftrightarrow\ ! \sim p. \ \Box && \{\text{df. `!'}\}
\end{aligned}
$$

In accordance with T 2.48 and T 2.49, the following two rules can be obtained.

(28) *Rule of negating a deontic functor of obligation* (denoted below by ´N!´):

$$\text{N!}: \quad \frac{\sim !\varphi}{\delta \sim \varphi}$$

(29) *Rule of negating a deontic functor of permission* (denoted below by ´Nδ´):

$$\text{N}\delta: \quad \frac{\sim \delta\varphi}{! \sim \varphi}$$

Thesis 2.50

$!(p \wedge q) \Leftrightarrow\ !p \wedge !q$

Proof:

$$
\begin{aligned}
!(p \wedge q) &\Leftrightarrow \Box (\sim (p \wedge q) \Rightarrow *) && \{\text{df. `!'}\} \\
&\Leftrightarrow \Box (\sim p \vee \sim q \Rightarrow *) && \{\text{NK, SR}\} \\
&\Leftrightarrow \Box ((\sim p \Rightarrow *) \wedge (\sim q \Rightarrow *)) && \{\text{AA, SR}\} \\
&\Leftrightarrow \Box (\sim p \Rightarrow *) \wedge \Box (\sim q \Rightarrow *) && \{\Box \text{K}\} \\
&\Leftrightarrow\ !p \wedge !q. \ \Box && \{\text{df. `!'}\}
\end{aligned}
$$

Thesis 2.51

$\delta (p \vee q) \Leftrightarrow \delta p \vee \delta q$

Proof:

$$
\begin{aligned}
\delta (p \vee q) &\Leftrightarrow \Diamond ((p \vee q) \wedge \sim *) && \{\text{df. `}\delta\text{'}\} \\
&\Leftrightarrow \Diamond (p \wedge \sim * \vee q \wedge \sim *) && \{\wedge \text{ is distributive over } \vee\} \\
&\Leftrightarrow \Diamond (p \wedge \sim *) \vee \Diamond (q \wedge \sim *) && \{\Diamond \text{A}\} \\
&\Leftrightarrow \delta p \vee \delta q. \ \Box && \{\text{df. `}\delta\text{'}\}
\end{aligned}
$$

Since $\wedge$ and $\vee$ are associative T 2.50 and T 2.51 can be generalised for more than two, but a finite number, arguments.

Thesis 2.52

$!p \vee !q \Rightarrow\ !(p \vee q)$

Proof:

$!p \lor !q$	$\Leftrightarrow$	$\Box(\sim p \Rightarrow *) \lor \Box(\sim q \Rightarrow *)$	{df. '!', SR}
	$\Rightarrow$	$\Box((\sim p \Rightarrow *) \lor (\sim q \Rightarrow *))$	{A□}
	$\Leftrightarrow$	$\Box(\sim(p \lor q) \Rightarrow *)$	{AA, SR}
	$\Leftrightarrow$	$!(p \lor q)$. □	{df. '!'}

Thesis 2.53

$\delta(p \land q) \Rightarrow \delta p \land \delta q$

Proof:

$\delta(p \land q)$	$\Leftrightarrow$	$\Diamond(p \land q \land \sim *)$	{df. 'δ' associativity for ∧, SR}
	$\Leftrightarrow$	$\Diamond((p \land \sim *) \land (q \land \sim *))$	{associativity and commutativity for ∧, idempotence for '∼ ∗', SR}
	$\Rightarrow$	$\Diamond(p \land \sim *) \land \Diamond(q \land \sim *)$	{◇K}
	$\Leftrightarrow$	$\delta p \land \delta q$. □	{df. 'δ'}

Thesis 2.54

$!p \Rightarrow \delta p$

Proof:

(1)	$!p$	{a}
(2)	$\sim \delta p$	{aip}
(3)	$! \sim p$	{N δ : 2}
(4)	$!p \land ! \sim p$	{+ K : 1,3}
(5)	$!(p \land \sim p)$	{T 2.50 : 4}
(6)	$\sim !(p \land \sim p)$	{is an axiom}
	contr. □	{5,6}

The following inference rules can be obtained (T 2.50 – T 2.54).

(30) *Rule of exchanging an obligation of conjunction by conjunction of obligations*
(denoted below by ´ ! K´):

$$!K : \quad \frac{!(\varphi \land \psi)}{!\varphi \,/\, !\psi \,/\, \begin{matrix} !\varphi \\ !\psi \end{matrix}}$$

(31) *Rule of exchanging a permission of disjunction by disjunction of permissions*
(denoted below by ´ δA´):

$$\delta A : \quad \frac{\delta(\varphi \lor \psi)}{\delta\varphi \lor \delta\psi}$$

(32) *Rule of disjunction of obligations*
(denoted below by ´ A!´):

$$A! : \frac{!\varphi \lor !\psi}{!(\varphi \lor \psi)}$$

(33) *Rule of permission of conjunction* (denoted below by ´δK´):

$$\delta K : \frac{\delta(\varphi \land \psi)}{\delta\varphi \,/\, \delta\psi \,/\, \begin{matrix}\delta\varphi \\ \delta\psi\end{matrix}}$$

(34) *Rule of changing an obligation deontic functor into permission deontic functor* (denoted below by ´ !/δ ´):

$$!/\delta : \frac{!\varphi}{\delta\varphi}$$

The deontic form of the Gödel's axiom (G2) is presented as follows.

Thesis 2.55

$!(p \Rightarrow q) \Rightarrow (!p \Rightarrow !q)$

Proof:

(1)	$!(p \Rightarrow q)$	{a}
(2)	$!p$	
(3)	$!(p \Rightarrow q) \land !p$	{+ K : 1,2}
(4)	$!((p \Rightarrow q) \land p)$	{!K : 3}
(5)	$!(p \land q)$	{$\models ((p \Rightarrow q) \land p \Leftrightarrow p \land q)$, SR : 4}
	$!q$. □	{!K : 5}

Thesis 2.56

$(\delta p \Rightarrow \delta q) \Rightarrow \delta(p \Rightarrow q)$

Proof:

(1)	$\delta p \Rightarrow \delta q$	{a}
(2)	$\sim \delta(p \Rightarrow q)$	{aip}
(3)	$! \sim (p \Rightarrow q)$	{ N δ : 2}
(4)	$!(p \land \sim q)$	{NC, SR : 3}
(5)	$!p$	{!K : 4}
(6)	$! \sim q$	
(7)	$\sim \delta q$	{ N δ : 6}
(8)	δp	{!/δ : 5}
(9)	δq	{– C : 1,8}
	contr. □	{7,9}

In accordance with the laws of exportation and importation, T 2.55 and the next thesis are equivalent (see T 1.12 of Subsection 1.3). By the way, another two proofs of T 2.57 are given below.

Thesis 2.57

$!(p \Rightarrow q) \land !p \Rightarrow !q$

Proof T 2.57(*by using T 2.50*)

(1)	$!(p \Rightarrow q)$	{a}
(2)	$!p$	
(3)	$!((p \Rightarrow q) \wedge p)$	{!K : 1,2}
(4)	$!(p \wedge q)$	{SR : 3, $(p \Rightarrow q) \wedge p \Leftrightarrow p \wedge q$}
	$!q.$ □	{!K : 4}

Proof T 2.57(*by using T 2.55*)

(1)	$!(p \Rightarrow q)$	{a}
(2)	$!p$	
(3)	$!p \Rightarrow !q$	{– C : T 2.55,1}
	$!q.$ □	{– C : 2,3}

Thesis 2.58

$!(p \Rightarrow q) \wedge \sim \delta q \Rightarrow \sim \delta p$

Proof:

(1)	$!(p \Rightarrow q)$	{a}
(2)	$\sim \delta q$	
(3)	$! \sim q$	{N δ : 2}
(4)	$!((p \Rightarrow q) \wedge \sim q)$	{+ K, !K : 1,3}
(5)	$!(\sim p \wedge \sim q)$	{$\models ((p \Rightarrow q) \wedge \sim q \Leftrightarrow \sim p \wedge \sim q)$, SR : 4}
(6)	$! \sim p$	{!K : 5}
	$\sim \delta p.$ □	{N δ : 6}

And hence, by using the forbiddance functor σ and SR, the last thesis can be presented equivalently as follows.

$!(p \Rightarrow q) \wedge \sigma q \Rightarrow \sigma p.$

Thesis 2.59

$!(p \Rightarrow q) \Rightarrow (\delta p \Rightarrow \delta q)$

Proof:

(1)	$!(p \Rightarrow q)$	{a}
(2)	δp	
(3)	$\sim \delta q$	{aip}
(4)	$! \sim q$	{N δ : 3}
(5)	$!((p \Rightarrow q) \wedge \sim q)$	{+ K, !K : 1,4}
(6)	$!(\sim p \wedge \sim q)$	{$\models ((p \Rightarrow q) \wedge \sim q \Leftrightarrow \sim p \wedge \sim q)$, SR : 5}
(7)	$! \sim p$	{!K : 6}
(8)	$\sim \delta p$	{N δ : 7}
	contr. □	{2,8}

Since $!(p \Rightarrow q) \Leftrightarrow \sim \delta \sim (p \Rightarrow q)$, the following thesis can be obtained (by using rules NC, SR and definition of σ: left to the reader).

Thesis 2.60

$!(p \Rightarrow q) \Leftrightarrow \sigma(p \wedge \sim q)$. □

De Morgan´s laws of deontic logic are presented in the next two theses.

Thesis 2.61

$\sim !(p \wedge q) \Leftrightarrow \sim !p \vee \sim !q$

Proof:

$$\begin{aligned} \sim !(p \wedge q) &\Leftrightarrow \delta(\sim p \vee \sim q) && \{N!, NK, SR\} \\ &\Leftrightarrow \delta \sim p \vee \delta \sim q && \{\delta A\} \\ &\Leftrightarrow \sim !p \vee \sim !q. \ \square && \{N!, SR\} \end{aligned}$$

Thesis 2.62

$\sim \delta(p \vee q) \Leftrightarrow \sim \delta p \wedge \sim \delta q$

Proof:

$$\begin{aligned} \sim \delta(p \vee q) &\Leftrightarrow !(\sim p \wedge \sim q) && \{N\delta, NA, SR\} \\ &\Leftrightarrow ! \sim p \wedge ! \sim q && \{!K\} \\ &\Leftrightarrow \sim \delta p \wedge \sim \delta q. \ \square && \{N\delta, SR\} \end{aligned}$$

In an equivalent way T 2.62 can be presented as follows.

$\sigma(p \vee q) \Leftrightarrow \sigma p \wedge \sigma q$.

Thesis 2.63

$!(p \Rightarrow r) \wedge !(q \Rightarrow r) \Leftrightarrow !(p \vee q \Rightarrow r)$

Proof:

$$\begin{aligned} !(p \Rightarrow r) \wedge !(q \Rightarrow r) &\Leftrightarrow !((p \Rightarrow r) \wedge (q \Rightarrow r)) && \{!K\} \\ &\Leftrightarrow !(p \vee q \Rightarrow r). \ \square && \{SR\} \end{aligned}$$

In fact, since $\models ((p \Rightarrow r) \wedge (q \Rightarrow r) \Leftrightarrow p \vee q \Rightarrow r)$ the above T 2.63 follows directly from the rule D4, !K and SR. In a similar way the following thesis can be obtained.

Thesis 2.64

$!(p \Rightarrow q) \wedge !(p \Rightarrow r) \Leftrightarrow !(p \Rightarrow q \wedge r)$. □

Thesis 2.65

$!(p \Rightarrow q) \wedge !(r \Rightarrow s) \Rightarrow (!(p \wedge r) \Rightarrow !(q \wedge s))$

Proof:

(1)	$!(p \Rightarrow q)$	
(2)	$!(r \Rightarrow s)$	{a}
(3)	$!(p \wedge r)$	
(4)	$!p \Rightarrow !q$	{– C : T 2.55, 1}
(5)	$!r \Rightarrow !s$	{– C : T 2.55, 2}
(6)	$!p$	{!K : 3}

(7)	!r	
(8)	!q	{– C : 4,6}
(9)	!s	{– C : 5,7}
	!(q ∧ s). □	{!K : 8,9}

The proof of T 2.65 can be also obtained by using the rule D1, !K, SR, T 2.55 and T 2.69 (given below: left to the reader).

Thesis 2.66

δ(p ⇒ q) ⇒ (!p ∨ δr ⇒ δ(q ∨ r))

Proof:

(1)	δ(p ⇒ q)	{a}
(2)	!p ∨ δr	
(3)	~ δ(q ∨ r)	{aip}
(4)	~ δq	{T 2.62, – K : 3}
(5)	~ δr	
(6)	δ ~ p ∨ δq	{CR, SR, δA : 1}
(7)	!p	{– A : 2,5}
(8)	δ ~ p	{– A : 4,6}
(9)	~ !p	{N! : 8}
	contr. □	{7,9}

Thesis 2.67

!p ⇒ !(p ∨ q)

Proof:

(1)	!p	{a}
(2)	~ !(p ∨ q)	{aip}
(3)	δ(~ p ∧ ~ q)	{N!, NA, SR : 2}
(4)	δ ~ p	{δK : 3}
(5)	~ !p	{N! : 4}
	contr. □	{1,5}

Thesis 2.68

σp ⇒ !(p ⇒ q)

Proof:

(1)	σp	{a}
(2)	~ !(p ⇒ q)	{aip}
(3)	~ δp	{ df. 'σ': 1}
(4)	δ(p ∧ ~ q)	{N!, NC, SR : 2}
(5)	δp	{δK : 4}
	contr. □	{3,5}

In accordance with the law of transitivity for implication TC, we have: $\models ((p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r))$. And hence by the obligation rule OBR and '– C' it follows that $\models !((p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r))$. And so, the following thesis is satisfied.

Thesis 2.69

$!(p \Rightarrow q) \wedge !(q \Rightarrow r) \Rightarrow !(p \Rightarrow r)$

Proof:

(1)	$!((p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	{TC, OBR,– C}
(2)	$!((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow !(p \Rightarrow r)$	{– C : T 2.55,1}
	$!(p \Rightarrow q) \wedge !(q \Rightarrow r) \Rightarrow !(p \Rightarrow r)$. □	{!K, SR : 2}

Obviously, the proof of T 2.69 can be also realised without using the obligation rule OBR. In fact, since $(p \Rightarrow q) \wedge (q \Rightarrow r)$ and $(p \Rightarrow r) \wedge \zeta$ are equivalent the following proof can be obtained (see the proof of T 2.33).

Proof T 2.69:

(1)	$!(p \Rightarrow q) \wedge !(q \Rightarrow r)$	{a}
(2)	$!((p \Rightarrow q) \wedge (q \Rightarrow r))$	{!K : 1}
(3)	$!((p \Rightarrow r) \wedge \zeta)$	{SR : 2, $(p \Rightarrow q) \wedge (q \Rightarrow r)$ and $(p \Rightarrow r) \wedge \zeta$ are equivalent}
	$!(p \Rightarrow r)$. □	{!K : 3}

Thesis 2.70

$!(p \Rightarrow q) \Rightarrow (!(p \vee r) \Rightarrow !(q \vee r))$

Proof:

(1)	$!((p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r))$	{$\models ((p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r))$, OBR,– C}
(2)	$!(p \Rightarrow q) \Rightarrow !(p \vee r \Rightarrow q \vee r)$	{– C : T 2.55,1}
(3)	$!(p \vee r \Rightarrow q \vee r) \Rightarrow (!(p \vee r) \Rightarrow !(q \vee r))$	{T 2.55}
	$!(p \Rightarrow q) \Rightarrow (!(p \vee r) \Rightarrow !(q \vee r))$. □	{T 2.69 : 2,3}

The proof of the next thesis is similar to the proof of the previous one and hence it is left to the reader.

Thesis 2.71 (*Hauber's deontic law of conversion of implications*)

$!(p \Rightarrow q) \wedge !(r \Rightarrow s) \wedge !(p \vee r) \wedge \sigma(q \wedge s) \Rightarrow !(q \Rightarrow p) \wedge !(s \Rightarrow r)$. □

Some derived rules related to the above proved theses are given below.

(35) *Rule of obligation of implication* (denoted below by '!C'):

$$!C : \frac{!(\varphi \Rightarrow \psi)}{!\varphi \Rightarrow !\psi}$$

(36) *Rule of implication of permissions* (denoted below by 'Cδ'):

$$C\delta : \frac{\delta\varphi \Rightarrow \delta\psi}{\delta(\varphi \Rightarrow \psi)}$$

(37) *Deontic rule* of *detachment for implication* (or *omitting an implication*, denoted below by ´– !C´):

$$-\,!C : \quad \frac{!(\varphi \Rightarrow \psi) \quad !\varphi}{!\psi}$$

(38) *Deontic obligation tollens* (denoted below by ´!-Toll´):

$$!\text{-Toll} : \quad \frac{!(\varphi \Rightarrow \psi) \quad \sim \delta\psi}{\sim \delta\varphi}$$

(39) *Deontic De Morgan´s law for obligation of conjunction* (denoted below by ´N!K´):

$$N!K : \quad \frac{\sim !(\varphi \wedge \psi)}{\sim !\varphi \vee \sim !\psi}$$

(40) *Deontic De Morgan´s law for permission of disjunction* (denoted below by ´NδA´):

$$N\delta A : \quad \frac{\sim \delta(\varphi \vee \psi)}{\sim \delta\varphi \;/\; \sim \delta\psi \;/\; \sim \delta\varphi \;\; \sim \delta\psi}$$

(41) *Deontic law of addition of antecedents* (denoted below by ´!AA´):

$$!AA : \quad \frac{!(\varphi \Rightarrow \chi) \quad !(\psi \Rightarrow \chi)}{!(\varphi \vee \psi \Rightarrow \chi)}$$

(42) *Deontic law of multiplication of consequents* (denoted below by ´!MC´):

$$!MC : \quad \frac{!(\varphi \Rightarrow \psi) \quad !(\varphi \Rightarrow \chi)}{!(\varphi \Rightarrow \psi \wedge \chi)}$$

(43) *Deontic rule of multiplication of the antecedents and consequents of two implications* (denoted below by ´!MAC´):

$$!MAC : \quad \frac{!(\varphi_1 \Rightarrow \psi_1) \quad !(\varphi_2 \Rightarrow \psi_2)}{!(\varphi_1 \wedge \varphi_2) \Rightarrow !(\psi_1 \wedge \psi_2)}$$

(44) *Deontic rule of joining a disjunction* (denoted below by ´+!A´):

$$+!A : \quad \frac{!\varphi}{!(\varphi \vee \psi)}$$

(45) *Deontic rule of transitivity for implication* (denoted below by ´!TC´):

$$!TC: \quad \frac{!(\varphi \Rightarrow \psi) \qquad !(\psi \Rightarrow \chi)}{!(\varphi \Rightarrow \chi)}$$

Let ψ be a logical consequence wrt $\varphi_1, \varphi_2, \ldots, \varphi_n$. Then by T 1.23 (see Subsection 1.5) it follows that $\vDash \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \Rightarrow \psi$. Using in turn rules OBR, – C, !C, !K, and SR we can obtain: $\vDash\ !\varphi_1 \wedge !\varphi_2 \wedge \ldots \wedge !\varphi_n \Rightarrow\ !\psi$. And hence, as in modal logic, any logical consequence involves some *deontic logical consequence*.

Besides the beginning of intensive investigations in deontic systems, there can be observed calls in question in regards to certain theses, i.e. any such thesis have been considered as non-intuitive and hence distinct from the common sense. As an example, the above thesis T 2.67 is a deontic representation of the well-known *Ross' paradox* for obligation* (Ross A.N.C. 1941). It was argued that a proposition of the form $!(p \vee q)$ is a confirmation of possessing a choice between realisation of two activities. The contra-argumentation was mainly based on the fact that the possessing of choice should be depending on some additional circumstances. And hence, the sense of the above proposition do not decide in advance if really such choice is necessary (similar considerations have been given wrt T 2.51, etc. (see: *The little encyclopaedia of logic* 1988). In fact, a good intuition should not necessary be a correctness criterion for a given proof.

Generally, there were presented various deontic systems, e.g. DKr, D, DM, DBr, etc. In particular, there have been made an attempt of constructing the deontic logical constants !, δ, and σ in a way more adequate to their common sense. And so, the interpretation difficulties seem to be the main reason in constructing relative deontic systems. Such extensions are based on the idea that a de re obligation may depend on circumstances in different situations. And this idea suggests that a conditional binary concept of obligation and similarly of permission may be more appropriate than the above monadic one (Cocchiarella N.B., in the late 1960s). For example, the propositional formula $!(\varphi / \psi)$, associated with the *conditional* (or *relative*) *obligation functor* $!(\cdots / \cdots)$, might be read as: “it is obligatory that φ given (or conditional on) the circumstances that ψ”, where φ and ψ may be arbitrary classical logic formulae (see Definition 1.1 of Subsection 1.1). Obviously, if ψ is a thesis then this functor becomes equivalent to the previous unary one. The rest two conditional deontic functors, i.e. $\delta(\cdots / \cdots)$ and $\sigma(\cdots / \cdots)$ can be introduced in a similar way. In fact, the *conditional forbiddance functor* can be defined as follows: $\sigma(\varphi / \psi) \Leftrightarrow_{df}\ !(\sim \varphi / \psi)$. But, in the case of conditional *permission functor* the corresponding definition $\delta(\varphi / \psi) \Leftrightarrow_{df} \sim \sigma(\varphi / \psi)$ will contribute a more specific sense of this notion (*The little encyclopaedia of logic* 1988). The following two *axioms* are presented in the last work (some example derived rules and theses are also given below).

(RD1) $!(\varphi \wedge \psi / \chi \vee \xi) \Leftrightarrow\ !(\varphi / \chi) \wedge !(\psi / \chi) \wedge !(\varphi / \xi) \wedge !(\psi / \xi)$

(RD2) $\sim (!(\varphi / \psi \vee \sim \psi) \wedge !(\sim \varphi / \psi \vee \sim \psi))$

Since φ, ψ, χ and ξ may be arbitrary in accordance with the laws of idempotence and SR, the following two *shortened versions of RD1* can be obtained.

($RD1_{sv\,/\,\psi =_{df} \varphi}$) $!(\varphi / \chi \vee \xi) \Leftrightarrow\ !(\varphi / \chi) \wedge !(\varphi / \xi)$

($RD1_{sv\,/\,\xi =_{df} \chi}$) $!(\varphi \wedge \psi / \chi) \Leftrightarrow\ !(\varphi / \chi) \wedge !(\psi / \chi)$

Moreover, since $\vDash (\psi \vee \sim \psi)$ (the Aristotelian law of excluded middle), by Thesis 2.50 it follows that RD2 is equivalent to the basic axiom: $\sim !(\varphi \wedge \sim \varphi)$.

Let $\vDash (\varphi \Rightarrow \psi)$ be a classical logic thesis. Then the following rules can be considered as derived:

---

* Alf Niels Christian Ross (1899 – 1979)

(R1) $!(\chi / \psi) \Rightarrow !(\chi / \varphi)$

(R2) $!(\varphi / \chi) \Rightarrow !(\psi / \chi)$

Similar rules can be derived by assuming the conditional deontic functors δ and σ (this is omitted: see *Formal logic. Encyclopedical outline with applications to informatics and linguistics* 1987). Some example theses are given below. Obviously, any such thesis should involve some new derived rules related to arbitrary classical logic formulae (the use of assumptional system style in the proofs given below is left to the reader).

Thesis 2.72

$!(p / q) \Rightarrow !(p / q \wedge r)$

Proof:

Since $\vDash (q \wedge r \Rightarrow q)$, corresponding to ′− K′, the proof follows directly from the above rule R1, by assuming: $\chi =_{df} p$, $\psi =_{df} q$, and $\varphi =_{df} q \wedge r$. ▫

The proof of the next thesis is similar wrt R2 and classical rule ′+ A′, since $\vDash (p \Rightarrow p \vee r)$.

Thesis 2.73

$!(p / q) \Rightarrow !(p \vee r / q)$. ▫

Thesis 2.74

$!(p \Rightarrow q / r) \Leftrightarrow !(\sim p \vee q / r)$

Proof: left to the reader. ▫

Thesis 2.75

$!(p \Rightarrow q / r) \wedge !(p / r) \Rightarrow !(q / r)$

Proof:

In accordance with the law of transitivity for implication TC, we can obtain.

$!(p \Rightarrow q / r) \wedge !(p / r)$	$\Rightarrow$	$!((p \Rightarrow q) \wedge p / r)$	$(RD1_{sv / \xi =_{df} \chi})$
	$\Rightarrow$	$!(p \wedge q / r)$	$\{\vDash ((p \Rightarrow q) \wedge p \Leftrightarrow p \wedge q)$, SR$\}$
	$\Rightarrow$	$!(p / r) \wedge !(q / r)$	$(RD1_{sv / \xi =_{df} \chi})$
	$\Rightarrow$	$!(q / r)$. ▫	$\{- K\}$

The following theses are also satisfied.

Thesis 2.76

$!(\sim p / q) \Rightarrow !(p \Rightarrow q / q \wedge r)$

Proof:

$!(\sim p / q)$	$\Rightarrow$	$!(\sim p \vee q / q)$	{T 2.73, $r =_{df} q$}
	$\Rightarrow$	$!(p \Rightarrow q / q)$	{T 2.74, SR, $r =_{df} q$}
	$\Rightarrow$	$!(p \Rightarrow q / q \wedge r)$. ▫	{T 2.72}

Thesis 2.77

$!(p \Rightarrow q / s) \wedge !(q \Rightarrow r / s) \Rightarrow !(p \Rightarrow r / s)$

Proof:

$!(p \Rightarrow q / s) \wedge !(q \Rightarrow r / s) \Rightarrow !((p \Rightarrow q) \wedge (q \Rightarrow r) / s)$ $(RD1_{sv / \xi =_{df} \chi})$

$\Rightarrow !((p \Rightarrow r) \wedge \zeta / s)$ {SR, it is assumed that there exists some formula ζ such that the formulae $(p \Rightarrow q) \wedge (q \Rightarrow r)$ and $(p \Rightarrow r) \wedge \zeta$ are equivalent: e.g., see the proof of T 2.33}

$\Rightarrow !(p \Rightarrow r / s).$ □

Thesis 2.78

$!(\sim (p \vee q) / r) \Leftrightarrow !(\sim p / r) \wedge !(\sim q / r)$

Proof:

$!(\sim (p \vee q) / r) \Leftrightarrow !(\sim p \wedge \sim q / r)$ {NA, SR}

$\Leftrightarrow !(\sim p / r) \wedge !(\sim q / r).$ □ $(RD1_{sv / \xi =_{df} \chi})$

Thesis 2.79

$!(p / \sim (q \wedge r)) \Leftrightarrow !(p / \sim q) \wedge !(p / \sim r)$

Proof:

$!(p / \sim (q \wedge r)) \Leftrightarrow !(p / \sim q \vee \sim r)$ {NK, SR}

$\Leftrightarrow !(p / \sim q) \wedge !(p / \sim r).$ □ $(RD1_{sv / \psi =_{df} \varphi})$

Any temporal logic system uses the notion of time. The next considerations are a brief introduction to such logic systems.

Temporal logics

Temporal logic (or in general: logics) is a deduction system for representing and reasoning about propositions qualified in terms of time. This logic is sometimes also used to refer to *tense logic*, a particular modal logic – based system of temporal logic, with reference to the grammatical tenses and introduced by Prior in the late 1950s (e.g. see: Prior A.N. 1957, 1967)[*]. And so, there was studied the possibility of using Diodorus Cronus' ideas (related to "strict" or "strong" implication) to contemporary works in modal logic by taking into consideration time. Subsequently, it was shown a possibility of defining the whole temporal functors in terms of "*since*" and "*until*" (by assuming a continuous linear ordering: Kamp H. 1968).

Typical examples of dependencies between modal notions having application also to temporal notions are given below (*Formal logic. Encyclopedical outline with applications to informatics and linguistics* 1987).

If always φ then φ	$\Box \varphi \Rightarrow \varphi$
If φ then sometimes φ	$\varphi \Rightarrow \Diamond \varphi$
If always φ then sometimes φ	$\Box \varphi \Rightarrow \Diamond \varphi$
It is not true that always φ iff sometimes not φ	$\sim \Box \varphi \Leftrightarrow \Diamond \sim \varphi$

[*] Arthur Norman Prior (1914 – 1969)

It can be observed that any relationship between the grammatical tenses is not possible to be described using classical propositional logic. An example of such inference may be the following: "*If you have finished this course then you were given a student visa*" (obviously, the opposite implication is not always satisfied). And hence, the above linguistic motivation was well-argued. In fact, there were also some philosophical motivations of using such logic (e.g. the notion of *determinism**).

Subsequently, temporal logic has been developed further in the area of computer science, in particular it has found an important application in *formal verification* to state requirements of *hardware* or *software* systems. For instance, one may wish to say that *whenever* a request is made, access to a resource is *eventually* granted, but it is *never* granted to two requestors simultaneously. Such a statement can conveniently be expressed in a temporal logic (*public domain*).

The use of *Petri nets* is another way for modelling such processes as above (in general: for modelling of *discrete event systems*, e.g. computer or communication networks, automated manufacturing systems or other large-scale plants, *reactive programs* such as computer operating systems, embedded and process control programs or other concurrent and real-time programs, office information systems and so on). These nets, as a general purpose mathematical model, were originally introduced for describing relations existing between conditions and events (Carl Adam Petri 1962: 1926 - 2010). The Petri nets has gained increased usage and acceptance as a basic tool for representation, analysis, and synthesis. So firstly, by using Petri nets we have a possibility to model and visualize types of behaviour involving parallelism, concurrency, synchronization and resource sharing. Secondly, the theoretical results are plentiful. The properties of these nets have been and still are extensively studied. There exist many net models (from special to higher net models, e.g. condition-event nets, place-transition nets, individual-token nets, etc. see: *High-level Petri Nets* 2000, 2005). Petri nets are conventionally represented in terms of sets and operations on sets (a more formal treatment is omitted in this part of the study).

The following two aspects of temporal logic systems are considered below. Initially it is considered the Prior's tense logic system (*Formal logic. Encyclopedical outline with applications to informatics and linguistics* 1987). An extension of tense logic in the area of computer science was originally given by Manna Z. and Pnueli A. (1992, 1995).The proposed temporal logic system was used in the specification and verification of reactive programs. A brief introduction to this excellent work is next presented.

Prior's tense logic

The logical features of the grammatical tenses can be described by means of introducing the following two primary tense functors: *′it was the case that...′* and *′it will be the case that...′* corresponding to the past and future tenses and denoted below by Ᵽ and Ƒ, respectively. Moreover, it is assumed that the lack of functor is associated with present tense. Let φ be arbitrary classical logic formula. The next two functors can be introduced as follows.

Ǥ$\varphi \Leftrightarrow_{df} \sim$ Ƒ $\sim \varphi$, i.e. *′it will always be the case that...′*

Ħ$\varphi \Leftrightarrow_{df} \sim$ Ᵽ $\sim\varphi$, i.e. *′it has always been the case that...′*

In accordance with the last two definitions, the conjunction Ǥ$\varphi \wedge$ Ħφ can be interpreted as: *′φ is eternal truth ′*. Moreover, by using the law of contraposition of equivalence CE and ′− N′ we can obtain: Ƒ $\sim \varphi \Leftrightarrow$

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* *Determinism* is the general philosophical thesis that states that for everything that happens there are conditions such that, given them, nothing else could happen. There are many versions of this thesis. Each of them rests upon various alleged connections, and interdependencies of things and events, asserting that these hold without exception. The wide variety of deterministic theories throughout the history of philosophy have sprung from diverse motives and considerations; some of which overlap considerably. All should be considered in the light of their historical significance, together with certain alternative theories that philosophers have proposed (see T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*).

$\sim G\varphi$. Since φ may be arbitrary, by assuming $\varphi \Leftrightarrow_{df} \sim \psi$ and using SR we have: $F\psi \Leftrightarrow \sim G \sim \psi$. In a similar way, $P\psi \Leftrightarrow \sim H \sim \psi$. And hence, F and P can be dually introduced by using G and H.

The following *minimal tense logic system* was introduced by Prior A.N. (1967).

(P1) $G(\varphi \Rightarrow \psi) \Rightarrow (G\varphi \Rightarrow G\psi)$

(P2) $H(\varphi \Rightarrow \psi) \Rightarrow (H\varphi \Rightarrow H\psi)$

(P3) $\varphi \Rightarrow HF\varphi$

(P4) $\varphi \Rightarrow GP\varphi$

In addition to the rule of detachment for implication, i.e. ′– C′, the following two rules have been also introduced.

(46) *Tense logic rule of joining G functor*
(denoted below by ′+ G′):

$$+G: \quad \frac{\varphi}{G\varphi}$$

(47) *Tense logic rule of joining H functor*
(denoted below by ′+ H′):

$$+H: \quad \frac{\varphi}{H\varphi}$$

The assumptional system style is illustrated below. As in the classical propositional calculus, the obtained theses are generalised for arbitrary tense logic formulae. The following example theses are satisfied.

Thesis 2.80

$G(p \Rightarrow q) \wedge Gp \Rightarrow Gq$

Proof:

(1)	$G(p \Rightarrow q)$	{a}
(2)	Gp	
(3)	$Gp \Rightarrow Gq$	{– C : P1,1}
	Gq. □	{– C : 2,3}

Thesis 2.81

$G(p \Rightarrow q) \wedge \sim Gq \Rightarrow \sim Gp$

Proof:

(1)	$G(p \Rightarrow q)$	{a}
(2)	$\sim Gq$	
(3)	$Gp \Rightarrow Gq$	{– C : P1,1}
	$\sim Gp$. □	{Toll : 2,3}

The proofs of the next two theses are similar and can be obtained by using axiom P2. Hence they are omitted.

Thesis 2.82

$\mathsf{H}(p \Rightarrow q) \wedge \mathsf{H}p \Rightarrow \mathsf{H}q$. □

Thesis 2.83

$\mathsf{H}(p \Rightarrow q) \wedge \sim \mathsf{H}q \Rightarrow \sim \mathsf{H}p$. □

Similar theses can be also proved in replacing formulae $\mathsf{G}(p \Rightarrow q)$ and $\mathsf{H}(p \Rightarrow q)$ in the antecedents of the main implications of T 2.80 – T 2.83 with the (present tense) implication $p \Rightarrow q$ and next using rules $+\mathsf{G}$ and $+\mathsf{H}$ (left to the reader).

The following two additional distributive axioms are assumed below.

(P5) $\mathsf{G}(\varphi \wedge \psi) \Leftrightarrow \mathsf{G}\varphi \wedge \mathsf{G}\psi$

(P6) $\mathsf{H}(\varphi \wedge \psi) \Leftrightarrow \mathsf{H}\varphi \wedge \mathsf{H}\psi$

For example, by P5 it follows that $\mathsf{G}(p \Leftrightarrow q) \Leftrightarrow \mathsf{G}((p \Rightarrow q) \wedge (q \Rightarrow p)) \Leftrightarrow \mathsf{G}(p \Rightarrow q) \wedge \mathsf{G}(q \Rightarrow p)$. Similarly for H using P6.

In accordance with CE, we have: $\mathsf{F} \sim \varphi \Leftrightarrow \sim \mathsf{G}\varphi$ and $\mathsf{P} \sim \varphi \Leftrightarrow \sim \mathsf{H}\varphi$. And hence, the next two distributive properties can be obtained (the proof of T 2.85 is left to the reader).

Thesis 2.84

$\mathsf{F}(p \vee q) \Leftrightarrow \mathsf{F}p \vee \mathsf{F}q$

Proof:

$\mathsf{F}(p \vee q)$	$\Leftrightarrow$	$\mathsf{F}\sim\sim(p \vee q)$	{– N, SR}
	$\Leftrightarrow$	$\sim \mathsf{G} \sim (p \vee q)$	$\{\mathsf{F} \sim \varphi \Leftrightarrow \sim \mathsf{G}\varphi\}$
	$\Leftrightarrow$	$\sim \mathsf{G}(\sim p \wedge \sim q)$	{NA, SR}
	$\Leftrightarrow$	$\sim (\mathsf{G} \sim p \wedge \mathsf{G} \sim q)$	{P5, SR}
	$\Leftrightarrow$	$\sim \mathsf{G} \sim p \vee \sim \mathsf{G} \sim q$	{NK}
	$\Leftrightarrow$	$\mathsf{F}p \vee \mathsf{F}q$. □	$\{\mathsf{F} \sim \varphi \Leftrightarrow \sim \mathsf{G}\varphi$, – N, SR}

Thesis 2.85

$\mathsf{P}(p \vee q) \Leftrightarrow \mathsf{P}p \vee \mathsf{P}q$. □

The following *De Morgan's laws* are satisfied. The proofs of T 2.86 and T 2.87 given below follow directly by P5, P6 and SR. And so, they are left to the reader.

Thesis 2.86

$\sim \mathsf{G}(p \wedge q) \Leftrightarrow \sim \mathsf{G}p \vee \sim \mathsf{G}q$. □

Thesis 2.87

$\sim \mathsf{H}(p \wedge q) \Leftrightarrow \sim \mathsf{H}p \vee \sim \mathsf{H}q$. □

The proofs of the next two theses are direct consequence of T 2.84 and T 2.85, respectively and they can be obtained by means of CE, NA and SR (left to the reader).

Thesis 2.88

$\sim \text{F}(p \lor q) \Leftrightarrow \sim \text{F}p \land \sim \text{F}q$. □

Thesis 2.89

$\sim \text{P}(p \lor q) \Leftrightarrow \sim \text{P}p \land \sim \text{P}q$. □

The following transitivity property is satisfied.

Thesis 2.90

$\text{G}(p \Rightarrow q) \land \text{G}(q \Rightarrow r) \Rightarrow \text{G}(p \Rightarrow r)$

Since $(p \Rightarrow q) \land (q \Rightarrow r)$ and $(p \Rightarrow r) \land \zeta$ are equivalent, where $\zeta = \zeta(p, q, r)$, the following proof can be obtained (see the proof of T 2.33).

Proof:

(1)	$\text{G}(p \Rightarrow q) \land \text{G}(q \Rightarrow r)$	{a}
(2)	$\text{G}((p \Rightarrow q) \land (q \Rightarrow r))$	{P5 : 1}
(3)	$\text{G}((p \Rightarrow r) \land \zeta)$	{SR : 2, $(p \Rightarrow q) \land (q \Rightarrow r)$ and $(p \Rightarrow r) \land \zeta$ are equivalent}
(4)	$\text{G}(p \Rightarrow r) \land \text{G}\zeta$	{P5 : 3}
	$\text{G}(p \Rightarrow r)$. □	{– K : 4}

The corresponding proof of the transitivity property for H functor is left to the reader. The obtained inference rules related to T 2.80 – T 2.90 are illustrated below.

(48) *Tense logic* G *rule* of *detachment for implication* (or *omitting an implication*, denoted below by ′– GC′):

$$-\text{GC}: \quad \frac{\text{G}(\varphi \Rightarrow \psi) \qquad \text{G}\varphi}{\text{G}\psi}$$

(49) *Tense logic* H *rule* of *detachment for implication* (or *omitting an implication*, denoted below by ′– HC′):

$$-\text{HC}: \quad \frac{\text{H}(\varphi \Rightarrow \psi) \qquad \text{H}\varphi}{\text{H}\psi}$$

(50) *Tense logic* G *tollens* (denoted below by ′ G-Toll′):

$$\text{G-Toll}: \quad \frac{\text{G}(\varphi \Rightarrow \psi) \qquad \sim \text{G}\psi}{\sim \text{G}\varphi}$$

(51) *Tense logic* H *tollens* (denoted below by ′ H-Toll′):

$$\text{H-Toll}: \quad \frac{\text{H}(\varphi \Rightarrow \psi) \qquad \sim \text{H}\psi}{\sim \text{H}\varphi}$$

(52) *Distributive property rule of* Ꞙ *functor* (denoted below by ´ꞘA´):

$$\text{ꞘA}: \quad \frac{\text{Ꞙ}(\varphi \vee \psi)}{\text{Ꞙ}\varphi \vee \text{Ꞙ}\psi}$$

(53) *Distributive property rule of* Ᵽ *functor* (denoted below by ´ᵽA´):

$$\text{ᵽA}: \quad \frac{\text{Ᵽ}(\varphi \vee \psi)}{\text{Ᵽ}\varphi \vee \text{Ᵽ}\psi}$$

(54) *Tense logic De Morgan´s law for* Ǥ *functor of conjunction* (denoted below by ´NǤK´):

$$\text{NǤK}: \quad \frac{\sim \text{Ǥ}(\varphi \wedge \psi)}{\sim \text{Ǥ}\varphi \vee \sim \text{Ǥ}\psi}$$

(55) *Tense logic De Morgan´s law for* Ħ *functor of conjunction* (denoted below by ´NĦK´):

$$\text{NĦK}: \quad \frac{\sim \text{Ħ}(\varphi \wedge \psi)}{\sim \text{Ħ}\varphi \vee \sim \text{Ħ}\psi}$$

(56) *Tense logic De Morgan´s law for* Ꞙ *functor of disjunction* (denoted below by ´NꞘA´):

$$\text{NꞘA}: \quad \frac{\sim \text{Ꞙ}(\varphi \vee \psi)}{\sim \text{Ꞙ}\varphi \;/ \sim \text{Ꞙ}\psi \;/ \begin{array}{l}\sim \text{Ꞙ}\varphi \\ \sim \text{Ꞙ}\psi\end{array}}$$

(57) *Tense logic De Morgan´s law for* Ᵽ *functor of disjunction* (denoted below by ´NᵽA´):

$$\text{NᵽA}: \quad \frac{\sim \text{Ᵽ}(\varphi \vee \psi)}{\sim \text{Ᵽ}\varphi \;/ \sim \text{Ᵽ}\psi \;/ \begin{array}{l}\sim \text{Ᵽ}\varphi \\ \sim \text{Ᵽ}\psi\end{array}}$$

(58) *Tense logic rule of* Ǥ*-transitivity for implication* (denoted below by ´ǤTC´):

$$\text{ǤTC}: \quad \frac{\begin{array}{l}\text{Ǥ}(\varphi \Rightarrow \psi) \\ \text{Ǥ}(\psi \Rightarrow \chi)\end{array}}{\text{Ǥ}(\varphi \Rightarrow \chi)}$$

(59) *Tense logic rule of* Ħ*-transitivity for implication* (denoted below by ´ĦTC´):

$$\text{ĦTC}: \quad \frac{\begin{array}{l}\text{Ħ}(\varphi \Rightarrow \psi) \\ \text{Ħ}(\psi \Rightarrow \chi)\end{array}}{\text{Ħ}(\varphi \Rightarrow \chi)}$$

It can be denoted that the Prior’s tense logic system is semantically complete under Definition 1.9 (see: Subsection 1.6). However, the corresponding proof would require the introduction of the following two additional conditions related to F and P, where $'\mathscr{M} \vDash \varphi_\tau'$ denotes ´the formula φ in time instance τ is satisfied in model $\mathscr{M}$´ and θ is a time instance set (a more formal treatment is omitted here).

(C1) $\mathscr{M} \vDash \mathsf{F}\varphi_\tau \Leftrightarrow_{df} \exists\, \nu \in \theta\, (\tau < \nu \wedge \mathscr{M} \vDash \varphi_\nu)$

(C2) $\mathscr{M} \vDash \mathsf{P}\varphi_\tau \Leftrightarrow_{df} \exists\, \nu \in \theta\, (\nu < \tau \wedge \mathscr{M} \vDash \varphi_\nu)$

Manna and Pnueli's temporal logic

Temporal logic, first proposed by Pnueli A. around 1976, see: (Pnueli A. 1977).* In the next research, this logic was used in the process of specification properties of reactive and concurrent systems (Manna Z. and Pnueli A. 1992, 1995)†. Some introductory notions related to Manna and Pnueli's temporal logic are given below‡. First, the notions of reactive program and fair transition system are briefly considered. Next it is illustrated the proposed language of temporal logic as a tool for *specification* of reactive systems (i.e. the description of the desired behaviour or operation of the system, while avoiding references to the method or details of its implementation).

From theoretical point of view, programs and systems that they control can be partitioned into transformational and reactive programs and systems. A *reactive program* is a program whose role is to maintain an ongoing interaction with its environment rather than to compute some final value on termination (as in the case of *transformational programs*). A fundamental element in reactive programs is that of *concurrency*. By definition, a reactive program runs concurrently with its environment. A reactive program is often strongly intertwined with the hardware system that it controls. The software component is then considered as an integrated part of the whole system referred to as "*embedded*". Hence, the notion of a reactive program is sometimes considered more generally as a *reactive system* and there is no preferred a sharp distinction between the program and the system that it controls. In fact, the most of proposed techniques, with very few changes, are applicable and have been successfully used for digital circuit specification and verification (Manna Z. and Pnueli A. 1992, 1995).

The main part of the Manna and Pnueli's generic model of reactive systems is given by a *basic transition system*. Any such system is considered as a quadruple including a *finite set of flexible state variables* (i.e. data or control variables which may assume different values in different states)§, a *set of states*, a *finite set of transitions*, and an *initial condition*. Next the above basic computational model is completed by introducing two different notions of transition fairness (called *weak* and *strong fairness*) representing some additional restrictions on the computations allowed by this model (*justice* and *compassion requirements*, respectively). This way, any computations that do not correspond to actual executions of real concurrent programs are excluded. The obtained model containing two additional subsets of transitions (*justice* and *compassion sets of transitions*) is said to be a *fair transition system*.

The *behavioural level* for describing semantics of reactive systems is assumed below. And hence, the semantics of any such system is identified with its behaviour and represented by the set of computations. Subsequently, any program is interpreted as a generator of a set of computations**.

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* The usefulness of (linear) temporal logic for specifying global properties of concurrent systems for the future fragment. The advisability of using past modalities was introduced in (Lichtenstein O. et al. 1985). Other earlier works: (Kroger F. 1977) and (Moszkowski B. 1983), see: file:///F:/Temporal%20predicate%20calculus/paper%206.pdf.

† Zohar Manna (1939 – 2018), Amir Pnueli (1941 – 2009).

‡ An extension of this system by additional axioms and rules, to deal with the first-order elements such as: variables, equality and quantification is given in Subsection 4.2 (temporal predicate calculus).

§ In fact the variables are partitioned into rigid and flexible ones. A *rigid variable* must have the same value in all states of computation. Rigid data variables do not appear in the program itself and they are used to relate values at different states in the sequence. All state variables of transition systems are flexible.

** This interpretation is related to the theory of Pawlak's machines (Pawlak Z. 1971)

The Manna and Pnueli's temporal logic can be considered as an appropriate, and at the same time convenient language for specifying the dynamic behaviour of reactive programs and describing their properties (the interaction either between a program and its environment or between concurrent processes within a program). This language defines predicates over infinite sequences of states. Hence, any temporal logic formula is (in general) satisfied by some sequences and not satisfied by some other sequences and this formula, interpreted over a computation, will express a property related to this computation.

Let p be a *property* required for a given *program* P. We shall say that p is a *valid property* of P iff p is true of all computations generated by P. It can be observed that the finite set of all required properties related to P can be considered as a specification of P. And this approach enjoys the important advantage of incrementality. Moreover, this notion viewed as a set (or list) of properties can be used to any other program requiring the same set. In fact, a specification rarely specifies an unique program. Next we shall say that P has a *valid specification* or equivalently that P is an *acceptable solution* iff all required properties are valid. And hence, in accordance with Theorem 1.23 of Subsection 1.5, the specification validity of a reactive program should be a logical consequence wrt the a priori required properties associated with this program.

Example 2.11 (*Manna Z. and Pnueli A. 1992*)

Consider a program P implementing *mutual exclusion* between two processes P_1 and P_2. The following example properties of computations of P can be expressed in temporal logic.

(p_0) For all states of computation , it is never the case that P_1 and P_2 occupy their critical sections at the same state.

(p_1) If a computation σ contains a state at position $j \geq 0$ in which P_1 is waiting to enter the critical section then σ also contains a state at position $k \geq j$ in which P_1 is inside the critical section.

(p_2) The same requirement as p_1 but for process P_2.

In other words, here two processes request one distributed resource. At any time, only one of the processes at the most is allowed to use the resource.

Let p_0, p_1 and p_2 be valid properties of a program P. Hence, the *compound property* $p_0 \wedge p_1 \wedge p_2$ is valid and P should be considered as an acceptable solution of the mutual exclusion problem. □

Let $Sat(p) =_{df} \{\sigma / \sigma \text{ satisfy } p\}$ and $Comp(P) =_{df} \{\sigma / \sigma \text{ is a computation of } P\}$. The following definition was introduced.

Definition 2.8

We shall say that P *implements* the single specification p or P *satisfies* p iff $Comp(P) \subseteq Sat(p)$.

Corollary 2.7

Let $\{p_0, p_1, \ldots , p_{n-1}\}$ be a specification of P. If this specification is implemented by P then $Comp(P) \subseteq \bigcap_{i=0}^{n-1} Sat(p_i)$.

Proof:

Assume that all single specifications p_i are implemented by P. And so, by Definition 2.8 it follows that $\forall i \in \{0,1, \ldots ,n-1\}$ $(Comp(P) \subseteq Sat(p_i))$. But the following implication is a thesis:* $\forall i \in \{0,1, \ldots ,n-1\}$ $(Comp(P) \subseteq Sat(p_i)) \Rightarrow Comp(P) \subseteq \bigcap_{i=0}^{n-1} Sat(p_i)$. Hence, the proof is obtained immediately by using '$-$ C'. □

* $\forall i\, (X \subseteq X_i) \Rightarrow X \subseteq \bigcap_i X_i$: see Subsection 5.3 of Chapter III.

The main advantage of the Manna and Pnueli's temporal logic is the possibility of obtaining a succinct and natural expression of frequently occurring program properties using a set of special operators (called below *functors*). The proposed *language of temporal logic* is built from a state language, used to construct state formulae and a set of logical and temporal functors. A state formula can be evaluated at a certain position $j \geq 0$ in a sequence σ and it expresses properties of the state s_j occurring at this position. The *vocabulary* $\mathscr{V}$ of the state language consists of a countable set of typed data and control variables. The data variables range over data domains provided in the programming language, such as Booleans, integers, lists and sets. The control variables assume as values locations in programs. The type of each variable indicates the domain over which this variable ranges. The Boolean variables are usually referred to as *propositions*. In addition to the variables of $\mathscr{V}$ there are also assumed constants, functions and predicates (considered as concrete individual elements). An example semantics of some introduced constructs is given below.

Let $V \subseteq \mathscr{V}$. A *state* s *over* V is defined as an interpretation that assigns to each variable $u \in V$ a value from the appropriate domain, denoted by $s[u]$. A *model* σ *over* V is an infinite sequence of the form σ: $s_0, s_1, s_2, \ldots$, where each s_i is a state over V.

For any state s and expression e over V, *the value of an expression* e at s, denoted by $s[e]$ is defined inductively as follows: (1) the value of any variable $x \in V$ (possibly Boolean) is $s[x]$ and (2) the value of an expression $f(e_1, e_2, \ldots, e_m)$, i.e. $s[f(e_1, e_2, \ldots, e_m)] =_{df} f(s[e_1], s[e_2], \ldots, s[e_m])$.

The logical value of a formula of classical logic φ over a state s, denoted by $s[\varphi]$, is defined as follows: (1) *the logical value of any atomic formula* $s[P(e_1, e_2, \ldots, e_m)] =_{df} P(s[e_1], s[e_2], \ldots, s[e_m])$ and (2) *the logical values of any propositional formulae* φ and ψ: $s[\sim \varphi] =_{df} \sim s[\varphi]$, $s[\varphi \circ \psi] =_{df} s[\varphi] \circ s[\psi]$, for $\circ \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$.

Let s and s' be two states over V and $x \in V$. We shall say that s' is a *x-variant* of s iff $s'[y] = s[y]$ (for any $y \in V - \{x\}$). Any (propositional) state formulae φ and ψ over V are evaluated as follows: $s \Vdash \varphi$ iff $s[\varphi] = \text{true}$, where '$s \Vdash \varphi$' denotes 'φ is *true* or *holds* at s'. Obviously, the last two-argument predicate can be interpreted as a binary relation (called a *satisfaction relation*). And hence, in the case that $s[\varphi] = \text{true}$ we shall say that s *satisfies* φ and s is said to be a *φ-state*. In a similar way: $s \Vdash \sim\varphi$ iff $s \nVdash \varphi$ and $s \Vdash (\varphi \vee \psi)$ iff $(s \Vdash \varphi) \vee (s \Vdash \psi)$. The remaining logical functors can be expressed in terms of negation and disjunction and this is left to the reader (the quantification is omitted here: see Subsection 4.2 of Chapter II). Next any formula φ is said to be a *state-satisfiable* iff $\exists\, s\ (s \Vdash \varphi)$. And φ is *state-valid* iff $\forall\, s\ (s \Vdash \varphi)$. We shall say that φ and ψ are *state-equivalent* iff $\forall\, s\ ((s \Vdash \varphi) \Leftrightarrow (s \Vdash \psi))$ (or equivalently: $\forall\, s\ (s \Vdash (\varphi \Leftrightarrow \psi))$). It is sufficient to consider in these definitions only states over V that contain only the variables appearing in φ.

Example 2.12

Let $a, b, c \in \mathbb{Z}$ (the set of integer numbers) and $x \in V$ be an integer variable. Assume that $b =_{df} a + 1$ and $c =_{df} b + 1$. The formula $\varphi =_{df} (a < x) \wedge (x < c)$, in short: $a < x < c$, is state-satisfiable for s: (x: b) since $s \Vdash \varphi$. On the other hand, the formula $\psi =_{df} \sim((a < x < b) \vee (b < x < c))$ is state-valid since the disjunction $(a < x < b) \vee (b < x < c)$ is false in all states. It can be observed that φ and $\chi =_{df} x = b$ are state-equivalent, i.e. $\varphi \Leftrightarrow \chi$ is state-valid. □

The above notions of state-satisfiability, state-validity and state-equivalence can be restricted to some subset of states. Assume that C is a set of sequences, e.g. C may be the set of computations *Comp*(P) related to a program P. We shall say that a state s is *C-accessible* iff $\exists\, \sigma \in C\ \exists\, j \geq 0\ ((\sigma = s_0, s_1, \ldots, s_j, \ldots) \wedge (s = s_j))$. Let now φ and ψ be two state formulae. We shall say that φ is *C-state-satisfiable* iff $\exists\, s$ ((s is C-accessible) $\wedge\ (s \Vdash \varphi)$). Similarly, φ is *C-state-valid* iff $\forall\, s$ ((s is C-accessible) $\Rightarrow (s \Vdash \varphi)$). The formulae φ and ψ are *C-state-equivalent* iff $\forall\, s$ ((s is C-accessible) $\Rightarrow ((s \Vdash \varphi) \Leftrightarrow (s \Vdash \psi))$).

In accordance with the proposed Manna and Pnueli's language of temporal logic, a *temporal formula* is constructed from state formulae to which are applied some temporal functors, Boolean connectives and quantification. The proposed set of temporal functors is partitioned into two classes: future and past functors[*]. The interpretation of any temporal formula is based on the notion of a formula φ *holding at a position* j, $j \geq 0$, *in a sequence* σ, denoted by $(\sigma,j) \vDash \varphi$, where σ: $s_0, s_1, \dots, s_j, \dots$. And then $(\sigma,j) \vDash \varphi$ iff $\exists\, j \geq 0$ $(s_j \Vdash \varphi)$. Hence: $(\sigma,j) \vDash \sim\varphi$ iff $(\sigma,j) \nvDash \varphi$ and $(\sigma,j) \vDash (\varphi \vee \psi)$ iff $(((\sigma,j) \vDash \varphi) \vee ((\sigma,j) \vDash \psi))$. The extension of these definitions to other logical functors is left to the reader.

A more formal inductive definition of the notion of a *temporal propositional formula* is omitted here. As an example, this definition would be very similar to Definition 2.6 (a modal propositional formula), having some modification in step (2) wrt the introduced below future and past functors. And so, the *future functors* are first presented.

Assume that φ is a temporal propositional formula. Then such formula is also $\circ\varphi$, read as '*next* φ'.The semantics of this formula is defined as follows: $(\sigma,j) \vDash \circ\varphi$ iff $(\sigma,j+1) \vDash \varphi$, i.e. $\circ\varphi$ holds at position j iff φ holds at the next position $j+1$[†].

An illustration of using the last temporal functor are the next two examples. As in the case of modal or deontic logic systems, the priorities of the new introduced temporal functors can be introduced in a natural way. Obviously, any such functor should bind more strongly than the classical two-argument conjunction, disjunction, implication and equivalence. The *elementary state formulae* are denoted below by p, q, etc. And any such formula is related to the notion of a propositional variable. The verification process described below is very similar to the classical Hilbert´s zero-one verification method (see Subsection 1.4). However, this verification will require the use of the above definition of '$\circ$' by considering a pair of adjacent positions in σ, e.g. $j = k$ and $j = k+1$. Provided there is no ambiguity, instead of 0 and 1 (reserved for positions) the logical constants F and T are used.

Example 2.13

state formula	k	k + 1
p	T	F
~ p	F	T
$\circ$p	F	
$\circ\sim$ p	T	=
$\sim\circ$p	T	

The following formula is satisfied: $\sim\circ p \Leftrightarrow \circ\sim p$. To check this property, we have $2 \cdot 2 = 4$ possible cases. Since any pair of values can be presented as vertex of a graph, we can obtain the clique $K_{2,2}$ (Berge C 1973). The verification process related to the case $p =_{df} T$ at $j = k$ and $p =_{df} F$ at $j = k+1$ is presented in the above given table. Any other case can be considered in a similar way (left to the reader). And this property can be generalised for any state formula φ. □

Example 2.14

Consider the following formula: $\circ(p \vee q) \Leftrightarrow \circ p \vee \circ q$. The set of all possible logical values for the above formulae p and q should be taken into account for each of these two adjacent positions. And hence, we have $4 \cdot 4 = 16$ possible cases to be analysed (corresponding to the clique $K_{4,4}$).

The verification related to the case $(p, q) =_{df} (T, F)$ at $j = k$ and $(p, q) =_{df} (F, T)$ at $j = k+1$ is presented in table given below. Any other case can be considered in a similar way (left to the reader). Since

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[*] As an extension of Prior's tense logic system in the area of computer science

[†] This approach is very similar to the notion of an iterative array model (for synchronous sequential machines) where positions are known as '*time frames*' (Breuer M.A. and Friedman A.D 1977).

disjunction is associative the above considered formula can be extended for more than two, but finite number of, arguments.

A similar analysis is related to the following state formula: $\circ(p \wedge q) \Leftrightarrow \circ p \wedge \circ q$ (left to the reader). And hence, the temporal functor $\circ$ is distributive wrt the classical disjunction and conjunction. And this can be generalised for arbitrary state formulae φ and ψ and also for arbitrary (finite) linear combinations of such connectives. □

state formula	k	k + 1
p	T	F
q	F	T
$p \vee q$	T	T
$\circ p$	F	
$\circ q$	T	
$\circ(p \vee q)$	T	=
$\circ p \vee \circ q$	T	

The following rules can be obtained.

(60) *Rule of negating a temporal next functor* (denoted below by ´N○´):

$$N\circ : \frac{\sim \circ \varphi}{\circ \sim \varphi}$$

(61) *Distributive property rule for temporal next functor of disjunction* (denoted below by ´○A´):

$$\circ A : \frac{\circ(\varphi \vee \psi)}{\circ\varphi \vee \circ\psi}$$

(62) *Distributive property rule for temporal next functor of conjunction* (denoted below by ´○K´):

$$\circ K : \frac{\circ(\varphi \wedge \psi)}{\circ\varphi \wedge \circ\psi}$$

The following *De Morgan's laws* are satisfied (the proof of T 2.92 is left to the reader).

Thesis 2.91

$\sim \circ(p \vee q) \Leftrightarrow \sim \circ p \wedge \sim \circ q$

Proof:

$\sim \circ(p \vee q)$	$\Leftrightarrow$	$\circ \sim (p \vee q)$	{N○}
	$\Leftrightarrow$	$\circ (\sim p \wedge \sim q)$	{NA, SR}
	$\Leftrightarrow$	$\circ \sim p \wedge \circ \sim q$	{○K}
	$\Leftrightarrow$	$\sim \circ p \wedge \sim \circ q$. □	{ N○, SR}

Thesis 2.92

$\sim \circ(p \wedge q) \Leftrightarrow \sim \circ p \vee \sim \circ q$. □

In accordance with the last two theses, the following rules are obtained.

(63) *De Morgan´s law for temporal next functor of disjunction* (denoted below by ´N○A´):

$$\text{N}\bigcirc\text{A}: \quad \frac{\sim \bigcirc(\varphi \vee \psi)}{\sim \bigcirc\varphi \wedge \sim \bigcirc\psi}$$

(64) *De Morgan´s law for temporal next functor of conjunction* (denoted below by ´N○K´):

$$\text{N}\bigcirc\text{K}: \quad \frac{\sim \bigcirc(\varphi \wedge \psi)}{\sim \bigcirc\varphi \vee \sim \bigcirc\psi}$$

The next two functors are very similar to the modal functors of necessity and possibility. Hence, provided there is no ambiguity, the same designations are used. The '*always*' functor (known also as '*henceforth*' or '*from now on*' functor) is first presented.

Let φ be a temporal propositional formula. Such formula is also $\Box\,\varphi$, read as '*always* φ' (or '*henceforth* φ'). Its semantics is defined as follows: $(\sigma,j) \vDash \Box\,\varphi$ iff $\forall\, k\, (k \geq j \Rightarrow ((\sigma,k) \vDash \varphi))$, i.e. $\Box\,\varphi$ holds at j iff φ holds at j and all following positions '*from now on*' and hence the set of positions satisfying $\Box\,\varphi$ in a sequence is *upwards closed*. Consequently, the following implication is satisfied (if $\Box\,\varphi$ holds at j): $\Box\,\varphi \Rightarrow \varphi$, read as '*if always* φ *then* φ' (similarly to the Gödel's axiom G1).

If φ is a temporal formula, then so is $\Diamond\varphi$, read as '*eventually* φ' (or '*sometimes* φ'). Its semantics is as follows: $(\sigma,j) \vDash \Diamond\varphi$ iff $\exists\, k \geq j\, ((\sigma,k) \vDash \varphi)$. The set of positions satisfying $\Diamond\varphi$ in a sequence is downwards closed, i.e. if $\Diamond\varphi$ holds at j, then it also holds at any k such that $0 \leq k \leq j$. As in modal logic, this functor is *dual* to the previous one, i.e. $\Diamond\varphi$ holds at j iff $\Box \sim \varphi$ does not hold there. And these two future functors are *mutually dual* and any of these two connectives can be expressed by the another one. In fact, directly by the above two definitions and in accordance with De Morgan's laws for quantifiers we have: $\sim \underset{k\geq j}{\exists} \phi(k) \Leftrightarrow \underset{k\geq j}{\forall} \sim \phi(k)$, where $\phi(k) =_{df}$ '$(\sigma,k) \vDash \varphi$' (see Subsection 3.3 of the next Chapter II). Here, for convenience, instead of the standard quantifiers their equivalent bounded versions are used, i.e. restricted to the range of k.

According to the existing dependencies between some modal and temporal expressions, some theses in modal logic can be in a natural way interpreted as theses in temporal logic, e.g. $\Box\,\varphi \Rightarrow \Diamond\varphi$, i.e. T 2.18, T 2.29 (G2), De Morgan's laws, and so on. An example is illustrated by the next thesis (here, the set of all k such that $k \geq j$ is considered as an universe and denoted by J).

Thesis 2.93

$\Box\,p \Rightarrow \Diamond p$

Proof:

(1)	$\Box\,p$	{a}
(2)	$\sim \Diamond p$	{aip}
(3)	$\underset{k\geq j}{\forall}\ \phi(k)$	{df. '□', SR : 1}
(4)	$\sim \underset{k\geq j}{\exists}\ \phi(k)$	{df. '◇', SR : 2}
(5)	$\underset{k\geq j}{\forall} \sim \phi(k)$	{N∃* : 4}

(6)	$k_0 \geq j \Rightarrow \phi(k_0)$	$\{- \forall^*: 3\}$
(7)	$k_0 \geq j \Rightarrow \sim \phi(k_0)$	$\{- \forall^*: 5\}$
(8)	$k_0 \geq j$	$\{k_0 \in J\}$
(9)	$\phi(k_0)$	$\{- C : 6, 8\}$
(10)	$\sim \phi(k_0)$	$\{- C : 7, 8\}$
	contr. □	$\{9, 10\}$

Thesis 2.94

$\Box(p \wedge q) \Leftrightarrow \Box p \wedge \Box q$

Proof:

Since $(\Box(p \wedge q) \Leftrightarrow \Box p \wedge \Box q) \Leftrightarrow ((\sigma,j) \models \Box(p \wedge q) \Leftrightarrow ((\sigma,j) \models \Box p \wedge (\sigma,j) \models \Box q)$ we can obtain:

$$(\sigma,j) \models \Box(p \wedge q) \Leftrightarrow \forall_{k \geq j} ((\sigma,k) \models (p \wedge q)) \qquad \{\text{df. '}\Box\text{'}\}$$

$$\Leftrightarrow \forall_{k \geq j} (((\sigma,k) \models p) \wedge ((\sigma,k) \models q)) \qquad \{\text{df. '}\Box(p \wedge q)\text{'}\}$$

$$\Leftrightarrow \forall_{k \geq j} ((\sigma,k) \models p) \wedge \forall_{k \geq j} ((\sigma,k) \models q) \qquad \{\models \forall_{A(x)} (B(x) \wedge C(x)) \Leftrightarrow \forall_{A(x)} B(x) \wedge \forall_{A(x)} C(x)\}$$

$$\Leftrightarrow (\sigma,j) \models \Box p \wedge (\sigma,j) \models \Box q. \ \Box \qquad \{\text{df. '}\Box\text{'}\}$$

Here $A(x)$, $B(x)$ and $C(x)$ denote one-argument predicates. The proofs of the next theses is very similar to the proof of the above T 2.94 and hence it is left to the reader. And so, the first-order predicate calculus should be used (*bounded quantifiers*: see: Subsection 3.3, Chapter II), e.g. the proof of T 2.99 given below is related to the following thesis: $(\exists_{A(x)} B(x) \Rightarrow \exists_{A(x)} C(x)) \Rightarrow \exists_{A(x)} (B(x) \Rightarrow C(x))$.

Thesis 2.95

$\Diamond(p \vee q) \Leftrightarrow \Diamond p \vee \Diamond q$. □

Thesis 2.96

$\Box p \vee \Box q \Rightarrow \Box(p \vee q)$. □

Thesis 2.97

$\Diamond(p \wedge q) \Rightarrow \Diamond p \wedge \Diamond q$. □

Thesis 2.98

$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$. □

Thesis 2.99

$(\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond(p \Rightarrow q)$. □

De Morgan's laws are presented as follows.

Thesis 2.100

$\sim \Box (p \wedge q) \Leftrightarrow \sim \Box p \vee \sim \Box q$. □

Thesis 2.101

$\sim \Diamond (p \vee q) \Leftrightarrow \sim \Diamond p \wedge \sim \Diamond q$. □

Obviously, as in the previous considerations, T 2.93 – T 2.101 can be generalised for any temporal formulae (this is omitted).

Let now φ and ψ be two temporal formulae. Then such formula is also $\varphi \mathcal{U} \psi$, read as 'φ *until* ψ'. Since $\Diamond\psi$ predicts the eventual occurrence of ψ and $\Box \varphi$ states that φ will hold continuously from now on, then the above *until-formula* will combine these two statements by predicting the eventual occurrence of ψ and stating that φ holds continuously at least until the (first) occurrence of ψ. The semantics of this formula is defined as follows: $(\sigma,j) \vDash \varphi \mathcal{U} \psi$ iff $\exists k \geq j ((\sigma,k) \vDash \psi) \wedge \forall i (j \leq i < k \Rightarrow (\sigma,i) \vDash \varphi)$. If $\varphi \mathcal{U} \psi$ holds at j then also $\Diamond\psi$ holds there, i.e. since '$\exists k \geq j ((\sigma,k) \vDash \psi) \wedge \forall i (j \leq i < k \Rightarrow (\sigma,i) \vDash \varphi) \Rightarrow \exists k \geq j ((\sigma,k) \vDash \psi)$' is a thesis, the following implication is satisfied.

Thesis 2.102

$p \mathcal{U} q \Rightarrow \Diamond q$. □

The last of the presented class of future functors is the *unless* (*waiting-for*) functor $\mathcal{W}$ introduced as follows: $(\sigma,j) \vDash \varphi \mathcal{W} \psi$ iff $((\sigma,j) \vDash \varphi \mathcal{U} \psi) \vee ((\sigma,j) \vDash \Box \varphi)$, where the formula $\varphi \mathcal{W} \psi$ is read as 'φ *unless* ψ' (or 'φ *waiting-for* ψ'). This weaker property states that φ holds continuously either until the next occurrence of ψ or throughout the sequence.

The above Manna and Pnueli's set of basic future functors can be completed by introducing the following one: φ / ψ, read as '*first* φ *then* ψ'. The proposed semantics can be as follows: $(\sigma,j) \vDash \varphi / \psi$ iff $\exists k \exists i ((j \leq k < i) \wedge ((\sigma,k) \vDash \varphi) \wedge ((\sigma,i) \vDash \psi))$. Since $\exists k \exists i ((j \leq k < i) \wedge ((\sigma,k) \vDash \varphi) \wedge ((\sigma,i) \vDash \psi))$ iff $\exists k \geq j ((\sigma,k) \vDash \varphi) \wedge \exists i > k ((\sigma,i) \vDash \psi)$* then by assuming $(\sigma,j) \vDash \varphi / \psi$ we can obtain $\exists k \geq j ((\sigma,k) \vDash \varphi)$ and hence the following thesis is satisfied.

Thesis 2.103

$(p / q) \Rightarrow \Diamond p$. □

The last functor is *transitive* and we have (the proof is left to the reader).

Thesis 2.104

$(p / q) \wedge (q / r) \Rightarrow (p / r)$. □

The above presented future temporal functors require analysis also of forward positions of the considered program computation σ. A similar analysis related also to the backward positions of σ corresponds to the past temporal functors. A brief presentation of the Manna and Pnueli's past functors is given below. It is first presented the one-argument functor *previous*, denoted by '$\ominus$'.

Let φ be a temporal propositional formula. And so, such formula is also $\ominus\varphi$, read as '*previously* φ'. The following semantics is introduced: $(\sigma,j) \vDash \ominus\varphi$ iff $(j > 0) \wedge (\sigma,j-1) \vDash \varphi$. And hence, $\ominus\varphi$ is false at position 0.

The properties of the temporal previous functor are very similar to these associated with the functor '*next*'. And so we can obtain (for any $j > 0$): $\sim \ominus p \Leftrightarrow \ominus \sim p$, $\ominus(p \vee q) \Leftrightarrow \ominus p \vee \ominus q$ and $\ominus(p \wedge q) \Leftrightarrow \ominus p \wedge$

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* $\exists x (A(x)) \wedge \exists y (B(y)) \Leftrightarrow \exists x \exists y (A(x) \wedge B(y))$: see Subsection 3.3 of Chapter II.

$\ominus q$, e.g. the verification process of the last formula related to the case $(p, q) =_{df} (T, F)$ at $j = k - 1$ and $(p, q) =_{df} (F, T)$ at $j = k$ $(k > 0)$ is presented in table given below.

state formula	k – 1	k
p	T	F
q	F	T
p ∧ q	F	F
⊖p		T
⊖q		F
⊖(p ∧ q)	=	F
⊖p ∧ ⊖q		F

The following *De Morgan's laws* are satisfied (for any $j > 0$: the proof is left to the reader).

Thesis 2.105

$\sim \ominus(p \vee q) \Leftrightarrow \sim \ominus p \wedge \sim \ominus q$. ▫

Thesis 2.106

$\sim \ominus(p \wedge q) \Leftrightarrow \sim \ominus p \vee \sim \ominus q$. ▫

The properties of the next two past functors are very similar to the corresponding future functors '*always*' and '*eventually*'. The functor '*has-always-been*', denoted by '⊡' is introduced as follows: $(\sigma,j) \models \boxdot\varphi$ iff $\forall k(0 \le k \le j \Rightarrow ((\sigma,k) \models \varphi))$, i.e. $\boxdot\varphi$ holds at j iff φ holds at j and all preceding positions. And hence, the set of positions satisfying $\boxdot\varphi$ is *downwards closed*.

The functor '*once*', denoted by '⟐' is specified as follows: $(\sigma,j) \models$ ⟐φ iff $\exists k ((0 \le k \le j) \wedge ((\sigma,k) \models \varphi))$. And so, ⟐$\varphi$ holds at j iff φ holds at j or some preceding position. And this functor is also *dual* to the previous one, i.e. ⟐φ holds at j iff $\boxdot \sim \varphi$ does not hold there, i.e. ⊡ and ⟐ are *mutually dual*.

Example 2.15

An example evaluation of the last two past functors is presented in table given below.

j	0	1	2	3	4	5	...
x	1	2	5	2	3	4	...
x = 3	F	F	F	F	T	F	...
⊡(x ≤ 3)	T	T	F	F	F	F	...
⟐(x ≤ 3)	T	T	T	T	T	T	...
~ ⊡(x ≤ 3)	F	F	T	T	T	T	...
⟐(x > 3)	F	F	T	T	T	T	...
⟐(x = 3)	F	F	F	F	T	T	...
⟐(x ≠ 3)	T	T	T	T	T	T	...
⊡(x = 3)	F	F	F	F	F	F	...

$\sim \boxminus(x = 3)$	T	T	T	T	T	T	...

It can be observed that $\sim \boxminus(x = 3) \Leftrightarrow \diamondminus(x \neq 3)$ and $\sim \boxminus(x \leq 3) \Leftrightarrow \diamondminus(x > 3)$. □

The semantics of the next past functor '*since*', denoted by 'S' is defined as follows: $(\sigma,j) \vDash \varphi S \psi$ iff $\exists k ((0 \leq k \leq j) \wedge ((\sigma,k) \vDash \psi)) \wedge \forall i ((k < i \leq j) \Rightarrow ((\sigma,i) \vDash \varphi))$. And directly by this definition we have.

Thesis 2.107

$pSq \Rightarrow \diamondminus q$. □

A weaker version of the above functor *since* is the functor '*back-to*', denoted by 'B' and defined as follows: $(\sigma,j) \vDash \varphi B \psi$ iff either $(\sigma,j) \vDash \boxminus\varphi$ or $(\sigma,j) \vDash \varphi S \psi$.

The above introduced version of the functor ' *previous* ' can be considered as a *strong* one, i.e. $\ominus\varphi$ is false at position 0. In a similar manner as in the case of the functors '*until*' and '*since*', the following weaker version of the functor '*previous*' was introduced, read as '*weak previous*': $(\sigma,j) \vDash \tilde{\ominus}\varphi$ iff either $j = 0$ or $((j > 0) \wedge (\sigma,j-1) \vDash \varphi)$. And hence, $\tilde{\ominus}\varphi$ is always true at the first position. At all other positions these two versions are equivalent, i.e. $\ominus\varphi \Leftrightarrow \tilde{\ominus}\varphi$ (for any temporal propositional formula φ).

The notion of the Manna and Pnueli's temporal propositional formula is summarised by the next inductive definition.

Definition 2.9

A *temporal propositional formula* is:

1. Any propositional variable,
2. If φ and ψ are some temporal propositional formulae, then such formulae are also: $\sim(\varphi)$, $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $(\varphi) \Rightarrow (\psi)$, $(\varphi) \Leftrightarrow (\psi)$, $\bigcirc(\varphi)$, $\ominus(\varphi)$, $\tilde{\ominus}(\varphi)$, $\square(\varphi)$, $\boxminus(\varphi)$, $\diamond(\varphi)$, $\diamondminus(\varphi)$, $(\varphi)\, U\, (\psi)$, $(\varphi)\, S\, (\psi)$, $(\varphi)\, W\, (\psi)$, and $(\varphi)\, B\, (\psi)$,
3. Every temporal propositional formula in this propositional calculus either is a propositional variable or is formed from propositional variables by a single or multiple application of rule (2).

The notions of satisfiability and validity (see Definition 1.5, Subsection 1.4) can be extended to the case of a given temporal formula φ (Manna Z. and Pnueli A. 1992). And so, we shall say that $(\sigma,j) \vDash \varphi$ iff the model σ satisfies φ at position j, j is said to be a *φ-position*. The model σ satisfies φ (or equivalently: the program computation σ is a *φ-model*), i.e. $\sigma \vDash \varphi$ iff $(\sigma,0) \vDash \varphi$. Let C be the set of computations *Comp*(P) related to a program P. Then we shall say that φ is *C-satisfiable* iff $\exists \sigma \in C (\sigma \vDash \varphi)$ or also *C-valid* iff $\forall \sigma \in C (\sigma \vDash \varphi)$. Moreover, we shall say that φ and ψ are *equivalent* iff $\varphi \Leftrightarrow \psi$ is valid (have the same truth value at the first position of every model) and also φ and ψ are *congruent* iff $\square(\varphi \Leftrightarrow \psi)$ is valid (have the same truth value in all positions of every model). The following abbreviations were introduced: $\varphi \Rightarrow> \psi$ for $\square(\varphi \Rightarrow \psi)$ and $\varphi <\Leftrightarrow> \psi$ for $\square(\varphi \Leftrightarrow \psi)$, known in modal logic as strict implication and strict equivalence, respectively[*].

Let $Sat_\sigma(\varphi) =_{df} \{ j / (\sigma,j) \vDash \varphi \}$. And so, we can obtain: σ satisfies $\varphi \Rightarrow> \psi$ iff $Sat_\sigma(\varphi) \subseteq Sat_\sigma(\psi)$ and σ satisfies $\varphi <\Leftrightarrow> \psi$ iff $Sat_\sigma(\varphi) = Sat_\sigma(\psi)$.

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[*] Originally denoted as: $\varphi \Rightarrow \psi$ for $\square(\varphi \rightarrow \psi)$ and $\varphi \Leftrightarrow \psi$ for $\square(\varphi \leftrightarrow \psi)$

The temporal interpretation of SR is related to the above two notions of formulae equivalence and formulae congruence.

And so, the corresponding *rules of substitution for equivalence* and *for congruence* are introduced as follows:

1. If φ and ψ are equivalent then $\chi(\varphi)$ and $\chi(\psi)$ are equivalent (the *state substitutivity* case) and
2. If φ and ψ are congruent then $\chi(\varphi)$ and $\chi(\psi)$ are congruent (the *temporal substitutivity* case).

For example, since $\Box\, q \;<\!\Leftrightarrow\!>\; \sim\Diamond \sim q$ then $p\ \mathsf{U}\,(\Box\, q) \;<\!\Leftrightarrow\!>\; p\ \mathsf{U}\,(\sim\Diamond \sim q)$, similarly: $\Diamond(p \,\vee\, (\sim \bigcirc q\,)) \;<\!\Leftrightarrow\!>\; \Diamond p \,\vee\, \Diamond(\bigcirc \sim q)$ for $\sim \bigcirc q \;<\!\Leftrightarrow\!>\; \bigcirc \sim q$, etc.

A state formula can be considered as a formula without any temporal functors. Moreover, a *past* (a *future*) *formula* can be considered as a formula that contains no future (past) functors. The Manna and Pnueli's basic set of temporal functors is presented as follows: $\sim, \vee, \bigcirc, \mathsf{W}, \tilde{\ominus}, \mathsf{B}$. And hence, the remaining temporal functors can be expressed using the basic ones as it is shown below.

$\ominus p \;<\!\Leftrightarrow\!>\; \sim \tilde{\ominus} \sim p$

$\Box\, p \;<\!\Leftrightarrow\!>\; p\,\mathsf{W}\,\mathrm{F}$

$\boxdot\, p \;<\!\Leftrightarrow\!>\; p\,\mathsf{B}\,\mathrm{F}$

$\Diamond p \;<\!\Leftrightarrow\!>\; \sim \Box \sim p$

$\text{◈} p \;<\!\Leftrightarrow\!>\; \sim \boxdot \sim p$

$p\,\mathsf{U}\,q \;<\!\Leftrightarrow\!>\; (p\,\mathsf{W}\,q) \,\wedge\, \Diamond q$

$p\,\mathsf{S}\,q \;<\!\Leftrightarrow\!>\; (p\,\mathsf{B}\,q) \,\wedge\, \text{◈} q$

And so, to prove a certain property it is sufficient to consider only the above set of basic functors and all of their Boolean combinations.

In accordance with the above considerations, the present is considered to be a part of both the future and the past. A strict basic set of temporal functors, in which the present is neither a part of the future nor of the past, was also presented. Moreover, some basic properties of these temporal functors and corresponding inference rules were proposed.

And so, the proposed by Manna Z. and Pnueli A. (1992, 1995) *proof system for temporal logic uses* some sets of future and past axioms, primitive (originally called '*basic*') and also derived inference rules. Hence, the axiomatic system style was used. To obtain a proof system with a small number of temporal functors, as basic the above presented four functors $(\bigcirc, \mathsf{W}, \tilde{\ominus}, \mathsf{B})$ were accepted. The following *future and past axioms* were proposed.

(FA1)	$\Box\, p \;\Rightarrow\; p$	(PA1)	$\ominus p \;\Rightarrow\!>\; \tilde{\ominus} p$
(FA2)	$\bigcirc \sim p \;<\!\Leftrightarrow\!>\; \sim \bigcirc p$	(PA2)	$\tilde{\ominus}(p \;\Rightarrow\; q) \;<\!\Leftrightarrow\!>\; \tilde{\ominus} p \;\Rightarrow\; \tilde{\ominus} q$
(FA3)	$\bigcirc(p \;\Rightarrow\; q) \;<\!\Leftrightarrow\!>\; \bigcirc p \;\Rightarrow\; \bigcirc q$	(PA3)	$\boxdot(p \;\Rightarrow\; q) \;\Rightarrow\!>\; (\boxdot p \;\Rightarrow\; \boxdot q)$
(FA4)	$\Box\,(p \;\Rightarrow\; q) \;\Rightarrow\!>\; (\Box\, p \;\Rightarrow \Box\, q)$	(PA4)	$\boxdot p \;\Rightarrow\; \boxdot \tilde{\ominus} p$
(FA5)	$\Box\, p \;\Rightarrow\; \Box \bigcirc p$	(PA5)	$(p \;\Rightarrow\!>\; \tilde{\ominus} p) \;\Rightarrow\; (p \;\Rightarrow\!>\; \boxdot p)$
(FA6)	$(p \;\Rightarrow\!>\; \bigcirc p) \;\Rightarrow\; (p \;\Rightarrow\!>\; \Box\, p)$	(PA6)	$p\,\mathsf{B}\,q \;<\!\Leftrightarrow\!>\; q \,\vee\, p \,\wedge\, \tilde{\ominus}(p\,\mathsf{B}\,q)$
(FA7)	$p\,\mathsf{W}\,q \;<\!\Leftrightarrow\!>\; q \,\vee\, p \,\wedge\, \bigcirc(p\,\mathsf{W}\,q)$	(PA7)	$\tilde{\ominus}\mathrm{F}$
(FA8)	$\Box\, p \;\Rightarrow\!>\; p\,\mathsf{W}\,q$		

For example, in accordance with FA1 if p holds at all positions, then in particular p holds at the first position. FA2 corresponds to the self-dual property of the next functor and this functor is also distributive wrt implication (FA3). FA5 (*if always p then always next p*) is in accordance with the used definitions for $\Box$ and $\bigcirc$. Axiom FA7 represents a future expansion formula, i.e. $p\,\mathsf{W}\,q$ holds at position j iff either q holds at j or p holds at j and $p\,\mathsf{W}\,q$ holds at $j + 1$. Similarly e.g. FA8 claims $\Box\, p$ as one of the ways to satisfy $p\,\mathsf{W}\,q$ at j (since p holds at j and all following positions). The above presented past axioms are almost symmetric wrt the future ones. For example, PA6 represents a past expansion formula, i.e. $p\,\mathsf{B}\,q$ holds at position j iff either q holds at j or p holds at j and p B q holds at j – 1, if it exists. Axiom PA7 states that the first position of every sequence satisfies $\tilde{\ominus}$F. By the above introduced semantics of the weak previous functor it follows that (σ,j) ⊨ $\tilde{\ominus}$F iff j = 0.

In general, a *future* (a *past*) *expansion formula* expresses the value of a future (past) functor at position j as a function of the values of its arguments at j and the value of the functor itself at j + 1 (at j – 1, if it exists). The following expansion formulae for the other future functors were given (Manna Z. And Pnueli A. 1992).

□p <⇔> p ∧ ○□p
◇p <⇔> p ∨ ○◇p
p U q <⇔> q ∨ p ∧ ○(p U q)

Some past expansion formulae for other past functors were also introduced, e.g. ⊡p <⇔> p ∧ $\tilde{\ominus}$⊡p, i.e. ⊡p holds at position j iff p holds at j and ⊡p holds at j – 1, if it exists. Similar expressions were given for ⟐p and p S q (are symmetric to the above future expansion formulae: left to the reader). The obtained expansion formulae are provable, e.g. the proof of ⊡p <⇔> p ∧ $\tilde{\ominus}$⊡p follows immediately in accordance with the used definitions for ⊡ and $\tilde{\ominus}$.

It can be observed that any such future (past) expansion formula is provable in this system (a more formal treatment is omitted).

The following two *mixed axioms* were also presented.

(FA9) p ⇒> ○⊖p
(PA8) p ⇒> $\tilde{\ominus}$○p

For example, in accordance with FA9, if p holds at position j, then going forwards one step to j + 1 and then backwards one step, the same property for p at j is obtained.

In a natural manner the above axioms can be generalised for any two temporal propositional formulae φ and ψ. The *state-tautology axiom* (denoted by TAU) was introduced as follows.

$$\text{TAU}: \quad \frac{\varphi}{\Vdash \varphi}$$

Let φ be a state-valid formula. Then as a new proof line we can introduce ⊩ φ. The last axiom shows the link between state validity and temporal validity in this deductive system. In fact, '⊩ φ' requires the *state validity* of φ, i.e. φ holds on every state. On the other hand, '⊨ φ' requires *validity* of φ, i.e. φ to hold at the first state (of any sequence of states σ: s_0, s_1, s_2, ..., i.e. of every model). Obviously, as in the case of modal logic, any thesis of the classical propositional calculus is a state-valid in this system.

Primitive rules[*]

Any state-valid formula ⊩ φ , obtained by axiom TAU can be transformed into temporally valid one. And hence the following *generalisation rule* (in short: GEN) was introduced.

$$\text{GEN}: \quad \frac{\Vdash \varphi}{\vDash \Box\, \varphi}$$

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[*] Originally called "*basic inference rules*"

And so, if φ is state-valid then $\Box\, \varphi$ is temporally valid. The opposite implication originally called "*specialisation*" (in short: SPEC) is also satisfied.

$$\text{SPEC :} \quad \frac{\vDash \Box\, \varphi}{\Vdash \varphi}$$

Since φ is state-valid iff $\Box\, \varphi$ is temporally valid, the main connective is two-sided binding. And hence, without loss of generality, GEN and SPEC can be reduced to only one rule.

$$\text{INST :} \quad \frac{\chi}{\chi[\varphi]}$$

This *instantiation rule* (in short: INST) allows to infer the instantiated formula $\chi[\varphi]$ from the more general χ. We observe that INST is related to RR (the rule of definitional replacement of one formula by another: Subsection 1.7).

And finally, the rule of omitting an implication '$-$ C' is used (i.e. modus ponens, originally denoted by 'MP'). Here the main implication is represented in a more general form (related to the notion of logical consequence: see T 1.23 of Subsection 1.5). An illustration of using '$-$ C' is the following thesis.

Thesis 2.108

$\Box\, p \Rightarrow p \mathcal{W} q$

Proof:

(1)	$\Box\, p \Rightarrow\!> p \mathcal{W} q$	{FA8}
(2)	$\Box\, (\Box\, p \Rightarrow p \mathcal{W} q)$	{df. '$\Rightarrow\!>$': 1}
(3)	$\Box\, (\Box\, p \Rightarrow p \mathcal{W} q) \Rightarrow (\Box\, p \Rightarrow p \mathcal{W} q)$	{ FA1, INST}
	$\Box\, p \Rightarrow p \mathcal{W} q.$ □	{$-$ C : 2,3}

In a similar way, e.g. $\circleddash p \Rightarrow \tilde{\circleddash} p$ (the proof is left to the reader). As in the case of classical propositional calculus, any proof in this system can be interpreted as a process of joining new lines by using some axioms, primitive or derived rules and/or other theses in accordance with the used assumptions. Some selected derived rules of the Manna-Pnueli proof system are presented below*.

Derived rules

The following *temporalisation rule* can be obtained by assuming $\Vdash p$ and then using in turn GEN, FA1 and '$-$ C' (in short: TEMP: this is left to the reader).

$$\text{TEMP :} \quad \frac{\Vdash \varphi}{\vDash \varphi}$$

The next derived *rule of particularisation* (in short: PAR) is represented as follows.

* Originally, the notions of "conditional proof" and "premise" were used (instead of the notions of assumptional proof and assumption, respectively).

$$\text{PAR}: \quad \frac{\Box\,\varphi}{\varphi}$$

The proof of PAR can be obtained by assuming $\Box$ p, FA1 and '– C' (left to the reader). PAR corresponds to the rule ′- $\Box$′ used in modal logic. For example, according to Toll (see T 1.6 of Subsection 1.3) the following implication is satisfied.

Thesis 2.109

$(\Box\, p \Rightarrow \Diamond q) \wedge \sim \Diamond q \Rightarrow \sim \Box\, p$

Proof:

(1)	$\Vdash (p \Rightarrow q) \wedge \sim q \Rightarrow \sim p$	{TAU}
(2)	$\Box\,((p \Rightarrow q) \wedge \sim q \Rightarrow \sim p)$	{GEN : 1}
(3)	$\Box\,((\Box\, p \Rightarrow \Diamond q) \wedge \sim \Diamond q \Rightarrow \sim \Box\, p)$	{INST : 2}
	$(\Box\, p \Rightarrow \Diamond q) \wedge \sim \Diamond q \Rightarrow \sim \Box\, p$. □	{PAR : 3}

Let now $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$ be propositional state formulae and ψ be a logical consequence wrt $\varphi_1, \varphi_2, \ldots, \varphi_n$[*]. Since $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \Rightarrow \psi$ is a thesis, i.e. state-valid, then having lines $\varphi_1, \varphi_2, \ldots, \varphi_n$ to our proof we can join ψ (using '+ K' and '– C'). To abbreviate this process, in the Manna-Pnueli proof system was introduced a special derived rule called "*propositional reasoning*" (in short: PR) as it is shown below.

$$\text{PR}: \quad \frac{\varphi_1, \varphi_2, \ldots, \varphi_n}{\psi}$$

The following formula is a classical logic thesis (the proof is left to the reader).

$(q \Leftrightarrow F \vee p \wedge r) \Rightarrow (q \Rightarrow p)$

And hence, the last formula is state-valid, i.e. $\Vdash (q \Leftrightarrow F \vee p \wedge r) \Rightarrow (q \Rightarrow p)$. It is used in the proof of the next thesis.

Thesis 2.110

$p\ \mathsf{B}\ F \Rightarrow p$

Proof:

(1)	$p\ \mathsf{B}\ F <\!\Leftrightarrow\!> F \vee p \wedge \tilde{\ominus}(p\ \mathsf{B}\ F)$	{PA6: $q =_{df} F$}
(2)	$p\ \mathsf{B}\ F \Leftrightarrow F \vee p \wedge \tilde{\ominus}(p\ \mathsf{B}\ F)$	{ df. '<⇔>', PAR : 1}
	$p\ \mathsf{B}\ F \Rightarrow p$. □	{PR : 2}

In accordance with T 2.110, the above text "PR : 2" is equivalent to the following one: "{$\Vdash (q \Leftrightarrow F \vee p \wedge r) \Rightarrow (q \Rightarrow p)$, $q =_{df} p\ \mathsf{B}\ F$, $r =_{df} \tilde{\ominus}(p\ \mathsf{B}\ F)$, TEMP, INST, – C : 2}". In a similar way '$p\ \mathsf{W}\ F \Rightarrow p$' (left to the reader).

The following thesis is satisfied.

Thesis 2.111

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[*] see T 1.23 of Subsection 1.5

$(p_1 \wedge p_2 \wedge ... \wedge p_n \Rightarrow> q) \Rightarrow (\Box p_1 \wedge \Box p_2 \wedge ... \wedge \Box p_n \Rightarrow \Box q)$

Proof:

(1)	$p_1 \wedge p_2 \wedge ... \wedge p_n \Rightarrow> q$	{a}
(2)	$\Box (p_1 \wedge p_2 \wedge ... \wedge p_n)$	
(3)	$\Box (p_1 \wedge p_2 \wedge ... \wedge p_n \Rightarrow q)$	{df. '⇒>': 1}
(4)	$\Box (p_1 \wedge p_2 \wedge ... \wedge p_n) \Rightarrow \Box q$	{– C : T 2.98 / $\varphi =_{df} p_1 \wedge p_2 \wedge ... \wedge p_n$ and $\psi =_{df} q$: 3}
	$\Box$ q. □	{– C : 2,4}

Obviously, proof line (4) of the last thesis can be obtained by using in turn FA4, df. '⇒>' and PAR (left to the reader). Consequently, the following derived rule of *entailment*[*] *omission of implication* is obtained. This rule was originally called "*entailment modus ponens*"(in short: E-MP). Equivalently bellow is used "E(– C)".

$$\text{E(– C)}: \quad \frac{\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n \Rightarrow> \psi \quad \Box \varphi_1 \quad \Box \varphi_2 \quad ... \quad \Box \varphi_n}{\Box \psi}$$

The following proof of the next rule "*entailment transitivity*" (in short: E-TRNS) was given. For convenience instead of E-TRNS, the abbreviation "ETC", i.e. *entailment transitivity for implication*, is used below.

Thesis 2.112

$(p \Rightarrow> q) \wedge (q \Rightarrow> r) \Rightarrow (p \Rightarrow> r)$

Proof:

(1)	$p \Rightarrow> q$	{a}
(2)	$q \Rightarrow> r$	
(3)	$\Box (p \Rightarrow q)$	{df. '⇒>': 1}
(4)	$\Box (q \Rightarrow r)$	{df. '⇒>': 2}
(5)	$\Vdash (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	{TAU}
(6)	$\Box ((p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r))$	{GEN : 5}
(7)	$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow> (p \Rightarrow r)$	{df. '⇒>': 6}
(8)	$\Box (p \Rightarrow r)$	{ E(– C) : 3,4,7}
	$p \Rightarrow> r$. □	{ df. '⇒>': 8}

The proof of T 2.112 can be realised without using E(– C): see T 2.33 (modal logic: left to the reader). Consequently, the following derived rule is obtained.

[*] The notion of *entailment* is used in (at least) three meanings: *implication connective* (having some properties), the name of the *logical system* characterising this connective as well as the *area* in which this system is defined (see Subsection 2.4: Relevance logic).

$$\text{ETC}: \quad \frac{\begin{array}{l} p \Rightarrow> q \\ q \Rightarrow> r \end{array}}{p \Rightarrow> r}$$

In accordance with rule PR, $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \Rightarrow \psi$ is a thesis (i.e. state-valid). Hence, using in turn GEN, FA4, df. '⇒>', PAR, T 2.94 (extended for more than two formulae) and SR, we can obtain.

$$\Box\, \varphi_1 \wedge \Box\, \varphi_2 \wedge \ldots \wedge \Box\, \varphi_n \Rightarrow \Box\, \psi$$

We observe that the last formula is similar to the notion of modal logical consequence (see modal logic). And so, as a result, the rule of "*entailment propositional reasoning*" (in short: E-PR) can be obtained. Below is used the abbreviation "EPR".

$$\text{EPR}: \quad \frac{\Box\, \varphi_1, \Box\, \varphi_2, \ldots, \Box\, \varphi_n}{\Box\, \psi}$$

And so, using EPR the proof of T 2.112 can be reduced. In fact, we have a possibility of omitting proof lines (5), (6) and (7).

The rule of omitting a strict equivalence in the Manna-Pnueli proof system is the same as in the case of modal logic (see rule ′– OSE′). In particular, the following thesis is satisfied.

Thesis 2.113

$(p <\Leftrightarrow> q) \Rightarrow (p \Rightarrow> q)$

Proof:

(1)	$p <\Leftrightarrow> q$	{a}
(2)	$\Box\,(p \Leftrightarrow q)$	{ df. '<⇔>': 1}
(3)	$\Box\,((p \Rightarrow q) \wedge (q \Rightarrow p))$	{– E, SR : 2}
(4)	$\Box\,(p \Rightarrow q) \wedge \Box\,(q \Rightarrow p)$	{T 2.94 : 3}
(5)	$\Box\,(p \Rightarrow q)$	{– K : 4}
	$p \Rightarrow> q$. ▫	{ df. '⇒>': 5}

Thesis 2.114

$\circ(p \Rightarrow q) \Leftrightarrow \circ p \Rightarrow \circ q$

Proof:

$\circ(p \Rightarrow q)$	$\Leftrightarrow$	$\circ(\sim p \vee q)$	{CR, SR}
	$\Leftrightarrow$	$\circ\sim p \vee \circ q$	{○A : see Example 2.14: $\circ(p \vee q) \Leftrightarrow \circ p \vee \circ q$}
	$\Leftrightarrow$	$\sim \circ p \vee \circ q$	{ N○, SR : see Example 2.13: $\sim \circ p \Leftrightarrow \circ \sim p$}
	$\Leftrightarrow$	$\circ p \Rightarrow \circ q$. ▫	{CR}

The following two theses are satisfied (the *monotonicity property* of a temporal next functor).

Thesis 2.115

$(p \Rightarrow> q) \Rightarrow (\circ p \Rightarrow> \circ q)$

Proof:

(1) $p \Rightarrow\!> q$ {a}
(2) $\Box (p \Rightarrow q)$ {df. '$\Rightarrow\!>$': 1}
(3) $\Box\bigcirc(p \Rightarrow q)$ {– C : FA5, 2 / $\varphi =_{df}$ '$p \Rightarrow q$'}
(4) $\Box(\bigcirc p \Rightarrow \bigcirc q)$ {T 2.114, SR : 3}
$\bigcirc p \Rightarrow\!> \bigcirc q$. □ {df. '$\Rightarrow\!>$': 4}

The proof of the next thesis is left to the reader.

Thesis 2.116

$(p <\!\Leftrightarrow\!> q) \Rightarrow (\bigcirc p <\!\Leftrightarrow\!> \bigcirc q)$. □

Any (primitive or derived) rule of the classical propositional calculus is a state-valid formula. And hence, the following two formulae are state-valid: $p \wedge q \Rightarrow p$ and $p \wedge q \Rightarrow q$ (in accordance with '– K': see Subsection 1.2). The following thesis is satisfied.

Thesis 2.117

$\bigcirc p \wedge \boxdot p \Rightarrow\!> \bigcirc p$

Proof:

(1) $\Vdash p \wedge q \Rightarrow p$ {TAU}
(2) $\Box (p \wedge q \Rightarrow p)$ {GEN: 1}
(3) $\Box (\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc p)$ {INST : 2 / $\varphi =_{df}$ '$\bigcirc$ p, $\psi =_{df} \boxdot$ p'}
$\bigcirc p \wedge \boxdot p \Rightarrow\!> \bigcirc p$. □ {df. '$\Rightarrow\!>$': 3}

In a similar way we can obtain: $\bigcirc p \wedge \boxdot p \Rightarrow\!> \boxdot p$ (left to the reader).

To end with this Subsection, it is given below a proof of the following example formula (Manna Z. and Pnueli A. 1992).

Thesis 2.118

$\bigcirc p \wedge \boxdot p \Rightarrow\!> \bigcirc\boxdot p$

Proof:

(1) $\ominus\boxdot p \Rightarrow\!> \tilde{\ominus}\boxdot p$ {PA1 / $\varphi =_{df}$ '$\boxdot$ p'}
(2) $\bigcirc\ominus\boxdot p \Rightarrow\!> \bigcirc\tilde{\ominus}\boxdot p$ {– C : T 2.115, 1 / $\varphi =_{df}$ '$\ominus\boxdot$ p, $\psi =_{df} \tilde{\ominus}\boxdot$ p'}
(3) $\boxdot p \Rightarrow\!> \bigcirc\ominus\boxdot p$ {FA9 / $\varphi =_{df}$ '$\boxdot$ p'}
(4) $\boxdot p \Rightarrow\!> \bigcirc\tilde{\ominus}\boxdot p$ {ETC : 2,3}
(5) $\bigcirc p \wedge \boxdot p \Rightarrow\!> \bigcirc p$ {T 2.117}
(6) $\bigcirc p \wedge \boxdot p \Rightarrow\!> \boxdot p$ {similarly as in 5}
(7) $\bigcirc p \wedge \boxdot p \Rightarrow\!> \bigcirc\tilde{\ominus}\boxdot p$ {ETC : 4,6}
(8) $(\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc p) \wedge (\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc\tilde{\ominus}\boxdot p) \Rightarrow (\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc p \wedge \bigcirc\tilde{\ominus}\boxdot p)$ {T 1.5b : generalised for arbitrary φ, ψ and η: by INST we have: $\varphi =_{df}$ '$\bigcirc p \wedge \boxdot p$', $\psi =_{df}$ '$\bigcirc p$' and $\eta =_{df}$ '$\bigcirc\tilde{\ominus}\boxdot$ p'}
(9) $(\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc p) \wedge (\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc\tilde{\ominus}\boxdot p) \Rightarrow\!> (\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc p \wedge \bigcirc\tilde{\ominus}\boxdot p)$ {GEN, df. '$\Rightarrow\!>$': 8 / 8 is state-valid}
(10) $\Box (\bigcirc p \wedge \boxdot p \Rightarrow \bigcirc p)$ {df. '$\Rightarrow\!>$': 5}

(11)	□ (○p ∧ ⊡p ⇒ ○⊖̃⊡p)	{df. '⇒>': 7}
(12)	□ (○p ∧ ⊡p ⇒ ○p ∧ ○⊖̃⊡p)	{ E(– C) : 9,10,11}
(13)	○p ∧ ⊡p ⇒> ○p ∧ ○⊖̃⊡p	{ df. '⇒>': 12}
(14)	p ∧ ⊖̃⊡p ⇒> ⊡p	{T 2.113 : ⊡ p <⇔> p ∧ ⊖̃⊡ p / a past expansion formula}
(15)	○(p ∧ ⊖̃⊡p) ⇒> ○⊡p	{– C : T 2.115,14 / $\varphi =_{df}$ 'p ∧ ⊖̃⊡ p', $\psi =_{df}$ '⊡ p' }
(16)	○p ∧ ○⊖̃⊡p ⇒> ○(p ∧ ⊖̃⊡p)	{T 2.115, Example 2.14: ○(p ∧ q) ⇔ ○p ∧ ○q / $\varphi =_{df}$ 'p, ' $\psi =_{df}$ '⊖̃⊡', GEN, df. '⇒>'}
(17)	○p ∧ ○⊖̃⊡p ⇒> ○⊡p	{ETC : 15,16}
	○p ∧ ⊡p ⇒> ○⊡p. □	{ETC : 13,17}

The above presented Prior's tense logic and the extension of this system in the area of computer science, i.e. the Manna-Pnueli proof system, can be considered as two basic temporal logic systems. Other important such systems have been also developed, e.g. interval temporal logic (reasoning about periods of time, see: Moszkowski B. 1986), μ-calculus and so on. Some introductory notions related to the theory of the modal μ-calculus are given below (for a more detailed information, see: Venema Y. 2008)*.

Modal μ-calculus and dynamic logic

We shall give first some introductory notions concerning sets and used below (see Subsection 5.2 of Chapter III of the last work: *Basic notions and definitions*).

Let $\rho \subseteq A \times B$ be a *binary relation*, where A and B are two sets and $a\rho b \Leftrightarrow_{df} (a,b) \in \rho$ (for any $a \in A$ and $b \in B$). By ρ^{-1} we shall denote the *converse* of ρ, i.e. $a\rho^{-1}b \Leftrightarrow_{df} b\rho a$. For $X \subseteq A$ we define $\rho[X] =_{df} \{b \in B \,/\, \exists_{a \in X} (a\rho b)\} \subseteq B$ and $\rho[x] =_{df} \rho[\{x\}]$ if $X =_{df} \{x\}$ is a singleton. For $Y \subseteq B$ the set $\langle\rho\rangle Y =_{df} \rho^{-1}[Y]$, while $[\rho]Y =_{df} \{a \in A \,/\, \forall_{b \in Y} (a\rho b)\} \subseteq A$. Obviously, by omitting all $a \in A$ related to $B - Y$ we can obtain all the elements $a \in A$ related to Y. And hence: $[\rho]Y = A - \langle\rho\rangle(B - Y)$, where '−' denotes set difference (the reader is invited to show this property).

The propositional modal μ-calculus (in short: *μ-calculus*) originates with Scott D.S. and J.W.de Bakker†, further developed by Hitchcock P. and Park D.M.R. (1973) and others. A more complete study was presented by Kozen D.C. (1983). The results of the last work were mostly inspired by the work of Pratt V.R. (1981), in particular the Pratt's propositional Pμ calculus.

In general, the propositional modal μ-calculus is an extension of the classical propositional modal logic and it can be considered as a multimodal system having two additional fixpoint operators: a *least fixpoint operator* μ and a *greatest fixpoint operator* ν (more strictly: ´μx´ and ´νx´, where the variable $x \in P$, see below). Such logic is said to be *fixed-point* and it is used for description and verification of a special class of systems, called *labelled transition systems*. Any such system can be considered as a process graph.

Let P be a set of *proposition letters* and D be a set of *atomic actions* (called also *labels*). The following definition can be introduced (by $\mathbb{P}(S)$ it is denoted the *power set* of S, i.e. the set of all subsets of S: obviously ∅, $S \in \mathbb{P}(S)$). The elements of P are denoted as: p, q, r, x, y, z, ... ,and the elements of D as: d, c, e, ... Below we shall assume that p, q, r, ... are propositional variables, called here *atomic propositions* (and also any such propositional formulae defined inductively: see Definition 1.1 of Subsection 1.1). On the other hand, the

* See also: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

† Jaco W.de Bakker (1939 – 2012)

propositional letters x, y, z, ... are here considered as some *free* or *bound variables* of a formula φ[*]. The last notions are similar to these ones, used in the first-order predicate logic wrt the introduced here (least and greatest) fixpoint operators.

Definition 2.10

A *(P,D)-(labelled) transition system* or *(P,D)-Kripke model* is the following triple $\mathbb{S} =_{df} (S,V,R)$, where: S is a set of objects called *states* or *points*, $V : P \to \mathbb{P}(S)$ is a valuation and $R =_{df} \{\rho_d \subseteq S \times S / d \in D\}$ is a family of binary *accessibility relations*[†]. The pair (P, D) is said to be a *type* of $\mathbb{S}$. The *set of all d-successors of s* (i.e. all states $t \in S$ which are out-incident to s in ρ_d) is defined as follows: $\rho_d[s] =_{df} \{t \in S \ / \ s \rho_d t\}$, for any $s \in S$. The system $\mathbb{S}$ is *finitely branching* (or *image-finite*) if $\rho_d[s]$ is finite (for any $d \in D$ and $s \in S$). The notion of a *pointed transition system* or *Kripke model* is introduced as a pair $(\mathbb{S}, s)$, where s is a designated state in $\mathbb{S}$.

For convenience, the following alternative, *coalgebraic* representation of $\mathbb{S}$ can be obtained (Venema Y. 2008): instead of V equivalently the map $\sigma_V : S \to \mathbb{P}(P)$ can be used, assigning to each state $s \in S$ the subset of atomic propositions that hold in s (i.e. which atomic propositions are true at each state). Similarly, any binary relation $\rho \subseteq S \times S$ can be represented as a map $\rho[\cdot] : S \to \mathbb{P}(S)$, mapping a state $s \in S$ to the subset $\rho[s]$ of its successors. And hence, the set R of binary accessibility relations can be seen as a map $\sigma_R : S \to \mathbb{P}(S)^D$, where $\mathbb{P}(S)^D$ is the set of all maps from D to $\mathbb{P}(S)$.

As a consequence, any transition system $\mathbb{S}$ can be equivalently defined as a pair (S, σ), where $\sigma : S \to \mathbb{P}(P) \times \mathbb{P}(S)^D$ such that $\sigma(s) =_{df} (\sigma_V(s), \sigma_R(s))$, for any $s \in S$. Next, by $K = K_{D,P}S =_{df} \mathbb{P}(P) \times \mathbb{P}(S)^D$ we shall denote the *Kripke functor associated with* D *and* P. Hence, $(X,Y) \in K$ iff $X \subseteq P$ and $Y =_{df} \{Y_d \subseteq S \ / \ d \in D\}$. Obviously, the last family Y corresponds in an unique way to some map $h_Y \in \mathbb{P}(S)^D$. In particular, the Kripke models are sometimes referred as *$K_{D,P}S$ - coalgebras* or *Kripke coalgebras*. And so, any state transition system $\mathbb{S}$ can be equivalently considered as a Kripke coalgebra.

Example 2.16

Let $D =_{df} \{d_1, d_2, d_3\}$ and $S =_{df} \{s_1, s_2\}$. Hence, $\mathbb{P}(S) = \{\varnothing, \{s_1\},\{s_2\},\{s_1,s_2\}\}$. Since D is finite, $h_Y \in \mathbb{P}(S)^D$ may be represented as a vector, e.g. $\underline{h}_Y =_{df} (\{s_2\},\{s_2\},\{s_1,s_2\})$. Then, the following multifamily can be obtained: $Y = \{\{s_2\},\{s_2\},\{s_1,s_2\}\}$, where $Y_{d_1} = Y_{d_2} = \{s_2\}$ and $Y_{d_3} = \{s_1,s_2\}$. □

The *polymodal logic in* D *and* P, in short: *PML(D,P)*, introduced here can be considered as a generalization of the *basic modal logic* system (called also *ordinary* or *monomodal logic*: see Definition 2.6, by assuming that D is a singleton). And so, in the case of polymodal logic the modal functors of necessity and possibility are *D-indexed*, i.e. instead of $\Box \varphi$ and $\Diamond\varphi$ we have: $\Box_d \varphi$ and $\Diamond_d \varphi$, respectively. The last two functors can be interpreted using the corresponding accessibility relation ρ_d. And hence, the notion of *truth* (or *satisfaction*) is defined as follows.

Definition 2.11

[*] Provided there is no ambiguity, the originally notion "propositional variable" used for the free or bound variables is omitted here. In fact, this notion is contradictory to the classical one, e.g. under Definition 1.1 of Subsection 1.1 (Słupecki J. and Borkowski L. 1967).

[†] It can be observed that the graph of $\mathbb{S}$ can be equivalently described by using the following ternary *labelled transition relation*: $\varsigma \subseteq S \times D \times S$, rather than the family R.

Let $\mathbb{S} = (S, \sigma)$ be a transition system of type (P, D). The *satisfaction relation* '$\Vdash$' between states of $\mathbb{S}$ and polymodal logic formulae can be defined inductively as follows (Venema Y. 2008)*.

$$
\begin{array}{lll}
\mathbb{S}, s \Vdash p & \Leftrightarrow_{df} & s \in V(p), \\
\mathbb{S}, s \Vdash \sim p & \Leftrightarrow_{df} & s \notin V(p), \\
\mathbb{S}, s \Vdash F & & \text{never}, \\
\mathbb{S}, s \Vdash T & & \text{always}, \\
\mathbb{S}, s \Vdash \varphi \wedge \psi & \Leftrightarrow_{df} & \mathbb{S}, s \Vdash \varphi \text{ and } \mathbb{S}, s \Vdash \psi, \\
\mathbb{S}, s \Vdash \varphi \vee \psi & \Leftrightarrow_{df} & \mathbb{S}, s \Vdash \varphi \text{ or } \mathbb{S}, s \Vdash \psi, \\
\mathbb{S}, s \Vdash \Box_d \varphi & \Leftrightarrow_{df} & \mathbb{S}, t \Vdash \varphi \text{ for all } t \in \rho_d[s], \\
\mathbb{S}, s \Vdash \Diamond_d \varphi & \Leftrightarrow_{df} & \mathbb{S}, t \Vdash \varphi \text{ for some } t \in \rho_d[s].
\end{array}
$$

Here, '$\mathbb{S}, s \Vdash \varphi$' denotes '$\varphi$ is *true* or *holds* at s'.

In accordance with the above considerations, we can obtain: $\mathbb{S}, s \Vdash \Box_d \varphi \Leftrightarrow \forall_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)$. In the same way: $\mathbb{S}, s \Vdash \Diamond_d \varphi \Leftrightarrow \exists_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)$. Hence, the following thesis (similar to T 2.93) can be obtained.

Thesis 2.119

$\mathbb{S}, s \Vdash \Box_d \varphi \Rightarrow \mathbb{S}, s \Vdash \Diamond_d \varphi$. □

Let now $[|\varphi|]^{\mathbb{S}} =_{df} \{s \in S / \mathbb{S}, s \Vdash \varphi\}$ be the *meaning* or *extension* of φ in $\mathbb{S}$. We have: $\mathbb{S}, s \Vdash \varphi$ iff $s \in [|\varphi|]^{\mathbb{S}}$. And hence, the above semantics of modal formulae was also (equivalently) given in terms of the meaning $[|\varphi|]^{\mathbb{S}}$. Such an approach is more suitable in the context of fixpoint operators.

$$
\begin{array}{ll}
[|p|]^{\mathbb{S}} & = V(p), \\
[|\sim p|]^{\mathbb{S}} & = S - V(p), \\
[|F|]^{\mathbb{S}} & = \varnothing, \\
[|T|]^{\mathbb{S}} & = S, \\
[|\varphi \wedge \psi|]^{\mathbb{S}} & = [|\varphi|]^{\mathbb{S}} \cap [|\psi|]^{\mathbb{S}}, \\
[|\varphi \vee \psi|]^{\mathbb{S}} & = [|\varphi|]^{\mathbb{S}} \cup [|\psi|]^{\mathbb{S}}, \\
[|\Box_d \varphi|]^{\mathbb{S}} & = [\rho_d][|\varphi|]^{\mathbb{S}}, \\
[|\Diamond_d \varphi|]^{\mathbb{S}} & = \langle p_d \rangle [|\varphi|]^{\mathbb{S}}.
\end{array}
$$

The state set S can be considered as an universe. Hence, in accordance with the last set equations, we can obtain (for any $s \in S$):

$$
\begin{array}{lll}
s \in [|p|]^{\mathbb{S}} & \Leftrightarrow_{df} & \mathbb{S}, s \Vdash p \\
 & \Leftrightarrow & s \in V(p). \ \Box
\end{array}
$$

---

* The *satisfaction relation* '$\Vdash$' is very similar to this one introduced into the Manna-Pnueli proof system. Also, provided there is no ambiguity and for convenience, instead of $\top$, $\bot$ and $\neg$ the following symbols are used: T, F and ~, i.e. the logical constants (called also: *constant formulae*) *'true'* , *'false'* and the symbol of negation, respectively.

$s \in [|\sim p|]^S \;\Leftrightarrow_{df}\; \mathbb{S}, s \Vdash \sim p$
$\Leftrightarrow\; s \notin V(p)$
$\Leftrightarrow\; s \in S - V(p).$ □

$s \in [|F|]^S \;\Leftrightarrow_{df}\; \mathbb{S}, s \Vdash F$
$\Leftrightarrow\; s \in S'$
$\Leftrightarrow\; s \in \varnothing.$ □

$s \in [|T|]^S \;\Leftrightarrow_{df}\; \mathbb{S}, s \Vdash T$
$\Leftrightarrow\; s \in S.$ □

$s \in [|\varphi \wedge \psi|]^S \;\Leftrightarrow_{df}\; \mathbb{S}, s \Vdash \varphi \wedge \psi$
$\Leftrightarrow\; (\mathbb{S}, s \Vdash \varphi) \wedge (\mathbb{S}, s \Vdash \psi)$
$\Leftrightarrow\; (s \in [|\varphi|]^S) \wedge (s \in [|\psi|]^S)$
$\Leftrightarrow\; s \in [|\varphi|]^S \cap [|\psi|]^S.$ □

$s \in [|\varphi \vee \psi|]^S \;\Leftrightarrow\; s \in [|\varphi|]^S \cup [|\psi|]^S.$ □ {left to the reader}

$s \in [|\Box_d \varphi|]^S \;\Leftrightarrow_{df}\; \mathbb{S}, s \Vdash \Box_d \varphi$
$\Leftrightarrow\; \forall_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)$

For any singleton {s} we have: $t \in \rho_d[s]$ iff $\mathbb{S}, t \Vdash \varphi$, and hence, iff $t \in [|\varphi|]^S$. So, we can obtain:

$[|\Box_d \varphi|]^S =_{df} \{s \in S / \mathbb{S}, s \Vdash \Box_d \varphi\}$
$= \{s \in S / \forall_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)\}$
$= \{s \in S / \forall_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)\}$
$= \{s \in S / \forall_{t \in [|\varphi|]^S} (\mathbb{S}, t \Vdash \varphi)\}$
$= [\rho_d][|\varphi|]^S.$ □

The proof of the set equality: $[|\Diamond_d \varphi|]^S = \langle \rho_d \rangle [|\varphi|]^S$ is left to the reader. The following property is satisfied.

Thesis 2.120

$[|\Box_d \varphi|]^S \subseteq [|\Diamond_d \varphi|]^S$

Proof:

$s \in [|\Box_d \varphi|]^S \;\Leftrightarrow\; \mathbb{S}, s \Vdash \Box_d \varphi$ {df. $[|\varphi|]^S$}
$\Leftrightarrow\; \forall_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)$ {df. $\mathbb{S}, s \Vdash \Box_d \varphi$}
$\Rightarrow\; \exists_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)$ {$\vDash \forall_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi) \Rightarrow \exists_{t \in \rho_d[s]} (\mathbb{S}, t \Vdash \varphi)$}
$\Leftrightarrow\; \mathbb{S}, s \Vdash \Diamond_d \varphi$ {df. $\mathbb{S}, s \Vdash \Diamond_d \varphi$}
$\Leftrightarrow\; s \in [|\Diamond_d \varphi|]^S.$ □ {df. $[|\varphi|]^S$}

Next we shall say that two transition systems $\mathbb{S}$ and $\mathbb{S}'$ are *similar* iff they have the same type (P,D). The notions of modal equivalence, bisimulation and bisimilarity are introduced as follows (Venema Y. 2008).

Definition 2.12

Let s and s' be two states in the similar transition systems $\mathbb{S}$ and $\mathbb{S}'$, respectively. We shall say that s and s' are *modally equivalent*, i.e. $\mathbb{S},s \rightsquigarrow_{(P,D)} \mathbb{S}',s'$ if for any polymodal logic formula φ we have: $\mathbb{S}, s \Vdash \varphi \Leftrightarrow \mathbb{S}', s' \Vdash \varphi$.

A transition system $\mathbb{S}$ is *deterministic* if $\rho_d[s]$ is a singleton, for any $s \in S$. This determinism does not allow $\rho_d[s] = \varnothing$, for any $s \in S$.

The above considered semantics was also interpreted as a two-person evaluation board game associated with a fixed formula ψ and a fixed labelled transition system $\mathbb{S}$. A match of the game consists of the two players moving a token from one position to another. Any position corresponds to some pair (φ,s), where φ is a subformula of ψ and $s \in S$. The first of the players is trying to show that φ is true at s, and the second one is trying to deduce that φ is false at s (a more formal treatment is omitted here).

The notion of bisimulation between two transitions systems is one of the most fundamental in the model theory of modal logic and it is introduced as follows.

Definition 2.13

Let $\mathbb{S}$ and $\mathbb{S}'$ be two similar transition systems and $\varnothing \neq \zeta \subseteq S \times S'$. We shall say that ζ is a *bisimulation* iff the following three conditions are satisfied, for any pair $(s,s') \in \zeta$*:

(*prop*) $\quad \forall_{p \in P} \; (s \in V(p) \Leftrightarrow s' \in V'(p)),$

(*forth*) $\quad \forall_{d \in D} \; \forall_{t \in \rho_d[s]} \; \exists_{t' \in \rho'_d[s']} (t \zeta t'),$

(*back*) $\quad \forall_{d \in D} \; \forall_{t' \in \rho'_d[s']} \; \exists_{t \in \rho_d[s]} (t \zeta t').$

Two states $s \in S$ and $s' \in S'$ are *bisimilar*, i.e. $\mathbb{S},s \;\underline{\leftrightarrow}\; \mathbb{S}',s' \Leftrightarrow_{df} \exists_{\zeta} (s \zeta s')$. And also, ζ is a *Q-bisimulation* if the (prop) clause is satisfied only for a subset $Q \subseteq P$. The corresponding relation of *Q-bisimilarity* is denoted by '$\underline{\leftrightarrow}_Q$'.

We observe that the bisimulation relation is very similar to the notion of machine isomorphism used in the theory of Pawlak's machines (Pawlak Z. 1971)†, see Example 1.5 of Subsection 1.3.

It can be shown that bisimilar states satisfy the same modal formulae, i.e. the following theorem is satisfied (Venema Y. 2008).

Thesis 2.121 (*bisimulation invariance*)

Let $\mathbb{S}$ and $\mathbb{S}'$ be two similar transition systems. Then:

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* Instead of Z it is used the small Greek letter ζ (zeta).

† Zdzislaw Pawlak (1926 – 2006)

$$\underset{(s,s')\in S\times S'}{\forall} (\mathbb{S},s \underline{\leftrightarrow} \mathbb{S}',s' \Rightarrow \mathbb{S},s \rightsquigarrow_{(P,D)} \mathbb{S}',s'). \square$$

Unfortunately, the above notions of modal equivalence and bisimilarity coincide only for some classes of models, satisfying the so-called *Hennessy-Milner property* (Hennessy M. and Milner R. 1985). In particular, this property holds in the class of finitely branching transition systems.

Thesis 2.122 (*Hennessy-Milner property*)

Let $\mathbb{S}$ and $\mathbb{S}'$ be two similar finitely branching transition systems. Then:

$$\underset{(s,s')\in S\times S'}{\forall} (\mathbb{S},s \underline{\leftrightarrow} \mathbb{S}',s' \Leftrightarrow \mathbb{S},s \rightsquigarrow_{(P,D)} \mathbb{S}',s'). \square$$

The above presented bisimilarity was considered as a kind of *behavioural equivalence*. In fact, as a more general, the notion of behavioural equivalence was also used in other areas of application, e.g. trace theory: *trace equivalence* (Mazurkiewicz A. 1995), *B-equivalence under a subset of distinguishable transitions* (André C. 1983: it was shown the set of all possible traces is not a sufficient condition to be described this kind of equivalence), *Petri net reduction rules* (Brams G.W. 1983), *Boolean interpreted Petri nets:behavioural equivalence of two nets*[*], and so on.

Next, by definition, $\mathbb{S}$ of type (P,D) is a *tree-like transition system* iff the structure $(S, \bigcup_{d\in D}\rho_d)$ is a tree. It can be observed that any transition system can be represented, i.e. restructured in a more ordered form, into a bisimilar (behavioural equivalent) tree-like model. Any such process of representation is said to be "*unravelling*". In fact, the following theorem is satisfied.

Thesis 2.123 (*the model property of modal logic*)

Let φ be a satisfiable modal formula. Then φ is satisfiable at the root of a tree-like model. □ {T 2.121}

It was shown that the above notion of bisimulation can be completely defined in terms of the *Egli - Milner lifting*[†] (Venema Y. 2008). The last relation, here denoted by $\overline{\mathbb{P}}(\zeta)$ or in short $\overline{\mathbb{P}}\zeta$, is defined as follows.

Definition 2.14

Let $\varnothing \neq \zeta \subseteq S \times S'$ and $\overline{\mathbb{P}}\zeta \subseteq \mathbb{P}(S) \times \mathbb{P}(S')$. Then:

$$\overline{\mathbb{P}}\zeta =_{df} \{ (X,X') \; / \; \underset{x\in X}{\forall} \; \underset{x'\in X'}{\exists} (x\zeta x') \;\wedge\; \underset{x'\in X'}{\forall} \; \underset{x\in X}{\exists} (x\zeta x') \}.$$

In a similar way, for the Kripke functor $K = K_{D,P}S$ the following relation can be introduced: $\overline{K}\zeta \subseteq KS \times KS'$ such that $\overline{K}\zeta =_{df} \{((X,Y),(X',Y')) \; / \; (X = X') \;\wedge\; \underset{d\in D}{\forall} ((Y_d, Y_d') \in \overline{\mathbb{P}}\zeta) \}$.

The above two relations $\overline{\mathbb{P}}\zeta$ and $\overline{K}\zeta$ are called the *lifting* of ζ wrt $\mathbb{P}$ and K, respectively.

The following proposition was shown (Venema Y. 2008).

---

[*] $\underset{N_1, N_2 \in \mathrm{BIPN}}{\forall} (N_1 \; b{\approx} \; N_2 \Leftrightarrow_{df} SM(N_1) \approx SM(N_2))$, where BIPN denotes the class of *Boolean interpreted Petri nets*, $SM(N_i)$ is the finite-state sequential machine corresponding to N_i (i = 1,2). Here, the *behavioral equivalence* and *state machine equivalence* relations are denoted by ' b≈ ' and ' ≈ ', respectively: see Definition 5.23 of Subsection 5.4. We observe some relation between the notions of state transition systems and Petri nets.

[†] As an example, another interpretation of this notion was early introduced in number theory, the so-called *Hasse - Davenport lifting relation* (Davenport H. and Hasse H. 1935).

Proposition 2.16

Let $\mathbb{S}$ and $\mathbb{S}'$ be two Kripke coalgebras for some Kripke functor K and $\varnothing \neq \zeta \subseteq \mathrm{S} \times \mathrm{S}'$ be some binary relation. Then ζ is bisimulation iff $\underset{(s,s')\in\zeta}{\forall} ((\sigma(s),\sigma'(s')) \in \overline{\mathrm{K}}\zeta)$. □

In accordance with the model theory of modal fixpoint logics, the modal μ-calculus is a natural generalisation of the basic modal logic system and it can be regarded as a bisimulation invariant, a fragment of the second order predicate logic. The use of fixpoint functors* can be considered as a very important extension of the modelling (or expressive) power of the above presented labelled transition systems.

A good illustration of the notion of a fixpoint functor can be obtained using *dynamic logic* developed by Pratt V. R. in 1974 (e.g. see: Pratt V.R. 1980). This system, used in the area of program verification, is related to *Hoare's logic*: reasoning about program correctness (Hoare C.A.R. 1969). A system similar to Hoare's logic was early presented by Floyd R.W. (1967)†.

Pratt's dynamic logic can be viewed as a refinement of *algorithmic logic* (Mirkowska G. and Salwicki A. 1987), and the *weakest-precondition predicate transformers* Dijkstra E.W. (1976)‡. Moreover, this logic becomes a good connection to the axiomatic and Kripke semantics of modal logic (Venema Y. 2008).
In dynamic logic, the basic modal functors '$\Box$' and '$\Diamond$' are extended by associating to every *action* 'a' the (two dual) modal functors '$[\mathrm{a}]$' and '$\langle \mathrm{a} \rangle$', where $[\mathrm{a}]\varphi$ and $\langle \mathrm{a} \rangle\varphi$ denote the facts that *after performing* a *it is necessarily* and *it is possible* the case that φ holds, respectively (for an arbitrary formula φ: there is one-to-one correspondence between $[\mathrm{d}]\varphi, \langle \mathrm{d} \rangle\varphi$ and the polymodal logic formulae $\Box_d \varphi$, $\Diamond_d \varphi$, respectively). And so, *De Morgan's laws* are satisfied: $\sim [\mathrm{a}]\varphi \Leftrightarrow \langle \mathrm{a} \rangle \sim \varphi$ and $\sim \langle \mathrm{a} \rangle\varphi \Leftrightarrow [\mathrm{a}] \sim \varphi$. Moreover, as in the case of the basic modal functors (see T 2.23 and T 2.24), $[\mathrm{a}]$ and $\langle \mathrm{a} \rangle$ are distributive wrt the logical connectives $\wedge$ and $\vee$, respectively. The *monotonicity rule* (in short: *MON*) is given as follows: $(\varphi \Rightarrow \psi) \Rightarrow ([\mathrm{a}]\varphi \Rightarrow [\mathrm{a}]\psi)$.

In particular, the following rule is also satisfied: $[\mathrm{a}](\varphi \Rightarrow \psi) \Rightarrow ([\mathrm{a}]\varphi \Rightarrow [\mathrm{a}]\psi)$, see axiom A0 given below. The corresponding proof is similar to this one given in T 2.29 (left to the reader). Also we have the following *necessitation rule*: $\vDash \varphi \Rightarrow \vDash [\mathrm{a}]\varphi$ (see the Gödel's rule GR). We shall say that this logic is *normal* if it follow A0 and the last necessitation rule.

The notion of a *propositional formula* is similar to this one used in the classical propositional calculus (see Definition 1.1 of Subsection 1.1), extended in step (2) by two new formulae: $[\mathrm{a}](\varphi)$ and $\langle \mathrm{a} \rangle(\varphi)$. Moreover, it is assumed that if ´a´ is an event, then such events are also: (a) ∘ (b), (a) ∪ (b), $(\mathrm{a})^*$, and ?(φ). An example axiomatic system used in *propositional dynamic logic* (in short: *PDL*) is given below, e.g. see: (Troquard N. and Balbiani P. 2015), (Fischer M.J. and Ladner R.E. 1979), (Başkent C. 2010), or the *Handbook of philosophical logic* (2002) §. Here p serves as metavariable.

A0. $[\mathrm{a}](p \Rightarrow q) \Rightarrow ([\mathrm{a}]p \Rightarrow [\mathrm{a}]q)$ {*Kripke axiom*}

A1. $[0]p$ {*the empty promise axiom*}

A2. $[1]p \Leftrightarrow p$ {*the identity function axiom*}

A3. $[\mathrm{a} \cup \mathrm{b}]p \Leftrightarrow [\mathrm{a}]p \wedge [\mathrm{b}]p$ {*union axiom*}

---

* Provided there is no ambiguity, instead of "fixpoint operator " , here the term "fixpoint functor" is used.

† Robert W. Floyd (1936 – 2001)

‡ Edsger Wybe Dijkstra (1930 – 2002)

§ See also: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

A4. $[a \circ b]p \Leftrightarrow [a][b]p$ {*concatenation axiom*}

A5. $[a*]p \Leftrightarrow p \wedge [a][a*]p$ {*fixed point axiom*}

A6. $[?p]q \Leftrightarrow p \Rightarrow q$ {*test axiom*}

A7. $[a*](p \Rightarrow [a]p) \Rightarrow (p \Rightarrow [a*]p)$ {*induction axiom*}

The axiomatisation based on axiom schemes A3, A4, A5, A6 and A7 were originally introduced by Segerberg, K. (1977). The above used symbols: $'\cup'$, $'\circ'$ and $'*'$ are the well-known algebraic operations: *union* (or *nondeterministic choice of events*, called also *regular expressions*), *concatenation* (called also *catenation*, *sequencing* or *composition*: sometimes $'\circ'$ is omitted, e.g. $'ab'$ instead of $'a \circ b'$) and the *Kleene star operator*, called *iteration* (Kleene S.C. 1952, 1956)*. The event '?p' , related to A6, means: "*test p and proceed only if true*". In accordance with the laws of exportation and importation (see T 1.12 of Subsection 1.2), A7, given e.g. in (Troquard N. and Balbiani P. 2015) or (Vetter B. 2015), can be equivalently transformed to the following form: $p \wedge [a*](p \Rightarrow [a]p) \Rightarrow [a*]p$, e.g. given in (Başkent C. 2010) or in the *Free Encyclopaedia*. The induction axiom A7 (in short: 'IND') is very similar to the third axiom (the principle of *mathematical induction*) of *Dedekind – Peano*† *axioms*. In fact, assume that p is true in the current state and assume further that after any number of iterations of a: if p is still true, then it will be true after one more iteration of a. Then p must remain true after any number next iterations, i.e. no matter how often we perform a.

To minimise the number of used parentheses in an expression, some priorities for logical connectives can be introduced. The following precedence of functors from highest to lowest is used below, e.g. (Başkent C. 2010): $\langle a \rangle$, $[a]$, $\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, ?, $*$, $\circ$, $\cup$.

The following property is satisfied (Kozen D.C. 1994): $a* \circ a* = a*$, where $a* =_{df} 1 \cup a \circ a*$. And hence, we can obtain (provided there is no ambiguity, below the use of Ai is denoted by a_i, for any i).

$$[a*][a*]p \Leftrightarrow [a* \circ a*]p \quad \{a_4\}$$
$$\Leftrightarrow [a*]p . \square$$

---

* Provided there is no ambiguity, instead of ′ ; ′ the concatenation operation is here denoted by $'\circ'$. The Kleene algebra of regular events will be presented in the next part of this work.

† Julius Wilhelm Richard Dedekind (1831 – 1916), Giuseppe Peano (1858 – 1932). An earlier formulation of this principle was also given by Pascal (Blaise Pascal, 1623 – 1662). The *principle of induction* concerns the arithmetic of natural numbers. In particular, this principle can be represented by means of the following rule:

$$\frac{F(0) \quad \forall_k (F(k) \Rightarrow F(k+1))}{\forall_n F(n)}$$ (the equivalent form of A7, represented as a rule: $$\frac{\varphi \quad [a*](\varphi \Rightarrow [a]\varphi)}{[a*]\varphi}$$, is very similar to this principle).

Here ′F(n)′ denotes an arbitrary formula or theorem. For convenience, it is assumed here that 0 is a natural number. Obviously, $k, n \in \mathbb{N}$.

Since $a** = a*$, in a similar way we can obtain: $[a**]p \Leftrightarrow [a*]p$.

Proposition 2.17 (*fixed point axiom*)

$[a*]p \Leftrightarrow p \wedge [a][a*]p$

Proof:

$$
\begin{aligned}
[a*]p &\Leftrightarrow [1 \cup a \circ a*]p && \{\text{df. } 'a*', \text{SR}\} \\
&\Leftrightarrow [1]p \wedge [a \circ a*]p && \{a_3\} \\
&\Leftrightarrow p \wedge [a][a*]p. \ \square && \{a_2, a_4, \text{SR}\}
\end{aligned}
$$

The following rule, called *loop invariance* (in short: 'LI'), was given in (Troquard N. and Balbiani P. 2015): $(p \Rightarrow [a]p) \Rightarrow (p \Rightarrow [a*]p)$. A more simpler and equivalent version of LI is given in the next thesis.

Thesis 2.124

$p \wedge [a]p \Rightarrow [a*]p \Leftrightarrow (p \Rightarrow [a]p) \Rightarrow (p \Rightarrow [a*]p)$

Proof T 2.124a:

(1)	$p \wedge [a]p \Rightarrow [a*]p$	{a}
(2)	$p \Rightarrow [a]p$	
(3)	$\sim (p \Rightarrow [a*]p)$	{aip}
(4)	p	{NC : 3}
(5)	$\sim [a*]p$	
(6)	$[a]p$	{– C : 2,4}
(7)	$p \wedge [a]p$	{ + K : 4,6}
(8)	$[a*]p$	{– C : 1,7}
	contr. □	{5,8}

Proof T 2.124b:

(1)	$(p \Rightarrow [a]p) \Rightarrow (p \Rightarrow [a*]p)$	{a}
(2)	$\sim (p \wedge [a]p \Rightarrow [a*]p)$	{aip}
(3)	$p \wedge [a]p$	{NC : 2}
(4)	$\sim [a*]p$	
(5)	$p \wedge (p \Rightarrow [a]p) \Rightarrow [a*]p$	{T 1.12 of Subsection 1.2 : 1}
(6)	$p \wedge (\sim p \vee [a]p) \Rightarrow [a*]p$	{CR, SR : 5}

(7) $p \wedge [a]p \;\Rightarrow\; [a*]p$ $\{\wedge$ is distributive over $\vee$, SR : 6$\}$

(8) $[a*]p$ $\{-\text{C} : 3{,}7\}$

contr. □ $\{4{,}8\}$

Since $\vDash \varphi \wedge \psi \Rightarrow (\varphi \Rightarrow \psi)$, the opposite implication, i.e. T 2.124b, can be shown by assuming: $\varphi =_{df} p$ $\psi =_{df} [a]p$ in line (3) and next using $'-\text{C}'$. Moreover, a more simplified proof of T 2.124 can be obtained by starting with the right side of T 2.124 and next using T 1.12 of Subsection 1.2, CR and SR (left to the reader).

Thesis 2.125 (*induction axiom*)

$$[a*](p \Rightarrow [a]p) \;\Rightarrow\; (p \Rightarrow [a*]p)$$

Proof:

(1) $[a*](p \Rightarrow [a]p) \;\Leftrightarrow\; (p \Rightarrow [a]p) \;\wedge\; [a][a*](p \Rightarrow [a]p)$ $\{\text{RR} : \; p \Rightarrow [a]p \;//\; p \;\text{ wrt } a_5\}$

(2) $[a*](p \Rightarrow [a]p) \;\Rightarrow\; (p \Rightarrow [a]p) \;\wedge\; [a][a*](p \Rightarrow [a]p)$ $\{-\text{E} : 1\}$

(3) $p \;\Rightarrow\; p$ $\{\vDash p \Rightarrow p\}$

(4) $(p \Rightarrow p) \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; (p \Rightarrow [a]p) \;\wedge\; [a][a*](p \Rightarrow [a]p)$ $\{+\text{K} : 2{,}3\}$

(5) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; p \wedge (p \Rightarrow [a]p) \;\wedge\; [a][a*](p \Rightarrow [a]p)$ $\{\text{MAC}, -\text{C} : (2), (3), (4) \;/\; \vDash (\varphi \Rightarrow \psi) \wedge (p \Rightarrow p) \Rightarrow (\varphi \wedge p \Rightarrow \psi \wedge p)\}$

(6) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; [a]p \;\wedge\; p \;\wedge\; [a][a*](p \Rightarrow [a]p)$ $\{\text{CR}, \wedge$ is distributive over $\vee$, also commutative and associative SR : 5$\}$

(7) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; [a](p \wedge [a*](p \Rightarrow [a]p))$ $\{[a]$ is distributive over $\wedge$, SR : 6$\}$

(8) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; [a*](p \wedge [a*](p \Rightarrow [a]p))$ $\{\text{LI}, -\text{C} : 7\}$

(9) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; [a*]p \;\wedge\; [a*][a*](p \Rightarrow [a]p)$ $\{[a*]$ is distributive over $\wedge$, SR : 8$\}$

(10) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; [a*]p \;\wedge\; [a*](p \Rightarrow [a]p)$ $\{\vDash [a*][a*]\varphi \Leftrightarrow [a*]\varphi$, SR : 9$\}$

(11) $[a*]p \wedge [a*](p \Rightarrow [a]p) \Rightarrow [a*]p$ $\{\vDash \varphi \wedge \psi \Rightarrow \varphi\}$

(12) $p \wedge [a*](p \Rightarrow [a]p) \;\Rightarrow\; [a*]p$ $\{\text{TC} : 10, 11\}$

$[a*](p \Rightarrow [a]p) \;\Rightarrow\; (p \Rightarrow [a*]p)$. □ $\{$T 1.12 of Subsection 1.2, exportation: 12$\}$

The following diamond version of the above fixed point axiom A5 can be obtained.

Thesis 2.126 (*dual fixed point axiom*)

$$\langle a*\rangle p \;\Leftrightarrow\; p \vee \langle a\rangle\langle a*\rangle p$$

Proof:

(1) $[a*]\sim p \;\Leftrightarrow\; \sim p \;\wedge\; [a][a*]\sim p$ $\{\text{RR} : \sim p \;//\; p \;\text{ wrt } a_5\}$

(2) $\sim[a*]\sim p \Leftrightarrow \sim(\sim p \wedge [a][a*]\sim p)$ {CE : 1}

$\langle a*\rangle p \Leftrightarrow p \vee \langle a\rangle\langle a*\rangle p$. □ $\{\vDash \sim[a]\varphi \Leftrightarrow \langle a\rangle \sim\varphi$, – N, NK, SR : 2}

The following properties are satisfied (Harel D., Kozen D.C. and Tiuryn J. 2000): some of the presented properties are in accordance with the above considerations, e.g. 5, 7, 9, 10 and 11 (the right-to-left implication).

Thesis 2.127

The following formulae are valid in PDL.

1. $[a*]p \Rightarrow p$
2. $p \Rightarrow \langle a*\rangle p$
3. $[a*]p \Rightarrow [a]p$
4. $\langle a\rangle p \Rightarrow \langle a*\rangle p$
5. $[a*]p \Leftrightarrow [a*\circ a*]p$
6. $\langle a*\rangle p \Leftrightarrow \langle a*\circ a*\rangle p$
7. $[a*]p \Leftrightarrow [a**]p$
8. $\langle a*\rangle p \Leftrightarrow \langle a**\rangle p$
9. $[a*]p \Leftrightarrow p \wedge [a][a*]p$
10. $\langle a*\rangle p \Leftrightarrow p \vee \langle a\rangle\langle a*\rangle p$
11. $[a*]p \Leftrightarrow p \wedge [a*](p \Rightarrow [a]p)$
12. $\langle a*\rangle p \Leftrightarrow p \vee \langle a*\rangle(\sim p \wedge \langle a\rangle p)$. □

Since 1 and 2 are valid, then so is: $[a*]p \Rightarrow \langle a*\rangle p$ (by using rules TC and – C: as in modal logic, see T 2.18). Moreover, from 1 and 3, by using + K, MC and – C, we can obtain: $[a*]p \Rightarrow p \wedge [a]p$. According to T 2.124, the opposite implication is equivalent to LI. The line 4 is obtained by 3 using rule CC.

The proof of the last formula 12 can be obtained immediately by 11 and it is given below.

(1) $[a*]p \Leftrightarrow p \wedge [a*](p \Rightarrow [a]p)$ {T 2.127 : 11}

(2) $\sim[a*]p \Leftrightarrow \sim(p \wedge [a*](p \Rightarrow [a]p))$ {CE : 1}

(3) $\langle a*\rangle \sim p \Leftrightarrow \sim p \vee \langle a*\rangle (p \wedge \langle a\rangle \sim p)$ $\{\vDash \sim[a]\varphi \Leftrightarrow \langle a\rangle \sim\varphi$, NK, NC, SR : 2}

$\langle a*\rangle p \Leftrightarrow p \vee \langle a*\rangle (\sim p \wedge \langle a\rangle p)$. □ {RR : p, ~ p // ~ p, p}

The *reflexive transitive closure rule* (in short: RTC) is given as follows.

$(p \vee \langle a \rangle q \Rightarrow q) \Rightarrow (\langle a* \rangle p \Rightarrow q)$

Semantically, the properties of the Kleene star operator $'*'$ follow directly from the classical notion of reflexive and transitive closure ρ^* of a binary relation ρ defined in a given set X. In fact, it can be shown that $a*$ is a reflexive and transitive relation containing a (see T 2.127, lines 2, 6 and 4, respectively: these three properties are also summarised in line 10). Moreover, it is necessary to show that $a*$ is the least such relation. There are several equivalent ways that this can be done: by using RTC, LI or IND, which are interderivable (Harel D., Kozen D.C. and Tiuryn J. 2000). As an example, the rule LI was used in the proof of IND (in line (7) of the proof of T 2.125), and vice versa: IND may be used in the proof of LI. This is illustrated by T 2.128, given below.

Let φ be valid. Then, in accordance with the necessitation rule and $-C$ we have: $[a]\varphi$ is valid. Assume now that $[a]\varphi$ is valid. Then by the necessitation rule and $-C$ it follows that $[a][a]\varphi$ is valid. Hence, according to A4, $[a \circ a]\varphi$ is valid, i.e. $[a^2]\varphi$, etc. Hence: $[a^i]\varphi$ is valid (for any $i > 0$). Since $[1]\varphi \Leftrightarrow \varphi$, from A2 and A3 it follows that a valid formula is the following: $[1 \cup a \cup a^2 \cup ...]\varphi$. Therefore, the following *modal generalisation* of the necessitation rule can be obtained (in short: *MGEN*): $\vDash \varphi \Rightarrow \vDash [a*]\varphi$

Thesis 2.128 (*loop invariance*)

$(p \Rightarrow [a]p) \Rightarrow (p \Rightarrow [a*]p)$

Proof:

(1)	$p \Rightarrow [a]p$	{a}
(2)	$[a*](p \Rightarrow [a]p)$	$\{\varphi =_{df} p \Rightarrow [a]p, -C : MGEN, 1\}$
(3)	$[a*](p \Rightarrow [a]p) \Rightarrow (p \Rightarrow [a*]p)$	{IND}
	$p \Rightarrow [a*]p$. □	$\{-C : 2,3\}$

The following equivalent dual form of the reflexive transitive closure rule, i.e. *dualisation* of RTC, can be obtained.

(1)	$(p \vee \langle a \rangle q \Rightarrow q) \Rightarrow (\langle a* \rangle p \Rightarrow q)$	{RTC}
(2)	$(p \vee \sim[a]\sim q \Rightarrow q) \Rightarrow (\sim[a*]\sim p \Rightarrow q)$	$\{\vDash \langle a \rangle \varphi \Leftrightarrow \sim[a]\sim\varphi$, SR : 1}
(3)	$\sim(p \vee \sim[a]\sim q) \vee q \Rightarrow \sim(\sim[a*]\sim p) \vee q$	{CR, SR : 2}
(4)	$\sim p \wedge [a]\sim q \vee q \Rightarrow [a*]\sim p \vee q$	{NA, $-$N, SR : 3}
(5)	$p \wedge [a]q \vee \sim q \Rightarrow [a*]p \vee \sim q$	{RR : p, q, ~ q // ~ p, ~ q, q}
	$(q \Rightarrow p \wedge [a]q) \Rightarrow (q \Rightarrow [a*]p)$. □	{CR, SR : 5}

The use of LI in the proof of dual RTC is illustrated in the next thesis.

Thesis 2.129 (*dual RTC*)

$(q \Rightarrow p \wedge [a]q) \Rightarrow (q \Rightarrow [a*]p)$

Proof:

(1)	$q \Rightarrow p \wedge [a]q$	{a}
(2)	q	
(3)	$q \Rightarrow p$	{MC, – K : 1}
(4)	$q \Rightarrow [a]q$	
(5)	$q \Rightarrow [a*]q$	{LI, – C : 4}
(6)	$[a]q \Rightarrow [a]p$	{MON : 3}
(7)	$[a]q$	{– C : 2,4}
(8)	$[a]p$	{– C : 6,7}
	$[a*]p$. □	{MGEN : 8}

In general, the use of PDL for formal verification of programs (e.g. describing correctness, termination and equivalence of programs) involves program models called "*regular programs*" because of their similarity to regular expressions. Some example standard block structured programming constructs are illustrated below, by definitional abbreviation (Fischer M.J. and Ladner R.E. 1979), (Başkent C. 2010), (Troquard N. and Balbiani P. 2015). Here ′$a \circ b$′, ′$a \cup b$′ and ′$a*$′ mean: ′*begin* $a \circ b$ *end*′, ′*nondeterministically do* a *or do* b′ and ′*repeat* a *a nondeterministically chosen number of times*′. The last two commands ′abort′ (′fail′ or ′blocked′) and ′skip′ correspond to: ′*immediately terminate*′ and ′*does no operation*′, respectively.

Program operator	*PDL syntax*
if p then a else b	$?p \circ a \cup ?\sim p \circ b$
while p do a	$(?p \circ a)* \circ ?\sim p$
repeat a until p	$a \circ (?\sim p \circ a)* \circ ?p$
abort	?0
skip	?1

In accordance with the Pratt's interpretation of Hoare's triples, PDL (extended with the first-order dynamic logic can be considered as a generalization of Hoare's calculus. In particular, the Hoare's partial correctness assertion $\{\varphi\}a\{\psi\}$ can be equivalently encoded by the following implication: $\varphi \Rightarrow [a]\psi$. Some rules of inference were used in this calculus, e.g. such as: *composition rule*, *conditional rule*, *iteration* (or *while*) *rule*, two *rules of consequence*, etc. As an illustration, the Hoare's composition rule is given as follows:

$$\frac{\{\varphi\}a\{\psi\},\{\psi\}b\{\chi\}}{\{\varphi\}a;b\{\chi\}}$$

This rule describes the elementary sequential composition of two programs ′a′ and ′b′. The first triple $\{\varphi\}a\{\psi\}$ is related to the assumption that when ′a′ is executed in a state satisfying φ, then it will finish in a state satisfying ψ, whenever it halts (similarly for $\psi\}b\{\chi\}$). The conclusion $\{\varphi\}a;b\{\chi\}$ of this rule concerns the partial correctness of the obtained program ′a;b′ (denotes: ′a′ *sequentially composed with* ′b′) and follows from the above two assumptions. We can conclude that if ´a;b´ is executed in a state satisfying φ, then it finishes in a state satisfying χ, whenever it halts (Troquard N. and Balbiani P. 2015).

The proof of the Hoare's composition rule is given below.

Thesis 2.130 (*composition rule*)

$$(p \Rightarrow [a]q) \wedge (q \Rightarrow [b]r) \Rightarrow (p \Rightarrow [a \circ b]r)$$

Proof:

(1) $p \Rightarrow [a]q$

(2) $q \Rightarrow [b]r$ {a}

(3) p

(4) $[a]q \Rightarrow [a][b]r$ {MON : 2}

(5) $[a]q$ {– C : 1,3}

(6) $[a][b]r$ {– C : 4,5}

$[a \circ b]r$. □ {A4 : 6}

A little more complex than the previous one, is the proof of the Hoare's rule of iteration. This rule is shown as follows.

$$\frac{\{\varphi \wedge \psi\}a\{\varphi\}}{\{\varphi\}\text{ while } \psi \text{ do } a\,\{\sim\psi \wedge \varphi\}}$$

A version of the proof of the Hoare's rule of iteration, given in (Troquard N. and Balbiani P. 2015) is illustrated below. Here, the program operator ´while q do a´ is equivalently represented by its corresponding PDL syntax, i.e. ´$(?q \circ a)* \circ ?\sim q$´. And so, we have the following thesis.

Thesis 2.131 (*iteration rule*)

$$(p \wedge q \Rightarrow [a]p) \Rightarrow (p \Rightarrow [(?q \circ a)* \circ ?\sim q](\sim q \wedge p))$$

Proof:

(1) $p \wedge q \Rightarrow [a]p$ {a}

(2) p

(3) $p \Rightarrow (q \Rightarrow [a]p)$ {T 1.12 of Subsection 1.2 : 1}

(4) $p \Rightarrow [?q][a]p$ {A6, SR : 3}

(5) $p \Rightarrow [?q \circ a]p$ {A4, SR : 4}

(6) $p \Rightarrow [(?q \circ a)*]p$ {LI, – C : 5}

(7) $p \Rightarrow (\sim q \Rightarrow \sim q \wedge p)$ $\{\vDash \varphi \Rightarrow (\psi \Rightarrow \psi \wedge \varphi)\}$

(8) $[(?q \circ a)*]p \Rightarrow [(?q \circ a)*](\sim q \Rightarrow \sim q \wedge p)$ {MON : 7}

(9) $p \Rightarrow [(?q \circ a)*](\sim q \Rightarrow \sim q \wedge p)$ {TC : 6,8}

(10) $p \Rightarrow [(?q \circ a)*][? \sim q](\sim q \wedge p)$ {A6, SR : 9}

(11) $p \Rightarrow [(?q \circ a)* \circ ? \sim q](\sim q \wedge p)$ {A4, SR : 10}

$[(?q \circ a)* \circ ? \sim q](\sim q \wedge p)$. ▫ {– C : 2,11}

It was shown by Pratt V.R. (1981) that Pμ subsumes PDL and extends the exponential-time decision procedure for PDL to Pμ. However, it was not known whether strictly PDL is contained in Pμ. Moreover, a deductive system was not given (Kozen D.C. 1983).

In general, the proposed and studied by Kozen D.C. (1983), calculus Lμ contains essentially of propositional modal logic with a "least fixpoint operator". It was shown that Lμ is syntactically simpler yet strictly more expressive than PDL.

In accordance with the above considerations, the use of fixpoint functors is a very important extension of the expressive power of any labelled transition system $\mathbb{S}$. Provided there is no ambiguity, below an event ´a´ is denoted by ´d´ and interpreted as an atomic action, $d \in D$ (the set of atomic actions or labels).

Let now consider the PDL formula $\langle d*\rangle p$. By definition, this formula holds at those states in $\mathbb{S}$ from which there exists a finite ρ_d-path, of unspecified length, leading to a state in which p is true, where: a *path* through $\mathbb{S}$ is defined as a non-empty sequence of the form $(s_0,d_1,s_1, \ldots ,d_n,s_n)$ such that $s_{i-1}\rho_{d_i}s_i$, for any $i \leq n$ (Venema Y. 2008). According to T 2.126, we have: $\mathbb{S} \Vdash \langle d*\rangle p \Leftrightarrow p \vee \langle d\rangle\langle d*\rangle p$, for any $\mathbb{S}$ (here, it is used ´$\langle d\rangle$´ rather than ´$\Diamond_d$´). And hence, the following example was presented.

Example 2.17

In accordance with the above dual fixed point axiom, $\langle d*\rangle p$ might be informally interpreted as a *fixed point* or solution of the following ´equation´: $x \Leftrightarrow p \vee \langle d\rangle x$.

Let ∞_d be a formula that is valid at those states of $\mathbb{S}$ from which an infinite ρ_d-path emanates. Then, the formula $\langle d*\rangle p \vee \infty_d$ is another such fixed point. Moreover, it can be shown that these two fixpoints are the smallest and the largest such possible solutions, respectively. ▫

Generally, the modal μ-calculus allows us to refer explicitly to such smallest and largest solutions, according to the last example: $\mu x.\, p \vee \langle d\rangle x$ and $\nu x.\, p \vee \langle d\rangle x$, respectively (Venema Y. 2008).

The polymodal logic PML(D,P) can be extended as a *polymodal fixpoint logic in D and P*, in short: *μPML(D,P)*, by introducing the following two additional formulae: $\mu x.\varphi$ and $\nu x.\varphi$, where ´μx´ and ´νx´ are called the *least* and *greatest fixpoint operators* ($x \in P$)*. The last two fixpoint formulae are called *μ-* and *ν-formulae*, respectively. Moreover, any fixpoint formulae are usually assumed to be in *positive normal form*, i.e. the only admissible occurrences of the negation symbol is in front of atomic formulae. And so, no occurrence of x in φ may be in the scope of the negation functor. The notions of *subformula* and *proper subformula* are similar, as in the classical case. So, they are omitted.

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* It can be observed that the used designations of these two fixpoints coincide with this one used in *λ-calculus* (Alonzo Church 1903 – 1995), in particular the notion of *λ-abstraction*: λx.t, binding the variable x in the term t, e.g. $\lambda x.\, 2x^3 + 5$ is a lambda abstraction for the function $f(x) =_{df} 2x^3 + 5$, see: (Church A. 1941) or e.g. (Cardone F. and Hindley J.R. 2006).

Syntactically, the fixpoint operators μx and νx are very similar to the quantifiers of first-order logic in the way that they bind variables (see Section 3 of Chapter II). Let $\eta \in \{\mu, \nu\}$, FV(φ) and BV(φ) be the *sets of free* and *bound variables of a formula* φ. The inductive definition of the last two sets and also the definitions of the notions of a clean formula φ, dependency order on BV(φ) and a guarded variable (or formula) are presented below (Venema Y. 2008). Here, for convenience, are exclusively assumed such formulae in which every bound variable uniquely determines a fixpoint operator binding it. Moreover, there is no overlap, i.e. partly cover, between free and bound variables.

Definition 2.15

The sets FV(φ) and BV(φ) are defined as follows.

FV(F)	$=_{df}$	$\varnothing$	BV(F)	$=_{df}$	$\varnothing$
FV(T)	$=_{df}$	$\varnothing$	BV(T)	$=_{df}$	$\varnothing$
FV(p)	$=_{df}$	{p}	BV(p)	$=_{df}$	$\varnothing$
FV(~ p)	$=_{df}$	{p}	BV(~ p)	$=_{df}$	$\varnothing$
FV($\varphi \wedge \psi$)	$=_{df}$	FV(φ) $\cup$ FV(ψ)	BV($\varphi \wedge \psi$)	$=_{df}$	BV(φ) $\cup$ BV(ψ)
FV($\varphi \vee \psi$)	$=_{df}$	FV(φ) $\cup$ FV(ψ)	BV($\varphi \vee \psi$)	$=_{df}$	BV(φ) $\cup$ BV(ψ)
FV($\Box_d \varphi$)	$=_{df}$	FV(φ)	BV($\Box_d \varphi$)	$=_{df}$	BV(φ)
FV($\Diamond_d \varphi$)	$=_{df}$	FV(φ)	BV($\Diamond_d \varphi$)	$=_{df}$	BV(φ)
FV($\eta x.\varphi$)	$=_{df}$	FV(φ) $-$ {x}	BV($\eta x.\varphi$)	$=_{df}$	BV(φ) $\cup$ {x}

Definition 2.16

Any $\varphi \in \mu$PML(D,P) is *clean* iff there are no two distinct occurrences of fixed point operators in φ that bind the same variable, and there is no variable that has both free and bound occurrences in φ.

Let x be a bound of the clean formula φ. By $\varphi_x =_{df} \eta_x x.\delta_x$ we shall denote the unique subformula of φ, where x is bound by η_x.

Let $\varphi \in \mu$PML(D,P) be a clean formula and x, y $\in$ BV(φ). The following partial ordering relation, called *dependency order* (or a *ranking relation*) on BV(φ) and denoted by ′$\leq_\varphi$′ , can be introduced.

Definition 2.17

$x \leq_\varphi y \Leftrightarrow_{df} \varphi_x \trianglelefteq \varphi_y$, where ′$\trianglelefteq$′ denotes: ′*is a subformula of*′.

In fact, if y ranks higher than x then the meaning of φ_x will depend on the meaning of φ_y. And finally, the notion of guardedness is presented as follows.

Definition 2.18

Let $\varphi, \chi \in \mu$PML(D,P). We shall say that a variable x is guarded in φ iff any occurrence of x in φ is in the scope of a modal functor. A formula χ is guarded iff for any subformula of χ of the form $\eta x.\delta$, x is guarded in δ.

The semantics of modal fixpoint formulae can be described algebraically or also by means of evaluation games (the more easier variant). In fact, the algebraic semantics of $\mu x.\varphi$ and $\nu x.\varphi$ in $\mathbb{S}$ is related to some algebraically defined meaning formulae. Any φ can be represented as an operation on the power set of S. And hence, it would be necessary to show the existence of a least and greatest fixpoint for any such operation (a more formal treatment is omitted here, see: Venema Y. 2008).

The above introductory notions related to modal fixpoint logics are an illustration of the last excellent work. Another topics concerning the modal μ-calculus were also presented, e.g. such as: deterministic and nondeterministic stream automata and their logical presentations, transition system theory, some methodological aspects with regard to the notions of decidability, expressive completeness, preservation results, axiomatisation and so on The existence of an effective procedure transforming a given alternating Kripke automaton into an equivalent nondeterministic one was presented (using bisimulation based on the notion of relation lifting, see Definition 2.14). And this can be considered as the most fundamental result concerning the automata-theoretic approach with respect to the modal μ-calculus.

In general, the modal μ-calculus can be considered as a very interesting and important theory in mathematics and theoretical computer science. This calculus is a well-behaved extension of the basic modal logic and at the same time by using mathematical structures that model processes, such as labelled transition systems, it can be considered as a good balance between computational efficiency and expressiveness.

The relationship between a program's syntactical structure and its behaviour is fundamental in program analysis. This relationship is often exploited to phrase program correctness problems in terms of the structure of a program rather than in terms of its behaviour. However, in the other direction, this relationship is much less understood and more complex: given a program behaviour, how can one capture the program structures that admit this behaviour? A natural way to capture both program structure and behaviour is the use of temporal logic formulae: structural properties are concerned with the textual sequencing of instructions in a program, while behavioural properties consider their executional sequencing (Gurov D. and Huisman M. 2013). Here, it was shown that every disjunction-free behavioural formula can be precisely characterised by a finite set of structural formulae: a program satisfies the behavioural formula if and only if it satisfies some structural formula from the set. Several extensions to behavioural formulae with disjunction were also considered. As a property specification language, it was used a fragment of the modal μ-calculus with boxes and greatest fixed points only. The obtained temporal logic system, called a *simulation logic* and capable of characterising simulation, was suitable for expressing safety properties. However, in accordance with the proof correctness requirements, sometimes the full modal μ-calculus was used. In fact, the considered simulation logic was defined as a restriction of the full logic, where negation was restricted to atomic propositions only. The last research paper is a good study concerning use of the modal μ-calculus in program verification and correctness. Some introductory notions are given below.

Let $E \subseteq \mathbb{S}$ be a set of *entry states*. The pair $(\mathbb{S}, E)$ is said to be an *initialised transition system*. For convenience, the set of proposition letters P is partitioned into the following two disjoint and non-empty subsets: the *set of atomic propositions* (e.g. p, q, r, ...) and the *set of free* or *bound variables* (e.g. x, y, z, ...), denoted below by A and X, respectively. It can be observed that the graph of $\mathbb{S}$ can be equivalently described by using the following ternary *labelled transition relation*: $\varsigma \subseteq S \times D \times S$, i.e. some subset of ordered triples, rather than the family R (used in Definition 2.10). Also, the valuation V, restricted to the set A of atomic propositions and denoted here by λ, is defined as follows: $\lambda : S \to \mathbb{P}(A)$. Simulation logic can be considered as a fragment with negation over atomic propositions only. The language of this logic is obtained recursively as follows: $\varphi =_{df} p \,/\, x \,/ \sim \varphi \,/\, \varphi \wedge \psi \,/\, \varphi \vee \psi \,/\, [d]\varphi \,/\, \nu x.\varphi$ $(p \in A, d \in D, x \in X)$. The considered labelled transition system $\mathbb{S}$ is said to be a *model**. And hence, this model can be represented as follows: $\mathbb{S} =_{df} (S, D, \varsigma, A, \lambda)$.

The semantics of the used simulation logic was defined in the standard fashion (Kozen D. 1983), through the notion $||\varphi||^{\mathbb{S}}_{\varepsilon}$, where φ is a formula relative to the model $\mathbb{S}$ and *environment* ε. The corresponding *meaning* or *extension* of φ in $\mathbb{S}$ wrt ε is given as follows (S is considered as an universe).

$$||p||^{\mathbb{S}}_{\varepsilon} = \{s \in S \,/\, p \in \lambda(s)\},$$
$$||x||^{\mathbb{S}}_{\varepsilon} = \varepsilon(x),$$
$$||\sim\varphi||^{\mathbb{S}}_{\varepsilon} = S - ||\varphi||^{\mathbb{S}}_{\varepsilon},$$

* Provided there is no ambiguity, instead of V, L, $\to$ and the model $\mathscr{M}$ (Gurov D. and Huisman M. 2013), the letters X, D, ς and $\mathbb{S}$ are used here, respectively. And also, Instead of big letters, in the next considerations we shall denote variables by small letters (e.g. x, x_1, x_2, etc. instead of X, X_1, X_2, etc.).

$$\|\varphi \wedge \psi\|^{S}_{\varepsilon} \quad = \quad \|\varphi\|^{S}_{\varepsilon} \cap \|\psi\|^{S}_{\varepsilon},$$

$$\|\varphi \vee \psi\|^{S}_{\varepsilon} \quad = \quad \|\varphi\|^{S}_{\varepsilon} \cup \|\psi\|^{S}_{\varepsilon},$$

$$\|[d]\varphi\|^{S}_{\varepsilon} \quad = \quad \{s \in S \ / \ \forall_{t \in S} ((s,d,t) \in \varsigma \Rightarrow t \in \|\varphi\|^{S}_{\varepsilon}\},$$

$$\|\nu x.\varphi\|^{S}_{\varepsilon} \quad = \quad \bigcup\{S' \subseteq S \ / \ S' \subseteq \|\varphi\|^{S}_{\varepsilon(S' - X)}\}.$$

In accordance with the above extensions, e.g. the dynamic logic formula $[d]\varphi$ holds in a state $s \in S$ if φ holds in all states accessible from s via an edge labelled by $d \in D$. The initialised transition system $(\mathbb{S}, E)$ *satisfies* a formula φ, i.e. $(\mathbb{S}, E) \Vdash \varphi$ iff all its entry states satisfy φ. It is assumed here that formulae have pair-wise distinct fixed-point binders, and unless stated otherwise, are *closed*, i.e. all variables $x \in X$ are in the scope of a fixed-point binder and *guarded*, i.e. every occurrence of a variable x is in the scope of a box-functor (Walukiewicz I. 1995)*.

For convenience, let $s \rightarrow_d t \Leftrightarrow_{df} (s,d,t) \in \varsigma$. We shall say that ´$s \rightarrow_d t$´ is an *usual* (or *strong*) *transition*. (for any $s, t \in S$ and $d \in D$). Let now $\alpha \in D$ be a *distinguished* (*silent* or *understood*) action. The notion of a *weak transition* is introduced as follows: $s \Rightarrow_\alpha t \Leftrightarrow_{df} s (\rightarrow_\alpha)^* t$ and $s \Rightarrow_d t \Leftrightarrow_{df} s \Rightarrow_\alpha \circ \rightarrow_d \circ \Rightarrow_\alpha t$ (for all $d \neq \alpha$, where: ´$\circ$´ denotes composition of relations† and ´$(\rightarrow_\alpha)^*$´ is the corresponding *transitive and reflexive closure* of ´$\rightarrow_\alpha$´). And hence, the observable behaviour of a system can be described by using only weak transitions. In fact, we have a possibility of interpretation the box modality of the logic over the weak transitions rather that the strong ones. And so $(\mathbb{S}, s) \Vdash_w [d]\varphi$ *holds* iff φ holds in all states accessible from $s \in S$ via an edge labelled by d , preceded and followed by an arbitrary number of α-steps (Gurov D. and Huisman M. 2013). Here, the existence of a standard translation of formulae interpreted over weak transitions into equivalent formulae interpreted over strong transitions, was also cited (Stirling C.P. 2001). This translation, denoted by ´δ´ has the following property: $(\mathbb{S}, s) \Vdash_w \varphi$ exactly when $(\mathbb{S}, s) \Vdash \delta(\varphi)$.

The presented in (Gurov D. and Huisman M. 2013) program model is control-flow based and there are considered two different views on programs: a structural and a behavioural one. Both these views are instantiations‡ of the general notion of model (see also: Soleimanifard S., Gurov D. and Huisman M. 2011). In particular, these instantiations yield a structural and a behavioural version of the logic. Any program structure, i.e. *program's flow graph*, is considered as a collection, i.e. disjoint union of control-flow graphs (called below "method graphs": one for each of the program's methods). Moreover, all data in the original program is abstracted away. The question of adding data to the program model is also discussed (related to the notion of a *Boolean flow graph*: Ball T.and Rajamani S.K. 2000 - *Boolean programs* were proposed as a model for software analysis, and in fact, as ''a starting point for investigating model checking of software'', in the context of an abstraction-refinement. In this model, Boolean variables, or sets of such variables, are used as an abstraction of the actual program variables. A Boolean program is essentially a C program over Boolean values; it has global and local variables and methods that take Boolean arguments and return Boolean values).

The next definition is an extension of the notion of a Boolean flow graph, that allows a faithful representation of Boolean programs based on models over finite sets of labels and atomic propositions (Gurov D. and Huisman M. 2013).

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* Provided there is no ambiguity, in the place of "box-operator" here it is used the term "box-functor".

† Only for convenience, sometimes ´$\circ$´ is omitted.

‡ The *principle of instantiation* or *exemplification*, known in ancient times (Socrates, Platon) is a concept in metaphysics and logic (e.g. see: Monaghan P.X. 2011 or T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.). In nowadays, the term "instantiation" have some different interpretations, e.g. an event can be considered as an exemplification of a property in an entity. And this identity is represented as an ordered triple of an entity, property type and time (see: *exemplification theory*). In the case of computer science, the notion of instantiation have applications e.g. in *generic programming* (algorithms are written in terms of types) or also in *object-oriented programming*, where instantiation is a process of creating of instance of an object from a class (and hence, as in the above given considerations).

Let *Meth* be a countably infinite set of method names, $M \subseteq$ *Meth* be finite and $m \in M$[*]. The notion of a method graph is introduced as follows (for convenience, ′S_m′ is used below, instead of ′V_m′, given in the original definition (obviously, there is one-to-one correspondence between states in S_m and *vertices*, called also *nodes* or *points* in V_m).

Definition 2.19

A *flow graph* for method $m \in$ *Meth* over a finite set $M \subseteq$ *Meth* of method names is a finite model $\mathcal{S}_m =_{df} (S_m, D_m, \varsigma_m, A_m, \lambda_m)$, where: S_m is the set of control nodes of m, $D_m =_{df} M \cup \{\alpha\}$, $A_m =_{df} \{m, r\}$ and $\lambda_m : S_m \rightarrow \mathbb{P}(A_m)$ so that $m \in \lambda_m(s)$ for all $s \in S_m$, i.e. each node is tagged with its method name. The nodes $s \in S_m$ with $r \in \lambda_m(s)$ are said to be *return points*. A *method graph* for method $m \in$ *Meth* over M is an initialised model $(\mathcal{S}_m, E_m)$ such that $\mathcal{S}_m$ is a flow graph for m over M and $\varnothing \neq E_m \subseteq S_m$ is a set of *entry points* of m.

In accordance with the above definition, there are assumed only two types of atomic propositions: related to method names, and to return points.

The control flow graph sequencing can be realised by means of some "interface", ensuring that these graphs can only be composed if their interfaces math. The notion of ′flow graph interface′ is given below.

Definition 2.20

Let $I^+, I^- \subseteq$ *Meth* be two finite sets of names of *provided* and externally *required* methods, respectively[†]. A *flow graph interface* is the pair $I =_{df} (I^+, I^-)$. Let now $I_1 =_{df} (I_1^+, I_1^-)$ and $I_2 =_{df} (I_2^+, I_2^-)$ be two such interfaces. The following binary operation, called *interface composition*, is introduced: $I_1 \sqcup I_2 =_{df} (I_1^+ \cup I_2^+, (I_1^- \cup I_2^-) - (I_1^+ \cup I_2^+))$.

In accordance with the above considerations, the program's flow graph is essentially a disjoint union of its method graphs. The following notion ′$(\mathcal{S}_1, E_1) \uplus (\mathcal{S}_2, E_2)$′ of *disjoint union of initialised models* is used below, where each state is tagged with 1 or 2, respectively and $(s, i) \rightarrow_{d/(\mathcal{S}_1,E_1) \uplus (\mathcal{S}_2,E_2)} (t, i) \Leftrightarrow_{df} s \rightarrow_{d/(\mathcal{S}_i,E_i)} t$ ($i = 1,2$: Gurov D. and Huisman M. 2013).

Definition 2.21

A *flow graph* G *with interface* I, written ′$G : I$′, is defined inductively by: (1) $(\mathcal{S}_m, E_m) : (\{m\}, M - \{m\})$ if $(\mathcal{S}_m, E_m)$ is a method graph for $m \in$ *Meth* over M and (2) $G_1 \uplus G_2 : I_1 \sqcup I_2$ if $G_1 : I_1$ and $G_2 : I_2$.

In accordance with the last work, the above flow graph is said to be *closed* if $I^- = \varnothing$, i.e. if G does not require any external methods. Satisfaction, instantiated to flow graphs and denoted by ′$\Vdash_s$′, is said to be *structural* one. And hence, $G \Vdash_s \varphi \Leftrightarrow_{df} G \Vdash \varphi$.

The instantiation of the initialised models on the behavioural level requires the use of the following three kinds of labels.

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[*] More precisely, any $m \in M$ should be understood as a method definition, consisting of a method name, the types of the return value and the parameters and its implementation: concerns the modular verification of temporal safety properties. *Modularity* at the procedure–level is a natural instantiation of the *modular verification paradigm*, where correctness of global properties is relativised on the local properties of the methods rather than on their implementations, and is based here on the construction of maximal models for a program model that abstracts away from program data (Soleimanifard S., Gurov D. and Huisman M. 2011).

[†] It is required I^- to contain the methods that are not provided by I^+.

(1) the *transition label* τ: to designate *internal transfer* of control,

(2) m_1 *call* m_2: for the *invocation* of method m_2 by method m_1 and

(3) m_2 *ret* m_1: for the corresponding *return* from the call.

The following definition was introduced (Gurov D. and Huisman M. 2013).

Definition 2.22 (*flow graph behaviour*)

Let $G =_{df} (\mathcal{S}, E) : I$ be a closed flow graph where $\mathcal{S} =_{df} (S, D, \varsigma, A, \lambda)$. The *behaviour* of G, denoted by $\mathbf{B}(G)$, is defined as follows: $\mathbf{B}(G) =_{df} (\mathcal{S}_B, E_B)$, where $\mathcal{S}_B =_{df} (S_B, D_B, \varsigma_B, A_B, \lambda_B)$ such that $S_B =_{df} S \times S^*$, i.e. elements of S_B are pairs of *control points* s and *stacks*[*] σ (also called *configurations*), $D_B =_{df} \{\tau, m_1 \text{ call } m_2,\ m_2 \text{ ret } m_1\}$ $(m_1, m_2 \in I^+)$, $A_B =_{df} A$, $\lambda_B((s, \sigma)) =_{df} \lambda(s)$, and $\varsigma_B \subseteq S_B \times D_B \times S_B$ is defined by the rules:

(*transfer*)	$(s, \sigma) \rightarrow_{\tau/B} (s', \sigma)$	if	$m \in I^+, s \rightarrow_{\alpha/m} s', s \Vdash \sim r$
(*call*)	$(s_1, \sigma) \rightarrow_{m_1 \text{call } m_2 / B} (s_2, s_1' \cdot \sigma)$	if	$m_1, m_2 \in I^+, s_1 \rightarrow_{m_2/m_1} s_1', s_1 \Vdash \sim r$, $s_2 \Vdash m_2, s_2 \in E$
(*return*)	$(s_2, s_1' \cdot \sigma) \rightarrow_{m_2 \text{ret } m_1 / B} (s_1, \sigma)$	if	$m_1, m_2 \in I^+, s_2 \Vdash m_2 \wedge r, s_1 \Vdash m_1$

Here $'\cdot'$ is a *stack push*. The set of initial configurations is defined as: $E_B =_{df} E \times \{\epsilon\}$, where $'\epsilon'$ denotes an *empty sequence* over S.

An alternative approach of describing the behaviour of a flow graph was also cited (using a *pushdown automaton*, i.e. a type of automaton extended with a stack (a more formal treatment is omitted).

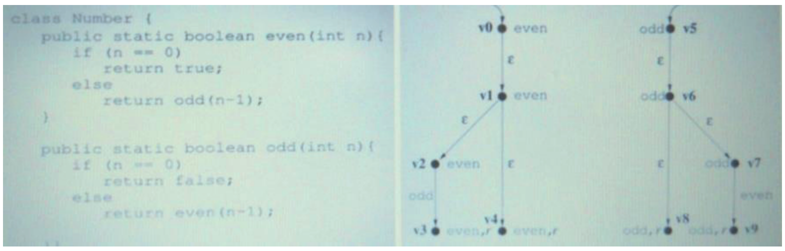

Figure 2.3 A simple Java class and its flow graph

Example 2.18 (Gurov D. and Huisman M. 2013)

A simple Java[†] class and the corresponding (simplified) flow graph that it induces are shown in the above Figure 2.3. The flow graph consists of two method graphs: one for method *even* and one for method *odd*. The entry nodes are depicted as usual through edges without source.

Let U be the set of edges of the above flow graph and u^+ and u^- be the *terminal vertices* of $u \in U$ (we shall assume u is connected from u^+ to u^- : Mayeda W. 1972). We shall say $\mu =_{df} (u_1, u_2, \ldots, u_k)$ is a

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[*] A *stack* or *LIFO* (last in, first out) is an abstract data type model considered as a collection of elements having two basic operations: *push* (insert element into stack), which adds an element to the collection and *pop* (remove element from stack), which removes the last added element (*The Free Encyclopaedia, The Wikimedia Foundation, Inc.* For a more detailed information, see: Knuth D.E. 1997: Donald Ervin Knuth, born 1938).

[†] Java (first designed by James Gosling: 1995) is one of the most popular today's programming languages because of its possibilities in the implementation of concurrence, application of classes, object orientation and platform independence.

(*directed*) *path* if all edges have the same orientation, i.e. $u_i^- = u_{i+1}^+$ $(i = 1,2, \ldots , k-1)$*. By $\mu =_{df} \mu[u_1^+, u_k^-]$ we shall denote a path having u_1^+ and u_k^- as *initial* and *terminal vertices*, respectively. Below by $l(u) \in D_B$ we shall denote the *label associated with* u.

According to Definition 2.22, the elements of S_B are ordered pairs of type (s, σ), considered as vertices in the above program's flow graph. An example execution of the program is illustrated by the following path: $\mu = \mu[(s_0, \in), (s_3, \in)]$. Moreover, instead of the original given branching structure a table representation is shown below. □

u^+	$l(u)$	u^-
$(s_0, \in)$	τ/B	$(s_1, \in)$
$(s_1, \in)$	τ/B	$(s_2, \in)$
$(s_2, \in)$	even call odd / B	(s_5, s_3)
(s_5, s_3)	τ/B	(s_6, s_3)
(s_6, s_3)	τ/B	(s_7, s_3)
(s_7, s_3)	odd call even / B	$(s_0, s_9 \cdot s_3)$
$(s_0, s_9 \cdot s_3)$	τ/B	$(s_1, s_9 \cdot s_3)$
$(s_1, s_9 \cdot s_3)$	τ/B	$(s_4, s_9 \cdot s_3)$
$(s_4, s_9 \cdot s_3)$	even ret odd / B	(s_9, s_3)
(s_9, s_3)	odd ret even / B	$(s_3, \in)$

The *behavioural satisfaction* related to G and denoted by ´$\Vdash_B$´ is introduced in a similar way, i.e. $G \Vdash_B \psi \Leftrightarrow_{df} B(G) \Vdash \psi$.

The above presented two kinds of satisfaction (structural and behavioural) will also involve and two kinds of formulae. For example, in accordance with the last example, the *structural formula* ´$\nu x.[even]r \wedge [odd]r \wedge [\alpha]x$´ expresses the property: "on every path from a program entry node, the first encountered call edge goes to a return node", in effect specifying that the program is *tail-recursive*†. Similarly, the *behavioural formula* ´*even* $\Rightarrow \nu x.[even\ call\ even]ff \wedge [\tau]x$´ expresses the property: "in every program execution, starting in method *even*, the first call is not to method *even* itself", where ´ff´ denotes the constant formula ´*false*´ (Gurov D. and Huisman M. 2013).

Method graphs allow return points to have outgoing edges. However, the proposed characterisation of behavioural properties by a set of structural formulae, defined in the last work, is only correct if the flow graph has no such edges and such graphs are called *clean*. The operation of cleaning, defined on method graphs, can be lifted to flow graphs. In fact, the following definition was introduced.

Definition 2.23 (*the unary operation of cleaning*)

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* Because of the one-to-one correspondence between vertices and states, instead of v_i the states s_i are used below $(i = 0, 1, \ldots, 9)$. Similarly, instead of ε the distinguished action is here denoted by α. Moreover, provided there is no ambiguity, the Greek letter ´μ´ also denotes a *path* (of a graph G, e.g. Berge C. 1973): depending on the context.

† In *computer programming*, a *subroutine* is a sequence of program instructions that perform a specific task. A *tail call* is a subroutine call performed as the final action of a procedure. If a tail call might lead to the same subroutine being called again later in the call chain, the subroutine is said to be *tail-recursive*, which is a special case of recursion. *Tail recursion* (called also: *tail-end recursion*) is particularly useful, and often easy to handle in implementations (see T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*). In accordance with the last considerations, any subroutine can be considered as a subprogram similar in most respects to a function. Moreover, recursion is generally favoured over iteration in very many languages because of its elegancy, minimal form, implementation with regular functions and easier to analyse formally. In particular recursion is preferred and promoted in *functional programming* (starting with *Church's λ-calculus*, as a theoretical framework for describing functions and their evaluation: Alonzo Church 1903 – 1995 and since the introduction of *Lisp*, in the late of 1950's: John McCarthy 1927 – 2011, many partially and fully functional languages have been developed, e.g. the *Haskell's language* in the late of 1980's: Haskell Brooks Curry 1900 – 1982). However, some languages may not have iteration, e.g. the general purpose logic programming language *Prolog* (developed in 1972, see: Colmerauer A. and Roussel P. 1996). Moreover, in the case of more functional calls and stack operations (context saving and restoration) recursion can also be less efficient.

Let $S_m =_{df} (S_m, D_m, \varsigma_m, A_m, \lambda_m)$ be a method graph. The unary operation of *cleaning* is defined as follows: $(S_m)^{\bullet} =_{df} (S_m, D_m, \{(s,d,t) \in \varsigma_m \ / \ r \notin \lambda_m(s)\}, A_m, \lambda_m)$.

In accordance with the last definition, we have: $(G^{\bullet})^{\bullet} = G^{\bullet}$ (the *idempotence*[*] of cleaning) and $G^{\bullet} \Vdash_B \psi \Leftrightarrow G \Vdash_B \psi$ (the *behaviour preservation*). Moreover, the following implication is satisfied: $(G^{\bullet},s) \Vdash_s r \Rightarrow \underset{d \in D}{\forall} \underset{\varphi}{\forall} ((G^{\bullet},s) \Vdash_s [d]\varphi)$. And hence, the structural box formulae are trivially satisfied by return points.

The mapping behavioural into structural properties was realised by a map Π, defined as follows (Gurov D. and Huisman M. 2013): $G \Vdash_B \psi \Leftrightarrow_{df} \underset{\varphi \in \Pi_I(\psi)}{\exists} (G \Vdash_s \varphi)$. It is assumed that the considered flow graphs are clean, and that behavioural properties are disjunction-free. An extension of Π to behavioural formulae with disjunction was also considered (though at the expense of completeness).

It was shown that Π computes a set of structural formulae that characterises ψ and I (for any behavioural property ψ and closed interface I, i.e. for any closed flow graph $G : I$ and ψ that only mentions labels that are in the behaviour of G: see D_B , Definition 2.22).

An implementation of the above explicit translation Π was developed in Ocaml[†] and it can be tested on-line. Moreover, this translation of simulation logic formulae also allows the label sequences to appear in box modalities. And so we have: $[\epsilon]\chi =_{df} \chi$ and $[d \circ \omega]\chi =_{df} [d][\omega]\chi$, where ω denotes a label sequence, χ is already a standard formula and $d \in D$. The map Π was defined by a finite *tableau construction*, in accordance with the fixed-point formulae of this logic. And hence, the obtained translation was based on a symbolic execution of the behavioural property by means of a such construction. When tracing a symbolic execution path, subformulae of ψ are tagged with unique constants related to the variables associated with this path. The set of all constants is denoted by $\mathscr{C}$. Let $\Psi =_{df} \{\psi \ / \ G \Vdash_B \psi\}$. The following global, injective map was introduced: $\mathscr{S} : \Psi \rightarrow \mathscr{C}$ to map formulae to their tags and this map was considered as an implicit parameter of the tableau construction. Moreover, this tableau construction operates on sequents[‡] of the shape $\vdash_{H,U,C} \psi$ parameterised on (Gurov D. and Huisman M. 2013):

(1) A non-empty *history stack* $H \in (I^+ \times (I^+ \cup \{\alpha\} \cup \mathscr{C})^*)^+$ is a pair consisting of the current method name and a sequence, called *frame* and denoted by F, of edge labels and constants abbreviating subformulae of ψ. By $\underline{F}$ it is denoted F cleaned from constants x[§] $\in \mathscr{C}$: $\underline{\epsilon} = \epsilon$, $\underline{m \circ \sigma} = m \circ \underline{\sigma}$, $\underline{\alpha \circ \sigma} = \alpha \circ \underline{\sigma}$ and $\underline{x \circ \sigma} = \underline{\sigma}$.

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[*] A property of some operations that can be applied multiple times without changing the result beyond the initial application (Benjamin Peirce 1809 – 1880): see T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[†] Objective Categorical Abstract Machine Language, in short: *Ocaml* is an extension of the *Caml programming language*, created by Xavier Leroy at all in 1996, with object-oriented constructs (see T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*). Not be confused with the concurrent programming language *Occam*, first developed by David May at all, 1984 (advised by Tony Hoare, *communicating sequential processes*: Charles Antony Richard Hoare, born: 1934 and named after William of Ockham of Occam's razor fame: 1285 – 1347). Occam was mainly considered as a programming language for transputer microprocessor systems, e.g. hardware realisation of data flow Petri nets.

[‡] see: Definition 1.14 of Subsection 1.8

[§] Provided there is no ambiguity and for convenience, the symbols used for variables are "overloaded": for reasons related to the notion of an *induced structural formula*, later considered in (Gurov D. and Huisman M. 2013). In addition, the *frame* $\tilde{F}$ we shall denote by the symbol $\underline{F}$.

(2) A *fixed-point stack* U, defining an environment for variables by a sequence of definitions of the form $x = \nu x.\chi$. An open formula ψ in a sequent parameterised by U can then be understood via a suitable notion of substitution. Let $\chi(\theta / x)$ be a substitution of a formula θ for a variable x in a formula χ. And so, the substitution of ψ under U is inductively defined as follows: $\psi[\epsilon] =_{df} \psi$ and $[(x = \nu x.\chi) \circ U] =_{df} (\psi(\nu x.\chi / x))[U]$.

(3) A *store* C, used for accumulating structural constants during symbolic execution (using stores can in principle be dispensed with, but simplifies the presentation of the extraction of structural formulae and the correctness proofs).

Let $\varnothing_{H,m}$, $\varnothing_U$ and $\varnothing_C$ be the single-element history stack (m, ϵ), the empty fixed-point stack and the empty store, respectively. The construction of a maximal tableau with *root* $\vdash_{\varnothing_{H,m}, \varnothing_U, \varnothing_C} \psi$ that induces a set of structural formulae through its leaves was described as below (for any given closed behavioural formula ψ and method m).

Let $\pi_m(\psi)$ be the set of induced structural formulae associated with ψ and m. The translation of ψ wrt a given interface I is defined as all possible conjunctions of the induced structural formulae for each method m that is proved by I: $\Pi_I(\psi) =_{df} \{ \wedge_{m \in I^+} \varphi_m \ / \ \varphi_m \in \pi_m(\psi) \}$.

During tableau construction the stacks H and U, and the store C are updated as follows (it is assumed that the current sequent is not a repeat of an earlier one, see: Gurov D. and Huisman M. 2013):

(1) First, if ψ is not a fixed-point formula, the constant $\mathscr{S}(\psi)$ tagging the behavioural property ψ of the current sequent is appended to the end of the frame of the top element of H (instead of using constants as tags, alternatively there exists a possibility of introducing fresh variables, and add their defining equations to U);

(2) Next,

(2.1) if ψ is a conjunction, both conjuncts are explored in two separate branches;

(2.2) if ψ is (the negation of) an atomic proposition, exploration terminates for this branch, and a set of structural constraints based on the atomic proposition and the current history stack are produced;

(2.3) if the behavioural property of ψ prescribes an internal transfer of the form $[\tau]\psi'$, then α is appended to the end of the frame of the top element of H and the symbolic execution is continued with formula ψ';

(2.4) if ψ prescribes a call from a to b of the form $[a \text{ call } b]\psi'$, and the top element of H is in method a, then b is added at the end of the frame of the top element of H, a new element (b, ϵ) is pushed onto H, and the symbolic execution is continued with formula ψ';

(2.5) if ψ prescribes a return from a to b of the form $[a \text{ ret } b]\psi'$, the top element of H is in method a and the next element is in method b, then a new structural constraint is added to C, reflecting the possibility of currently not being at a return point, the top element is popped from H, and the symbolic execution is continued with formula ψ';

(2.6) if ψ is a fixed-point formula $\nu x. \psi'$, then a new equation $x = \nu x. \psi'$ is pushed onto U, if not already there, this conditional addition being denoted by $(x = \nu x. \psi') \circledcirc U$, and the symbolic execution is continued with formula x;

(2.7) if ψ is a variable x for which there is an equation $x = \nu x. \psi'$ in U, then the symbolic execution is continued with formula ψ'. □

The tags were used to signal repetition in the symbolic execution of the considered formula and for ensuring termination of the tableau construction. The structural constraints and the elements in the call stack denote conditions under which the actual property holds. Moreover, each step of the symbolic execution adds new constraints. An informal illustration of symbolic execution is given below (Gurov D. and Huisman M. 2013).

Example 2.19

Consider the following behavioural property of a formula ψ: "invocation of a method cannot return without making a method call", formalised as follows: $\nu x. \sim r \wedge [\tau]x$. The obtained steps, when executing ψ symbolically for method a, are given below.

(1) It is started with the initial H, i.e. $(a, \in)$ and formula $\psi =_{df} \nu x. \sim r \wedge [\tau]x$;

(2) The equation $x = \nu x. \sim r \wedge [\tau]x$ is pushed onto U, and it is proceed with formula x;

(3) The definition of x is retrieved from U, and it is continued with formula $\sim r \wedge [\tau]x$;

(4) Each conjunct of the last conjunction is explored separately;

(5) The first conjunct $\sim r$ is the negation of an atomic proposition, and hence exploration terminates for this branch, producing a constraint that essentially requires $\sim r$ to hold;

(6) The second conjunct $[\tau]x$ appends α to the frame, i.e. H becomes (a, α), and it is proceed with formula x;

(7) This is recognised as a repeat of a situation that arose before (at step 3), therefore the exploration terminates, producing a constraint that essentially requires x to hold for all nodes that can be reached by passing a transfer edge. □

In accordance with the last example, the two constraints produced by this symbolic execution can be combined for obtaining the following recursive structural formula: $a \Rightarrow \nu x. \sim r \wedge [\alpha]x$ (see steps 5 and 7). Formally, this symbolic execution should require the introduction of the next notions given in (Gurov D. and Huisman M. 2013): tableau system, repeat conditions and structural formulae induced by a tableau. The last three notions are briefly presented below.

The *tableau system* is a proof system, considered as a *labelled tree* T with a set of goal-directed axioms called *rules* with an empty set of premises denoted by ′–′. In particular, the used in return rules condition *Ret*(i, a, b, H) $\Leftrightarrow_{df} (i = a) \wedge (H \neq \in) \wedge \underset{F,H'}{\exists}(H =_{df} (b, F) \circ H')$. The *labelling function*, denoted here by f, maps each node of T to a triple consisting of a sequent, a rule name (the rule applied to this sequent), and a set of triples of the form (i, F, q) where q are *literals* (i.e. atomic propositions in positive or negated form or variables $x \in X$)*. The triple sets are non-empty only at applications of axiom rules. Such leaves are termed *contributing*. The corresponding set of triples is depicted (by convention) as a premise to the rule. A *tableau for formula* ψ and *method* m (in short: $T_m(\psi)$) is a tree with root $\vdash_{\varnothing_{H,m}, \varnothing_U, \varnothing_C} \psi$ obtained by applying the rules. $T_m(\psi)$ is *maximal* if all its leaves are axioms.

Let $\vdash_{(i,F) \circ H, U, C} \chi$ be a leaf node for which there is an internal one $\vdash_{(i,F') \circ H', U', C'} \chi$ such that F′ is a prefix of F, U′ is a suffix of U, and $C' \subseteq C$. The former node is said to be a *pseudo-repeat* and any node of the latter

* In accordance with the above work, the used symbol λ denotes: a valuation, labelling function or a choice set (depending on the context). Provided there is no ambiguity, the labelling function and the choice set are here denoted by f and $\wedge$, respectively. Moreover, instead of $\rho(n)$, the return depth of a tableau node n is here denoted by *rd*(n).

kind is called a *companion*. An internal tableau node is said to be *stable* if all its descendant leaves are axioms or pseudo-repeats. A tableau is called *stable* if its root node is stable.

The process of *tableau construction* was realised as follows. A minimal stable tableau is first computed (i.e. pseudo-repeat nodes are not further explored). If all pseudo-repeats in this tableau satisfy some repeat condition for any of their companions (see below), the tableau is maximal and construction is complete. Otherwise, all pseudo-repeats that are not satisfying any of the repeat conditions are simultaneously unfolded, using a breadth-first exploration strategy, and tableau construction continues until the tableau is stable again, upon which the checking for the repeat conditions is repeated. As it was shown, this process is guaranteed to terminate, resulting in a finite maximal tableau. The tableau system is shown in Figure 2.4 given below (Gurov D. and Huisman M. 2013).

The following three *repeat conditions* were introduced into the above tableau system: internal, call and return repeat conditions, involving tree types of nodes. The *internal repeats* are related to recursion in structural formulae. In contrast, the other two *call* and *return repeat conditions* only recognise that a similar situation has been reached before. In fact, the first repeat condition requires only the examination of the top frame of H of the current sequent, the second one requires the examination of the whole path from the root to the pseudo-repeat, while the third one requires the examination of all remaining paths.

The *return depth* of a tableau node n, denoted by *rd*(n), was defined as the maximal difference between the number of applied return rules and the number of applied call rules on any path from n to a descendant node. By using the translation δ (of formulae interpreted over weak transitions into equivalent formulae interpreted over strong transitions) we have:

$$rd'(\epsilon) =_{df} 0, \qquad rd'(r \circ \delta) =_{df} \begin{cases} rd'(\delta) + 1, \text{ if } r \in \{ret_0, ret_1\} \\ rd'(\delta) - 1, \text{ if } r \in \{call_0, call_1\} \\ rd'(\delta), \text{ otherwise} \end{cases}$$

and

$$rd(n) =_{df} \max(\{rd'(rules(\mu)) \,/\, \mu \text{ is a path from } n \text{ to a descendant node}\} \cup \{0\}),$$

where r and δ range over rule names and sequences of rule names, respectively and *rules*(μ) denotes the sequence of rule names along a tableau path μ.

Let x be $\mathscr{S}(\psi)$, c be the companion node of the pseudo-repeat, and H_c be the history stack at c. More formally, the above three repeat conditions were defined as follows.

$$IntRep(x, (i,F) \circ H) \Leftrightarrow_{df} x \in F$$

$$CallRep(x, (i,F) \circ H, c) \Leftrightarrow_{df} (x \notin F) \wedge (take(rd(c) + 1, (i,F) \circ H) = take(rd(c) + 1, H_c))$$

$$RetRep(x, (i,F) \circ H, c) \Leftrightarrow_{df} (i,F) \circ H = H_c.$$

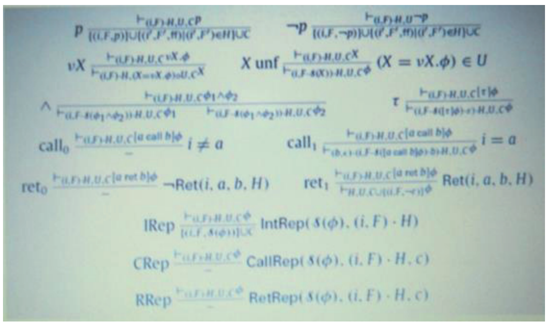

Figure 2.4 Tableau system

Let $T_m(\psi)$ be maximal. The *set* $\pi_m(\psi)$ of *induced structural formulae* was generated in the following manner (Gurov D. and Huisman M. 2013)

(1) Let $\mathscr{L}$ be the family of non-empty triple sets collected from the leaves of the tableau. Build a family of choice sets $\Lambda(\mathscr{L})$ by choosing one triple from each element in $\mathscr{L}$;

(2) For each *choice set* $\wedge \in \Lambda(\mathscr{L})$,

(2.1) Group the triples of $\wedge$ according to method names: for each $m \in I^+$, define $\Xi_m =_{df} \{(F,q) / (m,F,q) \in \wedge\}$;

(2.2) For any non-empty Ξ_m build a formula $m \Rightarrow \Omega(\Xi_m)$, where $\Omega(\Xi) =_{df} \bigwedge_{\psi \in \Omega'(\Xi)} \psi$ and $\Omega'(\Xi) =_{df} \{[a]\Omega(\Xi') / a \in I^+ \wedge \varnothing \neq \Xi' =_{df} \{(F,q) / (a \circ F, q) \in \Xi\} \cup \{\nu x.\Omega(\Xi') / x \in \mathscr{C} \wedge \varnothing \neq \Xi' =_{df} \{(F,q) / (x \circ F, q) \in \Xi\} \cup \{q / (\epsilon, q) \in \Xi\}$;

(2.3) The induced formula φ for $\wedge$ is the conjunction of the formulae obtained in the previous step;

(3) The set $\pi_m(\psi)$ is the set of induced formulae for $\wedge \in \Lambda(\mathscr{L})$. □

A formal symbolic execution of ψ of Example 2.19 is given in Figure 2.5 below.

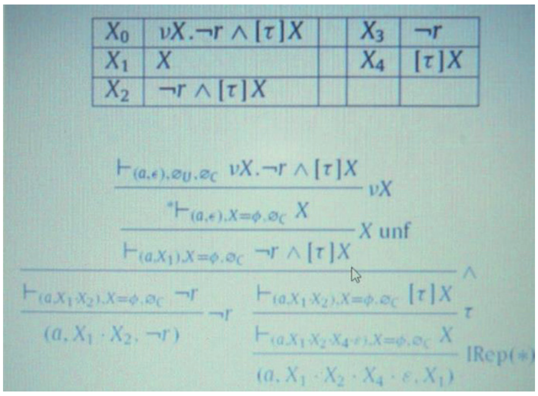

Figure 2.5 An example tableau $T_a(\nu x. \sim r \wedge [\tau]x)$

The non-emptiness of H and closedness of ψ are invariants during the construction of such a tableau system. In particular, it was shown that the proof tree induced by the unfolding of the tableau for behavioural formula ψ and method m generating the *set of structural formulae* $\mathscr{X}$ constitutes a proof that every flow graph satisfying some $\varphi \in \mathscr{X}$ also satisfies ψ. In fact, the following fundamental properties were shown: maximal tableaux are finite; $\mathscr{X} \vDash_{\varnothing_{H,m}, \varnothing_U, \varnothing_C} \psi$ holds whenever there is a proof with root $\mathscr{X} \vdash_{\varnothing_{H,m}, \varnothing_U, \varnothing_C} \psi$; translation Π from behavioural to structural formulae is sound and complete. A more formal treatment is omitted here (left to the reader).

In general, the above proposed tableau construction gives rise to a correctness argument that allows to view a maximal tableau as a proof that the structural formulae resulting from the tableau entail the original behavioural formula. The combination of this construction with the above properly translation Π provides a solution to the problem of computing maximal program structures from behavioural properties. Further, this translation can be used to reduce infinite-state verification of behavioural control flow properties to finite state verification of structural properties. Thus, tools for checking structural properties can in effect be used for verifying behavioural ones. Unfortunately, computing the set of induced structural formulae is exponential (i.e. the running time increases exponentially). And hence, some ad hoc simplifications were also used (e.g. see: *ProMoVer*:

Soleimanifard S., Gurov D. and Huisman M. 2011). The possibility of introducing more optimisations was also discussed. In particular, a considerable difficulty is presented by (greatest fixed-point) recursion in the behavioural formula ψ, which has to be captured by recursion in the structural ones. So, this recursion was handled by means of a tableau construction that maintains (during the symbolic execution) a symbolic "call stack" indicating which subformulae have been explored for which method m. The study of a parallel implementation of the above proof system seems to be an interesting topic for further research. Moreover, there exist problems that can be described by different algorithms (e.g. see Example 1.4 of Subsection 1.3, we shall say that these two algorithms are *functionally equivalent*)[*]. Each such algorithm should involve different program's behaviour and hence different program's syntactical structure (in consequence: different degrees of computational complexity).

An alternative solution, can be based on the theory of *nested words* (Alur R. and Madhusudan P. 2009)[†]. Using the results of this theory, a μ-calculus formula can be translated into an equivalent formula in a fixpoint calculus for nested words, and then in turn be translated into an equivalent alternating parity nested tree automaton. The latter automaton has a structural content that, in principle, can be used as a representation of program structure. In contrast, the above presented solution (Gurov D. and Huisman M. 2013) is "direct", in the sense that the presented symbolic execution directly follows the operational semantics of the program model, which relates structure with behaviour. This makes the obtained construction easy to adapt for variations and extensions of this model, as was explored in (Huisman M., Aktug I. and Gurov D. 2008).

Another interesting approach should be the use of high-level Petri nets, e.g. coloured Petri nets, introduced in (Jensen K. 1981), for software verification (a more formal treatment is omitted here)[‡].

A brief survey of the background and history of modal and temporal logics was given by Bradfield J. and Stirling C., see: http://www.dis.uniroma1.it/~degiacom/didattica/semingsoft/SIS05-06/materiale/3-servizi/altro%20materiale/bradfield-stirling-HPA-mu-intro.pdf. In particular, the modal μ-calculus was also discussed (a logic which subsumes most other commonly used logics: left to the reader).

2.4. Other non-classical systems

Below are briefly presented some other non-classical systems, such as: epistemic, game, intuitionistic and fuzzy intuitionistic, linear, (intuitionistic) computability, paraconsistent, relevant and non-monotonic logic systems. Some comments concerning fractal logic are also given.

Epistemic logic

The term '*episteme*' , used in the Ancient Greece philosophy, is etymologically derived from the word "ἐπιστήμη" to denote knowledge or science (e.g. Platon's term for *common belief* or *opinion* or also some aspects of the logic of knowledge and belief mentioned by Aristoteles). Aristoteles' insights were extended in the middle ages (e.g. Jean Buridan, John Duns Scotus and William of Ockham).

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[*] There exist also problems that cannot be algorithmically solved, and so their solution cannot be automatically found by means of computers (*public domain*).

[†] The model of nested words, introduced by Alur R. and Madhusudan P. (2004), was proposed for representing and querying data with dual linear-hierarchical structure. A *nested word* consists of a sequence of linearly ordered positions, augmented with *nesting edges* connecting calls to returns (or open-tags to close-tags). The edges do not cross creating a properly nested hierarchical structure, and we allow some of the edges to be pending. This nesting structure can be uniquely represented by a sequence specifying the types of positions (calls, returns, and internals). Words are nested words where all positions are internals. Ordered trees can be interpreted as nested words using the following traversal: to process an a-labeled node, first print an a-labeled call, process all the children in order, and print an a-labeled return. Note that this is a combination of top-down and bottom-up traversals, and each node is processed twice.

[‡] The c*oloured Petri nets*, the *numerical Petri nets* (developed originally in: Symons F.J.W. 1978) and other similar models were considered as a good theoretical basis for the introduction of a new international standard: the *high-level Petri nets*, as a well-defined semi-graphical technique for the specification, design and analysis of systems. This technique is mathematically defined, and may thus be used to provide unambiguous specifications and descriptions of applications. It is also an executable technique, allowing specification prototypes to be developed to test ideas at the earliest and cheapest opportunity. Specifications written in this technique may be subjected to analysis methods to prove properties about the specifications, before implementation commences, thus saving on testing and maintenance time (see: ISO/IEC 2000, 2005).

The contemporary epistemic logic is the logic of knowledge and belief. It provides insight into the properties of individual *knowers* (called also *agents*), has provided a means to model complicated scenarios involving groups of knowers and has improved our understanding of the dynamics of inquiry. The work (Wright G.H.von. 1951) was one of the most important initiation of the formal study of epistemic logic. This work were extended by Hintikka J. (1962), considered now as a classical (*The little encyclopaedia of logic* 1988, Hendrics V. and Symons J. 2006).

More formally, the following two unary *functors* were introduced (originally called "*operators*", as an syntactical augmentation of propositional logic): K_c and B_c, called *epistemic* and *doxastic* (from Ancient Greek term"*δοξα"*), respectively. Here, K_cp denotes "*Agent* c *knows* p" and B_cp denotes "*Agent* c *believes* p" , for some proposition p. Moreover, the following semantic interpretation was introduced (Hintikka J. 1962).

K_cp: *in all possible worlds compatible with what* c *knows, it is the case that* p, and

B_cp: *in all possible worlds compatible with what* c *believes, it is the case that* p.

The following axiomatic system was proposed (Lemmon E.J. and Scott D.S. 1977), next improved in (Bull R. and Segerberg K. 1984). The axiom abbreviations are under the first of these two works.

K $K_c(p \Rightarrow q) \Rightarrow (K_c p \Rightarrow K_c q)$

D $K_c p \Rightarrow \sim K_c \sim p$

T $K_c p \Rightarrow p$

4 $K_c p \Rightarrow K_c K_c p$

5 $\sim K_c p \Rightarrow K_c \sim K_c p$

.2 $\sim K_c \sim K_c p \Rightarrow K_c \sim K_c \sim p$

.3 $K_c(K_c p \Rightarrow K_c q) \vee K_c(K_c q \Rightarrow K_c p)$

.4 $p \Rightarrow (\sim K_c \sim K_c p \Rightarrow K_c p)$

The above axioms are very similar to the corresponding ones or also to some properties satisfied in modal logic (see Subsection 2.3: modal logic), e.g. K corresponds to Gödel's axiom (G2), D to T.2.18 ($\Diamond\varphi \Leftrightarrow_{df} \sim \Box \sim \varphi$, SR), T to the rule ´- □´, 4 to Gödel's axiom (G3), etc. Provided there is no ambiguity and for simplicity, sometimes the subscript ´c´ may be omitted (if c is known), e.g. that "*the Agent knows that p*" is written as *K*p, that "*She does not know that p and q*" as $\sim K(p \wedge q)$, that "*She knows whether or not q*" as $Kq \vee K\sim q$, that "*She knows that she does not know that if p, then q*" as $K \sim K(p \Rightarrow q)$, etc. Similarly, that "*the Agent* c *believes* p" is written as *B*p (Holiday W.H. 2016).

There exist various applications of epistemic logic. Some of them are cited below. As a natural way, applications to epistemology[*] are first cited.

In general, the contemporary *epistemology* is organized around two major goals: providing a *definition of knowledge* and modelling the *dynamics of epistemic and doxastic states*. The first of these goals, for the most part, concerns philosophers who rely on thought experiments, traditional conceptual analysis or intuitions-based methods of various kinds. By contrast, philosophers working with epistemic logic are pursued the second goal. However, the last two goals are related to a third, and possibly more general problem, namely the problem of *understanding the rationality of inquiry* (Hendrics V. and Symons J. 2006). A brief introduction to propositional epistemic logic and its applications to epistemology is also given in (Holiday W.H. 2016). Epistemic-logical topics, presented here, include the language and semantics of basic epistemic logic, multi-agent epistemic logic, combined epistemic-doxastic logic, and a quick look of dynamic epistemic logic. Epistemological topics

[*] The *Nyāya school of logic*, based on Hindu philosophy, can be considered as a form of epistemology (i.e. theory of knowledge) in addition to logic (see: the introduction of Section 1).

discussed include Moore-paradoxical phenomena[*], the surprise exam paradox, logical omniscience and epistemic closure, formalized theories of knowledge, debates about higher-order knowledge, and issues of knowability raised by Fitch'paradox[†]. Obviously, some considered topics are similar as in the previous work (a more formal treatment is left to the reader).

Epistemic logic has, in the past few decades, grown beyond its origins in philosophy, to be embraced in several other disciplines, including economics (e.g. game theory or also the study of various social and coalitional interactions), linguistics and computer science. Within computer science, its application can be found in several subdisciplines: artificial intelligence (e.g. applications in robotics, groups of knowers, i.e. multi-agent systems, for instance: grounding modal language in communications of artificial cognitive agent systems: Katarzyniak R.P. 2007), distributed computing and computer security, i.e. network security and cryptography (Meyden R. van der 2011).

The main purpose of the last work was identification of situations in which questions about the security of systems and communication protocol designs should be amenable to automated analysis. There were studied the decidability of logical problems into the following two distinct types: model checking and synthesis, where *model checking* involves the question of whether a given formula is satisfied in a given semantic structure and *synthesis* concerns the existence of strategies that lead to the satisfaction of a formula in a given context. The relevance of this in computer security is that an active adversary, who not only observes but also interferes with the system, can be viewed as seeking a strategy that satisfies certain conditions on the adversary's knowledge. Hence, the system is *secure* if no such strategy exists. Finally, there was presented a protocol designed to enable information to be transmitted anonymously: *Chaum's Dining Cryptographers*[‡] protocol (Meyden R. van der 2011).

Game logic

Game logic is an interesting extension of propositional dynamic logic, it does not use Kripke semantics but *neighbourhood semantics* (known as *Scott - Montague semantics*[§]: instead of accessibility relation a neighbourhood function was introduced), a generalisation of the standard Kripke semantics, invented independently by (Scoot D. 1970) and (Montague R. 1970). Game logic is a logic to reason about determined two-player games. It presents a good example of mathematical and practical application of propositional dynamic logic (Parikh R. and Pauly M. 2003). Dynamic logic can also express change in knowledge. And so, we can abstract variety of knowers and their respective knowledge, i.e. a *multi-agent system* (in terms of computer science and artificial intelligence) and formalise their interaction. The basic system of modal epistemic logic (Hintikka J. 1962) can be extended by incorporating a dynamic functor to express *knowledge update* (Plaza J.A. 1989).The corresponding epistemic states are updated as follows. It is assumed that knowers are exposed a truthful public announcement by an external agent. Then, they *update* their epistemic states in such a way that they get rid of the states and epistemic situations that do not agree with the announcement. Obviously, the syntax of this logic, known as *public announcement logic*, is an extension of the basic epistemic logic. The semantics is given under Kripke models, e.g. see (Başkent C. 2010).

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[*] Sentences of the Moorean form $p \wedge \sim Kp$ cannot be known (1942: G.E. Moore 1873 – 1958), e.g. "It is raining, but I don't believe that it is raining" or "It is raining but I believe that it is not raining", referred by Philosophers nowadays as the *omissive and commissive* vertions of *Moore's paradox*, more formally: $p \wedge \sim Bp$ or $p \wedge B \sim p$, respectively. There doesn't seem to be any logical contradiction between "It is raining" and "I believe that it is not raining": the former is a statement about the weather and the latter is a statement about a person's belief about the weather (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*, for a more formal treatment: Holiday W.H. 2016).

[†] *Fitch's paradox* of knowability (1963: F.B. Fitch 1908 – 1987) is another fundamental puzzle of epistemic logic. It provides a challenge to the *knowability thesis*, which states that every truth is, in principle, knowable (in correspondence with the *omniscience principle* ("having total knowledge", "knowing everything"), which asserts that every truth is known. Hence, Fitch's paradox asserts that the existence of an unknown truth is unknowable. And so, if all truths were knowable, it would follow that all truths are in fact known (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*). A more formal treatment is given in (Holiday W.H. 2016), where Fitch's thesis is represented as follows: $\forall_p \, (p \Rightarrow \Diamond Kp)$. Here, a question may be the interpretation of the last functor of possibility.

[‡] David Lee Chaum (born 1955) is the inventor of many cryptographic protocols , as well as such systems as: the *electronic cash system* (in short: *ecash*), *digital cash*, *blind signatures* and so on. His paper (Chaum D.L. 1981) laid the groundwork for the field of anonymous communications research (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*).

[§] Dana Stewart Scott, born 1932; Richard Montague, born 1930.

Quantum dynamic-epistemic logic

Quantum theory (also known as *quantum mechanics* or *quantum physics*) gradually arose from Max Planck's solution in 1900 to the *black-body radiation problem* (Max Karl Ernst Ludwig Planck 1858 – 1947) and Albert Einstein's 1905 paper which offered a quantum-based theory to explain the *photoelectric effect* (Albert Einstein 1879 – 1955)*. Some structural similarities of current research in the foundations of *quantum theory* and the foundations of computer science are given e.g. in (Coecke B: http://www.cs.ox.ac.uk/people/bob.coecke/SLS.pdf, a quick draft version of a course), see also (Lindley D. 1998: a guide incl. Quantum Computers), (Svozil K. 1998) or the *Handbook of quantum logic and quantum structures* (2009).

In the quantum computational framework, there are polynomial time solving algorithms for problems having exponential classical solutions. Obviously, the question is the existence of such algorithms for other similar problems. The most feasible implementation of *quantum algorithms* is based on the quantum circuit (gate network) model. *Quantum computing*† itself emerged from the attempt to simulate *quantum systems*. The simulation is often considered as a tool used in theoretical approaches. However, it could also be employed for quantum hardware design. Simulation of quantum computational systems is usually exponential. The quantum circuits make no difference to this rule. In quantum computation the circuits are prone to fail, and safe recovery is difficult. Therefore, a straightforward classically-inspired solution is not feasible because the overall fail rate will be given by ancillary *qubits* (i.e. quantum bits) preparation. The only conceivable solution to the safe recovery problem is to use structural redundancy (Udrescu-Milosav M. 2005)‡.

Dynamic epistemic logic can be also used to model and interpret *quantum behaviour* (Baltag A. and Smets S. 2005, 2006). Here, it was given a complete dynamic - logical characterisation of quantum systems The main thesis is that "all the non-classical properties of quantum systems are explainable in terms of the non-classical flow of quantum information". And hence, any real understanding of quantum behaviour should require non-classical *logical dynamics* of information, rather than having non-classical logical laws governing static information (such as in non-Boolean, non-distributive, partial or fuzzy logics etc.). As a fundamental action of this logical dynamics was considered the *quantum test* (corresponding to a successful yes-no measurement performed on a quantum system: a measurement of some yes-no property φ). Any such test can be considered as a form of information update and this form was compared with other dynamic informational functors (called also "operators") in logic, such as: the classical *test functor ´?´* in dynamic logic, the (*public) announcement functor* in dynamic epistemic logic and the *belief revision functor* (*Belief revision* is an action of revising a previously held belief "theory" T about the world after receiving some new information φ, that may contradict T. The *belief revision action* $*\varphi$ changes the state to a new information state, given by the *revised theory* $T*\varphi$. The classical AGM postulates express minimal rationality conditions that belief revision must satisfy: $T*\varphi$ must incorporate the new fact φ. The theory is left the same if $\varphi \in T$ implies $T*\varphi = T$. Otherwise, the theory is "minimally revised" in order to accommodate the new fact)§. Quantum tests share some common features with all these operators, but there are also important

* For a more information see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

† *Quantum computing* studies theoretical computation systems, called *quantum computers*, that make direct use of quantum-mechanical phenomena, such as superposition and entanglement, to perform operations on data. *Quantum superposition* states any two or more quantum states can be added together (i.e. "superposed") and the result will be another valid quantum state and vice versa, any quantum state can be represented as a sum of two or more other distinct states (similarly as a property of solutions of Schrödinger's partial differential equation, describes how the quantum state of a quantum system changes with time 1925 / 1926: Erwin Rudolf Josef Alexander Schrödinger 1887 – 1961). *Quantum entanglement* is a physical phenomenon that occurs when pairs or groups of particles are generated or interact in ways such that the quantum state of each particle cannot be described independently - instead, a quantum state must be described for the system as a whole (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*). In the case of quantum computational systems, the entanglement is a source of simulation complexity. When entanglement is detected in processed state, the circuit has to be described with a behavioural architecture and exponential resources must be used in this case (Udrescu-Milosav M. 2005). A very difficult problem seems to be quantum circuits testing (which is not possible directly). Some interesting results were given by Biamonte J. and Perkowski M. (2005): the "*gate insertion / removal*" model was used in place of the classical "*stuck-at*" model.

‡ The first universal quantum Turing machine was theoretically build by (Deutsch D. 1985) and the first quantum algorithms were given by (Shor P.W. 1994), able to solve polynomially: *discrete logarithms* and *integer factoring*.

§A formal definition of the notion of *public announcement* is given e.g. in (Pacuit E. 2013: *Definition 2.1,p.4*). The classical theory of belief revision (known also as *AGM theory*) was introduced in (Alchourrón C., Gärdenfors P. and Makinson D. 1985). A more information concerning belief revision functors is given e.g. in (Gabbay D.M., Rodrigues O.T. and Russo A. 2010) or also in (Perrussel L., Marchi J. and Zhang D. 2010). *Revision operators* are often judged based on whether they satisfy the well-known *AGM postulates*. These postulates are formulated for logically closed sets of formulae (*belief sets*), but they can be modified so as to apply to belief bases. The modified postulates (omitted for lack of space) are known as the *KM postulates* (Katsuno H. and Mendelzon A. 1991). The AGM/KM postulates have been differences, which make apparent the non-classical nature of quantum information flow (Baltag A. and Smets S. 2006). In accordance with the last work, in contrast to classical informational actions, quests typically change the *ontic state*[*] of the "observed" system. The considered here approach is logic-based and mainly concerned with *qualitative* (or *semantic*) information. And this is in contrast to the *syntactic*, i.e. *quantitative* approach (based on Shannon's *theory of information*: in quantum information, instead of the classical *Shannon entropy*, the notion of *von Neumann entropy* is used). However, the proposed in (Baltag A. and Smets S. 2006) approach intersects with the similarly qualitative such one, known as "*quantum logic*", in short: *QL* (Neumann J. von. 1932), (Birkhoff G. and Neumann J. von. 1936)[†]. In particular, it was observed that any testable property of a quantum system corresponds to a closed linear subspace of a *Hilbert space*, denoted below by $\mathcal{H}$ (David Hilbert, 1862 – 1943)[‡]. Moreover, the actual "*state*" of a system is given by an atomic such property, i.e. a one-dimensional subspace (a "*ray*"). And hence, states are represented by 'rays' in $\mathcal{H}$. The lattice of such closed linear subspaces (with inclusion as an ordering relation) does not form a Boolean algebra (i.e. a distributive lattice with complements). The *quantum disjunction* (called also "*join*") , defined as the closed subspace generated by the union, and the *quantum negation* (called bellow "*orthocomplement*")[§] are here two non-classical operations. The syntax of *QL*, i.e. the language of this logic, is obtained recursively as follows (here any proposition p denotes some basic testable property): $\varphi =_{df} F \,/\, p \,/\, \sim\varphi \,/\, \varphi \wedge \psi \,/\, \varphi \sqcup \psi$ ('$\wedge$' denotes the classical conjunction; note that the falsity of a proposition does not imply the truth of its orthocomplement and also the truth of '$\varphi \sqcup \psi$' does not imply that either 'φ' is true or 'ψ' is true). In accordance with the last formula $\varphi \sqcup \psi$, the current state of the system is defined as a superposition of states satisfying φ and states satisfying ψ. Moreover, '$\sqcup$' is not distributive over '$\wedge$' (i.e. $p \sqcup (q \wedge r)$ and $(p \sqcup q) \wedge (p \sqcup r)$ are not equivalent: generalised for any φ, ψ and χ). The *quantum implication* (known also as: "*Sasaki hook*", as an orthocomplement of the *Sasaki projection*: Sasaki U. 1964) is defined as follows: $\varphi \Rightarrow_S \psi =_{df} \sim\varphi \sqcup (\varphi \wedge \psi)$, introduced by Finch P.D. (1970) and Mittelstaedt P. (1970). As a consequence of the above nondistributivity, the classical deduction theorem (see: Theorem 1.31 of Subsection 1.7) fails for the quantum implication (Baltag A. and Smets S. 2006).

There are five different operations of implication in an orthomodular lattice[**] related to the classical implication in a distributive lattice (Kalmbach G. 1983). A comparative study of quantum implication algebras

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criticized for admitting revision operators that discard beliefs that have no real connection with the incoming information (Bienvenu M., Herzig A. and G. Qi 2006). An additional postulate for relevant revision was proposed in (Parikh R. 1999). *Belief revision* can be considered as a process of incorporating new pieces of information into a set of existing beliefs. It is usually assumed that the operation follows the following two principles: (i) the resulting belief set is consistent and (ii) the change on the original belief set is minimal (Perrussel L., Marchi J. and Zhang D. 2010). In the last work, agent's beliefs are represented in prime implicants and express agent's preference on beliefs as a pre-order over terms. It is shown that the introduced here belief revision operator satisfies the KM postulates for belief revision as well as Parikh's postulate for relevant revision (a more formal treatment is omitted here: left to the reader).

[*] *Ontic states* describe all properties of a physical system exhaustively. *Exhaustive* in this context means that an ontic state is precisely the way it is, without any reference to epistemic knowledge or ignorance (Atmanspacher H. 2001).

[†] George David Birkhoff (1884 – 1944), John von Neumann (1903 – 1957), Claude Elwood Shannon (1916 – 2001)

[‡] A *complete unitary* (or *inner product*) vector space (Bronstein I.N.. et al. 2001), a particular case of *Banach space* (Stefan Banach, 1892 – 1945). Sometimes, inner product spaces over the field of complex numbers are referred to as unitary. Also, inner products are referred as "*abstract scalar products*" (corresponding to the usual notion of a *scalar*, called also "*dot*" *product* in the case of an Euclidean space). The term "inner space" itself may also denotes, e.g. "the environment beneath the surface of the sea": *physical geography*. The above notion of a complete space is defined as follows: a given metric space (X, ρ) is said to be complete (or Cauchy space: Augustin-Louis Cauchy, 1759 – 1857) iff every *Cauchy sequence* of points $\{x_n\}_{n=1}^{\infty}$ in X converges in X, i.e. iff $\forall_{\varepsilon > 0} \exists_{n_0} \forall_{n,m > n_0} (\rho(x_n, x_m) < \varepsilon)$.

[§] An *orthocomplementation* on a complemented lattice is an (order-reversing) involution which maps each element to a complement. An *orthocomplemented lattice* (in short: *ortholattice*) is a bounded lattice with orthocomplementation. The ortholattices are most often used in quantum logic, where the closed subspaces of a *separable Hilbert space* (i.e. admitting a countable *orthonormal basis*: all unit vectors are orthogonal) represent quantum propositions. Ortholattices satisfy De Morgan's laws, like Boolean ones (e.g. T*he Free Encyclopaedia, The Wikimedia Foundation, Inc* or also: Barnum H. et al. 2014).

[**] *Lattice* is an algebraic system $\mathscr{L} =_{df} (L; \cup, \cap)$ having two binary (i.e. two-argument) operations which are at the same time *commutative*, associative and the following *law of absorption* is satisfied: $x \cup (x \cap y) = x = x \cap (x \cup y)$, for any $x, y \in L$. By this definition it follows that $x \cup y = x \Leftrightarrow x \cap y = y$ (left to the reader). Consider the partial ordering relation: $x \geq y \Leftrightarrow_{df} x \cup y = x \vee x \cap y = y$, for any $x, y \in L$. The above two binary operations can be introduced as supremum and *infimum* and $(L, \geq)$ is a lattice (and vice versa: the corresponding proofs will be given in Part II of this work). We shall say that $\mathscr{L}$ is *modular* (or *Dedekind*) *lattice* (Julius Wilhelm Richard Dedekind 1831 – 1916) iff $\forall_{x,y,z \in L} (x \geq z \Rightarrow x \cap (y \cup z) = (x \cap y) \cup z)$. It can be shown that any distributive lattice is modular (but not vice versa). However, the orthocomplemented lattices used in quantum logic would require the

(such as: *orthoimplication*, *orthomodular implication* , and *quasi-implication algebras*) was given in (Megill N.D. and Pavičić M. 2003): a more formal treatment is omitted here.

Another very important problem is the *quantum non-locality*, i.e. the appearance of non-trivial correlations between the results of measurements performed simultaneously on systems that are spacially remote (Baltag A. and Smets S. 2006). The corresponding problem, known as "*entanglement*" is usually modelled by representing a "compound" system S, composed of two subsystems S_1 and S_2 as a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the two Hilbert spaces. A global state s of S is *separated* iff $s =_{df} (s_1, s_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$. Hence, each of the subsystems S_i is in a well-defined ("*pure*") local state s_i. Otherwise s is called *entangled*.

In general, the *global state* s of S cannot be exactly separated into two *local states* s_1 and s_2. Hence, the notion of a "*mixed state*" is used (a higher-order representation, to capture all the available information concerning the local state of an entangled subsystem). But finding, a correct generally-agreed interpretation of mixed states is still a very much debated open problem (Baltag A. and Smets S. 2006). Here it is proposed an interpretation in terms of "*objectively imperfect information*" (rather than subjective "*ignorance interpretation*": one of the most popular previous) that an entangled subsystem has about its environment. Moreover, in accordance with the above problem of entanglement, it seems to be impossible to construct a general lattice-theoretic analogue of tensor product (Aerts D. 1981: Theorem 11 / p.395). A more formal, logical description of the above entanglement still remains an open problem.

It was shown that Hilbert spaces can be structured as non-classical relational models of propositional dynamic logic (Baltag A. and Smets S. 2005). Any such model, called *quantum transition system* consists of a *set of states* S (originally denoted by Σ and representing complete descriptions of possible states of a physical system) and a family of *basic transition relations*, which are binary relations between states in S (describing the changes of state induced by possible actions that may be performed on the system: similarly as in Definition 2.10). The *states* correspond to rays, i.e. one-dimensional subspaces of $\mathcal{H}$. The actions correspond to specific linear maps on $\mathcal{H}$. For quantum information change the actions vary from unitary evolutions (corresponding to the so-called "quantum gates" in quantum computation) to various types of measurements (Baltag A. and Smets S. 2006). The following two main types of basic actions were presented: *quantum tests* ′ ?φ ′ and *quantum gates* ′ U ′ (in short below: *qutests* and *qugates*)*. There were also introduced some spacial features related to the local properties of given subsystems of a quantum system. Hence, a given unitary action is associated with some type, depending on its location.

Let φ be tested and the obtained answer be "yes", then the state of the observed system collapses to a state satisfying this property φ: ′ ?φ ′ corresponds to a *projector*† onto the subspace of $\mathcal{H}$ generated by φ. Qugates U represent reversible evolutions of the observed system (corresponding to unitary transformations on $\mathcal{H}$). The program expressions π are interpreted as quantum programs. Hence, the language of the propositional dynamic logic is interpreted in a quantum transition system (keeping the classical interpretation of all other functors, in particular such as Boolean negation and conjunction, etc.). The resulting logic is called a *logic of quantum actions*, in short: *LQA* (Baltag A. and Smets S. 2006). Here, dynamic logic formulae denote possible properties of quantum states: any such property either holds at a given state or does not hold. And hence, this logic is *bivalent* satisfying all the classical laws of propositional logic (see Subsection 1.3). Since not all the expressible properties are "*testable*" (i.e. corresponding to an "experimental" property, e.g. the negation of a testable property might not be testable)‡, only negation-free formulae are assumed in this logic (Baltag A. and Smets S. 2006). Here, the following (dynamic) reinterpretation (of the non-classical connectives) of QL inside LQA was proposed. The *orthocomplement* ′ ∼ φ ′

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following special case: $y =_{df} \overline{z}$ (the complement of z is defined as follows: $z \cup \overline{z} = \vee$ and $z \cap \overline{z} = \wedge$, where ∨ and ∧ are the corresponding *maximal* and *minimal elements*). Since $x \cap (\overline{z} \cup z) = x \cap \vee = x$, the following implication should be satisfied: $x \geq z \Rightarrow (x \cap \overline{z}) \cup z = x$. Hence (in accordance with this modification of the above modularity law) an *orthomodular lattice* is defined as an orthocomplemented one, for which the last implication holds (for any $x, z \in L$).

* We shall denote qutests by ′ ?φ ′ instead of ′φ? ′ (in accordance with the previous considerations). More formally, the event ′ ?φ ′ means "*test φ and proceed only if true*".

† A self-adjoint idempotent linear operator (e.g. see Coecke B: http://www.cs.ox.ac.uk/people/bob.coecke/SLS.pdf, a quick draft version of a course or Bronstein I.N. et al. 2001).

‡ Provided there is no ambiguity, since ′ ∼ ′ denotes "orthocomplement", the classical symbol of negation is here denoted by ′ ¬ ′.

of a property φ is interpreted as the *impossibility of a successful test* and defined as: $\sim \varphi =_{df} [?\varphi]F$. In accordance with De Morgan´s laws, the *quantum disjunction* is next defined by: $\varphi \sqcup \psi =_{df} \sim(\sim \varphi \wedge \sim \psi)$. The *quantum implication* (*Sasaki hook*) is simply given by the weakest precondition of a qutest: $\varphi \Rightarrow_S \psi =_{df} [?\varphi]\psi$ (i.e. after performing qutest ´ $?\varphi$ ´ it is necessarily the case that ψ holds: see the classical '– C'). *Measurements* are here expressed as complex LQA programs (i.e. nondeterministic sums of qutests of mutually orthogonal properties). In accordance with De Morgan´s laws, the classical dual $\langle ?\varphi \rangle \psi =_{df} \neg [?\varphi] \neg \psi$ (expresses the possibility of actualising a property ψ by a successful qutest of φ). In particular, $\Diamond\varphi$ can be defined as the *possibility of testing for* φ and so: $\Diamond\varphi =_{df} \langle ?\varphi \rangle T$.

Consider the compound system S composed of two subsystems S_1 and S_2. The above presented quantum transition system will now also include, besides the *global actions* U affecting S, *i - local unitary actions* U_i only performed on S_i (i = 1,2). Let also $c =_{df} (c_1, c_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ be a (specially distinguished) *separated state*, designed by a constant symbol c. The notion of "*local state of an entangled subsystem* S_i" is internalised as follows (Baltag A. and Smets S. 2006).

Provided there is no ambiguity and for convenience, by GS we shall denote below *the set of all global states associated with* S. Let $\bar{i} \in \{1,2\} - \{i\}$ and $\rho_i \subseteq GS \times GS$ be an equivalence* defined as follows (i = 1,2): $\forall_{s,s' \in GS} (s\, \rho_i\, s' \Leftrightarrow_{df} \exists_{U_{\bar{i}}} (s' = U_{\bar{i}}(s)))$. Hence, the local state s_i of S_i in state $s \in GS$ can be defined as the *equivalence class* $[s]_{\rho_i} =_{df} \{ s' / s\, \rho_i\, s' \}$ of s, i.e. $s_i =_{df} [s]_{\rho_i} \subseteq GS$ (the subset of global states that are "possible" according to S_i, i.e. that are consistent with all the information available at location i: Baltag A. and Smets S. 2006). Hence, the local state s_i of (an entangled subsystem) S_i could be interpreted as an "*epistemic*" (or "*informational*") state. In fact, s_i encodes all the information that S_i "has" about (the global system) S. The following "*epistemic*" *functor* (originally called "operator", associated with the above indistinguishability relation ρ_i) was introduced (by ' t <u>*inv*</u> φ ' and ' s_i <u>*ass*</u> φ ' are denoted below: "the quantum state $t \in GS$ *is involved* by (property) φ" and "the equivalence class s_i *is associated with* φ", respectively)†:

$$K_i\varphi =_{df} \{s \in GS / \forall_{t \in GS} (t\, \rho_i\, s \Rightarrow t\ \underline{inv}\ \varphi) \} = \{s \in GS / s_i\ \underline{ass}\ \varphi \},$$ for every testable property φ of the quantum system S and every component i = 1,2.

In accordance with the above work, K_i has the formal properties of a "knowledge" functor , satisfying the axioms of the Lewis system S5 (see: Subsection 2.3: *Modal logic*). Hence, ´$K_i\varphi$´ could loosely be read as: "Subsystem S_i *knows* φ". The last text is only considered as potential local information (not as "subjective" knowledge by an actual "observer"). And so, the following better reading of ´$K_i\varphi$´ was also proposed: "The information that the global system S *satisfies* φ is potentially available at location i" (Baltag A. and Smets S. 2006). In particular, assuming $c = (c_1, c_2)$, the local state of S_1 , e.g. $\{s / s\, \rho_i\, c \} = \{(c_1,d) / d \in \mathcal{H}_2\}$ corresponds to a pure local state of S_1 ,i.e. c_1.Hence, a separated subsystem can be said to be in a well-defined *ontic state*.

Let φ be a testable property. We shall say that φ is *i - local* (denoted by φ_i) iff it entails i - separation and it can only hold when it is *known* to subsystem S_i. Hence: $\varphi_i \Leftrightarrow K_i\varphi$. Any *local measurement* on S_i is a qutest of the form ´$?\varphi_i$´. The local proposition variables (ranging over testable local properties) are denoted below by ´p_i´. The proposed language of this *quantum dynamic-epistemic logic* is obtained recursively as follows (Baltag A. and Smets S. 2006): in accordance with the previous considerations, only negation-free formulae (more precisely, wrt the quantum negation) are assumed below.

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* For convenience, instead of the original symbol '$\simeq_i$', the *equivalence relation* is here denoted by 'ρ_i'.

† $K_i\varphi =_{df} \{s : t \in \varphi$ for all $t \simeq_i s \} = \{s : s_i \subseteq \varphi \}$ (Baltag A. and Smets S. 2006).

$$\varphi =_{df} \sigma / \sigma_i / \neg \varphi / \varphi \wedge \psi / [\pi]\varphi / K_i\varphi$$

$$\pi =_{df} \alpha / \alpha_i / ?\varphi / \pi \cup \kappa / \pi \circ \kappa / \pi *$$

Here, σ is either a *propositional variable* p or the *constant state* c (a specially distinguished *separated state*) and α is either an *action variable* a or a *constant action* symbol (denoting some special qugate, from a given list of unitary evolutions). Similarly, φ and ψ (π and κ) denote some *properties* (some *program expressions*, interpreted as *quantum programs*), respectively.

Let $s \in GS$ be a separated state. It can be shown that S_i is separated in s iff $\exists_{s' \in GS} (s \rho_{\bar{i}} s' \wedge s' \rho_i c)$. The following characterisation of entanglement can be also obtained: s is entangled iff it satisfies $K_2K_1\neg$ c. In accordance with the last sentence two subsystems are entangled if they have some (specific and non-trivial) "*knowledge*" about each other, prior to any communication (Baltag A. and Smets S. 2006). Some more formal results related to this work are illustrated below.

Entanglement involve non-local ontic effects of local measurements and hence, two systems are entangled if every local measurement in the first system changes the other one. So $s \in GS$ is entangled iff $s \rho'_{\bar{i}} ?\varphi_i (s)$.

A more stronger property of entanglement (assumed in quantum theory) requires a *deterministic* non-local ontic impact, i.e. the existence of a deterministic correlation between the results of a local measurement on S_i and the subsequent ontic state of the environment $S_{\bar{i}}$ ($i \in \{1,2\}$). Let π_s be a deterministic program related to s. We have $\forall_{s \in GS} \exists_{\pi_s} (?p_i(s) \rho_{\bar{i}} \pi_s(p_i))$. The following dual representation was also given (below π_i denotes a deterministic program sending i - local states into $\bar{i}$ - local states): $\forall_{\pi_i} \exists_{s \in GS} (\pi_s = \pi_i)$. Below, by $\overline{\pi}_i$ it is denoted the state $s \in GS$ entangled in accordance with π_i. So we have: $?p_i(\overline{\pi}_i) \rho_{\bar{i}} \pi_i(p_i)$.

In the next considerations of the above cited work (more formally) are discussed such notions as: *entanglement*, *non-locality*, *Bell's states** and various known *quantum computing protocols* (e.g. *teleportation*, *super-dense coding*, *quantum secret sharing*, etc.). It was also presented a comparison of qutests with other forms of information update.

The above introductory notions (related to quantum dynamic-epistemic logic) are only an illustration of this excellent work. A more formal treatment is omitted here (this is left to the reader).

Intuitionistic and fuzzy intuitionistic logics

Intuitionistic logic encompasses the principles of logical reasoning used by Brouwer L.E.J. (Luitzen Egbertus Jan Brouwer: 1881 – 1966) in developing his *intuitionistic mathematics* (1907). An early basis of these principles was the Kant's philosophy of mathematics, in particular the laws of "*intuitive cognition*" and "*intuitive judgments*". So, the human *intuition* (a priori time and space) was considered as foundations of mathematics and hence the construction of any mathematical concept (Immanuel Kant: 1724 – 1804). A similar point of view was given by: Leopold Kronecker (1823 – 1891), Henri Poincaré (1854 – 1912), Félix Édouard Justin Émile Borel (1871 – 1956), Henri Lebesgue (1875 – 1941), Herman Klaus Hugo Weyl (1885 – 1955) and so on†. Because the above principles also underlie *Russian recursive analysis* (known also as *recursive constructive* mathematics: Andrei Andreevich Markov, Junior: 1903 – 1979) and the *constructive analysis* (Bishop E. 1967: Errett Bishop 1928 – 1983) and his followers, intuitionistic logic may be considered the logical basis of *constructive mathematics* (*Formal logic. Encyclopedical outline with applications to informatics and linguistics* 1987, Moschovakis J.R. 1999).

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* *Bell's states* (Bell J.S. 1964: John Stewart Bell: 1928 – 1990) are a concept in *quantum information science* and represent the most simple examples of entanglement (considered as four specific maximally entangled quantum states of two qubits)

† In fact, intuition is only necessary condition in any such construction.

An important stage in introduction of intuitionistic logic was the work given by Heyting A. (1930, Arend Heyting: 1898 – 1980, who was a student of Brouwer): an interpretation of propositional and (first-order) predicate formulae using proof terminology. Heyting's intuitionistic calculus *INT* is now considered as one of the best known constructive logics.

In fact, from the intuitionistic point of view, the ascertainment that φ is true requires the existence of a proof for φ. And so, we have:

(1) the proof π of ′$\varphi \wedge \psi$′ is considered as an ordered p9air (π_1,π_2) such that: π_1 is the proof of φ and π_2 is the proof of ψ ;

(2) the proof of ′$\varphi \vee \psi$′ is a construction, which selects one of the formulae φ, ψ and next gives the corresponding proof ;

(3) the proof π of ′$\varphi \Rightarrow \psi$′ is a construction such that, for each proof κ of φ, it is obtained a corresponding proof $\pi(\kappa)$ of ψ together with a further reference that this condition is in fact satisfied by π ;

(4) since $\models \sim\varphi \Leftrightarrow \varphi \Rightarrow F$ (is a thesis: the rule ′– N′ is not satisfied in this system), the proof of ′$\sim \varphi$′ corresponds to the assumption that the logical value ′$\varphi \Rightarrow F$′ = T. But then, we shall obtain a contradiction with any proof of φ (wrt: ′$T \Rightarrow F$′ = F) ;

(5) the proof π of ′$\exists_x A(x)$′ is a construction which selects an object $a \in \mathscr{U}$ (the universe under consideration) and gives a proof $\pi(a)$ of $A(\bar{a})$, where $\bar{a}$ is a name for a ;

(6) the proof π of ′$\forall_x A(x)$′ is a construction such that for each object $a \in \mathscr{U}$ it is assigned a proof $\pi(a)$ of $A(\bar{a})$ together with the ascertainment that the above conditions are satisfied by π.

We observe a logical consistency between cases (5,6) and the corresponding *primitive rules of joining the existential and universal quantifiers*, i.e. ′$+\exists$′ and ′$+\forall$′, respectively (See: Classical first-order and higher order predicate logics: Chapter II, Subsection 3.3).

An approach, similar to the above presented, was proposed by Kolmogorov A.N. (1925, Andrej Nikolaevich Kolmogorov: 1903 – 1987): in accordance with Kolmogorov's known thesis ('*INT* = logic of problems'), the above formulae are here interpreted as ′*problems*′ (e.g. the solution of the *problem* ′$\varphi \wedge \psi$′ is considered as an ordered pair of solutions corresponding to the *problems* associated with φ and ψ, respectively).

The axiomatic system (of the *intuitionistic propositional logic*), introduced by Heyting A. (1930), is shown below*.

(A1) $p \Rightarrow p \wedge p$
(A2) $p \wedge q \Rightarrow q \wedge p$
(A3) $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$
(A4) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(A5) $q \Rightarrow (p \Rightarrow q)$
(A6) $p \wedge (p \Rightarrow q) \Rightarrow q$
(A7) $p \Rightarrow p \vee q$
(A8) $p \vee q \Rightarrow q \vee p$
(A9) $(p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$
(A10) $\sim p \Rightarrow (p \Rightarrow q)$
(A11) $(p \Rightarrow q) \wedge (p \Rightarrow \sim q) \Rightarrow \sim p$

* Brouwer's ideas on the formalisation of intuitionistic logic were studied also by Kolmogorov A.N. (1925) and Glivenko V. (1929). Heyting's system was universally accepted.

We observe that any of the above axioms is a thesis of the classical propositional calculus, but not vice versa. As an example, the following formulae are not satisfied in the above system: $p \vee \sim p$ (the Aristotelian law of the excluded middle), $\sim\sim p \Rightarrow p$, $\sim(p \wedge q) \Rightarrow \sim p \vee \sim q$ (are the *if-implications* of T 1.3 and T 1.8, respectively: Subsection 1.3), $(\sim p \Rightarrow p) \Rightarrow p$, $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$, etc. In particular, the following equivalence was shown (Glivenko V. 1929: Valery Ivanovich Glivenko 1897 – 1940, Gödel K. 1930: Kurt Gödel 1906 – 1978): $\vDash \varphi$ (is a thesis in *classical logic*) iff $\vDash \sim\sim \varphi$ (is a thesis in *intuitionistic logic*). An embedding of classical logic into intuitionistic one was also given (Gödel K. 1930).

In general, the expressive power of intuitionistic logic involves various applications, in particular in the field of discrete mathematics and computer science. Some brief considerations on the *fuzzy intuitionistic logic* (known also as: *intuitionistic fuzzy logic*)* are given below.

Fuzzy intuitionistic logic was introduced by Takeuti G. and Titani S. (1984) as the logic corresponding to intuitionistic fuzzy set theory: as in classical propositional logic, this logic has its corresponding set theory (see also: Atanassov K.T. 1986, 1988† and Turunen E. 1989, 1991‡). Fuzzy intuitionistic logic also coincides with the first-order Gödel's logic (Gödel K. 1933) based on the truth-values set [0,1]. Here, one of the basic t-norm logics is used (Hájek P. 1998), see Subsection 2.2.

The validity of *modus ponens* (i.e. the *rule of omitting an implication* '– C': Heyting's axiom A6) is a critical aspect for any form of knowledge inference, as it guarantees the correctness of reasoning. In classical propositional logic this rule is indeed a valid one. However, fuzzy intuitionistic logic introduces more expressivity, in a gradual logic framework: it allows not only to define degrees of truth, but also to simultaneously observe degrees of falsehood (Atanassov K.T. 1986). As observed in some particular cases, the validity of the modus ponens is not guaranteed for most fuzzy intuitionistic implications, e.g. see (Detyniecki M. et al. 2014, Rushdi A.M. et al. 2015)§. In particular, there was proposed in (Detyniecki M. et al. 2014) to interpret the last observation as due to the fact that the classical fuzzy intuitionistic tautology (Atanassov K.T. and Gargov G. 1998) is too optimistic. And so, as an alternative there were introduced several more strict definitions. It was also presented a more intuitive approach to tautology wrt *ignorance* processing (through the expressivity allowed by the 2 degrees, of truth and falsehood, intuitionistic fuzzy logic makes it possible to model distinct levels of knowledge and in particular *ignorance*: as *neither positive*, i.e. formula with degree of truth greater than degree of falsehood *nor negative*, i.e. formula with degree of truth lower than degree of falsehood). The considered tautology definitions make it possible to express the notion of being "*certainly true*", whereas the classical definition considers ignorance as a tautology. In this way we have a more intuitive approach wrt ignorance processing (Detyniecki M. et al. 2014). Some results, given in the last work, are presented below. In particular, it is also shown below that the *generalised Łukasiewicz's fuzzy t-norm* (Tabakow I.G. 2010, 2014), redefined as *generalised Łukasiewicz's intuitionistic fuzzy t-norm* is a t-representable intuitionistic fuzzy t-norm**.

The notion of intuitionistic fuzzy set is introduced as follows (Atanassov K.T. 1986): here, μ_A and ν_A are two maps from X *(the universe)* to [0,1].

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* The notions "*fuzzy intuitionistic logic*" and "*intuitionistic fuzzy sets*" are here used in their chronological order (wrt intuitionistic logic and fuzzy sets).

† An earlier was the work, given by Atanassov K.T. (born 1954): *Intuitionistic fuzzy sets*, VII ITKR's Session, Sofia, Deposed in Central Sci-Techn. Library of Bulg Acad of Sci. 1697/84 (1983), in Bulgarian. *Fuzzy sets* are special cases of *intuitionistic fuzzy sets* and the last have better modeling power than the fuzzy sets.

‡ The starting point in fuzzy intuitionistic logic is to fuzzify truth. There are accepted formulae as have different truth values (instead of the classical true-false-dualism). In fuzzy intuitionistic logic a half true expression is not always half false. Since we are not interested in the false sentences of a theory we let the falsehood be crisp. There is only one falsehood in fuzzy intuitionistic logic. Here, similarly as in classical first-order predicate logic, it is introduced a set $\mathcal{F}$ of *well formed formulae*, a *partially ordered set* L of truth values (assuming some binary operations defined in L, corresponding to the classical one) and an interpretation T , represented as the map $T : \mathcal{F} \rightarrow L$ (satisfying some required properties).

§ As in fuzzy logic, some formulae may be or not be satisfied, depending on the used t-norm, e.g. Ł-BL, G-BL and π-BL considered in Subsection 2.2.

** In particular, the generalised Łukasiewicz's fuzzy t-norm, redefined as *generalised Łukasiewicz's intuitionistic fuzzy t-norm*, is t-representable and hence this t-norm seems to be an interesting application in *intuitionistic fuzzy rough sets* (Tabakow I.G. 2014).

Definition 2.24 (*intuitionistic fuzzy set*)[*]

Let a set X be fixed. An *intuitionistic fuzzy set* A in X is defined as: $A =_{df} \{(x, \mu_A(x), \nu_A(x)) / x \in X\}$, which assigns to each element x a *membership degree* $\mu_A(x)$ and a *non-membership degree* $\nu_A(x)$, where $\mu_A(x), \nu_A(x) \geq 0$, with the condition: $\underset{x \in X}{\forall} (\mu_A(x) + \nu_A(x) \leq 1)$.

This definition is an extension of the classical one (Zadeh L. A. 1965). Provided there is no ambiguity and for simplicity, let $\mu_a =_{df} \mu_A(x)$ and $\nu_a =_{df} \nu_A(x)$, $x \in X$. By $\chi_a =_{df} 1 - \mu_a - \nu_a$ it is denoted the *hesitancy degree* or an *intuitionistic index* of x to A, which represents the indeterminacy degree of x to A (Szmidt E. and Kacprzyk J. 2000). Each pair of (μ_a, ν_a) in A is called an *intuitionistic fuzzy number*, in short: IFN. Here, (1,0) and (0,1) are the *largest* and *smallest* IFN's, respectively. Obviously, each IFN has a physical interpretation (e.g. (0.7,0.2): $\mu_a = 0.7$, $\nu_a = 0.2$ and $\chi_a = 0.1$, which can be interpreted as: 'the vote for resolution is seven in favour, two against, and one abstention'). For convenience, IFN is denoted by $\mathbf{a} =_{df} (\mu_a, \nu_a)$, where $\mu_a, \nu_a \geq 0$, $\mu_a + \nu_a \leq 1$. And also, by $s_a =_{df} \mu_a - \nu_a$ and $h_a =_{df} \mu_a + \nu_a$ are denoted the *score* and the *accuracy degree* of the IFN **a**, respectively (Xu Z.S. and Yager R.R. 2006, 2011).

An intuitionistic fuzzy set can be represented algebraically on the *complete lattice*[†] $\mathscr{L} =_{df} (L, \leq_L)$ defined as follows (Detyniecki M. et al. 2014).

Definition 2.25

The *intuitionistic fuzzy lattice* $\mathscr{L} =_{df} (L, \leq_L)$, where $L =_{df} \{(x,y) \in [0,1]^2 / x + y \leq 1\} \subsetneq [0,1]^2$ is the *set of elements* of $\mathscr{L}$ and $\leq_L \subseteq L \times L$ is a *binary relation* (originally called: "*comparison operator*") defined as: $\underset{x,y,z,t \in [0,1]}{\forall} ((x,y) \leq_L (z,t) \Leftrightarrow_{df} (x \leq z) \wedge (y \geq t))$.

We observe that L corresponds to the set of points of the rectangular equilateral triangle defined by the inequalities: $x \geq 0$, $y \geq 0$ and $x + y \leq 1$. Moreover, it can be observed that the above relation $'\leq_L'$ is a *partial order*, i.e. a *reflexive*, *weak antisymmetric* and *transitive*. In fact, for any $x, y, z, t, u, v \in [0,1]$ we have (the proof is left to the reader)[‡]:

$(x,y) \leq_L (x,y)$,

$(x,y) \leq_L (z,t) \wedge (z,t) \leq_L (x,y) \Rightarrow (x = z) \wedge (y = t)$ and

$(x,y) \leq_L (z,t) \wedge (z,t) \leq_L (u,v) \Rightarrow (x,y) \leq_L (u,v)$. □ {Df.2.25}

Corollary 2.8

$\underset{x,y \in [0,1]}{\forall} ((0,1) \leq_L (x,y) \leq_L (1,0))$. □

The above two ordered pairs (0,1) and (1,0) correspond to the *minimal* and *maximal* elements of this lattice. The last two pairs are denoted below by 0_L and 1_L, respectively. Another extreme point of special interest is the pair (0,0) denoted below by U_L (by definition, $(1,1) \notin L$).

The notions of *intuitionistic fuzzy t-norm* and *t-representable intuitionistic fuzzy t-norm* are given in the next two definitions.

---

[*] The characterisation of fuzzy sets is that the range of truth value of the membership relation is the closed interval [0,1] of real numbers. And hence, a more general approach (by using L. S. Hay's extention of Łukasiewicz's logic) was presented in (Takeuti G. Titani S. 1984).

[†] $\mathscr{L}$ is *complete* iff $\underset{\text{empty} \neq B \subseteq L}{\forall} \; \underset{a,b \in L}{\exists!} (a = \sup(B), b = \inf(B))$: Definition 2.25 is a version of the original one, given in (Detyniecki M. et al. 2014). Moreover, lattice theory is omitted in this part of study: will be presented in Part II of this book.

[‡] Provided there is no ambiguity and for simplicity, it is assumed below that $'\leq_L'$ binds more strongly than the symbol of conjunction $'\wedge'$. We observe that: $x + y \leq 1$ iff $x \leq 1 - y = y'$ (the *Łukasiewicz's fuzzy negation*).

Definition 2.26

Let $\otimes_{if}$ be a binary operation defined in L with the following properties (for any (x,y), (z,t), (u,v), (p,q) $\in$ L $\subsetneq$ $[0,1]^2$):

$(x,y) \otimes_{if} (z,t) = (z,t) \otimes_{if} (x,y)$	*commutative*
$(x,y) \otimes_{if} (u,v) \ {}_L\!\geq (z,t) \otimes_{if} (p,q)$ for $(x,y) \ {}_L\!\geq (z,t)$ and $(u,v) \ {}_L\!\geq (p,q)$	*monotonic*
$(x,y) \otimes_{if} ((z,t) \otimes_{if} (u,v)) = ((x,y) \otimes_{if} (z,t)) \otimes_{if} (u,v)$	*associative*
$(x,y) \otimes_{if} 1_L = (x,y)$	has 1_L as *unit element*

We shall say that $\otimes_{if}$ is an *intuitionistic fuzzy t-norm*[*].

The notion of an *intuitionistic fuzzy t-conorm*, denoted here by $\oplus_{if}$, can be introduced in a similar way (having 0_L as unit element: left to the reader). Among the intuitionistic fuzzy t-norms, the category of t-representable t-norms is of special interest.

Definition 2.27

For any (x,y), (z,t) $\in$ L $\subsetneq$ $[0,1]^2$:

$\otimes_{if}$ is *t-representable* wrt $\otimes$ and $\oplus$ iff $(x,y) \otimes_{if} (z,t) = (x \otimes z , y \oplus t)$ and

$\oplus_{if}$ is *t-representable* wrt $\oplus$ and $\otimes$ iff $(x,y) \oplus_{if} (z,t) = (x \oplus z , y \otimes t)$.

Obviously, the last definition assumes the existence of such fuzzy norms $\otimes$ and $\oplus$. It can be observed that any such t-representable intuitionistic fuzzy t-norm (e.g. $\otimes_{if}$) is considered as a binary operation related to the *direct product* of the two *Abelian*[†] *monoids* $\mathscr{A} =_{df} ([0,1]; 1 ; \otimes)$ and $\mathscr{B} =_{df} ([0,1]; 0 ; \oplus)$, i.e. $\mathscr{A} \times \mathscr{B}$, where $\otimes_{if}$ is the obtained binary operation restricted to L and defined in the corresponding algebraic subsystem of this direct product (*preserving the monotonicity property*)[‡].

In applications the often used as a t-norm is the classical Łukasiewicz's such one because the notion of fuzzy t-equivalence relation is dual to that of a pseudo-metric. The generalised Łukasiewicz's fuzzy t-norm was used to represent and improve fuzzy rough approximations (Tabakow I.G. 2014). It is shown below that the *generalised Łukasiewicz's intuitionistic fuzzy t-norm* (denoted as: $\hat{\otimes}_{if}$) satisfies the last two definitions. Moreover, $\hat{\otimes}_{if}$ is well-defined, i.e. closed in L.

Consider the *direct product* $\mathscr{A} \times \mathscr{B}$ of the following two similar *Abelian*[§] *monoids* $\mathscr{A} =_{df} ([0,1]; 1 ; \hat{\otimes})$ and $\mathscr{B} =_{df} ([0,1]; 0 ; \hat{\oplus})$, where ′ $\hat{\otimes}$ ′ and ′ $\hat{\oplus}$ ′denote the generalised Łukasiewicz's fuzzy t-norm and t-conorm,

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[*] Equivalently, the algebraic system $(L ; 1_L ; \otimes_{if})$ is *Abelian monoid* with the monotonicity property for $\otimes_{if}$. Similarly, for $\oplus_{if}$ wrt 0_L.

[†] Any algebraic system $\mathscr{A} =_{df} (A ; \circ)$ is said to be *groupoid*, where ′$\circ$′ is a *binary operation*, i.e. $\circ : A^2 \rightarrow A$. The system $\mathscr{A}$ is *Abelian* if ′$\circ$′ is commutative (Niels Henrik Abel: 1802 – 1829).

[‡] A more formal treatment concerning the notion of direct product of two algebraic systems will be given in the second part of this book. In particular, $\mathscr{A}$ and $\mathscr{B}$ are *similar*, i.e. are of the same *type*: (0,2).

[§] Any algebraic system $\mathscr{A} =_{df} (A ; \circ)$ is said to be *groupoid*, where ′$\circ$′ is a *binary operation*, i.e. $\circ : A^2 \rightarrow A$. The system $\mathscr{A}$ is *Abelian* if ′$\circ$′ is commutative (Niels Henrik Abel: 1802 – 1829).

respectively*. By definition, $\mathscr{A} \times \mathscr{B} =_{df} ([0,1] \times [0,1]; (1,0); \overline{\otimes})$, where $\overline{\otimes}$ is the new obtained binary operation, defined in $[0,1]^2$ and $\overline{\otimes}((x_1,y_1),(x_2,y_2)) =_{df} (\hat{\otimes}(x_1,x_2), \hat{\oplus}(y_1,y_2))$, for any $x_1, x_2, y_1, y_2 \in [0,1]$. Provided there is no ambiguity and for simplicity, below instead of x_1, x_2, y_1, y_2, etc. we shall denote variables by x, y, z, t, etc. And also, instead of $\hat{\otimes}(x_1,x_2)$ and $\hat{\oplus}(y_1,y_2)$, we shall use e.g. $x \hat{\otimes} y$ and $z \hat{\oplus} t$.

Let now $\mathscr{C} =_{df} (L; (1,0); \hat{\otimes}_{if}) \sqsubseteq \mathscr{A} \times \mathscr{B}$ be the corresponding algebraic subsystem defined for $L \subsetneq [0,1]^2$, where $\hat{\otimes}_{if}$ is the binary operation $\overline{\otimes}$ restricted to L, i.e. $dom(\hat{\otimes}_{if}) =_{df} dom(\overline{\otimes}) \cap L$. Now it is necessary to show that $\hat{\otimes}_{if}$ is t-representable wrt $\hat{\otimes}$ and $\hat{\oplus}$. Hence, in accordance with Definition 2.27, it is necessary to show that $(x \hat{\otimes} z, y \hat{\oplus} t) \in L$, for any $(x,y), (z,t) \in L$, i.e. that $\hat{\otimes}_{if}$ is closed in L.

Let $x \hat{\otimes} z =_{df} \max\{0, x^\alpha + z^\alpha - 1\}^{1/\alpha}$ and $y \hat{\oplus} t =_{df} \min\{1, y^\alpha + t^\alpha\}^{1/\alpha}$. The following proposition is satisfied.

Proposition 2.18 ($\hat{\otimes}_{if}$ is *t-representable*)

$\forall_{x,y,z,t \in [0,1]} ((x + z \le 1) \wedge (y + t \le 1) \Rightarrow \max\{0, x^\alpha + z^\alpha - 1\}^{1/\alpha} + \min\{1, y^\alpha + t^\alpha\}^{1/\alpha} \le 1)$, for any $\alpha \ge 1$.

Proof:

Let $x + z \le 1$ and $y + t \le 1$ $(x, y, z, t \in [0,1])$. Assume that $\alpha \ge 1$. Since $x + z \le 1$, $x \ge x^\alpha$ and $z \ge z^\alpha$ then $1 \ge x + z \ge x^\alpha + z^\alpha$. Hence: $x^\alpha + z^\alpha - 1 \le 0$. In a similar way we can obtain: $y^\alpha + t^\alpha \le 1$. Therefore, $\max\{0, x^\alpha + z^\alpha - 1\}^{1/\alpha} + \min\{1, y^\alpha + t^\alpha\}^{1/\alpha} = (y^\alpha + t^\alpha)^{1/\alpha} \le 1$. □

Proposition 2.19 (the *intuitionistic fuzzy t-norm* $\hat{\otimes}_{if}$)

Let $\hat{\otimes}$ be the generalised Łukasiewicz's fuzzy t-norm. Then $\hat{\otimes}_{if}$ defined in L is an intuitionistic fuzzy t-norm.

Proof:

Let $(x,y), (z,t), (u,v), (p,q) \in L \subsetneq [0,1]^2$. We have:

1. $\hat{\otimes}_{if}$ is *commutative*.

$$\begin{aligned}(x,y) \hat{\otimes}_{if} (z,t) &=_{df} (x \hat{\otimes} z, y \hat{\oplus} t)\\ &= (z \hat{\otimes} x, t \hat{\oplus} y)\\ &=_{df} (z,t) \hat{\otimes}_{if} (x,y).\end{aligned}$$

2. $\hat{\otimes}_{if}$ is *monotonic*, i.e. $(x,y) \hat{\otimes}_{if} (u,v) \ge_L (z,t) \hat{\otimes}_{if} (p,q)$ for $(x,y) \ge_L (z,t)$ and $(u,v) \ge_L (p,q)$.

Let $(z,t) \le_L (x,y)$ and $(p,q) \le_L (u,v)$. So, we have: $z \le x$, $t \ge y$, $p \le u$ and $q \ge v$. Since $(z,t) \hat{\otimes}_{if} (p,q) =_{df} (z \hat{\otimes} p, t \hat{\oplus} q)$ and $(x,y) \hat{\otimes}_{if} (u,v) =_{df} (x \hat{\otimes} u, y \hat{\oplus} v)$ we can obtain: $z \hat{\otimes} p \le x \hat{\otimes} u$ and $t \hat{\oplus} q \ge y \hat{\oplus} v$. Hence: $(z \hat{\otimes} p, t \hat{\oplus} q) \le_L (x \hat{\otimes} u, y \hat{\oplus} v)$. Then $(z,t) \hat{\otimes}_{if} (p,q) \le_L (x,y) \hat{\otimes}_{if} (u,v)$.

3. $\hat{\otimes}_{if}$ is *associative*.

* See: Subsection 2.2: A generalisation of the Łukasiewicz's BL system.

$$(x,y)\ \hat{\otimes}_{if}\ ((z,t)\ \hat{\otimes}_{if}\ (u,v)) =_{df} (x,y)\ \hat{\otimes}_{if}\ (z\,\hat{\otimes}\,u,\ t\,\hat{\oplus}\,v)$$
$$= (x\,\hat{\otimes}\,(z\,\hat{\otimes}\,u),\ y\,\hat{\oplus}\,(t\,\hat{\oplus}\,v))$$
$$= ((x\,\hat{\otimes}\,z)\,\hat{\otimes}\,u,\ (y\,\hat{\oplus}\,t)\,\hat{\oplus}\,v)$$
$$= (x\,\hat{\otimes}\,z,\ y\,\hat{\oplus}\,t)\ \hat{\otimes}_{if}\ (u,v)$$
$$=_{df} ((x,y)\ \hat{\otimes}_{if}\ (z,t))\ \hat{\otimes}_{if}\ (u,v).$$

4. $\hat{\otimes}_{if}$ has 1_L as *unit element*.

$$(x,y)\ \hat{\otimes}_{if}\ 1_L =_{df} (x,y)\ \hat{\otimes}_{if}\ (1,0)$$
$$= (x\,\hat{\otimes}\,1,\ y\,\hat{\oplus}\,0)$$
$$= (x,y).\ \square\ \{\text{Df.2.25, Df.2.26,Df.2.27,Prop.2.18}\}$$

Let now $\sigma_{fi} : L \times L \rightarrow L$ be a map such that:

1. σ_{fi} satisfies the following four *boundary conditions*: $\sigma_{fi}(0_L,0_L) = 1_L$, $\sigma_{fi}(0_L,1_L) = 1_L$, $\sigma_{fi}(1_L,0_L) = 0_L$ and $\sigma_{fi}(1_L,1_L) = 1_L$*,
2. σ_{fi} is decreasing in its first component: $\mathbf{b}\ {}_L\!\geq\ \mathbf{a} \Rightarrow \sigma_{fi}(\mathbf{b},\mathbf{c}) \leq_L \sigma_{fi}(\mathbf{a},\mathbf{c})$ and
3. σ_{fi} is increasing in its second component: $\mathbf{d}\ {}_L\!\geq\ \mathbf{c} \Rightarrow \sigma_{fi}(\mathbf{a},\mathbf{d})\ {}_L\!\geq\ \sigma_{fi}(\mathbf{a},\mathbf{c})$,

where $\mathbf{a} =_{df} (x,y)$, $\mathbf{b} =_{df} (z,t)$, $\mathbf{c} =_{df} (u,v)$ and $\mathbf{d} =_{df} (p,q)$, for any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in L$.

Definition 2.28 (the *fuzzy intuitionistic R-implication ´$\Rightarrow_{fi}$´*)†

Let σ_{fi} be as above and $\mathbf{a} \Rightarrow_{fi} \mathbf{b} =_{df} \sigma_{fi}(\mathbf{a},\mathbf{b})$. The fuzzy intuitionistic R-implication is introduced as follows (for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in L$): $\mathbf{a} \Rightarrow_{fi} \mathbf{b} =_{df} \sup\{\mathbf{c} \in L\ /\ \mathbf{a}\ \otimes_{if}\ \mathbf{c} \leq_L \mathbf{b}\}$.

In accordance with the last definition, there exists a possibility of obtaining some different fuzzy intuitionistic R-implications depending on the required intuitionistic fuzzy t-norms (Cornelis, C. and Deschrijver G. 2001). An illustration of the above considerations are the next two examples. Here, the intuitionistic fuzzy t-norm $\hat{\otimes}_{if}$ is used.

Example 2.20 ($\sigma_{fi}(1_L,0_L) = 0_L$)

$$\sigma_{fi}(1_L,0_L) =_{df} \sup\{\mathbf{c} \in L\ /\ (1,0)\ \hat{\otimes}_{if}\ \mathbf{c} \leq_L (0,1)\}$$
$$= \sup\{\mathbf{c} \in L\ /\ (1,0)\ \hat{\otimes}_{if}\ (u,v) \leq_L (0,1)\}$$
$$= \sup\{\mathbf{c} \in L\ /\ (1\,\hat{\otimes}\,u,\ 0\,\hat{\oplus}\,v) \leq_L (0,1)\}$$
$$= \sup\{\mathbf{c} \in L\ /\ (\max\{0,\ 1^{\alpha} + u^{\alpha} - 1\}^{1/\alpha},\ \min\{1,\ 0^{\alpha} + v^{\alpha}\}^{1/\alpha}) \leq_L (0,1)\}$$
$$= \sup\{\mathbf{c} \in L\ /\ (u,v) \leq_L (0,1)\}$$
$$= \sup\{\mathbf{c} \in L\ /\ \mathbf{c} \leq_L 0_L\}$$
$$= 0_L.\ \square\ \{\text{Coroll. 2.8}\}$$

Example 2.21 (*σ_{fi} is decreasing in its first component*)

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* According to Corollary 2.8: $\sigma_{fi}(\mathbf{a},\mathbf{b}) = 1_L$ iff $\mathbf{a} \leq_L \mathbf{b}$ ($\mathbf{a}, \mathbf{b} \in \{0_L, 1_L\}$), similarly as in classical logic, where $p \Rightarrow q = 1$ iff $p \leq q$, $\{p, q \in \{0,1\}\}$: see Subsection 1.4.

† See (Detyniecki M. et al. 2014) and also Subsection 2.2 of this book.

Let K $=_{df}$ {0, 1/4, 1/2, 3/4, 1}. Consider the set Q $=_{df}$ $\{(x,y) \in K^2 / x + y \leq 1\} \subsetneq L$, $|Q| = 15$. Suppose that **a** $=_{df}$ (0,1/2), **b** $=_{df}$ (1/2,1/4) and **c** $=_{df}$ (1/4,3/4). We have: **a** = (0,1/2) $\leq_L$ (1/2,1/4) = **b**, i.e. **b** $_L\geq$ **a**. Assume the intuitionistic fuzzy t-norm $\hat{\otimes}_{if}$, e.g. for $\alpha =_{df} 2$. We have:

$$
\begin{aligned}
\sigma_{fi}(\mathbf{b},\mathbf{c}) &=_{df} \sup\{\mathbf{d} \in Q / \mathbf{b} \,\hat{\otimes}_{if}\, \mathbf{d} \leq_L \mathbf{c}\} \\
&= \sup\{(p,q) \in Q / (1/2,1/4) \,\hat{\otimes}_{if}\, (p,q) \leq_L (1/4,3/4)\} \\
&= \sup\{(p,q) \in Q / (1/2 \,\hat{\otimes}\, p, 1/4 \,\hat{\oplus}\, q) \leq_L (1/4,3/4)\} \\
&= \sup\{(p,q) \in Q / (\max\{0, (1/2)^2 + p^2 - 1\}^{1/2}, \min\{1, (1/4)^2 + q^2\}^{1/2}) \leq_L (1/4,3/4)\}. \\
&= \sup\{(p,q) \in Q / (\max\{0, 1/4 + p^2 - 1\} \leq 1/16 \text{ and } \min\{1, 1/16 + q^2\} \geq 9/16\} \\
&= \sup\{(0,1), (0,3/4)\} \\
&= (0,3/4).
\end{aligned}
$$

In a similar way, we can obtain (left to the reader):

$$
\begin{aligned}
\sigma_{fi}(\mathbf{a},\mathbf{c}) &= \sup\{(0,1), (0,3/4), (1/4,3/4)\} \\
&= (1/4,3/4).
\end{aligned}
$$

And hence: $\sigma_{fi}(\mathbf{b},\mathbf{c}) = (0,3/4) \leq_L (1/4,3/4) = \sigma_{fi}(\mathbf{a},\mathbf{c})^*$. □ {Df.2.25,Prop.2.18,Df.2.28}

In formal logic, the term '*tautology*'† translates the idea of universal truth, i.e. a formula that is true in every possible interpretation (see: Definition 1.5 of Subsection 1.4). However, in accordance with Definition 2.28, any 'logical value' of the fuzzy intuitionistic R-implication is represented by ordered pair in L. And hence, it is necessary to know how such pair is related to the above notion of tautology.

In the case of fuzzy intuitionistic logic, the following definition was proposed: confusing a formula with its truth value (Atanassov K.T. and Gargov G. 1998).

Definition 2.29 (*fuzzy intuitionistic tautology*)

An ordered pair **a** $=_{df}$ (x,y) $\in$ L is a fuzzy intuitionistic tautology iff $x \geq y$.

According to the last definition, it is assumed that the degree of truth should be greater than the degree of falsehood for an intuitionistic pair to be considered as tautology. And hence, the validity of modus ponens (i.e. the rule '– C', in terms of truth values) was characterised as follows (Cornelis, C. and Deschrijver G. 2001).

Definition 2.30 (the validity of *modus ponens*)

Let **a** $\Rightarrow_{fi}$ **b** be a fuzzy intuitionistic R-implication (**a**, **b** $\in$ L). Assume that this implication is tautology and the antecedent **a** is tautology. Then the consequent **b** is tautology.

By the last definition it follows that the validity of the above fuzzy intuitionistic implication directly depends upon the definition of the tautology‡.

In accordance with Definition 2.29, fuzzy intuitionistic tautology can be any pair (x,y) $\in$ L such that $x \geq y$. This way U_L = (0,0) becomes a tautology, which is counterintuitive. In fact, the both truth and falsehood degrees equal 0, is closer to the notion '*unknown*', modelling total ignorance, rather than to the one of

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* As in the case of *real numbers*: $X \subseteq Y \Rightarrow \sup X \leq \sup Y$ $(X,Y \subseteq \mathbb{R})$.

† Called also, equivalently: *thesis*, *satisfied formula, true formula* or *valid formula*.

‡ In accordance with Definition 2.28, the 'logical value' of **a** $\Rightarrow_{fi}$ **b** corresponds to some ordered pair, e.g. **c** and hence the consequent **b** is tautology if **a** and **c** are also tautologies (**a**, **b**, **c** $\in$ L).

'true' (Detyniecki M. et al. 2014). Some remarks were also given by (Cornelis, C. and Deschrijver G. 2001, El-Hakeim K. and Zeyada F. 2000, etc.).

As an improvement of the above Definition 2.29 (i.e. to exclude some cases from being considered as true), the following two definitions were introduced (Detyniecki M. et al. 2014).

Definition 2.31 (*certain fuzzy intuitionistic tautology*)

An ordered pair $\mathbf{a} =_{df} (x,y) \in L$ is a certain fuzzy intuitionistic tautology iff $x \geq y$ and $1 - x - y \leq 0.5$[*].

Definition 2.32 (*truth fuzzy intuitionistic tautology*)

An ordered pair $\mathbf{a} =_{df} (x,y) \in L$ is a truth fuzzy intuitionistic tautology iff $x \geq 0.5$.

By Definition 2.31 it follows that $x \geq y$ and $0.5 \leq x + y \leq 1$.

Corollary 2.9

Let $(x,y) \in L$. If $x \geq 0.5$ then $x \geq y$.

Proof:

Let $(x,y) \in L$. Assume that $x \geq 0.5$. Then $x + y \geq 0.5 + y$. Since $1 \geq x + y$ then $1 \geq 0.5 + y$ and so $y \leq 0.5$. Hence $x \geq y$. □

And so, in accordance with the above considerations, it follows that Definition 2.29 gives a necessary but not sufficient condition for tautology.

The validity experimental study was also presented (Detyniecki M. et al. 2014), i.e. a study concerning the validity of modus ponens wrt the considered 18 fuzzy intuitionistic implications for 10 000 pairs $\mathbf{a}, \mathbf{b} \in L$ (a random generation, using the uniform distribution).

In general, the contemporary investigations in fuzzy intuitionistic logic and intuitionistic fuzzy sets have both theoretical and practical aspects. (i.e. involving some research and development). There were studied, e.g. fuzzy intuitionistic lattices and fuzzy intuitionistic Boolean algebras (Tripathy B.K. et al. 2013) or also an algebraic structure for fuzzy intuitionistic logic (Eslami E. 2012), new modal predicate logics: *global intuitionistic* and *global fuzzy intuitionistic logics* (Ciabattoni A. 2004), *complex intuitionistic fuzzy classes* (Ramot D. et al. 2002, Ali M. et al. 2016)[†], quantum logic and (different forms of) nonclassical logics (Cattaneo G. et al., see: https://pdfs.semanticscholar.org/9e4a/2d6900cec7671dd68adc5de6c44aa33858a9.pdf, *neutrosophic logic*[‡]: considered as a generalisation of the fuzzy intuitionistic logic (Smarandache F. 2002), intuitionistic fuzzy Bonferroni means[§] (Xu Z.S. and Yager R.R. 2006, 2011) and so on. An implementation-oriented research was also presented, e.g. concerning the use of fuzzy intuitionistic logic techniques in image processing (Rajarajeswari P. and Uma N. 2013), the use of intuitionistic fuzzy classifiers in intrusion detection systems (Kavitha B. et al. 2011), noise removal from images using fuzzy intuitionistic logic controllers (Radhika C. et al. 2016), techniques for developing large scale fuzzy logic systems (Ervin J.C. and Altekin S.E. 2008), fuzzy intuitionistic logic applied to real-time traffic (Alodat M. 2015), etc.

Linear logic

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[*] Corresponds to the notion of *crisp set* A^* *associated with a fuzzy set* A (using a *threshold value* 0.5): $\mu_{A^*}(x) =_{df}$ if $\mu_A(x) \geq 0.5$ then 1 else 0 (for any $x \in X$ – the domain: see Chapter III, Section 7 of this book).

[†] See also Subsection 9.1.

[‡] This logic is based on the *non-standard analysis,* originated in the early 1960s by Abraham Robinson (1918 – 1974): a formalization of analysis and a branch of mathematical logic, that in particular rigorously defines the infinitesimals (i.e. the infinitely small numbers: x is *infinitesimal* iff $|x| < 1/n$, for any $n \in \mathbb{N}$). Here, the notions of 'non-standard real subsets' and 'non-standard unit interval' are used (Smarandache F. 2002).

[§] A mean-type aggregation operator called the *Bonferroni mean* (Bonferroni C. 1950: Carlo Emilio Bonferroni 1892 - 1960).

Linear logic, introduced by J.- Y. Girard[*] in 1987, is considered as a refinement of classical and intuitionistic logics. Instead of emphasizing *truth*, as in classical logic, or *proof*, as in intuitionistic logic, linear logic emphasizes the role of formulae as *resources*. This logic does not allow the usual structural rules of contraction an weakening to apply to all formulae but only those formulae marked with certain modalities. Linear logic contains a fully involutive negation while maintaining a strong constructive interpretation. Linear logic can be considered as a bold attempt to reconcile the beauty and symmetry of the systems for classical logic with the quest for constructive proofs that had led to intuitionistic logic (Di Cosmo R. and Miller D. 2016). It was shown how linear intuitionistic logic can be explained as a logic of resources (Lafont Y. 1993). And hence, this logic becomes an attractive in computer science, because of its logical way of coping with resources and resource control (Braüner T. 1996).

The linear logic is weaker than the classical one, i.e. some formulae provable in classical logic may not be provable in linear logic..On the other hand, the linear logic is useful precisely. The classical logic is '*logic about truth*' (if some formula is true, then it is always true). The linear logic is a '*logic about resources*' (resources are finite and consumable: if we use a resource, then we cannot use it again). In linear logic assumptions (a.k.a. *hypotheses*) correspond to consumable resources. Assumptions cannot be arbitrarily duplicated, nor can they be discarded without being used, see: https://www.cs.ucsb.edu/~benh/162_S16/handouts/handout8-LFOL.pdf.

The following Gentzen's style structural rules are mainly used for obtaining substructural logics (e.g. such as: affine, linear, ordered, relevance and other ones:).

$$\frac{\Gamma,\Delta \vdash A}{\Delta,\Gamma \vdash A}\ (\textit{exchange}), \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (\textit{contraction}) \ \text{ and } \ \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ (\textit{weakening})$$

The first two rules follow directly from the commutative and idempotent axioms for conjunction and SR. The last rule follows directly from T 1.12 (exportation of implication: Subsection 1.3). The corresponding formal proofs are left to the reader (see Subsection 1.8 and also *Relevance logic*, considered in this Subsection 2.4)[†].

The *language of linear classical logic* can be defined recursively as follows: $\varphi =_{df} p \,/\, {\sim}\varphi \,/\, \varphi \otimes \psi \,/\, \varphi \oplus \psi \,/\, \varphi \,\&\, \psi \,/\, \varphi \parr \psi \,/\, 1 \,/\, 0 \,/\, \top \,/\, \bot \,/\, !\,\varphi \,/\, ?\,\varphi$. The logical connectives $\otimes$, $\parr$, 1 and $\bot$ are said to be *multiplicatives*, similarly $\&$, $\oplus$, $\top$ and 0 are called *additives.* The last two connectives: ! and ? are called *modals* or *exponentials* (used to give controlled access to weakening and contraction), $\top$ and $\bot$ are pronounced as: '*top*' and '*bottom*', respectively. The following terminology is used here (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.).

Logical connective	Name / Is pronounced
$\otimes$	*multiplicative conjunction* (*times* , *tensor*), $\varphi \otimes \psi$: ´*both φ and ψ*´ or ´*φ tensor ψ*´
$\oplus$	*additive disjunction* (*plus*), $\varphi \oplus \psi$: ´(*at least one of*) *φ or ψ*´
&	*additive conjunction* (*with*), $\varphi \,\&\, \psi$: ´*choose from φ and ψ*´ or ´*φ with ψ*´
$\parr$	*multiplicative disjunction* (*par*)
!	*escape* (*of course*, *bang*), $!\varphi$: ´*of course φ*´ or ´*bang φ*´

[*] Jean-Yves Girard , born 1947.

[†] In fact, the above Gentzen's formulae (related to the antecedents of the main sequent connectives) can be also represented as follows: $\frac{\Delta, A, B, \Gamma \vdash \Theta}{\Delta, B, A, \Gamma \vdash \Theta}$, $\frac{\Gamma, A, A, \Delta \vdash \Theta}{\Gamma, A, \Delta \vdash \Theta}$ and $\frac{\Gamma, \Delta \vdash \Theta}{\Gamma, A, \Delta \vdash \Theta}$, corresponding to *exchange*, *contraction* and *weakening*, respectively.

?	*why not*

The following '*polarity*' classification is also used: $\otimes$, $\oplus$, 1, 0, ! are called *positive* and their duals $⅋$, & , $\bot$, T , ? are called *negative*.

The *linear implication* is denoted by ′$\multimap$′ , where ′$\varphi \multimap \psi$′ is pronounced as: ′*consuming* φ *yields* ψ′ or ′φ *lollipop* ψ′. We have: $\varphi \multimap \psi =_{df} \sim\varphi ⅋ \psi$. The intuitionistic implication $\varphi \Rightarrow \psi$ can be defined as: $!\varphi \multimap \psi$. In particular, the following equivalences are also satisfied: $!(\varphi \,\&\, \psi) \equiv !\varphi \otimes !\psi$ (known as: '*exponential isomorphism*') and $?(\varphi \oplus \psi) \equiv ?\varphi ⅋ ?\psi$. Moreover, $!\top \equiv 1$ and $?0 \equiv \bot$. The linear logic equivalence $\varphi \equiv \psi$ means that the formula $(\varphi \multimap \psi) \,\&\, (\psi \multimap \varphi)$ is derivable in this logic. It can be observed that any formula φ has a dual $\sim\varphi$, e.g. $\sim(\varphi \otimes \psi) \equiv \sim\varphi ⅋ \sim\psi$ and $\sim(\varphi ⅋ \psi) \equiv \sim\varphi \otimes \sim\psi$ or $\sim(\varphi \oplus \psi) \equiv \sim\varphi \,\&\sim\psi$ and $\sim(\varphi \,\&\, \psi) \equiv \sim\varphi \oplus \sim\psi$, etc. In particular, T 1.3 of Subsection 1.3: $\sim\sim\varphi \equiv \varphi$ is also satisfied, i.e. ′$\sim$′ is involutive (Di Cosmo R. and Miller D. 2016)*.

In addition to the above De Morgan's dualities, the following formulae are also satisfied (Cockett J.R.B. and Seely R.A.G. 1997: see also the above exponential isomorphism): $\varphi \otimes (\psi \oplus \chi) \equiv (\varphi \otimes \psi) \oplus (\varphi \otimes \chi)$: ′$\otimes$′ is distributive wrt ′$\oplus$′, $(\varphi \otimes (\psi \,\Xi\, \chi)) \multimap ((\varphi \otimes \psi) \,\Xi\, \chi)$: $\Xi \in \{\otimes, \oplus, \&, ⅋\}$. The last linear implications, called *linear distributions*, are fundamental in the proof theory of linear logic. In particular, we have: if $\varphi \multimap \psi$, $\varphi \multimap \chi$ then $\varphi, \varphi \multimap \psi \otimes \chi$.

A good introduction to linear classical and linear intuitionistic logics is given in (Braüner T. 1996). An illustration of the difference between intuitionistic logic and linear intuitionistic logic is also presented. The correspondence between the linear intuitionistic logic and the linear λ-calculus is also considered (by using the original *Curry-Howard isomorphism*†, see: Howard W.A. 1980) as well the *Girard translation* (which embeds intuitionistic logic into linear intuitionistic one). And finally, a brief introduction to some example models of linear intuitionistic logic are given. This technical report is a fundamental study on proof-theory and computational interpretation of proofs, in particular: the interpretation of proofs as programs and reduction (*cut-elimination*) as evaluation (corresponds to the fundamental idea of the proofs-as-programs paradigm). A more formal treatment is omitted here, see also: (Lafont Y. 1999/2017).

Traditionally, the linear logic proof techniques are related to the classical sequent calculus (see Subsection 1.8) in which uses of the structural rules contraction and weakening are carefully controlled. Example inference rules used in this logic are given below: see: https://www.cs.ucsb.edu/~benh/162_S16/handouts/handout8-LFOL.pdf‡.

$$+\otimes : \quad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad\qquad -\otimes : \quad \frac{\Gamma \vdash A \otimes B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

---

* We shall assume that $\otimes$ and $⅋$ bind more strongly than the symbol of equivalence. See also: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*. Obviously, ′$\sim\varphi$′ is understood as an abbreviation for ′$\varphi \multimap \bot$′. Sometimes, the 0-ary ′$\bot$′ is denoted by ′$′ and known as "*intuitionistic absurd*".

† The original *Curry-Howard isomorphism* relates the natural deduction formulation of intuitionistic logic to the λ-calculus: formulae correspond to types, proofs to terms, and reduction of proofs to reduction of terms: Haskell Brooks Curry (1900 – 1982) and William Alvin Howard (born 1926).

‡ The corresponding *introduction* and *reduction* (called also *elimination*) *rules* are here denoted by ′+′ and ′−′, respectively. Moreover, the above presentation is characterized by the presence of two different forms of rules for some connectives. Then, the used indexes ′L′ and ′R′ are related to the left hand side and to the right hand side of the considered connective. A more complete systems of introduction rules related to the linear classical, linear intuitionistic logics and the linear λ-calculus are given in (Braüner T. 1996). Another interesting case seems to be the linear μ-calculus, e.g. (Bradfield J.C. and Stirling C. 2001).

$$+\oplus: \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\,L \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,R \qquad\qquad -\oplus: \quad \frac{\Gamma \vdash A \oplus B \quad \Delta, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta \vdash C}$$

$$+\&: \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad\qquad -\&: \quad \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash A}\,L \qquad \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash B}\,R$$

$$+!: \quad \frac{\Gamma \vdash A \quad !\Gamma}{\Gamma \vdash !A} \qquad\qquad -!: \quad \frac{\Gamma \vdash !A \quad \Gamma, \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

$$+\multimap: \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad\qquad -\multimap: \quad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

In particular, a special case seems to be the rule $'-\exists'$, if the predicate A is more than one argument. Then, some *Skolemian functions* should be introduced[*]. A more formal treatment is omitted here (see: Classical first-order and higher order predicate logics: Chapter II, Subsection 3.3). A modified version of $'-\exists'$ and corresponding proof are given below.

$$+\forall: \quad \frac{\Gamma \vdash A}{\Gamma \vdash \underset{x}{\forall} A} \qquad\qquad -\forall: \quad \frac{\Gamma \vdash \underset{x}{\forall} A}{\Gamma \vdash A(x/\xi) \,/\, \Gamma \vdash A(x)}$$

$$+\exists: \quad \frac{\Gamma \vdash A(x/\xi)}{\Gamma \vdash \underset{x}{\exists} A} \qquad\qquad -\exists: \quad \frac{\Gamma \vdash \underset{x}{\exists} A \quad \Delta, \underset{x}{\exists} A \vdash B}{\Gamma, \Delta \vdash B}$$

For example, in accordance with $'-\exists'$, the following formula should be proved (see Subsection 1.8).

$$(p \Rightarrow \underset{x}{\exists} A(x)) \wedge (q \wedge \underset{x}{\exists} A(x) \Rightarrow r) \Rightarrow (p \wedge q \Rightarrow r)$$

Proof:

(1)	$p \Rightarrow \underset{x}{\exists} A(x)$	
(2)	$q \wedge \underset{x}{\exists} A(x) \Rightarrow r$	{a}
(3)	p	
(4)	q	
(5)	$\underset{x}{\exists} A(x)$	{– C : 1,3}
(6)	$q \wedge \underset{x}{\exists} A(x)$	{+ K : 4,5}
	r. □	{– C : 2,6}

Intuitionistic computability logic

[*] Thoralf Albert Skolem (1887 – 1963).

Computability logic introduced by Japaridze G[*]. (2003) is a formal theory of computability in the sense as classical logic is a formal theory of truth. It understands formulae as interactive computational problems (defined as games played by a *machine* against the *environment*:), and their 'truth' as algorithmic solvability (meaning existence of a machine that always wins the game)..This logic is a systematic formal theory of computational tasks and resources, which, in a sense, can be seen as a semantics-based alternative to (the syntactically introduced) linear logic.

In accordance with the above work, a relatively modest fragment of computability logic, called *intuitionistic computability logic*, was considered and it was conjectured that the (set of all valid formulae of the) resulting fragment is described by Heyting's intuitionistic calculus INT. A verification of the soundness part of that conjecture was given in (Japaridze G. 2007). A fragment of this work is presented bellow.

The above two players, i.e. machine and environment, are denoted below by $\top$ and $\bot$, respectively. Here, $\top$ is a mechanical device with a a fully determined algorithmic behaviour, but there are no restrictions on the behaviour of $\bot$. A problem / game is considered (*algorithmically*) *solvable* / *winnable* iff there is a machine that wins the game no matter how the environment acts.

Logical operators (i.e. functors) are understood as *operations* on games / problems. An important group of such operations, called *choice operations*, includes: $\sqcap$, $\sqcup$, Π and $\amalg$ corresponding to the intuitionistic operators of *conjunction* and *disjunction*, and the *universal* and *existential quantifiers*, respectively (it is used Peirce´s[†] interpretation of the last two quantifiers: $\sqcap$ and $\sqcup$ are associative and hence can be generalised as finite argument, see Chapter II).

As an example, consider the following formula / problem (Japaridze G. 2007): $\varphi_1 \sqcap \varphi_2 \sqcap \ldots \sqcap \varphi_n$[‡]. This formula is interpreted as a game, where the first legal move (i.e. "*choice*"), which should be one of the elements of $\{1, 2, \ldots, n\}$, is by the environment. After such a move / choice ´i´ is made, the play continues and the winner is determined according to the rules of φ_i: $\bot$ loses if a choice is never made.

Another basic operations having no official intuitionistic counterparts were also proposed (Japaridze G. 2007): comprises negation ´$\sim$´and the so called *parallel operations*: ´$\wedge, \vee, \Rightarrow$´, e.g. applying ´$\sim$´ to a formula / game φ interchanges the rules of the two players: $\top$'s moves and wins become $\bot$'s moves and wins, and vice versa, etc. A more formal treatment is omitted here.

Paraconsistent logic

The *paraconsistent logic* is a logical system that attempts to deal with contradictions in a discriminating way. As the forerunner of this logic is reckoned N.A. Vasiliev[§] about 1910 (there was proposed a modified Aristotelian syllogistic including statements of the form: ψ is both φ and $\sim\varphi$)[**], see: (Arruda A.I. 1984, Bazhanov V.A. 1994). The first axiomatisation of a paraconsistent logic (concerning the relevant logic) was given by I.E. Orlov[††]. A more formal approach was given by S. Jaśkowski (1948: Stanisław Jaśkowski 1906 – 1965, who was a student of Łukasiewicz), There was introduced a system, originally called *discussive logic* (known also as: *discursive* one), where the law of Duns Scotus (see T 1.22 of Subsection 1.3) was not satisfied. There was presented a

[*] Giorgi Japaridze, born 1961.

[†] Charles Sanders Peirce (1839 – 1914).

[‡] Provided there is no ambiguity, instead of A_i, formulae are here denoted by φ_i. $i \in \{1, 2, \ldots, n\}$.

[§] Nikolai Alexandrovich Vasiliev (1880 – 1940): with "*imaginary non-Aristotelian logics*", also reckoned as one of the forerunners of multi-valued logics.

[**] In accordance with T 1.22 (the rule DS: see Subsection 1.3), using the law of importation, i.e. T 1.12b, we can obtain: $p \wedge \sim p \Rightarrow q$, or in general: $\varphi \wedge \sim \varphi \Rightarrow \psi$ (latin: *ex false quodlibet*). And hence, in accordance with Subsection 1.5 we have: $\psi \in Cn(\{\varphi, \sim \varphi\})$ or equivalently $\varphi, \sim \varphi \models \psi$. So, a contradictory proposition has any proposition as its consequence (i.e. everything follows from a contradiction: involving "*twofold standards*" and hence any formula becomes a thesis).

[††] Ivan Efimovich Orlov (1886 – 1936).

formalisation of discussive logic by means of modelling a discourse in modal logic using S5 (see Subsection 2.3). A brief history concerning systems of paraconsistent logic is given in (Priest G. et al. 2013)*: left to the reader.

Alternatively paraconsistent logic can be considered as a subfield of logic concerned with studying and developing paraconsistent (or "*inconsistency-tolerant*") systems of logic. Inconsistency-tolerant logics have been discussed since at least 1910 (and arguably much earlier, e.g. in the writings of Aristoteles), however the term *paraconsistent* ("*beside the consistent*") was not coined until 1976, by F.M. Quesada† (Priest G. et al. 2013). *Dialectical logic* is the system of laws of thought, developed within the Hegelian and Marxist traditions‡, which seeks to supplement or replace the laws of formal logic. The main consensus among dialecticians is that dialectics do not violate the law of contradiction of formal logic, although attempts have been made to create a paraconsistent logic (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.). An integrated discussion of all major topics in the area of paraconsistent logic (philosophical and historical aspects, major developments and real-world applications) was also presented at the second world congress on paraconsistency (*Paraconsistency: the logical way to the inconsistent*. 2002).

It was observed classical logic, and most standard non-classical logics, e.g. such as intuitionistic logic, are explosive. Inconsistency, according to received wisdom, cannot be coherently reasoned about. Paraconsistent logic challenges this orthodoxy. And hence, the above (logical consequence) relation ´$\vDash$´ is said to be *paraconsistent* if it is not explosive. Paraconsistent logic accommodates inconsistency in a sensible manner that treats inconsistent information as informative. Sometimes paraconsistent logic is erroneously interpreted as *dialetheism* (the view that there are true contradictions).The view that a (logical) consequence relation should be paraconsistent does not entail the view that there are true contradictions. Paraconsistency is a property of an inference relation whereas dialetheism is a view about truth (Priest G. et al. 2013).

Let ´$\vDash$´ be a relation of *logical consequence*. We shall say $\vDash$ is *explosive* if it validates $\varphi, \sim\varphi \vDash \psi$, for any φ and ψ, i.e. from contradiction follows anything (latin: *ex contradictione sequitur quodlibet*, abbreviated below as: ECSQ): known as the *principle of explosion*. It is easily to describe this principle as direct proof from assumptions. This is shown below (see: Subsection 1.3).

(1)	$\varphi \wedge \sim\varphi$	{a}
(2)	φ	{– K : 1}
(3)	$\sim\varphi$	
(4)	$\varphi \vee \psi$	{+ A : 2}
	ψ. □	{– A : 3,4}

And hence, an abandoning of the principle of explosion should require the rejection at least one of the above two rules: ´+ A´ or ´– A´.

By rejecting ´+ A´ ,assuming ´– A´ and transitivity, the most of the natural deduction rules hold, e.g. such as: double negation and also associativity, commutativity, distributivity, De Morgan's laws, idempotence (for conjunction and disjunction), etc. But, e.g. the Aristotelian law of the excluded middle, i.e. $\varphi \vee \sim\varphi$, does not hold.

Let now ´– A´ be rejected. Hence, having $\sim\varphi$, there will not be possible to infer ψ from $\varphi \vee \psi$ (this case may be interesting from the perspective of dialetheism).

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* 'In the metaphysics, Aristotle called it *βεβαιοτάτη πασῶνἀρχή*, "The firmest of all principles" - *firmissimum omnium principiorum*, the medieval theologians said. They referred to the principle that was to be known as the *law of non-contradiction* (LNC). They called it *firmissimum*, for in the western philosophical tradition the LNC was regarded as the most fundamental principle of knowledge and science. According to Thomas Reid (1710 – 1796) the law, in the form: "No proposition is both true and false", was also a cornerstone of common sense, together with other basic truths that shape our experience ("Every complete sentence must have a verb", for instance, or "Those things really happened which I distinctly remember"). Nevertheless, today the LNC has found itself under logical attack by so-called *strong paraconsistency*, also called *dialetheism*. Paraconsistency is the doctrine according to which there are theories, that is, sets of sentences closed under logical consequence, that are inconsistent but non-trivial. The logical consequence at issue, then, must be such that $\{\varphi, \sim\varphi\} \nvDash \psi$ - the inference from inconsistent premises to an arbitrary conclusion (often called *ex falso quodlibet*) is invalid. A logic invalidating ex falso is called *paraconsistent* in its turn' (Berto F. 2012).

† Francisco Miró Quesada Cantuarias, born 1918.

‡ Georg Wilhelm Friedrich Hegel (1770 – 1831), Karl Marx (1818 – 1883)

The last case, rejecting both: ′+ A′ and ′− A′, will involve use of two separate disjunctive connectives (corresponding to ′+ A′ and ′− A′), e.g. as in linear logic as well as in relevant logic.

In general, there were proposed many paraconsistent logics. A natural way of generating a paraconsistent logic (perhaps the simplest one) is the use of many-valued logic. Such a system was first proposed in 1954 by Asenjo F.G. in his PhD dissertation, see also LP: *logic of paradox* (Asenjo F.G. 1966, 1999). Here, the set of the following logical constants was considered: {F. B, T}, where ′B′ denotes '*both*' (i.e. false and true). In accordance with the requirements of this logic, there was used Kleene's ternary logic system (it was used ′B′, instead of the original Kleene's symbol 'U': *unknown*)[*], e.g. the *logical value* of $\sim$ B $=_{df}$ B, in a similar way $\sim$ F $=_{df}$ T, F $\vee$ B $=_{df}$ max{F,B} $=_{df}$ B, F $\wedge$ B $=_{df}$ min{F,B} = F, B $\wedge$ T $=_{df}$ min{B,T} = B, B $\vee$ T $=_{df}$ max{B,T} = T, T $\Rightarrow$ B $=_{df}$ B, B $\Rightarrow$ F $=_{df}$ B, F $\Rightarrow$ B $=_{df}$ T, B $\Rightarrow$ B $=_{df}$ B, etc.

Paraconsistent logic has a significant part in common with many-valued logic, however not all paraconsistent logics are many-valued (and vice versa). On the other hand, intuitionistic logic allows (the Aristotelian law of the excluded middle) $\varphi \vee \sim \varphi$ not to be satisfied, i.e. not to be equivalent to true, while paraconsistent logic allows $\varphi \wedge \sim \varphi$ not be equivalent to false. And hence, it seems natural to regard paraconsistent logic as the "*dual*" of the intuitionistic one[†].

Paraconsistent logic becomes an important part in applications concerning such areas as: computer science and engineering, quantum theory, in particular the problem of entanglement (see Subsection 2.4: quantum dynamic-epistemic logic), etc. On the other hand, to invalidate ECSQ, there were introduced various systems of paraconsistent logic (accepting the validity of classical inferences in consistent contexts). As an example such system, the relevance logic is briefly presented below.

Relevance logic

Relevance (or equivalently: relevant) logic[‡] systems were developed as attempts to avoid some paradoxes related to material and strict implications. Such logic systems were first initiated in (Anderson A.R. and Belnap N.D. 1975) as a study of relevance of the conclusion wrt the premises: see also (Anderson A.R. et al. 1992) as well as (Dunn J.M. and Restall G., *Relevance logic*. http://consequently.org/papers/rle.pdf) and (Mirek R. 2011). The considered here conception of logical implication (or more general: inference) was of fundamental importance. A similar suggestion was early given by Ackermann W.F.[§]. (1956).

Assume now that $\vDash \varphi \Rightarrow \psi$. Consider the implicative proposition ′$\varphi \Rightarrow \psi$′, where ψ *follows from* (*is a logical consequence of / is deducible by*) φ. Such term as ′ *follows from*′ can be interpreted as a binary relation between ψ and φ. The opposite relation associated with this one is denoted by ′*entail*′. And hence, we shall say that φ *entail* ψ $\Leftrightarrow_{df}$ ψ *follows from* φ (Moore G.E. 1920)[**]. In general, with this conception of implication are associated the following two approaches.

The first one, was based on Diodorus Cronus' ideas of "strict" or "strong" implication (died c.284 b.c.) and continued by Lewis (Clarence Irving Lewis 1883 – 1964): modal notions and strict implication systems (see Subsection 2.3). This way, there were eliminated some paradoxes only concerning the material implication. And hence, it was observed a necessity of considering the term '*implication*' as a primary one (instead of using Lewis' modal logics).

The second approach was based on Kant's observations concerning the notion of "*analyticity*": some implicative proposition is said to be an *analytical proposition* if the significance (the meaning or sense) of the

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[*] Kleene's ternary logic (1952: Stephen Cole Kleene 1904 – 1994): Kleene's ternary implication differs in its definition in that 'U implies U' is 'U' (instead of 'T', as in the case of Łukasiewicz's one). The corresponding definitions for negation, conjunction and disjunction connectives are the same as in Łukasiewicz's system (1918: Jan Łukasiewicz 1878 – 1956), concerning the weak conjunction and disjunction connectives, see Subsection 2.1 (instead of $W_3 =_{df} \{0, 1/2, 1\}$, there was used the set {F, U, T}, where 0, 1/2 and 1 correspond to F, U and T, respectively).

[†] T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[‡] Called "*relevance logics*" in North America and "*relevant logics*" in Britain and Australasia.

[§] Wilhelm Friedrich Ackermann (1896 – 1962).

[**] George Edward Moore (1873 – 1958).

consequent is included in the significance of the antecedent of this proposition (Immanuel Kant 1724 – 1804). In accordance with the above observations, there was proposed an *analytical implication* system (Parry W.T. 1932, 1933)*, where the set of propositional variables of the consequent (let be here denoted by: ´C´) is a subset of the set of the propositional variables associated with the antecedent (say: ´A´): i.e. $C \subseteq A$ (for each implicative thesis $\varphi \Rightarrow \psi$ of this system).† Here, the minimal condition of existence a semantical consistency between φ and ψ was interpreted as impossibility of realisation the so called '*error of inconsistency*': related to the case $C \cap A = \varnothing$ (for any thesis $\varphi \Rightarrow \psi$). So, this *minimal* (*relevance*) *condition* was used in systems known as 'e*ntailment*'‡. In accordance with the last condition, the obtained logics of the systems of type entailment can be considered as subsystems of the basic systems of strict implication S3, S4 and S5 (obviously considering only the implicative fragments of such systems).

It can be observed that the above notion of *entailment* is usable in (at least) three meanings: *implication connective* (having some properties), the name of the *logical system* characterising this connective as well as the *area* in which this system is defined.

Among the most important systems of type entailment are the following: E (the basic system *entailment*), T (*ticket entailment*), EM (entailment-mingle), R (relevance), in particular $R_{\rightarrow}$ (*pure implicative fragment*), etc. The last implicative fragment is considered by the relevance logicians as the most important. As an example, in the implicative system E (*pure calculus of entailment*), in its natural deduction version, are used the following rules.

(1) Hyp The antecedents of the main implication, i.e. the primary assumptions (see Subsection 1.2: direct proof from assumptions) are introduced successively, in the next proof steps, in such a way that to each assumption is assigned one element set {k}, where the natural number ´k´, called *index*, denotes the actual proof's step.

(2) Rep Any formula can be repeated in a corresponding proof's step without changing its index.

(3) Reit Any implicative formula can be relocated to a next proof's step without changing its index.

(4) $\rightarrow$E By φ_a and $(\varphi \Rightarrow \psi)_b$ we can obtain $\psi_{a \cup b}$ (a, b denote index sets).

(5) $\rightarrow$I From the proof's step ψ_a and the assumption $\varphi_{\{k\}}$ we can obtain $(\varphi \Rightarrow \psi)_{a - \{k\}}$, if $k \in a$.

The system T can be obtained from E by using the following restriction for the above rule (4):

$(4)_T$ $\rightarrow$E By φ_a and $(\varphi \Rightarrow \psi)_b$ we can obtain $\psi_{a \cup b}$, if $\max(b) \leq \max(a)$.

The implicative system EM is an extension of E wrt the following additional rule.

(6) Mgl By $(\varphi \Rightarrow \psi)_a$ and $(\varphi \Rightarrow \psi)_b$ we can obtain $(\varphi \Rightarrow \psi)_{a \cup b}$.

Example 2.23

Consider the following *rule of reduction*: $(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q)$. As an illustration, there are shown below two kinds of proofs: the classical one (a direct proof from assumptions) and the proof using the basic system E.

Proof (classical version):

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* William Tuthill Parry (1908 – 1988).

† In general, ´⇒´ corresponds to the *main implication* of the generalised form of an expression. In fact, this generalised form can be transformed in a form similar to the right side of Theorem 1.23: using (n – 2) times T 1.12 (since logical equivalence is transitive).

Let now $\{\varphi\}$ be the set of propositional variables associated with φ. Assume that ψ is a logical consequence wrt $\varphi_1, \varphi_2, \ldots, \varphi_n$. Then: $\models \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \Rightarrow \psi$ (see Theorem 1.23 of Subsection 1.5). And hence, the following condition should be satisfied: $\{\psi\} \subseteq \bigcup_{i=1}^{n}\{\varphi_i\}$, e.g. $\models (p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$ and $\{q,r\} \subseteq \{p,q,r\}$.

‡ *Formal logic. Encyclopedical outline with applications to informatics and linguistics* (1987): 338 – 359.

(1)	$p \Rightarrow (p \Rightarrow q)$	{1,2 / a}
(2)	p	
(3)	$p \Rightarrow q$	{– C : 1,2}
	p. □	{– C : 3,2}

Proof (E version)[*]*:*

(1)	$p \Rightarrow (p \Rightarrow q)_{\{1\}}$	Hyp;
(2)	$p_{\{2\}}$	
(3)	$p \Rightarrow (p \Rightarrow q)_{\{1\}}$	1,Reit;
(4)	$p \Rightarrow q_{\{1,2]}$	2,3, → E;
(5)	$q_{\{1,2]}$	2,4, → E;
(6)	$p \Rightarrow q_{\{1\}}$	2,5, → I;
(7)	$(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q)$	1,6, → I. □

Consider now the relevance logic system R. A formalisation of R can be realised (as in classical logic systems) into two versions: axiomatic and assumptional (natural deduction methods).

The axiomatic system of R can be obtained by completion the axioms of the *Moh Shaw-Kwei and Church's implicative system* (Moh Shaw-Kwei 1950, Church A. 1951)[†]:

(A1) $p \Rightarrow p$
(A2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$
(A3) $(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q)$
(A4) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r))$

with the following formulae characterising conjunction, disjunction and negation[‡]:

(A5) $p \wedge q \Rightarrow p$
(A6) $p \wedge q \Rightarrow q$
(A7) $(p \Rightarrow q) \wedge (p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r)$
(A8) $p \Rightarrow p \vee q$
(A9) $q \Rightarrow p \vee q$
(A10) $(p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$
(A11) $p \wedge (q \vee r) \Rightarrow p \wedge q \vee r$
(A12) $(p \Rightarrow \sim q) \Rightarrow (q \Rightarrow \sim p)$
(A13) $\sim\sim p \Rightarrow p$

The assumptional version of R is an extension of the basic system E , i.e. rules (1) – (5), by adding the following rules.

(6) ~ ~ E From $\sim\sim \varphi_a$ we can obtain φ_a.
(7) ~ ~ I From φ_a we can obtain $\sim\sim \varphi_a$.
(8) ∧ E From $(\varphi \wedge \psi)_a$ we can obtain φ_a.
(9) From $(\varphi \wedge \psi)_a$ we can obtain ψ_a.

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[*] According to the used below designations: ´Hyp´ corresponds to ´{a}´, ´2,3, → E´ may be denoted as: {→ E : 2,3}, etc.

[†] Moh Shaw-Kwei (1917 – 2011), Alonzo Church (1903 – 1995).

[‡] *Formal logic. Encyclopedical outline with applications to informatics and linguistics* (1987): 338 – 359. In accordance with the used priorities for logical connectives some parentheses may be omitted, e.g. $p \wedge q \Rightarrow p$ instead of: $(p \wedge q) \Rightarrow p$, etc., see Subsection 1.1.

(10)	$\wedge$ I	From φ_a and ψ_a we can obtain $(\varphi \wedge \psi)_a$.
(11)	$\vee$ I	From φ_a we can obtain $(\varphi \vee \psi)_a$.
		From ψ_a we can obtain $(\varphi \vee \psi)_a$.
(12)	$\wedge\vee$	From $(\varphi \wedge (\psi \vee \chi))_a$ we can obtain $((\varphi \wedge \psi) \vee \chi)_a$.
(13)	~ E	From $\sim\psi_a$ and $(\varphi \Rightarrow \psi)_b$ we can obtain $\sim\varphi_{a \cup b}$.
(14)	~ I	From $(\varphi \Rightarrow \sim\varphi)_a$ we can obtain $\sim\varphi_a$.
(15)	$\vee$ E	From $(\varphi \vee \psi)_a$, $(\varphi \Rightarrow \chi)_b$ and $(\psi \Rightarrow \chi)_b$ we can obtain $\chi_{a \cup b}$.

Example 2.24

Since conjunction and disjunction are mutually distributive*, in particular the following implication is satisfied: $p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)$. There are shown below two kinds of proofs: the classical one (a ramified direct proof from assumptions) and the corresponding proof using the assumptional version of R.

Proof (classical version):

(1)	$p \vee q \wedge r$	{a}
(1.1)	p	(ada}
(1.2)	$p \vee q$	{+ A : 1.1}
(1.3)	$p \vee r$	
(1.4)	$(p \vee q) \wedge (p \vee r)$	{+ K : 1.2,1.3}
(2.1)	$q \wedge r$	{ada}
(2.2)	q	{− K : 2.1}
(2.3)	r	
(2.4)	$p \vee q$	{+ A : 2.2}
(2.5)	$p \vee r$	{+ A : 2.3}
(2.6)	$(p \vee q) \wedge (p \vee r)$	{+ K : 2.4,2.5}
	$(p \vee q) \wedge (p \vee r)$. □	{1.4,2.6}

Proof (R version):

(1)	$p \vee q \wedge r_{\{1\}}$	Hyp;
(2)	$p_{\{2\}}$	Hyp;
(3)	$p \vee q_{\{2\}}$	2, $\vee$I;
(4)	$p \vee r_{\{2\}}$	2, $\vee$I;
(5)	$(p \vee q) \wedge (p \vee r)_{\{2\}}$	3,4, $\wedge$I;
(6)	$p \Rightarrow (p \vee q) \wedge (p \vee r)$	2,5, $\rightarrow$ I;
(7)	$q \wedge r_{\{3\}}$	Hyp;
(8)	$q_{\{3\}}$	7, $\wedge$E;
(9)	$r_{\{3\}}$	7, $\wedge$E;
(10)	$p \vee q_{\{3\}}$	8, $\vee$I;
(11)	$p \vee r_{\{3\}}$	9, $\vee$I;
(12)	$(p \vee q) \wedge (p \vee r)_{\{3\}}$	10,11, $\wedge$I;
(13)	$q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)$	7,12, $\rightarrow$ I;
(14)	$(p \vee q) \wedge (p \vee r)_{\{1\}}$	1,6,13, $\vee$E;
(15)	$p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)$	1,14, $\rightarrow$ I. □

The reader is invited to give an assumptional R-version proof of the following formula: $(p \wedge q) \vee (p \wedge r) \Rightarrow p \wedge (q \vee r)$.

---

* In fact, conjunction and disjunction satisfy the *commutative*, *associative*, *absorptive*, *idempotent*, and *distributive* axioms (see Subsection 1.3).

The extended system RM, in natural deduction version, uses the following additional rule.

(16) Mgl From φ_a and φ_b we can obtain $\varphi_{a \cup b}$.

The above briefly considered relevance logic systems have also representation by using the Gentzen's* sequent calculus. Here, the only usable is the following axiom (known as *identity*): $A \vdash A$. In particular, starting with this axiom, the following four rules are sufficient in proof representation related to $T_{\rightarrow}$, $E_{\rightarrow}$, $R_{\rightarrow}$, the modal logic system $S4_{\rightarrow}$ and Heyting's intuitionistic system $H_{\rightarrow}$. The proof of the introduction rule ´$+ C_a$´ was presented in Subsection 1.8 (*Example 1.14*). The proof of ´$+ C_c$´ is left to the reader. The last two introduction rules are also known as: '*arrow on the left*' and '*arrow on the right*', respectively. The proofs of the rest two rules (*contraction* and *weakening*) are also left to the reader.

Id	$A \vdash A$	*identity*
Cont$_a$	$\dfrac{\Gamma, A, A, \Delta \vdash \Theta}{\Gamma, A, \Delta \vdash \Theta}$	*contraction*
Weak$_a$	$\dfrac{\Gamma, \Delta \vdash \Theta}{\Gamma, A, \Delta \vdash \Theta}$	*weakening*
$+ C_a$	$\dfrac{\Delta \vdash \Lambda, A \quad B, \Gamma \vdash \Theta}{A \Rightarrow B, \Delta, \Gamma \vdash \Lambda, \Theta}$	*arrow on the left*
$+ C_c$	$\dfrac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \Rightarrow B}$	*arrow on the right*†

Example 2.25 (*the logic system $R_{\rightarrow}$*)

Consider the following thesis: $p \Rightarrow (q \Rightarrow r) \Leftrightarrow q \Rightarrow (p \Rightarrow r)$, known as '*commutation rule*' (Słupecki J. and Borkowski L. 1967). The proof of the *if-implication* is given below. The proof of the *only-if-implication* is similar: left to the reader.

Proof ($R_{\rightarrow}$ version):

(1)	$q \vdash q$	{Id}
(2)	$r \vdash r$	
(3)	$p \vdash p$	{Id}
(4)	$q, q \Rightarrow r \vdash r$	$\{+ C_a\text{: } 1,2\}$‡
(5)	$p \Rightarrow (q \Rightarrow r), q, p \vdash r$	$\{+ C_a\text{: } 3,4\}$
(6)	$p \Rightarrow (q \Rightarrow r), q \vdash p \Rightarrow r$	$\{+ C_c\text{: } 5\}$
(7)	$p \Rightarrow (q \Rightarrow r) \vdash q \Rightarrow (p \Rightarrow r)$	$\{+ C_c\text{: } 6\}$
(8)	$\vdash (p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r))$. □	$\{+ C_c\text{: } 7\}$

It can be observed that systems such as $T_{\rightarrow}$, $E_{\rightarrow}$ and $S4_{\rightarrow}$ should be require (in using) some additional conditions, e.g. either $\Delta \neq \lambda$ or in sequence Θ should be an implicative formula (*Weak*$_a$: related to $S4_{\rightarrow}$), etc. A more formal treatment is omitted here.

Example 2.26

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*Gerhard Karl Erich Gentzen (1909 – 1945).

†A more simplified version of this rule is the following one: $\dfrac{A, \Gamma \vdash B}{\Gamma \vdash A \Rightarrow B}$, where Θ is interpreted as an empty formula (or sequence of formulae), i.e. λ. The proof of the last sequent follows directly from Thesis 1.12 (exportation of implication: see Subsection 1.3).

‡ For example, this line is obtained from lines 1,2 by assuming in $+ C_a$: $\Delta = A = q$, $B = \Theta = r$ and $\Lambda = \Gamma = \lambda$.

Consider the following *modified version* of ´+ C_a´: $\frac{\Delta,\Lambda \vdash A \quad \Gamma,B,\Theta \vdash \Pi}{\Delta,\Gamma,A \Rightarrow B,\Lambda,\Theta \vdash \Pi}$. In accordance with Subsection 1.8, the following formula can be obtained.

$(p \wedge q \Rightarrow r) \wedge (s \wedge t \wedge u \Rightarrow w) \Rightarrow (p \wedge s \wedge (r \Rightarrow t) \wedge q \wedge u \Rightarrow w)$

Proof:

(1)	$p \wedge q \Rightarrow r$	
(2)	$s \wedge t \wedge u \Rightarrow w$	
(3)	p	
(4)	s	{1- 7 / a}
(5)	$r \Rightarrow t$	
(6)	q	
(7)	u	
(8)	~ w	{aip}
(9)	$p \wedge q$	{+ K : 3,6}
(10)	r	{− C : 1,9}
(11)	t	{− C : 5,10}
(12)	$s \wedge t \wedge u$	{+ K : 4,7,11}
(13)	w	{− C : 2,12}
	contr. □	{8,13}

Let now ´+ C_a^m´ be the condition: ´either $\Lambda \neq \lambda$ or A is implicative´ and ´+ C_a^t´ be the condition: ´$\Lambda \neq \lambda$´. So, ´+ C_a^m´ is an additional condition for $S4_{\rightarrow}$. In a similar way, ´+ C_a^m´ and ´+ C_a^t´ are requirements for $E_{\rightarrow}$ and $T_{\rightarrow}$, respectively*. □

And finally, to complete the above considerations, there are given below the remaining Gentzen's *introduction rules*: + N_a, + N_c, + K_a, + K_c, + A_a, +A_c as well as some *structural rules* (Glushkov V.M. 1964). The corresponding proofs are left to the reader (the proof of the last rule is given below).

$+N_a: \frac{\Gamma \vdash \Theta,A}{\sim A,\Gamma \vdash \Theta}$ $\qquad$ $+N_c: \frac{A,\Gamma \vdash \Theta}{\Gamma \vdash \Theta,\sim A}$

$+K_a: \frac{A,\Gamma \vdash \Theta \quad B,\Gamma \vdash \Theta}{A \wedge B,\Gamma \vdash \Theta}$ $\qquad$ $+K_c: \frac{\Gamma \vdash \Theta,A \quad \Gamma \vdash \Theta,B}{\Gamma \vdash \Theta,A \wedge B}$

$+A_a: \frac{A,\Gamma \vdash \Theta \quad B,\Gamma \vdash \Theta}{A \vee B,\Gamma \vdash \Theta}$ $\qquad$ $+A_c: \frac{\Gamma \vdash \Theta,A \quad \Gamma \vdash \Theta,B}{\Gamma \vdash \Theta,A \vee B}$

*Cont*a (as above) $\qquad$ *Cont*c : $\frac{\Gamma \vdash \Theta,A,A,\Delta}{\Gamma \vdash \Theta,A,\Delta}$

*Weak*a (as above) $\qquad$ *Weak*c : $\frac{\Gamma,\Delta \vdash \Theta}{\Gamma,\Delta \vdash \Theta,A}$

$Perm_a: \frac{\Lambda,A,B,\Delta \vdash \Gamma}{\Lambda,B,A,\Delta \vdash \Gamma}$ $\qquad$ $Perm_c: \frac{\Gamma \vdash \Lambda,A,B,\Delta}{\Gamma \vdash \Lambda,B,A,\Delta}$

$Cut\ rule: \frac{\Delta \vdash \Lambda,A \quad A,\Gamma \vdash \Theta}{\Delta,\Gamma \vdash \Lambda,\Theta}$.

* *Formal logic. Encyclopedical outline with applications to informatics and linguistics* (1987).

In accordance with the last rule, any formula A appearing at the same time in the consequent of the first sequent and also in the antecedent of the second can be omitted. It was shown (Gentzen G.K.E. 1934, 1935) that any proof making use of this rule can be transformed to a *normal form proof*, i.e. without using this rule (the '*cut-elimination theorem'*: known as one of the most important results in proof theory). The proof of this rule is related to the proof of the following formula.

$$(p \Rightarrow q \vee r) \wedge (r \wedge s \Rightarrow t) \Rightarrow (p \wedge s \Rightarrow q \vee t)$$

Proof (Cut rule):

(1)	$p \Rightarrow q \vee r$	
(2)	$r \wedge s \Rightarrow t$	{1- 4 / a}
(3)	p	
(4)	s	
(5)	$\sim (q \vee t)$	{aip}
(6)	$\sim q$	{6,7 / NA : 5}
(7)	$\sim t$	
(8)	$q \vee r$	{– C : 1,3}
(9)	r	{– A : 6,8}
(10)	$r \wedge s$	{+ K : 4,9}
(11)	t	{– C : 2,10}
	contr. □	{7,11}

Provided there is no ambiguity and for convenience, it is used the same interpretation of commas (in the antecedent or the consequent of a given sequent) as in Subsection 1.8 (for a more formal treatment see: Dunn J.M. and Restall G., *Relevance logic*. http://consequently.org/papers/rle.pdf). And hence, e.g. the following additional rules may be introduced.

$$Comma_a : \frac{\Gamma, A, B \vdash A, \Delta}{\Gamma, A \wedge B \vdash A, \Delta} \qquad Comma_c : \frac{\Gamma, A \vdash A, B, \Delta}{\Gamma, A \vdash A \vee B, \Delta}$$

Example 2.27

The proof of the *if-implication* of *De Morgan′s law of negating a disjunction* is given below (see: Thesis 1.7 of Subsection 1.3). The proof of the *only-if-implication* is similar: left to the reader.

(1)	$p \vdash p$	{Id}
(2)	$q \vdash q$	
(3)	$p \vdash p, q$	{$Weak_c$: 1}
(4)	$q \vdash p, q$	{$Weak_c$, $Perm_c$: 2}
(5)	$p \vdash p \vee q$	{$Comma_c$: 3}
(6)	$q \vdash p \vee q$	{$Comma_c$: 4}
(7)	$\sim (p \vee q) \vdash \sim p$	{+ N_a, +N_c : 5}
(8)	$\sim (p \vee q) \vdash \sim q$	{+ N_a, +N_c : 6}
(9)	$\sim (p \vee q) \vdash \sim p \wedge \sim q$	{+ K_c : 7,8}
(10)	$\vdash \sim (p \vee q) \Rightarrow \sim p \wedge \sim q$. □	{+ C_c: 9}

Non-monotonic logic

The classical logic is a *monotonic* system, i.e. the following implication is satisfied.

$\psi \in Cn(\{\varphi_1,\varphi_2, \dots , \varphi_n\}) \Rightarrow \psi \in Cn(\{\varphi_1,\varphi_2, \dots , \varphi_n,\chi\})$ (for any $\varphi_1, \varphi_2, \dots ,\varphi_n, \psi, \chi$ and $n \in \mathbb{N}$)

The *non-monotonic* reasoning deals with the problem of deriving plausible conclusions, but not infallible, from a *knowledge base* (denoted below by KB: a set of formulae). Since the conclusions are not certain, it must be possible to retract some of them if new information shows that they are wrong (Olivetti N. 2010 / 11). Historically, the first more important research concerning non-monotonic logics was given in (McCarthy J. 1980)[*], (McDermott D.[†] and Doyle J. 1980), (Reiter R. 1980)[‡] and (Moore R.C. 1984). Some notions given in (Olivetti N. 2010 / 11) are briefly presented below: see also the proof-theoretic reconstruction (known as: '*analytic sequent calculi*) given in (Bonatti P.A. and Olivetti N. 2002) as well as (Horty J.F. 2001).

Consider the statement '*typically* A'. This statement can be interpreted as follows: '*in the absence of information to the contrary, assume* A'. A more precise meaning of the last text could be the following one: '*there is nothing in KB that is inconsistent with assumption* A'. Obviously, other interpretations are also possible: they will give rise to different non-monotonic logics. The inadequacy of classical logic for description of such cases is illustrated in the next example given in (Olivetti N. 2010 / 11)[§].

Example 2.28

Consider the following rule '*typically birds fly*'. More formally, this rule can be represented as follows.

$\forall_{x \in \mathscr{U}} (\mathit{bird}(x) \wedge \sim \mathit{exception}(x) \Rightarrow \mathit{fly}(x))$, where $\mathscr{U}$ is the *universe of birds*.

The one argument predicate ´*exception*(x)´ is defined as:

$\forall_{x \in \mathscr{U}} (\mathit{exception}(x) \Leftrightarrow_{df} \mathit{penguin}(x) \vee \mathit{ostrich}(x) \vee \dots)$[**].

Unfortunately, all exceptions are not known in advance and cannot be pre-approved. In accordance with the laws CE (*transposition* or *contraposition of equivalence*), NA and SR, given in Subsection 1.3, we can obtain:

$\forall_{x \in \mathscr{U}} (\sim \mathit{exception}(x) \Leftrightarrow_{df} \sim \mathit{penguin}(x) \wedge \sim \mathit{ostrich}(x) \wedge \dots)$.

Assume that '*Tweety is a bird*' belongs to KB $=_{df}$ {*Typically birds fly*, *Penguins do not fly*, *Tweety is a bird*}. To conclude that '*Tweety*' fly, we should prove that '*Tweety*' is not an exception, i.e. ~ *penguin*(*tweety*), ~ *ostrich*(*tweety*), etc.

On the contrary, we would like to prove that '*Tweety*' flies because we cannot conclude that it is an exception, not because can prove that it is not an exception. □

A basic understanding of *database*[*] logic is that only positive information is represented explicitly. If a positive fact is not presented in this database (in short: DB) then it is assumed that its negation holds (this is the so called '*closed word assumption'*, in short: CWA). Formally (Olivetti N. 2010 / 11):

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[*] John McCarthy (1927 – 2011).

[†] Drew McDermod, born 1949.

[‡] Raymond Reiter (1939 – 2002).

[§] In general, the following problems are associated with classical logic (Ginis A. 2008): the *size of theory* necessary to describe real situations is overwhelmingly large, this *logic is very weak* in the face of incomplete knowledge and also this logic is *too rigid* to deal with new (conflicting) knowledge.

[**] *Conjunction* and *disjunction* are two finite argument logical operations.

DB $\nvDash \varphi \Rightarrow \vDash_{CWA} \sim \varphi$.

Obviously, the last inference is not valid in classical logic. In accordance with the above work, it was next discussed the problem of *dynamical world representation*: in particular, representation of objects that are not affected by state change (using *frame axioms*[†]). Below is considered only the fragment related to *default reasoning* (first introduced by Reiter R. 1980).

Default logic is an extension of classical logic by non-standard inference rules (these rules allows one to express default properties)[‡].

Example 2.29 (*default rules*)

Consider the following inference rule: $\dfrac{bird(\mathrm{x}) : fly(\mathrm{x})}{fly(\mathrm{x})}$. This rule is interpreted as follows: '*if* x *is a bird and we can consistently assume that* x *flies then we can infer that* x *flies*'. A generalised form of the last rule can be represented as follows: $\dfrac{\mathrm{A(x)} : \mathrm{B(x)}}{\mathrm{C(x)}}$ and this rule can be interpreted as: '*if* A(x) *holds and* B(x) *can be consistently assumed then we can conclude* C(x)'. Here: A(x) is said to be the *prerequisite*, B(x) - the *justification* and C(x) - the *consequent*[§]. □ (Olivetti N. 2010 / 11).

A *default theory* (or more exactly: *theory of default reasoning*, considered as a deductively closed set of logical formulae)[**] is a pair (D,W), where D is a (non-empty) set of *default rules*[††] and W is a (non-empty) set of *first-order predicate logic formulae*. The set W represents the stable (but incomplete) *knowledge of the world* and D represents *rules for extending the knowledge* W by plausible (but defeasible) conclusions. The theory obtained by extending W wrt rules in D is said to be an *extension* of (D,W). It was shown that this theory may have zero, one or many extensions (Olivetti N. 2010 / 11). In particular, the following definitions were given.

Definition 2.33 (*default theory extension: propositional case*)

A set of formulae E is an *extension* of $\Delta =_{df}$ (D,W) if E is deductively closed, i.e. E = Th(E) and all applicable default rules wrt E have been applied, i.e. for each $\dfrac{\mathrm{A(x)} : \mathrm{B(x)}}{\mathrm{C(x)}} \in \mathrm{D} : \mathrm{A(x)} \in \mathrm{E} \wedge \sim \mathrm{B(x)} \notin \mathrm{E} \Rightarrow \mathrm{C(x)} \in \mathrm{E}$.

Definition 2.34 (*default theory extension: semi-inductive definition*[‡‡])

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[*] An important difference between *knowledge bases* and *databases* is that the former require a semantic theory for the interpretation of their contents, while the latter require a computational theory for their efficient implementation on physical machines (Brodie M.L. and Mylopoulos J. eds. 1986).

[†] In general: the *frame problem*: using first-order predicate logic for expressing facts about (behaviour of) a robot in the world and hence the problem of finding adequate *frame axioms*.

[‡] *Default logic* can express facts like "by default, something is true"; by contrast, standard (i.e. classical) logic can only express that something is true or that something is false. This is a problem because reasoning often involves facts that are true in the majority of cases but not always. A classical example is: "birds typically fly". This rule can be expressed in standard logic either by "all birds fly", which is inconsistent with the fact that penguins do not fly, or by "all birds that are not penguins and not ostriches and ... fly", which requires all exceptions to the rule to be specified. Default logic aims at formalizing inference rules like this one without explicitly mentioning all their exceptions (T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*).

[§] Provided there is no ambiguity and for convenience, instead of the originally used α(x), β(x) and γ(x), these one argument predicates are here denoted by A(x), B(x) and C(x), respectively: as in (Słupecki J. and Borkowski L. 1967).

[**] In general, any (*formal*) *theory* is a set of sentences (i.e. *formulae* φ) in a *formal language* L. Let $\Delta =_{df}$ (D,W). Then Δ can be considered as a deductively closed set of logical formulae, more formally: Δ = Th(Δ), where $\mathrm{Th}(\Delta) =_{df} \{\varphi \in \mathrm{L} / \Delta \vDash \varphi\}$. This notion is very similar to Tarski's third axiom: Cn(Cn(A)) = Cn(A), introduced in classical logic systems (see: Subsection 1.5).

[††] In short: *defaults*.

[‡‡] There exists a reference to the whole E in step(2).

A set of formulae E is an *extension* of $\Delta =_{df} (D,W)$ if it can be obtained as below.

(1) $S_0 =_{df} W$,

(2) $S_{i+1} =_{df} Th(S_i) \cup \{C(x) \,/\, \frac{A(x) : B(x)}{C(x)} \in D, A(x) \in S_i, \sim B(x) \notin E\}$ and

(3) $E =_{df} \bigcup_i S_i$.

In particular, the following example was given in (Olivetti N. 2010 / 11): provided there is no ambiguity, instead of b, p and f below are used the propositional variables p, q and r, respectively..

Example 2.30 (*semi-inductive extension*)

Let $\Delta =_{df} (\{p, q \Rightarrow \sim r\}, \{\frac{p:r}{r}\})$. There exists an unique extension $E = Th(\{p, q \Rightarrow \sim r, r\})$ such that: $S_0 =_{df} \{p, q \Rightarrow \sim r\}$ and $S_1 = S_0 \cup \{r\}$, since $S_0 \models p$ and $\sim r \notin E$. □

Let d be a default rule. We shall say that $d \in D$ is *normal* if has the form: $\frac{A(x) : B(x)}{B(x)}$ and that Δ is a *normal default theory* if all $d \in D$ are normal. The following property is satisfied (Olivetti N. 2010 / 11).

Theorem 2.1

Let Δ be a normal default theory. Then Δ has always an extension. □

At the conclusion of the above considerations,, it can be observed that the Achilles' heel of any such logic seems to be the notion of *justification* (B(x): *can be consistently assumed*): the process of justification and hence the obtained sufficiency of this justification may be subjective or not sufficient.

In particular, some moral aspects are related to deontic logic (Horty J.F. 1994)*. In general, the importance of this logic depends on the area of application and the boundary conditions in each particular case. A more formal treatment and/or study is omitted here: left to the reader.

Fractal logic

Nature observations and inferences are basically changing, turning to multimode, temporality and complexity. *Fractals* (lat: "frāctus", meaning "broken" or "fractured"), first introduced by Mandelbrot in 1975† (Benoit B. Mandelbrot: 1924 – 2010), are abstract objects used to describe and simulate naturally occurring objects. In general, fractals can be grouped in the following three categories (depending on how they are defined and generated): *iterative*, *recursive* and *random* ones. Moreover, in accordance with their self-similarity property, fractals can be classified as having: *exact*, *quasi* and *stochastic* self-similarity. It can be observed that there does not exist an object of type 'exact fractal' in nature. On the other hand, many such objects have properties very similar to these concerning fractals (however, in a bounded version). Artificially created fractals commonly exhibit similar patterns at increasingly small scales. Fractals are not readily definable. But the following definition (related to general measure theory) covers many classes of fractals (Bjorvand A.T. 1995).

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* This paper is to establish some formal connections between deontic and nonmonotonic logics, and to suggest some ways in which the techniques developed in the study of nonmonotonic reasoning and the issues confronted there might help to illuminate deontic ideas. These two subjects have evolved within different disciplines. The field of deontic logic was developed by philosophers and legal theorists as a high level framework for describing valid patterns of normative reasoning. The study of nonmonotonic logic was initiated, much more recently, by researchers in artificial intelligence who felt that ordinary logical techniques could not be applied properly to a number of practical problems arising within that area---most notably, problems involving planning and action, such as the frame problem. By linking the subject of deontic logic to this research, it may be possible also to relate the idealized study of moral reasoning typical of the field to a more robust treatment of practical deliberation.

† Objects, in nowadays called "fractals" were known as far back as 1872: the *Weierstrass function* (Karl Theodor Wilhelm Weierstrass: 1815 – 1897). Like fractals, this function exhibits self-similarity and has the property of being continuous everywhere but differentiable nowhere.

Definition 2.35 (Mandelbrot B.B. 1983)

A *fractal* is a set for which the *Hausdorff - Besicovith dimension*[*] strictly exceeds the *topological dimension*[†].

Besides the fractal computer graphics, e.g. (Pickover C.A. ed. 1998), fractal modelling is suggested e.g. in quantum physics and cosmology (El-Naschie M.S. 2016), in classical parallel computing (Craus M. et al. 2016), in intelligent systems theory (Bjorvand A.T. 1995), nonlinear dynamics, chaos theory and complex systems modelling, data compression and modelling, use of recursion to determine logical operations and circuits (http://www.mathpages.com/home/kmath140/kmath140.htm), and so on. In particular, the following main areas of investigation were considered in (Bjorvand A.T. 1995): the modelling of the processes of the human brain, using rough set theory for decision support based on fractal models and also using the genetic algorithms for the evolution or learning within these models (left to the reader).

Universal logic is the field of logic that studies the common features of all logical systems, aiming to be to logic what universal algebra is to algebra. A number of approaches to universal logic have been proposed since the twentieth century, using model theoretic, and categorical approaches. As a precursor of this logic it is reckoned Alfred Tarski (1901 – 1983). A more formal approach was given in the 1990s by Jean-Yves Béziau, e.g. see: (Béziau J-Y. 2007). The term 'universal logic' was also separately used by Richard Sylvan (1935 – 1996) and Ross Brady, see (Brady R.T. 2006: to refer a new type of *weak relevance logic*)[‡].

In accordance with the contemporary requirements in computer science, information technology and artificial intelligence, universal logic was born which stress on the relational flexibility of different objects (tasks, propositions, connectives). This logic is considered as a common architecture of logical theory system, which is *self-contained* (puts up the common characteristics of all kinds of logic and also gives an application oriented logic builder) and *exoteric* (the possibility of joining a new logical system into its frame). And this logic puts forward the following essential features of all kinds of logics: propositional connectives, relational quantifier, set of common rules and appropriate inferential model. An exploration of fractal and chaos logics based on universal logic is briefly presented below (Chen Z-H. et al. 2010). In this paper there are considered the basic characteristics of fractal phenomenon and the correlation of fractal, chaos and universal logic. Next, there are fully discussed the necessary and probability of setting up fractal logic and chaos logic. And finally, there is given a compendium of fractal logic and the way of the implementing of chaos logic.

In accordance with the last work, the *characteristics* of classical logic systems, some non-standard logics, e.g. many-valued logic or any other '*rigid*' application of formal logic in mathematics can be summarised as follows: an *integer truth-value* of any proposition, *correlation-independent* propositions (i.e. every proposition is independent from others: otherwise, the inferential march becomes difficult) and the *uniqueness* of the operating model *of logical connectives* (the operating rules of connectives are strictly defined and cannot be changed in the process of reasoning).

The basic *features* and *relationship* of chaos and fractal can be summarised as follows (Chen Z-H. et al. 2010):

Chaos:	internal randomicity, fractional dimension, orderliness in disorder,
Fractal:	self-similar, infinite complex and subtle, fractional dimension and
Relationship:	from chaos to order, from order to similitude, from similitude to fractal.

According to the last work, the following four *factors* of the fractal logic can be considered as important:

Research field:	all things that have fractal features, logical inferential system based on fractal mathematics,
Proposition connectives:	introduction of basic proposition connectives and their operation models,

[*] Felix Hausdorff (1868 – 1942), Abram Samoilovitch Besicovith (1891 – 1970): for a more information see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[†] Known also as: *Lebesgue covering dimension* (Henri Lebesgue: 1875 – 1941).

[‡] T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

Relational quantifier:[*]	the generalised quantifiers are useful in fractal logic and
Rules and inference model:	to construct a set of common rules and appropriate reasoning model.

The notion of *fractal dimension* is of fundamental sense. This notion is used in definitions of the propositional connectives of fractal logic. The concept of fractal dimension involves unconventional views of *scaling* and *dimension*. Usually, these two terms are illustrated by using traditional notions of geometry. This is briefly presented below.

Let n, s and d be the *number of sticks*, the *scaling factor* and the *dimension* of a fractal, respectively. It can be observed that assuming d = 1, 2, 3 (a geometric line, quadrate and cube) and e.g. s = 1/3 = *const*, the number of sticks n increases along with increasing the number d: n = 3, 9, 27. In fact: $n \propto s^{-d}$ ('$\propto$': denotes: '*is proportional to*')[†]. And hence, the value of d can be found by rearranging this proportionality as follows: $\lg_s(n) = -d$. So, $d = -\frac{\lg(n)}{\lg(s)}$, e.g. for n = 4 and s = 1/3 we can obtain:

$$d = -\frac{\lg(4)}{\lg(1/3)}$$

$$\approx \frac{1.386294}{1.098612}$$

$$\approx 1.26186. \ \square$$

In general, the concept of fractal dimension is more complicated, based on such notions as e.g. approximation, estimation, regression, etc. A more formal treatment is omitted here. Different types of fractal dimension are the following (*The Free Encyclopaedia, The Wikimedia Foundation, Inc.*): *box counting* (or *capacitive*) *dimension*[‡] (Hermann Minkowski: 1864 – 1909), *information dimension* (Alfréd Rényi: 1921 – 1970), *correlation dimension*, *generalised* or *Rényi dimensions*, *Higuchi dimension*, *multifractal dimensions*, *uncertainty exponent*, *Hausdorff dimension*[§] (Felix Hausdorff: 1868 – 1942), *packing dimension* (in some sense dual to Hausdorff), A*ssouad dimension, local connected dimension*, etc.: left to the reader.

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[*] Known also as *polyadic quantifiers*: having several *argument-places*. As a precursor of *generalised quantifiers* it is reckoned Andrzej S. Mostowski (1957: 1913 – 1975): it was suggested that we might generalise the standard (*predicative*) notion of logical quantifier along two dimensions, syntactic and semantic (Sher G. 2015). A more formal treatment is given in the next chapter of this work: see Subsection 3.8.

[†] For convenience, instead of the original N, $\in$ and D, there are used n, s and d, respectively

[‡] Let A be a *compact set* in the *metric space* (M, ρ) and $N_\varepsilon(A)$ be the minimal number of sets having *diameter* ≤ ε, required for *covering* A. The *upper* and the *lower capacitive dimensions* of A are defined as follows:

$\overline{d}_c(A) = \lim_{\varepsilon \to 0} \sup \frac{\ln N_\varepsilon(A)}{\ln \frac{1}{\varepsilon}}$ and $\underline{d}_c(A) = \lim_{\varepsilon \to 0} \inf \frac{\ln N_\varepsilon(A)}{\ln \frac{1}{\varepsilon}}$. If $\overline{d}_c(A) = \underline{d}_c(A) = d_c(A)$ then $d_c(A)$ is said to be *capacitive dimensions* of A (Bronstein I.N.. et al. 2001). A similar result was obtained by Georges Louis Bouligand (1889 – 1979).

[§] Hausdorff dimension is based on measure theory and defined on the ground of Lebesgue measure (Henri Lebesgue: 1875 – 1941). Let $A \subsetneq \mathbb{R}^3$. It is required a *cover* of A by a finite number of spheres B_{r_i} with radius $r_i \le \varepsilon$ such that $\bigcup_i B_{r_i} \supsetneq A$. So, roughly, the volume of A is defined as: $\sum_i \frac{4}{3}\pi r_i^3$. Let now $\{\sum_i \frac{4}{3}\pi r_i^3 \,/\, r_i \le \varepsilon\}$ be the set of all such finite covers of A and $\eta_\varepsilon(A) =_{df} \inf\{\sum_i \frac{4}{3}\pi r_i^3 \,/\, r_i \le \varepsilon\}$. The *external Lebesgue measure* $\overline{\lambda}(A) =_{df} \lim_{\varepsilon \to 0} \eta_\varepsilon(A)$. If A is measurable then $\overline{\lambda}(A) = vol(A)$. Let now M be $\mathbb{R}^n$ or in general: a *separable* metric space (contains a *countable* and *dense* subset, e.g. the set of rational numbers $\mathbb{Q}$ is both countable and dense in $\mathbb{R}$: in general, in a metric space (X, ρ), $Y \subsetneq X$ is a *dense* set if $\forall_{x \in X} \forall_{\varepsilon > 0} \exists_{y \in Y} (\rho(x,y) < \varepsilon)$). Assume that $A \subsetneq M$. For any parameter $d \ge 0$ and $\varepsilon > 0$, we can define $\eta_{d,\varepsilon}(A) =_{df} \inf\{\sum_i (diam\ B_i)^d \,/\, A \subsetneq \bigcup_i B_i,\ diam\ B_i \le \varepsilon\}$, where $B_i \subsetneq M$ are arbitrary having $diam\ B_i =_{df} \sup_{x, y \in B_i} \rho(x, y)$. The *external*

The *true-value field of proposition* is considered as a multi-dimension and super order space, based on $[0,r]^d$ (the *radix* or *base* space, $r \in \mathbb{R}$, d - *fractal dimension*). Moreover, it is also used some (proposition or predication) *accessorial characteristics* ´< α >´. Some operation models of the *proposition connectives* were also presented, such as: *negation*, *conjunction*, *disjunction*, *implication*, *equivalence* and also three new connectives, i.e. the *average*, *combinatorial* and *series* ones: see below (Chen Z-H. et al. 2010).

Connective	*Denotation*	*Operation model*
negation: Not	$\sim_k$	$N(x,k) =_{df} D(r - D^{-1}(x))$
conjunction: And	$\wedge_h$	$A(x,y,h) =_{df} \Gamma^r(D((D^{-1}(x)^m + D^{-1}(y)^m - r)^{1/m}))$
disjunction: Or	$\vee_h$	$O(x,y,h) =_{df} \Gamma^r(D(r - ((r - D^{-1}(x))^m + (r - D^{-1}(y))^m - r)^{1/m}))$
implication: I	$\Rightarrow_h$	$I(x,y,h) =_{df} \Gamma^r(D((r - D^{-1}(x)^m + D^{-1}(y)^m)^{1/m}))$
equivalence: E	$\Leftrightarrow_h$	$E(x,y,h) =_{df}$ *ite*$\{D((r + \lvert D^{-1}(x)^m - D^{-1}(y)^m \rvert)^{1/m}) \;/\; m \le 0 \;;\; D((r - \lvert D^{-1}(x)^m - D^{-1}(y)^m \rvert)^{1/m})\}$
average: V	Ⓟ$_h$	$V(x,y,h) =_{df} D(r - (((r - D^{-1}(x))^m + (r - D^{-1}(y))^m) / 2)^{1/m})$
combinatorial: C	©^e_h	$C^e(x,y,h) =_{df}$ *ite*$\{\Gamma^e(D((D^{-1}(x)^m + D^{-1}(y)^m - e^m)^{1/m})) \;/\; D^{-1}(x) + D^{-1}(y) \le 2e \;;\; D(r - (\Gamma^{r-e}((r - D^{-1}(x))^m + (r - D^{-1}(y))^m - (r - e)^m)^{1/m})) \;/\; D^{-1}(x) + D^{-1}(y) > 2e\}$
series: R	Ⓡ$_h$	$R(x,y,h) =_{df} D(r - (((r - D^{-1}(x))^{mn} + (r - D^{-1}(y))^{mn}))^{1/mn})$

According to the above table: $D(x)$ calculates the dimension of fractal object ´x´, $D^{-1}(x)$ constructs the fractal object with dimension ´x´ ($D(x)$ and $D^{-1}(x)$ denote two reverse operating processes, but not the converse mathematical operators). By h and k are denoted the *general correlative* and *self-correlative coefficients* (Huacan H. 2001), $m =_{df} \frac{3-4h}{4h(1-h)}$ and n denotes the *dispersion* of series. It is assumed that any argument x of $\Gamma^r(x)$ is restricted in [0,r] (i.e. if $x > r$ then $x =_{df} r$ / if $x < 0$ then $x =_{df} 0$). The used abbreviation "*ite*" denotes: *if ... then ... else*, e.g. $s =$ *ite*$\{b / \alpha ; c\}$ denotes: '*if* α is true, *then* $s = b$, *else* $s = c$[*].

The above introductory notions are only an illustration of this excellent work. However, in general, the introduction of inference rules and proof methods (*axiomatic system proof* / *assumptional system style*) seems to be more difficult and at the same time the most important task in this logic (e.g. if *De Morgan's law* ´$\sim_k(x \wedge_h y) \Leftrightarrow_h \sim_k x \vee_h \sim_k y$´ is satisfied: left to the reader). And so, much remains to be done.

– ∵ –

In this chapter were considered classical and also various non-standard logic systems. However, all these systems are not sufficient for describing all theoretical as well as practical computer science oriented research. In particular, it is required a ´*knowledge base*´ concerning also such notions as: *predicates*, *sets*, *algebraic systems, morphisms*, *graphs*, and so on. Predicates are presented in the next chapter.

Hausdorff dimension wrt A and (the dimension) d is defined as: $\eta_d(A) =_{df} \lim_{\varepsilon \to 0} \eta_{d,\varepsilon}(A) = \sup_{\varepsilon > 0} \eta_{d,\varepsilon}(A)$. And hence, *Hausdorff dimension* $d_H(A)$ is an univocal critical value of Hausdorff measure defined as follows: $d_H(A) =_{df}$ if $\forall_{d \ge 0} (\eta_d(A) \ne 0)$ then ´$+\infty$´ else $\inf\{d \ge 0 \;/\; \eta_d(A) = 0\}$ (Bronstein I.N.. et al. 2001).

[*] For convenience, proposition letters are here denoted by x, y, ... , as in other logic systems, e.g. *modal μ-calculus*.

II. Predicates

Use of predicates in mathematics (mathematical analysis or discrete mathematics) and theoretical computer science is of fundamental sense. In particular, we have a possibility of an exact / strict description as well as study of various definitions, processes, models, etc. Example notions involving predicates may be the following: *limit of a sequence* (*Cauchy's ε-definition*), *continuous function* (*Weierstrass' definition*), *continuous t-norm*, *metrical spaces*, *magic graphs*, *bounded*, *live* and *reversible Petri nets*, *Petri net k-distinguishability* and *D-partition*, *algebraic systems* and *direct products* of such systems, *test generation* for logic circuits, *D-algebra* and so on.

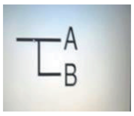
The first-order predicate calculus is initially presented below (Słupecki J. and Borkowski L. 1967). Some well-known basic notions related to the classical predicate logic are first introduced. As an illustration, by using such formulae, some example mathematical and/or computer science definitions are also described and a set of primitive rules is then presented. Next, a carefully selected subset of theses is proved. The corresponding formal proofs are based on assumptions. The notion of the existential uniqueness quantifier is next presented and some properties are also given. The Gentzen's sequent calculus is also illustrated. As in the previous chapter, the corresponding rules are proved. Some new proofs and/or theses, mainly concerning bounded (or equivalently: restricted) quantifiers, are also given. In the next considerations the skolemisation, resolution and interpretation techniques are discussed (Chang C.-L. and Lee R.C.-T. 1973). The resolution technique is considered as a construction of ramified indirect proofs with joined additional assumptions and natural numbers are used in the formula interpretation rules. Next, the higher order predicate logic is briefly considered. Basic notions related to the genralised quantifier theory are also presented, mainly under (Pogonowski J. and Smigerska J. 2008). Some non-classical systems, such as fuzzy, modal and temporal predicate calculus are considered in the second section of this chapter.

3. Classical first-order and higher order predicate logics

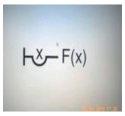
Predicate calculus known also as *predicate logic*, *quantifier calculus* or *quantifier logic* is an extension of propositional calculus. Expressions of predicate calculus may include additional symbols as well as additional primitive rules, theses and derived rules.

Modern quantifiers were first introduced by Frege G. (1879)*. The quantifiers *´All´* and *´Some´* were already recognised as logical operators by Frege's predecessors, going all the way back to Aristoteles (384 b.c. – 322 b.c.), e.g. Aristoteles' syllogisms, operators like *´and´* and *´or´*: the ancient Stoic philosophers, in the middle ages (much effort to the semantics statements, essentially restricted to syllogistic form): William of Ocham (1285 – 1349: *Summa Logicae* 1320), Albert of Saxony (1320 – 1390: *Perutilis Logica Magistri Alberti de Saxonia Logic*, Venice 1522 and Hildesheim 1974, reproduction). Quantifiers in beginning predicate logic were studied mainly by Peirce, Peano and Russell†, see: (Sher G. 2015, Peters S. and Westerståhl D. 2002).

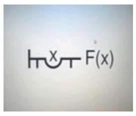
$\sim A$



$B \Rightarrow A$



$\forall_x F(x)$



$\exists_x F(x) \Leftrightarrow_{df} \sim \forall_x \sim F(x)$

Figure 3.1 Frege's designations

* Friedrich Ludwig Gottlob Frege (1848 – 1925).

† Charles Sanders Peirce (1839 – 1914), Giuseppe Peano (1858 – 1932), Bertrand Arthur William Russell (1872 – 1970).

A more simplified form of the above Frege's designations, used for quantifiers (universal and existential), was first given by Peirce (about 1885): $\prod_x F(x)$ and $\sum_x F(x)$, respectively. Some other designations were also proposed. The most usable are these ones, given by Hilbert and also by Kuratowski*. The quantifier symbols, given by Hilbert, are shown in Figure 2.6 (on the right hands of the corresponding pictures: related to German words *´Ales´* and *´Eine´*, respectively). The following designations were proposed by Kuratowski: $\bigwedge_x F(x)$ and $\bigvee_x F(x)$: are based on Peirce interpretation / analogy concerning quantifiers, i.e. $\forall_x F(x) \backsim F(x_1) \wedge F(x_2) \wedge \ldots$ and $\exists_x F(x) \backsim F(x_1) \vee F(x_2) \vee \ldots$; in fact, the universe can be finite or not, but ´$\wedge$´ and ´$\vee$´ are finite argument logical connectives (equivalence is obtained only if universe is finite). There was also proposed the use of *´All´* and *´Exist´*, and so on (see p.9: The used designations). However, the most commonly-used seems to be Hilbert's designations. And so, these designations are used in this book.

3.1. Symbols and formulae

Any expression of predicate logic, in addition to propositional logic symbols, may also include:

(1) *individual* (known also: *quantified*) *variables*: $x, y, z, x_1, y_1, z_1, \ldots$, replaceable by names of individuals,

(2) variables representing (proposition generating) functors and having names as arguments, i.e. expressions generating propositions wrt some names. Any such functor is said to be (one-or-more, but finite argument) *predicate*. By $A, B, C, A_1, B_1, C_1, \ldots$ we shall denote variables representing one argument predicates. The letters $P, Q, R, S, P_1, Q_1, R_1, S_1, \ldots$ are reserved for more than one argument predicates and

(3) two constants: $\forall, \exists$, i.e. the *universal* and *existential quantifiers*. These two most common quantifiers mean: "*for all*" and "*there exists*", respectively.

As an example, a one argument predicate is the phrase "*is even number*" in the following proposition: "6 is even number". Similarly, a two argument predicate is the phrase "*is greater than*" in: " 6 is greater than 5" (i.e. $6 > 5$). In general, any one argument predicate express a *property*. A more than one argument predicate express some *relation* (Carnap R. 1954)†.

The above presented symbols (1 – 3) together with the symbols of the propositional calculus exhaust all symbols considered in classical first-order predicate logic‡. The following inductive definition is generalisation of Definition 1.1, given in Subsection 1.1 (Słupecki J. and Borkowski L. 1967).

Definition 2.36 (*expression of predicate logic*)

(1a) propositional variables are propositional expressions of predicate logic ,

(1b) expressions obtained by variable representing n argument predicates and the successive sequence of n individual variables ($n \geq 1$), included in parentheses ,

(2a) If φ and ψ are some expressions of predicate logic, then such expressions are also: $\sim(\varphi)$, $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $(\varphi) \Rightarrow (\psi)$, and $(\varphi) \Leftrightarrow (\psi)$,

* David Hilbert (1862 – 1943), Kazimierz Kuratowski (1896 – 1943).

†Rudolf Carnap (1891 – 1970).

‡ Provided there is no ambiguity, instead of '*classical first-order predicate logic*', for simplicity, there is used phrase '*predicate logic*'.

(2b) If φ is an expression of predicate logic, while a is an individual variable, then $\underset{a}{\forall}(\varphi)$ and $\underset{a}{\exists}(\varphi)$ are also such expressions and

(3) Every expression of predicate logic either is a propositional variable or *molecular expression*, i.e. an expression formed from rule (1b), or also formed from these basic expressions by a single or multiple application of rules (2a) and (2b).

Example 2.31 (*molecular expression*)

Some example *molecular* (called also *atomic*) *expressions* are the following: A(x), B(y), $C(x_1)$, $Q(x_1, x_2, \dots, x_n)$, R(x,y), etc. In the last case, R(x,y) is sometimes written as: xRy. In accordance with *Carnap's interpretation*, e.g. the molecular expression A(x) denotes that: A is a *property* of x. Similarly xRy denotes that: R is a binary relation between x and y. For convenience, instead xRy, it is also used the notion: xρy (x and y are in ρ). □

According to Definition 2.36(2b), if φ is a molecular expression the parentheses are omitted, e.g. instead of $\underset{x}{\forall}(A(x))$, we have: $\underset{x}{\forall}A(x)$ .Similarly, we have: $\underset{x}{\forall}\underset{y}{\exists}xRy$, instead of: $\underset{x}{\forall}(\underset{y}{\exists}xRy)$, etc. However, in a more complicated expressions some additional parentheses may be useful.

The *scope* of a quantifier is the portion of the formula that is controlled or governed by the quantifier. As an example, the scope of ′∃′ in $\underset{x}{\forall}\underset{y}{\forall}(xRy \Rightarrow \underset{z}{\exists}(xRz \wedge zRy))$ is expression ′$xRz \wedge zRy$′, the scope of ′∀′wrt y is ′$xRy \Rightarrow \underset{z}{\exists}(xRz \wedge zRy)$′ and the scope of ′∀′wrt x is ′$\underset{y}{\forall}(xRy \Rightarrow \underset{z}{\exists}(xRz \wedge zRy))$′.

A variable occurring in a quantifier and in a propositional function within the scope of the quantifier is said to be a *bound variable*. Some variable of φ not occurring under the quantifier is a *free variable* iff it is not bound. Here, a *propositional* (or *sentential*) *function* is an expression including free variables and from which we can obtain propositions after substituting all variables by constants. The notion of propositional function was introduced by Russell B. (in 1903, see: Russell B. 1938).

Example 2.32 (*bound and free variables, propositional function*)

(1) The variable x is free in the expression: $\underset{y}{\exists}xRy$. Consider the following expression: $\underset{x}{\forall}A(x) \Rightarrow A(x)$. Now, the variable x occurring on the left side of implication is bound, but x on the right side of this implication is free.

(2) Some example propositional functions having one, two or three free variables may be the following: "x *is even number*", "x *is greater than* y" and "x is the *greatest common divisor of two integers* y and z", e.g. "6 *is even number*" (the proposition's logical value is true), "5 *is even number*" (the proposition's logical value is *false*), etc. The mathematical equations are well-known examples of propositional functions. The term "*condition*" is often used as a such function.

(3) Consider the following expression: $\underset{x}{\forall}\underset{y}{\exists}xRy$. By substituting "xRy" for the propositional function "x = y + 1" we can obtain the proposition: $\underset{x}{\forall}\underset{y}{\exists}(x = y + 1)$. □

It can be observed that propositional functions correspond to some expressions, but mathematical functions correspond to some relations.

In addition to the above presented two kinds of quantifiers, there exist also bounded quantifiers. These quantifiers are often used as an abbreviated description of various definitions and mathematical theorems. As an example, $\underset{x>0}{\forall} |x| = x$ and $\underset{x\neq 0}{\exists} a \cdot x = a$ are abbreviations of the following two expressions: $\underset{x}{\forall}(x > 0 \Rightarrow$

$|x| = x)$ and $\exists_{x} (x \neq 0 \wedge a \cdot x = a)$. In general, the following two *bounded quantifiers* can be introduced (Słupecki J. and Borkowski L. 1967): $\forall_{\varphi(x)} \psi(x) \Leftrightarrow_{df} \forall_{x} (\varphi(x) \Rightarrow \psi(x))$ and $\exists_{\varphi(x)} \psi(x) \Leftrightarrow_{df} \exists_{x} (\varphi(x) \wedge \psi(x))$. The use of these two quantifiers is illustrated in the next example.

Example 2.33 (*some mathematical / computer science definitions*)

limit of a sequence	$\lim_{n\to\infty} a_n = g \Leftrightarrow_{df} \forall_{\varepsilon > 0} \exists_{m} \forall_{n > m} \lvert a_n - g \rvert < \varepsilon$
continuous function	f(x) is *continuous* $\Leftrightarrow_{df} \forall_{\varepsilon > 0} \forall_{x} \forall_{y} \exists_{\delta > 0} (\lvert x - y \rvert < \delta \Rightarrow \lvert f(x) - f(y) \rvert < \varepsilon)$
fault detecting test	$\forall_{x \in U} \forall_{\alpha \in F} (x$ *detects* $\alpha \Leftrightarrow_{df} \exists_{j \in K} (f_j(x) \neq f_j^{\alpha}(x)))$
bounded, live and reversible P/T net	The *Petri net* N is *bounded* iff $\exists_{n} \forall_{p \in P} \forall_{M \in [M_0\rangle} (M(p) \leq n)$
	The *Petri net* N is *live* iff $\forall_{t \in T} \forall_{M \in [M_0\rangle} \exists_{M' \in [M\rangle} (t \in T(M'))$
	The *Petri net* N is *reversible* iff $\forall_{M \in [M_0\rangle} (M_0 \in [M\rangle)$

In particular, *boundedness*, *liveness* and *reversibility* are the three most important and required properties in modelling of *discrete event systems*. These properties are independent of each other and generalised also for High-level Petri* nets (see: *High-level Petri Nets* 2000, 2005). □

3.2. Primitive rules

The set of primitive rules in predicate logic consists of:

(1) All primitive rules, i.e. − C, ± K, ± A, and ± E, of the classical propositional calculus, generalised for use in predicate logic. Now the symbols φ and ψ, engaged in the above rule schemes, represent expressions of predicate logic. Similarly are interpreted $\phi_1, \phi_2, \ldots, \phi_n$ occurring in the generalised form of an expression (given in Subsection 1.2). The process of joining new lines (by using some primitive or derived rules and/or other theses in accordance with the used assumptions) is extended to predicate logic.

(2) Rules of joining and omitting the universal and existential quantifiers: $\pm\forall, \pm\exists$.

In accordance with $\pm\forall, \pm\exists$, it is used a special symbol $\varphi(x/\xi)$ denoting an expression formed from φ by substituting the *individual variable* x for the *expression* ξ: in all places where x is free. Moreover, if x is in the scope of a quantifier $Q \in \{\forall, \exists\}$ and y is a variable occurring in Q then ξ should not depend on y. The quantifier rules are given below: we shall require in (2) that x is not free in the corresponding proof assumptions†.

(1) *Rule of omitting an universal quantifier*
(denoted below by ′−∀′):

* Carl Adam Petri (1926 – 2010).

† Consider the following thesis: $x > 0 \Rightarrow x + y > y$ ($x,y \in \mathbb{R}$). Assume that: $x > 0$. By using ′+ A′ we can obtain: ′$x > 0 \vee x = 0$′. However, this is a contradiction with our assumption about x. And hence, the obtained expression: $\forall_{x} (x > 0 \vee x = 0)$ should be not correct (Słupecki J. and Borkowski L. 1967).

$$-\forall: \quad \frac{\forall_{x} \varphi(x)}{\varphi(x/\xi)}$$

(2) *Rule of joining an universal quantifier* (denoted below by ´+∀´)*:

$$+\forall: \quad \frac{\varphi(x)}{\forall_{x} \varphi(x)}$$

(3) *Rule of joining an existential quantifier* (denoted below by ´+∃´):

$$+\exists: \quad \frac{\varphi(x/\xi)}{\exists_{x} \varphi(x)}$$

(4) *Rule of omitting an existential quantifier* (denoted below by ´−∃´):

$$-\exists: \quad \frac{\exists_{x} \varphi(x)}{\varphi(x/\xi_{\beta_1,\beta_2,\ldots,\beta_n})}$$

Example 2.34 (*the rule ´−∃´*)

Consider the expression $\exists_{y} P(x,y,z)$: "*for any two different points x and z, there exists a point y that lies on the line defined by x and z*". By omitting this quantifier, we can obtain the expression: $P(x,a_{x,z},z)$. And hence, the point $a_{x,z}$ is associated with x and z (Słupecki J. and Borkowski L. 1967). In fact, the use of *Skolemian type functions*† is also possible (see Subsection 3.5).

Let now x, y and z be three points that are not *collinear* (i.e. they do not all lie on a single line). Give an example use of the rule ´−∃´: left to the reader. ▫

The bounded quantifier rules are presented below.

(5) *Rule of omitting a bounded universal quantifier* (denoted below by ´−∀*´):

$$-\forall^*: \quad \frac{\forall_{\varphi(x)} \psi(x)}{\varphi(x/\xi) \Rightarrow \psi(x/\xi)}$$

(6) *Rule of joining a bounded universal quantifier* (denoted below by ´+∀*´):

$$+\forall^*: \quad \frac{\varphi(x) \Rightarrow \psi(x)}{\forall_{\varphi(x)} \psi(x)}$$

(7) *Rule of joining a bounded existential quantifier* (denoted below by ´+∃*´):

$$+\exists^*: \quad \frac{\varphi(x/\xi) \quad \psi(x/\xi)}{\exists_{\varphi(x)} \psi(x)}$$

---

* Known also as *rule of generalisation*: x should not be free in the corresponding proof assumptions.

† Thoralf Skolem (1887 – 1963).

(8) *Rule of omitting a bounded existential quantifier* (denoted below by ′$-\exists^*$′):

$$-\exists^*: \quad \frac{\underset{\varphi(x)}{\exists}\, \psi(x)}{\begin{array}{l}\varphi(x/\xi_{\beta_1,\beta_2,\dots,\beta_n})\\ \psi(x/\xi_{\beta_1,\beta_2,\dots,\beta_n})\end{array}}$$

3.3. Theses and derived rules

Thesis 2.132

$\underset{x}{\forall} A(x) \;\Rightarrow\; A(y)$

Proof:

(1) $\underset{x}{\forall} A(x)$ {a}

$A(y)$. □ $\{-\forall : 1\}$

Thesis 2.133

$A(y) \;\Rightarrow\; \underset{x}{\exists} A(x)$

Proof:

(1) $A(y)$ {a}

$\underset{x}{\exists} A(x)$. □ $\{+\exists : 1\}$

Corollary 2.10

(a) $A(x) \;\Rightarrow\; \underset{x}{\exists} A(x)$

(b) $\sim A(x) \;\Rightarrow\; \underset{x}{\exists} \sim A(x)$. □ {T 2.133}

Thesis 2.134

$\underset{x}{\forall} A(x) \;\Rightarrow\; \underset{x}{\exists} A(x)$. □ {+ K, TC, − C: T 2.132, T 2.133}

An illustration of De Morgan's laws are the next two theses.

Thesis 2.135

$\sim \underset{x}{\forall} A(x) \;\Leftrightarrow\; \underset{x}{\exists} \sim A(x)$

ProofT2.135a:

(1) $\sim \underset{x}{\forall} A(x)$ {a}

(2) $\sim \underset{x}{\exists} \sim A(x)$ {aip}

(3) $A(x)$ {Toll: Coroll. 2.10b, 2}

(4)	$\forall_x A(x)$	{+∀ : 3}
	contr. □	{1,4}

ProofT2.135b:

(1)	$\exists_x \sim A(x)$	{a}
(2)	$\forall_x A(x)$	{aip}
(3)	$\sim A(a)$	{−∃ : 1}
(4)	$A(a)$	{−∀ : 2}
	contr. □	{3,4}

Thesis 2.136

$\sim \exists_x A(x) \Leftrightarrow \forall_x \sim A(x)$

ProofT2.136a:

(1)	$\sim \exists_x A(x)$	{a}
(2)	$A(x) \Rightarrow \exists_x A(x)$	{Coroll. 2.10a}
(3)	$\sim A(x)$	{Toll: 2,1}
	$\forall_x \sim A(x)$. □	{+∀ : 3}

Another possible (but indirect) proof of the last if-implication may be the following.

(1)	$\sim \exists_x A(x)$	{a}
(2)	$\sim \forall_x \sim A(x)$	{aip}
(3)	$\exists_x A(x)$	{T 2.135, SR,− N}
	contr. □	{1,3}

ProofT2.136b:

(1)	$\forall_x \sim A(x)$	{a}
(2)	$\exists_x A(x)$	{aip}
(3)	$A(a)$	{−∃ : 2}
(4)	$\sim A(x)$	{−∀ : 1}
	contr. □	{3,4}

According to T 2.135 and T 2.136, the following rules of *negating an universal quantifier* (N∀) and *negating an existential quantifier* (N∃) are obtained:

N∀: $\dfrac{\sim \underset{x}{\forall} \varphi(x)}{\underset{x}{\exists} \sim \varphi(x)}$ and N∃: $\dfrac{\sim \underset{x}{\exists} \varphi(x)}{\underset{x}{\forall} \sim \varphi(x)}$.

De Morgan's laws for bounded quantifiers are introduced as follows.

$$\begin{aligned}
\sim \underset{\varphi(x)}{\forall} \psi(x) &\Leftrightarrow \sim \underset{x}{\forall} (\varphi(x) \Rightarrow \psi(x)) && \{\text{df. } \forall^*\}\\
&\Leftrightarrow \underset{x}{\exists} (\varphi(x) \wedge \sim \psi(x)) && \{\text{N}\forall\text{, NC, SR}\}\\
&\Leftrightarrow \underset{\varphi(x)}{\exists} \sim \psi(x)\ .\ \square && \{\text{df. } \exists^*\}
\end{aligned}$$

In a similar way we can obtain.

$$\begin{aligned}
\sim \underset{\varphi(x)}{\exists} \psi(x) &\Leftrightarrow \sim \underset{x}{\exists} (\varphi(x) \wedge \psi(x)) && \{\text{df. } \exists^*\}\\
&\Leftrightarrow \underset{x}{\forall} (\sim \varphi(x) \vee \sim \psi(x)) && \{\text{N}\exists\text{, NK, SR}\}\\
&\Leftrightarrow \underset{x}{\forall} (\varphi(x) \Rightarrow \sim \psi(x)) && \{\text{CR, SR}\}\\
&\Leftrightarrow \underset{\varphi(x)}{\forall} \sim \psi(x)\ .\ \square && \{\text{df. } \forall^*\}
\end{aligned}$$

The following rules of *negating a bounded universal quantifier* $(\text{N}\forall^*)$ and *negating a bounded existential quantifier* $(\text{N}\exists^*)$ are obtained:

N∀*: $\dfrac{\sim \underset{\varphi(x)}{\forall} \psi(x)}{\underset{\varphi(x)}{\exists} \sim \psi(x)}$ and N∃*: $\dfrac{\sim \underset{\varphi(x)}{\exists} \psi(x)}{\underset{\varphi(x)}{\forall} \sim \psi(x)}$.

An illustration of the last two rules is the proof of the next thesis. We shall first present the direct version of this proof, given in (Słupecki J. and Borkowski L. 1967).

Thesis 2.137

$$\underset{A(x)}{\exists} \underset{B(y)}{\forall} R(x,y) \Rightarrow \underset{B(y)}{\forall} \underset{A(x)}{\exists} R(x,y)$$

Proof:

(1)	$\underset{A(x)}{\exists} \underset{B(y)}{\forall} R(x,y)$	{a}
(2)	A(a)	{−∃*: 1}
(3)	$\underset{B(y)}{\forall} R(a,y)$	
(4)	B(y) ⇒ R(a,y)	{−∀*: 3}
(1.1)	B(y)	{ada}
(1.2)	R(a,y)	{− C : 4, 1.1}
(1.3)	$\underset{A(x)}{\exists} R(x,y)$	{+∃*: 2,1.2}
(5)	$B(y) \Rightarrow \underset{A(x)}{\exists} R(x,y)$	{+ C : 1.1 ⇒ 1.3}

$\forall_{B(y)} \exists_{A(x)} R(x,y)$. □ $\{+\forall^*: 5\}$

ProofT2.137(indirect version):

(1)	$\exists_{A(x)} \forall_{B(y)} R(x,y)$	{a}
(2)	$\sim \forall_{B(y)} \exists_{A(x)} R(x,y)$	{aip}
(3)	$\exists_{B(y)} \forall_{A(x)} \sim R(x,y)$	$\{ N\forall^*,\ N\exists^*: 2\}$
(4)	$A(a)$	$\{-\exists^*: 1\}$
(5)	$\forall_{B(y)} R(a,y)$	
(6)	$B(b)$	$\{-\exists^*: 3\}$
(7)	$\forall_{A(x)} \sim R(x,b)$	
(8)	$B(b) \Rightarrow R(a,b)$	$\{-\forall^*: 5\}$
(9)	$A(a) \Rightarrow \sim R(a,b)$	$\{-\forall^*: 7\}$
(10)	$R(a,b)$	{– C : 6,8}
(11)	$\sim R(a,b)$	{– C : 4,9}
	contr. □	{10,11}

A particular case of the last T 2.137 is the following thesis (the proof is left to the reader).

$\exists_x \forall_y R(x,y) \Rightarrow \forall_y \exists_x R(x,y)$. □

Several theses (mainly for unbounded quantifiers) are cited below (Słupecki J. and Borkowski L. 1967). The corresponding proofs are left to the reader (here: $Q \in \{\forall, \exists\}$, Q' is a "*complement*" of Q, e.g. if $Q =_{df} \forall$ then $Q' =_{df} \exists$, and $\bullet \in \{\wedge , \vee\}$: Q and $\bullet$ are interpreted in the same manner in a given predicate formula)*.

---

* In particular, the following formulae: $\forall_x (A(x) \wedge B(x)) \Leftrightarrow \forall_x A(x) \wedge \forall_x B(x)$, $\forall_x A(x) \vee \forall_x B(x) \Rightarrow \forall_x (A(x) \vee B(x))$, $\exists_x(A(x) \vee B(x)) \Leftrightarrow \exists_x A(x) \vee \exists_x B(x)$ and $\exists_x(A(x) \wedge B(x)) \Rightarrow \exists_x A(x) \wedge \exists_x B(x)$ are satisfied using bounded quantifiers, e.g. the proof of the following thesis: $\forall_{A(x)} B(x) \vee \forall_{A(x)} C(x) \Rightarrow \forall_{A(x)} (B(x) \vee C(x))$, is given below.

(1)	$\forall_{A(x)} B(x) \vee \forall_{A(x)} C(x)$	{a}
(2)	$\sim \forall_{A(x)} (B(x) \vee C(x))$	{aip}
(3)	$\exists_{A(x)} (\sim B(x) \wedge \sim C(x))$	$\{ N\forall^*$, NA, SR : 2}
(4)	$A(a)$	
(5)	$\sim B(a)$	$\{-\exists^*$, – K : 3}
(6)	$\sim C(a)$	
(1.1)	$\forall_{A(x)} B(x)$	{ada}
(1.2)	$A(a) \Rightarrow B(a)$	$\{-\forall^* : 1.1\}$
(1.3)	$B(a)$	{– C : 4, 1.2}
	contr.	{5, 1.3}

$\forall_x (A(x) \wedge B(x)) \Leftrightarrow \forall_x A(x) \wedge \forall_x B(x)$

$\exists_x (A(x) \vee B(x)) \Leftrightarrow \exists_x A(x) \vee \exists_x B(x)$

$\forall_x A(x) \vee \forall_x B(x) \Rightarrow \forall_x (A(x) \vee B(x))$

$\exists_x (A(x) \wedge B(x)) \Rightarrow \exists_x A(x) \wedge \exists_x B(x)$

$\forall_x (A(x) \Rightarrow B(x)) \Rightarrow (Q_x A(x) \Rightarrow Q_x B(x))$

$Q_x (p \Rightarrow A(x)) \Leftrightarrow p \Rightarrow Q_x A(x)$

$Q_x (A(x) \Rightarrow p) \Leftrightarrow Q'_x A(x) \Rightarrow p$

$Q_x A(x) \bullet Q_x B(x) \Leftrightarrow Q_x Q_y (A(x) \bullet B(y))$

$\exists_x A(x) \wedge \forall_x B(x) \Rightarrow \exists_x (A(x) \wedge B(x))$

$Q_x (p \bullet A(x)) \Leftrightarrow p \bullet Q_x A(x)$

$\forall_{B(x)} (A(x) \Rightarrow p) \Leftrightarrow \exists_{B(x)} A(x) \Rightarrow p$

$\exists_{B(x)} (p \wedge A(x)) \Leftrightarrow p \wedge \exists_{B(x)} A(x)$

The following theses are also satisfied. The proof of the first one is given below (the rest proofs are left to the reader).

$\exists_x A(x) \wedge \forall_x B(x) \Leftrightarrow \exists_x \forall_y (A(x) \wedge B(y))$

$\forall_x B(x) \wedge \exists_x A(x) \Leftrightarrow \forall_x \exists_y (B(x) \wedge A(y))$

$\forall_x A(x) \vee \exists_x B(x) \Leftrightarrow \forall_x \exists_y (A(x) \vee B(y))$

$\exists_x B(x) \vee \forall_x A(x) \Leftrightarrow \exists_x \forall_y (B(x) \vee A(y))$

The proof of the first thesis (*if-implication*):

(1) $\exists_x A(x)$

(2) $\forall_x B(x)$ {a}

(3) $\sim \exists_x \forall_y (A(x) \wedge B(y))$ {aip}

(2.1) $\forall_{A(x)} C(x)$ {ada}

(2.2) $A(a) \Rightarrow C(a)$ $\{-\forall^* : 2.1\}$

(2.3) $C(a)$ $\{-C : 4, 2.2\}$

contr. □ {6, 2.3}

(4) $\forall_x \exists_y (\sim A(x) \lor \sim B(y))$ {N∃, N∀, NK, SR : 3}

(5) $A(a)$ {–∃ : 1}

(6) $\exists_y (\sim A(a) \lor \sim B(y))$ {–∀ : 4}

(7) $\sim A(a) \lor \sim B(b)$ {–∃ : 6}

(8) $\sim B(b)$ {– A : 5,7}

(9) $B(b)$ {–∀ : 2}

contr. □ {8,9}

The proof of the first thesis (*only-if-implication*):

(1) $\exists_x \forall_y (A(x) \land B(y))$ {a}

(2) $\sim (\exists_x A(x) \land \forall_x B(x))$ {aip}

(3) $\forall_y \sim A(x) \lor \exists_x \sim B(x)$ {N∃,NK, N∀, SR : 2}

(4) $\forall_y (A(a) \land B(y))$ {–∃ : 1}

(1.1) $\forall_y \sim A(x)$ {ada}

(1.2) $A(a)$ {$Q_x (p \bullet A(x)) \Leftrightarrow p \bullet Q_x A(x)$, – K : 4}

(1.3) $\sim A(a)$ {–∀ : 1.1}

contr.

(2.1) $\exists_x \sim B(x)$ {ada}

(2.2) $\sim B(b)$ {–∃ : 2.1}

(2.3) $B(b)$ {–∀, – K : 4}

contr. □ {2.2,2.3}

3.4. The existential uniqueness quantifier

In addition to the propositional logic constants and quantifiers, it can be also introduced the symbol ′=′ known as *identity* or *equality* (used in most mathematical expressions). Moreover, only there is added the following axiom.

A1: $x = x$

It is also used the primitive rule EI (*extensionality for identity*), given as follows*.

---

* EI is very similar to the derived *rule of extensionality* ER , given in Subsection 1.3.

$$\text{EI:}\quad \frac{\begin{matrix} x = \mathrm{y} \\ \varphi(\mathrm{x}) \end{matrix}}{\varphi(y\,//\,\mathrm{x})}$$

According to A1, (interpreted as a binary relation) identity is reflexive. It can be shown that identity is also symmetric and transitive (Słupecki J. and Borkowski L. 1967). The addition of A1 and EI to predicate logic as well as the generalization of the corresponding rules wrt propositional expressions including identity is an extension of predicate logic called (*classical first-order*) *predicate logic with identity*. This logic allows to introduce the following definition (here ′$\exists_{\mathrm{x}}!\varphi(\mathrm{x})$′ denotes: "*there exists exactly one x such that φ(x)*").

Definition 2.37 (*existential uniqueness quantifier*)

$$\exists_{\mathrm{x}}!\ \varphi(\mathrm{x}) \Leftrightarrow_{\mathrm{df}} \exists_{\mathrm{x}}\ \varphi(\mathrm{x}) \ \wedge \ \forall_{\varphi(x)}\ \forall_{\varphi(y)}\ (\,\mathrm{x} = \mathrm{y}\,).$$

Thesis 2.138

$$\forall_{\varphi(x)}\ \forall_{\varphi(y)}\ \mathrm{R(x,y)} \ \Leftrightarrow \ \forall_{\mathrm{x}}\ (\varphi(\mathrm{x}) \ \Rightarrow \ \forall_{\mathrm{y}}\ (\varphi(\mathrm{y}) \ \Rightarrow \ \mathrm{R(x,y)})).\ \square$$

Thesis 2.139

$$\forall_{\mathrm{x}}\ (\varphi(\mathrm{x}) \ \Rightarrow \forall_{\mathrm{y}}\ (\varphi(\mathrm{y}) \ \Rightarrow \ \mathrm{R(x,y)})) \ \Leftrightarrow \forall_{\mathrm{x}}\ \forall_{\mathrm{y}}\ (\varphi(\mathrm{x}) \ \wedge \ \varphi(\mathrm{y}) \ \Rightarrow \ \mathrm{R(x,y)}).\ \square$$

Thesis 2.140

$$\forall_{x}\ \forall_{\mathrm{y}}\ (\varphi(\mathrm{x}) \ \wedge \ \varphi(\mathrm{y}) \ \Rightarrow \ \mathrm{R(x,y)}) \ \Leftrightarrow \ \forall_{\varphi(x)}\ \forall_{\varphi(y)}\ \mathrm{R(x,y)}.\ \square$$

In accordance with the above three theses, the corresponding three formulae are logically equivalent: the proofs are left to the reader (it is sufficient to prove cyclic only 3 of the whole 6 implications). Obviously, the considered properties can be generalised for any two or more (but finite argument) predicate. It can be observed that T 2.139 is very similar to the laws of exportation and importation (see: T 1.12, Subsection 1.3). And so, the above Definition 2.37 can be presented as follows.

Definition 2.38 (*existential uniqueness quantifier*)

$$\exists_{\mathrm{x}}!\ \varphi(\mathrm{x}) \Leftrightarrow_{\mathrm{df}} \exists_{\mathrm{x}}\ \varphi(\mathrm{x}) \ \wedge \ \forall_{x}\ \forall_{\mathrm{y}}\ (\varphi(\mathrm{x}) \ \wedge \ \varphi(\mathrm{y}) \ \Rightarrow \ \mathrm{x} = \mathrm{y}).$$

Predicate logic with identity allows use of descriptors*. Any such *descriptor* can be considered as an individual expression of the form: $\iota_{x}\varphi(\mathrm{x})$, where ′ι′ is *descriptor's operator*, x is a free individual variable and φ(x) - a propositional function, i.e. the *scope* of ′ι′ in this expression. For example, ′$\iota_{x} A(\mathrm{x})$′ denotes: "*one and only one x such that A(x)*".

The following rule of *joining a descriptor's operator* (or omitting an existential uniqueness quantifier, denoted by: ′− ∃!′) can be introduced.

$$-\exists!:\quad \frac{\exists_{\mathrm{x}}!\varphi(\mathrm{x})}{\varphi(\mathrm{x}\,/\,\iota_{x}\varphi(\mathrm{x}))}$$

We shall use Definition 2.38, the above presented descriptor and rule ′− ∃!′ in the proof of the next thesis.

* Similar notion is used in *computer data processing*: use of *descriptors* from *thesaurus*.

Thesis 2.140 (*Słupecki J. and Borkowski L. 1967*).

$$\underset{x}{\exists!}\ \varphi(x) \Leftrightarrow \underset{\varphi(x)}{\exists}\ \underset{\varphi(y)}{\forall}\ (x = y)$$

ProofT2.140a:

(1)	$\underset{x}{\exists!}\ \varphi(x)$	{a}
(2)	$\sim \underset{\varphi(x)}{\exists}\ \underset{\varphi(y)}{\forall}\ (x = y)$	{aip}
(3)	$\underset{\varphi(x)}{\forall}\ \underset{\varphi(y)}{\exists}\ (x \neq y)$	{N∃*, N∀*, SR : 2}*
(4)	$\varphi(x/\underset{x}{\iota}\varphi(x))$	{− ∃! : 1}
(5)	$\varphi(x/\underset{x}{\iota}\varphi(x)) \Rightarrow \underset{\varphi(y)}{\exists}\ (\underset{x}{\iota}\varphi(x) \neq y)$	{−∀* : 3}
(6)	$\underset{\varphi(y)}{\exists}\ (\underset{x}{\iota}\varphi(x) \neq y)$	{− C : 4,5}
(7)	$\varphi(a)$	{−∃* : 6}
(8)	$\underset{x}{\iota}\varphi(x) \neq a$	
	contr. □	{7,8}

ProofT2.140b:

(1)	$\underset{\varphi(x)}{\exists}\ \underset{\varphi(y)}{\forall}\ (x = y)$	{a}
(2)	$\sim \underset{x}{\exists!}\ \varphi(x)$	{aip}
(3)	$\sim (\underset{x}{\exists}\ \varphi(x) \wedge \underset{x}{\forall}\ \underset{y}{\forall}\ (\varphi(x) \wedge \varphi(y) \Rightarrow x = y))$	{Df.2.38, SR : 2}
(4)	$\varphi(a)$	{−∃* : 1}
(5)	$\underset{\varphi(y)}{\forall}\ (a = y)$	
(6)	$\underset{x}{\forall} \sim \varphi(x) \vee \underset{x}{\exists}\underset{y}{\exists}\ (\varphi(x) \wedge \varphi(y) \wedge x \neq y)$	{N∃, NK, N∀, NC, SR : 3}
(1.1)	$\underset{x}{\forall} \sim \varphi(x)$	{ada}
(1.2)	$\sim \varphi(a)$	{−∀ : 1.1}
	contr.	{4,1.2}
(2.1)	$\underset{x}{\exists}\underset{y}{\exists}\ (\varphi(x) \wedge \varphi(y) \wedge x \neq y)$	{ada}
(2.2)	$\varphi(b)$	
(2.3)	$\varphi(c)$	{−∃ : 2.1}
(2.4)	$b \neq c$	

* $x \neq y \Leftrightarrow_{df} \sim (x = y)$.

(2.5)	$\varphi(b) \Rightarrow a = b$	$\{-\forall^{*} : 5\}$
(2.6)	$\varphi(c) \Rightarrow a = c$	$\{-\forall^{*} : 5\}$
(2.7)	$a = b$	$\{-C : 2.2, 2.5\}$
(2.8)	$a = c$	$\{-C : 2.3, 2.6\}$
(2.9)	$b = c$	{prop. of ´=´: 2.7, 2.8}
	contr. □	{2.4, 2.9}

In general, there exist three kinds of existential quantifiers: $\exists$ (“*there exists at least one*”, corresponding to: $\exists_{\geq 1}$), $\exists!$ (“*there exists exactly one*”, corresponding to: $\exists_{=1}$) and $\exists_{\leq 1}$ (“*there exists at most one*”). The last quantifier is denoted below by: ´$\exists_{*}$´ and defined as follows (Megill N.D. 2005).

Definition 2.39 (*existential uniqueness quantifier*)

$$\underset{x}{\exists_{*}} \varphi(x) \Leftrightarrow_{df} \underset{x}{\exists} \varphi(x) \Rightarrow \underset{x}{\exists!} \varphi(x)$$

In particular, several other definitions were also presented (Megill N.D. 2005)., e.g. $\underset{x}{\exists_{*}} \varphi(x) \Leftrightarrow_{df} \underset{y}{\exists} \underset{x}{\forall} (\varphi(x) \Rightarrow x = y)$. It is shown below that these two definitions are logically equivalent.

Thesis 2.141

$$\underset{x}{\exists} \varphi(x) \Rightarrow \underset{x}{\exists!} \varphi(x) \Leftrightarrow \underset{y}{\exists} \underset{x}{\forall} (\varphi(x) \Rightarrow x = y)$$

ProofT2.141a:

(1)	$\underset{x}{\exists} \varphi(x) \Rightarrow \underset{x}{\exists!} \varphi(x)$	{a}
(2)	$\sim \underset{y}{\exists} \underset{x}{\forall} (\varphi(x) \Rightarrow x = y)$	{aip}
(3)	$\underset{y}{\forall} \underset{x}{\exists}(\varphi(x) \wedge x \neq y)$	$\{N\exists, N\forall, NC, SR : 2\}$
(4)	$\sim \underset{x}{\exists} \varphi(x) \vee \underset{x}{\exists!} \varphi(x)$	{CR : 1}
(1.1)	$\sim \underset{x}{\exists} \varphi(x)$	{ada}
(1.2)	$\underset{x}{\exists}(\varphi(x) \wedge x \neq a)$	$\{-\forall : 3\}$
(1.3)	$\varphi(b)$	$\{-\exists , -K : 1.2\}$
(1.4)	$\underset{x}{\forall} \sim \varphi(x)$	$\{N\exists : 1.1\}$
(1.5)	$\sim \varphi(b)$	$\{-\forall : 1.4\}$
	contr.	{1.3,1.5}
(2.1)	$\underset{x}{\exists!} \varphi(x)$	{ada}
(2.2)	$\underset{x}{\exists} \varphi(x) \wedge \underset{x}{\forall} \underset{y}{\forall} (\varphi(x) \wedge \varphi(y) \Rightarrow x = y)$	{Df.2.38 : 2.1}
(2.3)	$\underset{x}{\exists} \varphi(x)$	$\{-K : 2.2\}$
(2.4)	$\underset{x}{\forall} \underset{y}{\forall} (\varphi(x) \wedge \varphi(y) \Rightarrow x = y)$	
(2.5)	$\varphi(c)$	$\{-\exists : 2.3\}$
(2.6)	$\varphi(d)$	$\{-\forall, -\exists, -K : 3\}$
(2.7)	$d \neq c$	

(2.8)	$\varphi(c) \wedge \varphi(d)$	{+ K : 2.5,2.6}
(2.9)	$\varphi(c) \wedge \varphi(d) \Rightarrow c = d$	{−∀ : 2.4}
(2.10)	$c = d$	{− C : 2.8,2.9}
(2.11)	$d = c$	{prop. of ´=´: 2.10}
	contr. □	{2.7,2.11}

ProofT2.141b:

(1)	$\exists_y \forall_x (\varphi(x) \Rightarrow x = y)$	
(2)	$\exists_x \varphi(x)$	{a}
(3)	$\sim \exists!_x \varphi(x)$	{aip}
(4)	$\sim (\exists_x \varphi(x) \wedge \forall_x \forall_y (\varphi(x) \wedge \varphi(y) \Rightarrow x = y))$	{Df.2.38, SR : 3}
(5)	$\forall_x \sim \varphi(x) \vee \exists_x \exists_y (\varphi(x) \wedge \varphi(y) \wedge x \neq y)$	{N∃, NK, N∀, NC, SR : 4}
(1.1)	$\forall_x \sim \varphi(x)$	{ada}
(1.2)	$\varphi(a)$	{−∃ : 2}
(1.3)	$\sim \varphi(a)$	{−∀ : 1.1}
	contr.	{1.2,1.3}
(2.1)	$\exists_x \exists_y (\varphi(x) \wedge \varphi(y) \wedge x \neq y)$	{ada}
(2.2)	$\varphi(b)$	
(2.3)	$\varphi(c)$	{−∃ : 2.1}
(2.4)	$b \neq c$	
(2.5)	$\forall_x (\varphi(x) \Rightarrow x = d)$	{−∃ : 1}
(2.6)	$\varphi(b) \Rightarrow b = d$	{−∀ : 2.5}
(2.7)	$\varphi(c) \Rightarrow c = d$	{−∀ : 2.5}
(2.8)	$b = d$	{− C : 2.2,2.6}
(2.9)	$c = d$	{− C : 2.3,2.7}
(2.10)	$b = c$	{prop. of ´=´: 2.8,2.9}
	contr. □	{2.4,2.10}

3.5. Sequent calculus

The sequent calculus and (first order) predicate logic can be considered as two equivalent approaches in proof theory. In fact, any formula φ is deducible in the predicate logic iff $\vdash \varphi$ is deducible in the sequent calculus. The fundamental theorem of Gentzen (or the *normalisation theorem*) is fundamental in the proof of this assertion.

The following theorem was given: If $\Gamma \vdash \Delta,\Theta$ and $\Theta,\Gamma \vdash \Delta$ are deducible in the sequent calculus, then so is $\Gamma \vdash \Delta$. The obtained derivation rule, called *cut rule* (or *cut-elimination rule*, in short: 'CER') is presented as follows:

$$\text{CER} : \frac{\Gamma \vdash \Delta,\Theta \quad \Theta,\Gamma \vdash \Delta}{\Gamma \vdash \Delta}.$$

The proof of this rule is given below.

Proof:

(1)	$\Gamma \Rightarrow \Delta \vee \Theta$	{a}
(2)	$\Theta \wedge \Gamma \Rightarrow \Delta$	
(3)	Γ	
(4)	$\sim \Delta$	{aip}
(5)	$\sim \Theta \vee \sim \Gamma$	{Toll : 2,4}
(6)	$\sim \Theta$	{– A : 3,5}
(7)	$\Delta \vee \Theta$	{– C : 1,3}
(8)	Θ	{– A : 4,7}
	contr. □	{6,8}

The normalisation theorem asserts that the cut rule is admissible in the sequent calculus, i.e. does not change the collection of deducible sequents. In view of this, Gentzen's theorem is also called the *cut-elimination theorem* (Encyclopedia of Mathematics 2002).

The following four derivation rules (concerning quantifiers) are used: here, the variable ´b´ occurring in ´$+\forall_c$´ and ´$+\exists_a$´ rules is called the *eigenvariable*[*]. This variable should not occur as free in the lower sequents of the respective rules. In fact, ´b´ may be arbitrary, if A(x) do not contain x. And then, A(b) is equivalent to A(x) (Glushkov V.M. 1964).

$$+\forall_a : \quad \frac{A(t),\ \Gamma \vdash \Theta}{\underset{x}{\forall} A(x),\ \Gamma \vdash \Theta}$$

$$+\exists_a : \quad \frac{A(b),\ \Gamma \vdash \Theta}{\underset{x}{\exists} A(x),\ \Gamma \vdash \Theta}$$

$$+\forall_c : \quad \frac{\Gamma \vdash \Theta,\ A(b)}{\Gamma \vdash \Theta,\ \underset{x}{\forall} A(x)}$$

$$+\exists_c : \quad \frac{\Gamma \vdash \Theta,\ A(t)}{\Gamma \vdash \Theta,\ \underset{x}{\exists} A(x)}$$

As an example, the proof of ´$+\exists_a$´ rule is given below. The remaining proofs are left to the reader.

Proof ´ $+\exists_a$ ´ :

[*] The usual formalisation of first-order logic needs to distinguish between letters that stand for things that can be substituted by any term and letters that are taken to be particular terms.

(1) $A(b) \wedge \Gamma \Rightarrow \Theta$

(2) $\exists_x A(x)$ {a}

(3) Γ

(4) $\sim \Theta$ {aip}

(5) $\sim A(b) \vee \sim \Gamma$ {Toll, NK, SR : 1,4}

(6) $\sim A(b)$ {– A : 3,5}

(7) $A(a)$ {–∃ : 2}

(8) $\forall_x \sim A(x)$ {+∀ : 6 / $A(b) =_{df} A(x)$ }

(9) $\sim A(a)$ {–∀ : 8}

contr. □ {7,9}

The use of the above four derivation rules is shown in the next two examples.

Example 2.35($\forall_x A(x) \Rightarrow \exists_x A(x)$)

(1) $A(x) \vdash A(x)$ {Id}

(2) $A(x) \vdash \exists_x A(x)$ {$+\exists_c$: 1}

(3) $\forall_x A(x) \vdash \exists_x A(x)$. □ {$+\forall_a$: 2}

Example 2.36($\exists_x \forall_y R(x,y) \Rightarrow \forall_y \exists_x R(x,y)$)*

(1) $R(x,y) \vdash R(x,y)$ {Id}

(2) $\forall_y R(x,y) \vdash R(x,y)$ {$+\forall_a$: 1}

(3) $\forall_y R(x,y) \vdash \exists_x R(x,y)$ {$+\exists_c$: 2}

(4) $\exists_x \forall_y R(x,y) \vdash \exists_x R(x,y)$ {$+\exists_a$: 3}

(5) $\exists_x \forall_y R(x,y) \vdash \forall_y \exists_x R(x,y)$. □ {$+\forall_c$: 4}

In particular, according to the last example, it can be observed that any transition between two adjacent lines is a thesis, e.g. $\vDash$ (3) $\Rightarrow$ (4): the proof is given below.

(1) $\forall_y R(x,y) \Rightarrow \exists_x R(x,y)$

(2) $\exists_x \forall_y R(x,y)$ {a}

(3) $\sim \exists_x R(x,y)$ {aip}

(4) $\sim \forall_y R(x,y)$ {Toll : 1,3}

---

* *The Free Encyclopaedia, The Wikimedia Foundation, Inc*: $\exists_x \forall_y R(x,y) \Rightarrow \forall_y \exists_x R(x,y)$ coincides with this one, considered in this *Encyclopaedia,* if we assume that: $p(x,y) \Leftrightarrow_{df} \sim R(x,y)$, after using rule ´CC´: left to the reader.

(5) $\underset{y}{\exists} \sim R(x,y)$ {N∀ : 4}

(6) $\sim R(x,a)$ {−∃ : 5}

(7) $\underset{x}{\forall} \sim R(x,a)$ {+∀ : 6}

(8) $\underset{y}{\forall} R(b,y)$ {−∃ : 2}

(9) $\sim R(b,a)$ {−∀ : 7}

(10) $R(b,a)$ {−∀ : 8}

contr. □ {9,10}

Gentzen's system was used by Hao Wang (1921 − 1995) in automated theorem proving. Using generalised procedures, there were proved (on an IBM-704) 350 theorems, within 8.5 minutes, concerning the first nine chapters of Principia mathematica (Whitehead A. N.* and Russell B. 1913). Since predicate logic is *undecidable* in general (Kurt Gödel 1906 – 1978) , it is clear that a procedure for decidability for an arbitrary (well-defined) predicate logic formula does not exist.

3.6. Skolemisation, resolution and interpretation

Let φ be a formula of (first-order) predicate logic. We shall say that φ is represented in a *prenex normal form* if it is written as a string of quantifiers and bound variables, called *prefix*, followed by a quantifier-free part, called the *matrix*. Assume that the matrix of φ is represented by ψ. More formally, the following prenex normal form can be obtained: $\varphi \Leftrightarrow_{df} \underset{x_1}{Q_1}\underset{x_2}{Q_2}\cdots\underset{x_n}{Q_n}\psi(x_1, x_2, \ldots, x_n)$, where $Q_i \in \{\forall, \exists\}$ (for $I = 1, 2, \ldots, n$). The *quantifier rank* of this form φ, denoted by $qr(\varphi) = n$†.

The next two examples are an illustration of this form.

Example 2.37($\underset{x}{\exists}\underset{y}{\forall} R(x,y) \Rightarrow \underset{y}{\forall}\underset{x}{\exists} R(x,y)$: *prenex normal form*)

$\underset{x}{\exists}\underset{y}{\forall} R(x,y) \Rightarrow \underset{y}{\forall}\underset{x}{\exists} R(x,y) \Leftrightarrow \sim\underset{x}{\exists}\underset{y}{\forall} R(x,y) \vee \underset{y}{\forall}\underset{x}{\exists} R(x,y)$ {CR}

$\Leftrightarrow \underset{x}{\forall}\underset{y}{\exists} \sim R(x,y) \vee \underset{y}{\forall}\underset{x}{\exists} R(x,y)$ {N∃, N∀, SR}

$\Leftrightarrow \underset{x}{\forall}\underset{y}{\exists} \sim R(x,y) \vee \underset{z}{\forall}\underset{t}{\exists} R(t,z)$ {the right side of disjunction: $y =_{df} z$, $x =_{df} t$}

$\Leftrightarrow \underset{x}{\forall}\underset{z}{\forall}\underset{y}{\exists}\underset{t}{\exists} (\sim R(x,y) \vee R(t,z))$ {quantifier theses: Subsection 3.3}

$\Leftrightarrow \underset{x}{\forall}\underset{z}{\forall}\underset{y}{\exists}\underset{t}{\exists} (R(x,y) \Rightarrow R(t,z))$. □ {CR}

The considered formula is a thesis. And hence, the obtained prenex normal form is also a thesis. The proof of this form is given as follows.

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* Alfred North Whitehead (1861 – 1947): Russell was a former student of Whitehead.

† The term "*prenex*" comes from the Latin "*praenexus*" (tied or bound up in front). The *quantifier rank* of a formula is the depth of nesting of its quantifiers. This rank, denoted by $qr(\varphi)$, is defined as follows: (1) $qr(\varphi) = 0$, if φ is atomic, (2) $qr(\varphi_1 \wedge \varphi_2) = qr(\varphi_1 \vee \varphi_2) = \max\{qr(\varphi_1), qr(\varphi_2)\}$, (3) $qr(\sim\varphi) = qr(\varphi)$ and (4) $qr(\underset{y}{\exists}\varphi) = qr(\varphi) + 1$. For example: $qr(A(x) \vee R(x,y)) = 0$, $qr(A(x) \wedge \underset{u}{\forall} R(x,u)) = 1$, $qr(\varphi)$ for the initial form of φ of Example 2.37 equals to 2, but in the obtained prenex form this rank is 4, etc., see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

(1)	$\sim \forall_x \forall_z \exists_y \exists_t (R(x,y) \Rightarrow R(t,z))$	{aip}
(2)	$\exists_x \exists_z \forall_y \forall_t (R(x,y) \wedge \sim R(t,z))$	{N∀, N∃, NC, SR : 1}
(3)	$\forall_y \forall_t (R(a,y) \wedge \sim R(t,b))$	{–∃ : 2}
(4)	$R(a,b)$	{–∀, – K : 3}
(5)	$\sim R(a,b)$	
	contr. □	{4,5}

Example 2.38($\forall_x (A(x) \vee B(x)) \Rightarrow \forall_x A(x) \vee \forall_x B(x)$*: prenex normal form*)

$$\forall_x (A(x) \vee B(x)) \Rightarrow \forall_x A(x) \vee \forall_x B(x) \Leftrightarrow \sim \forall_x (A(x) \vee B(x)) \vee \forall_x A(x) \vee \forall_x B(x) \quad \{CR\}$$

$$\Leftrightarrow \exists_x (\sim A(x) \wedge \sim B(x)) \vee \forall_x \forall_y (A(x) \vee B(y))^{*}$$

$$\Leftrightarrow \exists_x \forall_y \forall_z (\sim A(x) \wedge \sim B(x) \vee A(y) \vee B(z)). \ \square$$

In accordance with the last example, the considered formula and also the obtained prenex form are not theses. However, a thesis is the following logical equivalence. The corresponding proofs of if- and only-if-implications are illustrated below.

$$\forall_x (A(x) \vee B(x)) \Rightarrow \forall_x A(x) \vee \forall_x B(x) \Leftrightarrow \exists_x \forall_y \forall_z (\sim A(x) \wedge \sim B(x) \vee A(y) \vee B(z))$$

Proof (if-implication):

(1)	$\forall_x (A(x) \vee B(x)) \Rightarrow \forall_x A(x) \vee \forall_x B(x)$	{a}
(2)	$\sim\exists_x \forall_y \forall_z (\sim A(x) \wedge \sim B(x) \vee A(y) \vee B(z))$	{aip}
(3)	$\forall_x \exists_y \exists_z ((A(x) \vee B(x)) \wedge \sim A(y) \wedge \sim B(z))$	{N∃, N∀, NA, – N, SR : 2}
(4)	$\forall_x ((A(x) \vee B(x)) \wedge \sim A(f(x)) \wedge \sim B(g(x)))$	{–∃, SR : 3}†
(5)	$\forall_x ((A(x) \vee B(x)) \wedge \forall_x \sim A(f(x)) \wedge \forall_x \sim B(g(x))$	{∀-thesis: 4}
(6)	$\forall_x ((A(x) \vee B(x))$	
(7)	$\forall_x \sim A(f(x))$	{– K : 5}
(8)	$\forall_x \sim B(g(x))$	
(9)	$\forall_x A(x) \vee \forall_x B(x)$	{– C : 1,6}
(1.1)	$\forall_x A(x)$	{ada}
(1.2)	$\sim A(f(a))$	{–∀ : 7}

* quantifier theses: Subsection 3.3

† The rule of omitting an existential quantifier, given (Słupecki J. and Borkowski L. 1967) can be represented in a more convenient way by using *Skolem functions* (Albert Thoralf Skolem 1887 – 1963). As an example, consider a line L defined by two different points x and y. For any such x and y, there exists a point z in L such that z is between x and y, in short: L(x,y,z). So, by omitting the existential quantifier in the expression $\exists_z L(x,y,z)$ we can obtain: $L(x,y,a_{xy})$, where a_{xy} is determined by x and y (Słupecki J. and Borkowski L. 1967). Here, the indexes x and y in a, correspond to β_1 and β_2. Since x and y may be arbitrary, then a = a(x,y) or more formally a = f(x,y), i.e. a_{xy} can be interpreted as some Skolem function f(x,y). Hence, by omitting this quantifier we can obtain: L(x,y,f(x,y)).

(1.3)	A(f(a))	{–∀ : 1.1}
	contr.	{1.2,1.3}
(2.1)	$\forall_x$ B(x)	{ada}
(2.2)	~ B(g(b))	{–∀ : 8}
(2.3)	B(g(b))	{–∀ : 2.1}
	contr. □	{2.2,2.3}

Proof (only-if-implication):

(1)	$\exists_x \forall_y \forall_z$ (~ A(x) ∧ ~ B(x) ∨ A(y) ∨ B(z))	{a}
(2)	~ ($\forall_x$ (A(x) ∨ B(x)) ⇒ $\forall_x$ A(x) ∨ $\forall_x$ B(x))	{aip}
(3)	$\forall_x$ (A(x) ∨ B(x))	
(4)	$\exists_x$ ~ A(x)	{NC, N∀, NA, – K, SR : 2}
(5)	$\exists_x$ ~ B(x)	
(6)	$\forall_y \forall_z$ (~ A(a) ∧ ~ B(a) ∨ A(y) ∨ B(z))	{–∃ : 1}
(7)	~ A(a) ∧ ~ B(a) ∨ $\forall_y \forall_z$ (A(y) ∨ B(z))	{∀-thesis: 6}
(8)	~ (A(a) ∨ B(a)) ∨ $\forall_y \forall_z$ (A(y) ∨ B(z))	{NA, SR : 7}
(9)	A(a) ∨ B(a)	{–∀ : 3}
(10)	$\forall_y \forall_z$ (A(y) ∨ B(z))	{– A : 8,9}
(11)	~ A(b)	
(12)	~ B©	{–∃ : 4,5}
(13)	A(b) ∨ B(c)	{– ∀ : 10}
(14)	BI	{– A : 11,13}
	contr. □	{12,14}

Let consider the following formula: $\forall_x$ A(x) ∨ $\exists_x$ B(x). The obtained prenex normal form is as follows: $\forall_x \exists_y$ (A(x) ∨ B(y)). Since disjunction is *commutative*, we have: $\forall_x$ A(x) ∨ $\exists_x$ B(x) ⇔ $\exists_x$ B(x) ∨ $\forall_x$ A(x) . The following prenex form of the right side of this equivalence can be obtained: $\exists_x \forall_y$ (B(x) ∨ A(y)). And so, the last two prenex normal forms are also logically equivalent, i.e: ⊨ $\forall_x \exists_y$ (A(x) ∨ B(y)) ⇔ $\exists_x \forall_y$ (B(x) ∨ A(y)). The proof is given below.

Proof (if-implication):

(1)	$\forall_x \exists_y$ (A(x) ∨ B(y))	{a}

(2)	$\sim \exists_x \forall_y (B(x) \vee A(y))$	{aip}
(3)	$\forall_x \exists_y (\sim B(x) \wedge \sim A(y))$	{N∃, N∀, NA, SR : 2}
(4)	$\forall_x (A(x) \vee B(f(x)))$	{−∃ : 1}
(5)	$\forall_x (\sim B(x) \wedge \sim A(g(x)))$	{−∃ : 3}
(6)	$\forall_x \sim B(x)$	{∀-thesis, − K : 5}
(7)	$\forall_x \sim A(g(x))$	
(8)	$A(g(a)) \vee B(f(g(a)))$	{−∀, $x =_{df} g(a)$: 4}
(9)	$\sim A(g(a))$	{−∀ : 7}
(10)	$B(f(g(a)))$	{− A : 8,9}
(11)	$\sim B(f(g(a)))$	{−∀, $x =_{df} f(g(a))$: 6}
	contr.	{10,11}

Proof (only-if-implication):

(1)	$\exists_x \forall_y (B(x) \vee A(y))$.	{a}
(2)	$\sim \forall_x \exists_y (A(x) \vee B(y))$	{aip}
(3)	$\exists_x \forall_y (\sim A(x) \wedge \sim B(y))$	{N∀, N∃, NA, SR : 2}
(4)	$\forall_y (B(a) \vee A(y))$	{−∃ : 1}
(5)	$B(a) \vee \forall_y A(y)$	{∀-thesis: 4}
(6)	$\forall_y (\sim A(b) \wedge \sim B(y))$	{−∃ : 3}
(7)	$\sim A(b) \wedge \forall_y \sim B(y)$	{∀-thesis: 6}
(8)	$\sim A(b)$	{− K : 7}
(9)	$\forall_y \sim B(y)$	
(10)	$\sim B(a)$	{−∀ : 9}
(11)	$\forall_y A(y)$	{− A : 5,10}
(12)	$A(b)$	{−∀ : 11}
	contr. □	{8,12}

In a similar way, we have: $\forall_x A(x) \wedge \exists_x B(x) \Leftrightarrow \forall_x \exists_y (A(x) \wedge B(y))$ and $\exists_x B(x) \wedge \forall_x A(x) \Leftrightarrow \exists_x \forall_y (B(x) \wedge A(y))$. Since conjunction is *commutative*, we can obtain: $\models \forall_x \exists_y (A(x) \wedge B(y)) \Leftrightarrow \exists_x \forall_y (B(x) \wedge A(y))$. The proof is left to the reader.

In general, every formula of classical logic is logically equivalent to some formula represented in prenex normal form. Any formula φ of (first-order) predicate logic represented in prenex normal form having only universal quantifiers is said to be a *Skolem normal form*: not changing the satisfiability of φ. Any such process is called *232ulfilment232232n**. The last process (of removing existential quantifiers, often performed as a first step) plays

* Albert Thoralf Skolem (1887 – 1963).

a fundamental role in automated deduction methods. As an example, a brief presentation of the *Robinson's resolution method* is given below (Robinson J.A. 1963, 1965)*.

Example 2.39(*Robinson's resolution method*)

Consider the following prenex normal form: $\varphi \Leftrightarrow_{df} \forall_u \exists_v \forall_w \forall_x \exists_y (R(v,w) \vee R(v,x) \Rightarrow R(u,y))$. Hence, in accordance with general concept of the ramified indirect proof from assumptions (see Subsection 1.3) , as input data to this method is used $\sim \varphi$, i.e: $\exists_u \forall_v \exists_w \exists_x \forall_y ((R(v,w) \vee R(v,x)) \wedge \sim R(u,y))$.

The followimg Skolem normal form (known also as a *standard form*) is obtained: $\forall_v \forall_y ((R(v,f(v)) \vee R(v,g(v))) \wedge \sim R(a,y))$. And hence, for any v and y, we have: $(R(v,f(v)) \vee R(v,g(v))) \wedge \sim R(a,y)$. Since conjunction and disjunction are mutually distributive, in particular we can obtain: $R(v,f(v)) \wedge \sim R(a,y) \vee R(v,g(v)) \wedge \sim R(a,y)$. Now, we have two additional assumptions.

By assuming $R(v,f(v))$ and $\sim R(a,y)$ we can obtain a contradiction for $v =_{df} a$ and $y =_{df} f(a)$. In a similar way, assuming $R(v,g(v))$ and $\sim R(a,y)$ we have a contradiction for $v =_{df} a$ and $y =_{df} g(a)$. □

Consider the expression: $(R(v,f(v)) \vee R(v,g(v))) \wedge \sim R(a,y)$, obtained in the last example. The notion of a *disjunct* is introduced as a disjunction of *literals*. So, in accordance with this *conjnctive normal form*, we have two disjucts: 2-literal and 1-literal ones. The obtained *set of all disjuncts* $S =_{df} \{(R(v,f(v)) \vee R(v,g(v))), \sim R(a,y)\}$. In this earlier work, given by Davis M. and Putnam H†. (1960), the study of unsatisfiability was based on a special domain H, known as *Herbrand's universum*‡: a sufficient condition of any such study (Chang C.-L. and Lee R.C.-T. 1973). In this example we have: $H_0 = \{a\}$, $H_1 = \{a, f(a), g(a)\}$, $H_2 = \{a, f(a), g(a), f(f(a)), f(g(a)), g(f(a)), g(g(a))\}, \ldots , H =_{df} H_\infty = \{a, f(a), g(a), f(f(a)), f(g(a)), g(f(a)), g(g(a)), \ldots\}$.

Some attempts of search for general proof procedures were first given by Leibniz (Gottfried Wilhelm Leibniz: 1646 – 1716). Similar attempts are some works given by Peano (Giuseppe Peano: 1858 – 1932) and also in Hilbert's School (David Hilbert: 1862 – 1943). But finally, it was shown independently by Church A. (1936) and Turing A.M. (1936)§ the lack of any such procedures or algorithms for this logic.

The work given by Herbrand J. (1930) was a very important approach in proof theory and automated deduction methods. Let φ be a first-order predicate logic formula such that φ is not a thesis (see: Definition 1.5, Subsection 1.4). It was proposed an algorithm that selects such an interpretation for which φ is not satisfied. However, the main problem here was the obtained computational complexity of this algorithm. And hence, the Robinson's *resolution method* can be considered as an improving of the last approach, by omitting some *basic disjuncts* (Chang C.-L. and Lee R.C.-T. 1973). In accordance with the used standard (conjnctive normal) form, one of the basic rules used in this method is the following one, called *resolvent rule* (denoted here by ´RES´). This rule can be considered as a generalisation of the rule ´– A´ (for $\chi =_{df} \lambda$: *empty formula*).

$$\text{RES: } \frac{\begin{array}{c}\varphi \vee \psi \\ \sim\varphi \vee \chi\end{array}}{\psi \vee \chi}$$

The proof of ´RES´ corresponds to the proof of the following law: $\models (p \vee q) \wedge (\sim p \vee r) \Rightarrow q \vee r$: left to the reader. The use of thi rule is illustrated in Example 2.40, shown below (the construction of the used prenex normal form is first presented).

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* John Alan Robinson (1930 – 2016).

† Hilary Whitehall Putnam (1926 – 2016).

‡ Jacques Herbrand (1908 – 1931).

§ Alonzo Church (1903 – 1995), Alan Mathison Turing (1912 – 1954).

Consider the formula: $\underset{x}{\forall}(A(x) \Rightarrow B(x)) \wedge \underset{x}{\exists}(A(x) \wedge C(x)) \Rightarrow \underset{x}{\exists}(B(x) \wedge C(x))$. This formula is a thesis (the proof is left to the reader) and it is logically equivalent to the following one (by using CR): $\sim(\underset{x}{\forall}(A(x) \Rightarrow B(x)) \wedge \underset{x}{\exists}(A(x) \wedge C(x))) \vee \underset{x}{\exists}(B(x) \wedge C(x))$. In a similar way , next we have: $\underset{x}{\exists}(A(x) \wedge \sim B(x)) \vee \underset{x}{\forall}(\sim A(x) \vee \sim C(x)) \vee \underset{x}{\exists}(B(x) \wedge C(x))$. In accordance with quantifier theses we can obtain: $\underset{x}{\exists}(A(x) \wedge \sim B(x) \vee B(x) \wedge C(x)) \vee \underset{x}{\forall}(\sim A(x) \vee \sim C(x))$. The last formula is logically equivalent to the following one (the *prenex normal form*): $\underset{x}{\exists}\underset{y}{\forall}(A(x) \wedge \sim B(x) \vee B(x) \wedge C(x) \vee \sim A(y) \vee \sim C(y))$.

The original formula $\underset{x}{\forall}(A(x) \Rightarrow B(x)) \wedge \underset{x}{\exists}(A(x) \wedge C(x)) \Rightarrow \underset{x}{\exists}(B(x) \wedge C(x))$ and the obtained prenex form are logically equivalent.

Proof (if-implication):

(1)	$\underset{x}{\forall}(A(x) \Rightarrow B(x)) \wedge \underset{x}{\exists}(A(x) \wedge C(x)) \Rightarrow \underset{x}{\exists}(B(x) \wedge C(x))$	{a}
(2)	$\sim\underset{x}{\exists}\underset{y}{\forall}(A(x) \wedge \sim B(x) \vee B(x) \wedge C(x) \vee \sim A(y) \vee \sim C(y))$	{aip}
(3)	$\underset{x}{\forall}\underset{y}{\exists}((\sim A(x) \vee B(x)) \wedge (\sim B(x) \vee \sim C(x)) \wedge A(y) \wedge C(y))$	{N∃, N∀, NK, NA, – N, SR : 2}
(4)	$\underset{x}{\forall}((\sim A(x) \vee B(x)) \wedge (\sim B(x) \vee \sim C(x)) \wedge A(f(x)) \wedge C(f(x)))$	{–∃ : 3}
(5)	$\underset{x}{\forall}(\sim A(x) \vee B(x))$	
(6)	$\underset{x}{\forall}(\sim B(x) \vee \sim C(x))$	{∀-thesis,– K : 4}
(7)	$\underset{x}{\forall}A(f(x))$	
(8)	$\underset{x}{\forall}C(f(x))$	
(9)	A(f(a))	{–∀ : 7,8}
(10)	C(f(a))	
(11)	$A(f(a)) \wedge C(f(a))$	{+ K : 9,10}
(12)	$\underset{x}{\exists}(A(x) \wedge C(x))$	{+∃, $x =_{df} f(a)$: 11}
(13)	$\underset{x}{\forall}(A(x) \Rightarrow B(x))$	{CR : 5}
(14)	$\underset{x}{\forall}(A(x) \Rightarrow B(x)) \wedge \underset{x}{\exists}(A(x) \wedge C(x))$	{+ K : 12.13}
(15)	$\underset{x}{\exists}(B(x) \wedge C(x))$	{– C : 1,14}
(16)	B(b)	{– E : 15}
(17)	C(b)	
(18)	$\sim B(b) \vee \sim C(b)$	{–∀ : 6}
(19)	$\sim C(b)$	{– A : 16,18}
	contr. □	{17,19}

Proof (only-if-implication):

(1)	$\underset{x}{\exists}\underset{y}{\forall}(A(x) \wedge \sim B(x) \vee B(x) \wedge C(x) \vee \sim A(y) \vee \sim C(y))$	{a}
(2)	$\sim(\underset{x}{\forall}(A(x) \Rightarrow B(x)) \wedge \underset{x}{\exists}(A(x) \wedge C(x)) \Rightarrow \underset{x}{\exists}(B(x) \wedge C(x)))$	{aip}

(3) $\underset{x}{\forall}$ (A(x) $\Rightarrow$ B(x))

(4) $\underset{x}{\exists}$ (A(x) $\wedge$ C(x)) {NC, – K : 2}

(5) ~ $\underset{x}{\exists}$ (B(x) $\wedge$ C(x))

(6) $\underset{y}{\forall}$ (A(a) $\wedge$ ~ B(a) $\vee$ B(a) $\wedge$ C(a) $\vee$ ~ A(y) $\vee$ ~ C(y)) {–∃ : 1}

(7) $\underset{x}{\forall}$ (~ B(x) $\vee$ ~ C(x)) {N∃, NK, SR : 5}

(8) A(b)

(9) C(b) {–∃, – K : 4}

(10) A(b) $\Rightarrow$ B(b) {–∀ : 3}

(11) B(b) {– C : 8,10}

(12) ~ B(b) $\vee$ ~ C(b) {–∀ : 7}

(13) ~ C(b) {– A : 11,12}

contr. □ {9,13}

Example 2.40(*Robinson's resolution method: use of ´RES´*)

Let φ $\Leftrightarrow_{df}$ $\underset{x}{\exists}\ \underset{y}{\forall}$ (A(x) $\wedge$ ~ B(x) $\vee$ B(x) $\wedge$ C(x) $\vee$ ~ A(y) $\vee$ ~ C(y)). Then: ~ φ $\Leftrightarrow_{df}$ $\underset{x}{\forall}\ \underset{y}{\exists}$ ((~ A(x) $\vee$ B(x)) $\wedge$ (~ B(x) $\vee$ ~ C(x)) $\wedge$ A(y) $\wedge$ C(y)). By omitting the existential quantifier, the following standard form is obtained: $\underset{x}{\forall}$ ((~ A(x) $\vee$ B(x)) $\wedge$ (~ B(x) $\vee$ ~ C(x)) $\wedge$ A(f(x)) $\wedge$ C(f(x))). Here, we have two 2-literal disjuncts and two 1-literal ones, associated in a conjunctive normal form. In accordance with the corresponding quantifier thesis and ´– K´, the above standard form can be decomposed as follows.

$\underset{x}{\forall}$ (~ A(x) $\vee$ B(x)),

$\underset{x}{\forall}$ (~ B(x) $\vee$ ~ C(x)),

$\underset{x}{\forall}$ A(f(x)) and

$\underset{x}{\forall}$ C(f(x)).

According to the last decomposition, it can be observed that all literals of the above standard form can be treted independently wrt the selected values for x. And so, by assuming x $=_{df}$ a for 1-literal disjuncts and x $=_{df}$ f(a) for 2-literal disjuncts we can obtain.

(1) ~ A(f(a)) $\vee$ B(f(a))

(2) ~ B(f(a)) $\vee$ ~ C(f(a))

(3) A(f(a))

(4) C(f(a))

(5) ~ A(f(a)) $\vee$ ~ C(f(a)) {RES,´$\vee$´ *is commutative*: 1,2}

(6) ~ C(f(a)) {– A : 3,5}

contr. □ {4,6}

As distinct from propositional logic (see Subsection 1.4), an *interpretation* of a given formula φ in first-order predicate logic should require: a finite *domain* of interpretation and also corresponding value definition related to all *constants*, *functional* and *predicate symbols* involved in φ. The following interpretation is used below (Chang C.-L. and Lee R.C.-T. 1973).

Domain	$D =_{df} \{1,2\}$*	interpretation†
universal quantifier	$\underset{x}{\forall}\ \varphi(x)$	$\underset{x\in D}{\forall}\ \varphi(x) \Leftrightarrow_{df} \varphi(1) \wedge \varphi(2)$
existential quantifier	$\underset{x}{\exists}\ \varphi(x)$	$\underset{x\in D}{\exists}\ \varphi(x) \Leftrightarrow_{df} \varphi(1) \vee \varphi(2)$
predicate	$P(x_1, x_2, \dots, x_n)$	$P : D^n \rightarrow \{T,F\}$
Skolem function	$f(x_1, x_2, \dots, x_m)$	$f : D^m \rightarrow D$
constants	a, b, …	$a, b, \dots \in D$

The next two examples are an illustration of this method.

Example 2.41(*formula interpretation*)

Let $\varphi \Leftrightarrow_{df} \underset{x}{\forall} (A(x) \vee B(x)) \Rightarrow \underset{x}{\forall} A(x) \vee \underset{x}{\forall} B(x)$. We have the following prenex normal form for φ: $\underset{x}{\exists}\ \underset{y}{\forall}\ \underset{z}{\forall} (\sim A(x) \wedge \sim B(x) \vee A(y) \vee B(z))$. And hence, by omitting the existential quantifier we ca obtain: $\underset{y\in\{1,2\}}{\forall}\ \underset{z\in\{1,2\}}{\forall} (\sim A(a) \wedge \sim B(a) \vee A(y) \vee B(z))$. Provided there is no ambiguity and for simplicity, assume that $\psi(y,z) \Leftrightarrow_{df} \sim A(a) \wedge \sim B(a) \vee A(y) \vee B(z)$. Then, the last restricted standard form will be logically equivalent to the following conjunction: $\psi(1,1) \wedge \psi(1,2) \wedge \psi(2,1) \wedge \psi(2,2)$.

Let now the constant $a =_{df} 1 \in D$. The one argument predicates $A(y)$, $B(z)$ and $\psi(y,z)$ are interpreted as follows:

A(1)	A(2)		B(1)	B(2)		ψ(y,z)	1	2
T	F		F	T		1	T	T
						2	F	T

Since $\psi(2,1)$ = ´F´, the original formula φ is not a tesis. ▫

Example 2.42(*formula interpretation*)

Consider the following prenex normal form: $\underset{x}{\exists}\ \underset{y}{\forall}\ \underset{z}{\exists} (A(x) \Rightarrow (R(x,y) \Rightarrow S(y,z)))$. By omitting the first existential quantifier we can obtain: $\underset{y}{\forall}\ \underset{z}{\exists} (A(a) \Rightarrow (R(a,y) \Rightarrow S(y,z)))$. And hence, we have the following restricted standard form: $\underset{y\in\{1,2\}}{\forall} (A(a) \Rightarrow (R(a,y) \Rightarrow S(y, f(y))))$. For convenience, let $\varphi(y) \Leftrightarrow_{df} A(1) \Rightarrow (R(1,y) \Rightarrow S(y,f(y)))$. Then, this form is logically equivalent to the conjunction: $\varphi(1) \wedge \varphi(2)$.

Asume that $a =_{df} 1 \in D$. The following interpretation is used.

A(1)	A(2)		f(1)	f(2)		R	1	2		S	1	2
T	F		2	1		1	F	T		1	T	F
						2	T	F		2	F	T

Since $\varphi(2)$ = ´F´, the original formula is not a tesis. ▫

* In general, the set $D \subsetneq \mathbb{N}$ (the set of natural numbers). Usually it is used the subset $\{1,2\}$.

† Based on Peirce interpretation for quantifiers (Charles Sanders Peirce 1839 – 1914).

3.7. The higher order predicate logic

In addition to propositional variables (p, q, …), in first-order predicate logic we have also only individual variables (x, y, …) and variables representing predicates (A, B, … , P, Q, …). In fact, it is possible to introduce a more general form of the notion of a molecular expression (see: Example 2.31).

For convenience, consider the following example definitions given in (Słupecki J. and Borkowski L. 1967).

Example 2.43(*α, ϱ, σ: symbol definition*)

(Df.1) $\alpha(A) \Leftrightarrow_{df} \exists!_{x}\, A(x)$,

(Df.2) $A\varrho B \Leftrightarrow_{df} \sim \exists_{x} (A(x) \Leftrightarrow B(x))$ and

(Df.3) $x\sigma A \Leftrightarrow_{df} A(x) \wedge \alpha(A)$.

In accordance with Carnap's interpretation, by Df.1 it follows that A is a property associated with exactly one individual variable (i.e. *object*). From Df.2 it follows that these two properties A and B are not associated simultaneously with any one object. According to Df.3, it follows that x is the only one such object having A. □

The above predicate variables are also known as *first-order functors*. And hence, the corresponding molecular expressions are said to be *first-order molecular expressions*.

The *second-order functors* are defined in a similar way. They have as arguments individual variables or first-order functors (at least one such functor). The obtained molecular expressions are said to be *second-order molecular expressions*. As an example, such expressions are: α(A), AϱB and xσA, introduced in the last example. The second-order functors are denoted below by bold letters, e.g. **A**, **B**, **C**, … are one argument functors and **P**, **Q**, **R**, … denote more than one (but finite argument) functors. And so, another srcond-order molecular expressions are, e.g. **B**(A), **R**(x,A) or equivalently: x**R**A, **P**(A,x,y), etc.

In accordance with this calculus, the quantifiers bound only first-order entities, i.e. individual variables or first-order functors. As an illustration, some example theses are given below (Słupecki J. and Borkowski L. 1967). The corresponding rules related to quantifiers are quite similar to these ones used in first-order predicate logic. And so, the proofs of the next theses are omitted. T 2.147 is a modified version of the original one: by using rules ´MC´ and ´SR´ (see T 1.5 of Subsection 1.3: left to the reader).

Thesis 2.142

$\forall_{A}\, \mathbf{A}(A) \Rightarrow \exists_{A}\, \mathbf{A}(A)$. □

*Thesis 2.143**

$\forall_{A} (\mathbf{A}(A) \wedge \mathbf{B}(A)) \Rightarrow \forall_{A}\, \mathbf{A}(A) \wedge \forall_{A}\, \mathbf{B}(A)$. □

Thesis 2.144

$\exists_{A} \forall_{B}\, \mathbf{R}(A,B) \Rightarrow \forall_{B} \exists_{A}\, \mathbf{R}(A,B)$. □

Thesis 2.145

$\forall_{x} \exists_{A}\, A(x)$. □

Thesis 2.146

$\exists_{R} (R(x,y) \Leftrightarrow R(y,x))$. □

* The *if-implication* of the corresponding thesis given in Subsection 3.3.

Thesis 2.147

$\exists_{A} \forall_{B} \forall_{x} (A(x) \Rightarrow B(x) \wedge \sim B(x))$. □

Frequently, in the proofs of this calculus, some definitions may be introduced. As an example, the proof of T 2.145 is based on the following definition.

Definition 2.40 (*property definition*)

Let $'*<x>'$ be one argument predicate depending on x. Then: $*<x>(y) \Leftrightarrow_{df} x = y$.

According to the last definition, it is assumed that the above expression indicate a property associated with y iff $y = x$ (i.e. a property distinctive for x).

By Definition 2.40 and the axiom A1 of Subsection 3.4 it follows that: $*<x>(x)$. This property is used, e.g. in the proof of T 2.145 as follows.

(1)	$*<x>(x)$	{a}
(2)	$\exists_{A} A(x)$	$\{+\exists : 1\}$
	$\forall_{x} \exists_{A} A(x)$. □	$\{+\forall : 2\}$

The proof of T 2.146 is based on the following property: $x = y \Leftrightarrow y = x$ (left to the reader). The following important thesis is satisfied (Słupecki J. and Borkowski L. 1967).

Thesis 2.148

$$x = y \Leftrightarrow \forall_{A} (A(x) \Leftrightarrow A(y))$$

Proof(if-implication):

(1)	$x = y$	{a}
(2)	$A(x) \Leftrightarrow A(x)$	$\{\vDash p \Leftrightarrow p\}$
(3)	$A(x) \Leftrightarrow A(y)$	{EI : 1,2}[*]
	$\forall_{A} (A(x) \Leftrightarrow A(y))$. □	$\{+\forall : 3\}$

Proof(only-if-implication):

(1)	$\forall_{A} (A(x) \Leftrightarrow A(y))$	{a}
(2)	$*<x>(x) \Leftrightarrow *<x>(y)$	$\{-\forall : 1\}$
(3)	$*<x>(y)$	{DE : 2, $*<x>(x)$}[†]
	$x = y$. □	{DE : Df. 2.40,3}

According to T 2.148, identical objects have the same properties. And hence, this thesis can be considered as a definition of the notion of identity. Similar considerations, from intuitive point of view, were first presented by Aristoteles (384 b.c. – 322 b.c.) and next by: Tommaso d'Aquino (1225 – 1274) and Gottfried Wilhelm Leibniz (1646 – 1716). A more formal definition was given by Peirce in 1885 (Charles Sanders Peirce: 1839 – 1914) and published by Whitehead A.N. and Russell B (1913).

The *third-order predicate logic* can be introduced in a similar way, where the *third-order functors* have as arguments individual variables and first- and second-order functors (at least one such functor). The corresponding

[*] See: Subsection 3.4.

[†] See: T 1.16 of Subsection 1.3.

third-order molecular expressions are introduced in a similar way (left to the reader). In this logic we have a possibility of an extension of the above identity relation to some properties, e.g. A = B, by using the axiom A1′: A = A.

3.8. Generalised quantifiers

The ordinary quantifiers "*for same*" and "*for all*" are not sufficient for expressing some basic mathematical concepts. Generalised quantifiers, such as: "*for infinitely many*" and "*for uncountably many*" (as a part of mathematical logic) were first introduced by Mostowski A.S. (1957)[*]. Such and other similar quantifiers were intensively studied by logicians[†]. Researches in *descriptive complexity theory*[‡] (a branch of *computational complexity* and *finite model theories*) and *natural language semantics* were looking at ways of formalise expressions such as: "*for at least half*" or "*for an even number*" (Väänänen J. 1997).

The above introduced quantifiers were used implicit in *Montague Grammar* (Montague R. 1974) and also used in its force in (Barwise K. J. and Cooper R. 1981) and (Keenan E.L. and Stavi J. 1986). Generalised quantifier theory can be considered as a logical semantic theory which studies the interpretation of *noun phrases* and *determiners*, i.e. terms for any kind at (mostly) non-lexical element preceding a noun in a noun phrase (see: *Glottopedia, the free encyclopedia of linguistics*).

It was suggested a generalisation of this notion in two dimensions: syntactic and semantic. *Syntactically,* a logical quantifier is a variable binding operator that generates new formulae from old formulae. *Semantically,* a logical quantifier over a universe $\mathscr{U}$ is a cardinality function from subsets of $\mathscr{U}$ to a truth value, satisfying a certain invariance condition. Let $\exists_{\mathscr{U}}$ and $\forall_{\mathscr{U}}$ be the quantifiers $\exists$ and $\forall$ over $\mathscr{U}$. Assume that $\mathscr{V} \subseteq \mathscr{U}$. Then $\exists_{\mathscr{U}}(\mathscr{V}) = \mathrm{T}$ iff the cardinality $|\mathscr{V}| > 0$ and $\forall_{\mathscr{U}}(\mathscr{V}) = \mathrm{T}$ iff $|\mathscr{V}'| = 0$, where the complement $\mathscr{V}' =_{df} \mathscr{U} - \mathscr{V}$. In general, a quantifier Q on $\mathscr{U}$ is invariant under all permutations of $\mathscr{U}$. The following criterion for logical quantifiers was given by Mostowski: Q is *logical* iff for any $\mathscr{U} \neq \varnothing$, $Q_{\mathscr{U}}$ is invariant under all permutations of $\mathscr{U}$. In particular, a generalisation of the last criterion to first-order predicates and quantifiers of all types, proposed by Lindström[§], was also presented (Sher G. 2015).

The following characterisation of standard first order logic was given (*Lindström's theorem*): standard first-order logic is the strongest logic that has both *completeness* (see Subsection 1.6) and the *Löwenheim-Skolem properties*[**]. Next, it was shown that *completeness* is not limited only to this logic (Keisler H.J. 1970).

Generalised quantifiers can be introduced by using such notions as: *signature* (denoted by σ : a finite sequence of relation symbols and constant symbols) and a *finite ordered structure over* σ. It is assumed (without loss of generality) that the universe of every structure is always an initial set of natural numbers. Let Struct(σ) be the set of all finite ordered structures over σ. It can be shown that every class of structures $\subseteq$ Struct(σ) over a signature σ defines the first-order Lindström quantifier (Vollmer H. 1999).

Some introductory notions concerning the notion of a generalised quantifier are briefly presented below (Pogonowski J. and Smigerska J. 2008)[††].

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[*] Andrzej S. Mostowski (1913 – 1975).

[†] *Quantifiers* are words which show how many things or how much of something we are talking about, e.g. much, many, (a) little, (a) few, a lot (of), some, any, no, none, both, all, either, neither, each, every, (the) other(s), another, etc. (file:///C:/Users/user/Documents/QUANTIFIERS%20JEZYKOWO.pdf)

[‡] The goal of descriptive complexity theory is to classify problems, not according to how much resources they need when solved by a Turing machine, but according to how powerful logical languages are necessary for describing the problems (Väänänen J. 1997).

[§] Per Lindström (1936 – 2009: see Lindström P. 1966).

[**] Any noncontradictory theory in a countable language (i.e. a language with a countable number of formulae) has a finite or countable model (Leopold Löwenheim: 1878 – 1957, Thoralf Skolem: 1887 – 1963). See: *Formal logic. Encyclopedical outline with applications to informatics and linguistics* (1987).

[††] In fact, a lot of important research was done, eg. see: (Henkin L. 1961, Benthem J. van 1986, Väänänen J. 1997, Vollmer H. 1999, D'Alfonso D. 2011, Westerståhl D. 2001, 2011), etc.: left to the reader.

The quantifiers introduced by Mostowski are quantitative. The *local quantifier* defined on $\mathscr{U}$ is considered as a set of subsets of $\mathscr{U}$. The *global quantifier* is considered as a functor Q assigning to each non-empty $\mathscr{U}$ the quantifier $Q_{\mathscr{U}}$ defined on $\mathscr{U}$[*]. Some examples of such quantifiers are given below (indexsed, the first letter of hebranian alphabet $\aleph_\alpha$ denotes *aleph*-α for arbitrary *ordinal number* α). The last two quantifiers are known as Rescher's and Chang's quantifiers, respectively[†].

$\forall_{\mathscr{U}} =_{df} \{\mathscr{U}\}$,
$\exists_{\mathscr{U}} =_{df} \{X \subseteq \mathscr{U} / X \neq \varnothing\}$,
$(\exists_{\geq n})_{\mathscr{U}} =_{df} \{X \subseteq \mathscr{U} / |X| \geq n\}$,
$(Q\alpha)_{\mathscr{U}} =_{df} \{X \subseteq \mathscr{U} / |X| \geq \aleph_\alpha\}$,
$(Q_R)_{\mathscr{U}} =_{df} \{X \subseteq \mathscr{U} / |X| > |\mathscr{U} - X|\}$,
$(Q_C)_{\mathscr{U}} =_{df} \{X \subseteq \mathscr{U} / |X| = |\mathscr{U}|\}$, etc.

Let $f : \mathscr{U} \to \mathscr{U}'$ be a bijection. Since the elements of $\mathscr{U}$ are not distinguished, the following condition is satisfied (assumed in the next works concerning generalised quantifiers).

ISOM $X \in Q_{\mathscr{U}} \Leftrightarrow f(X) \in Q_{\mathscr{U}'}$.

The introduced by Mostowski quantifiers do not involved such quantifiers, e.g. as the binary quantifier "*most*" in propositions of the form: "Most φ are ψ", corresponding to a binary relation between subsets of $\mathscr{U}$. And so, we have:

$(Q_{most})_{\mathscr{U}} =_{df} \{(X,Y) \in \mathscr{U}^2 / X \rho Y\}$,

where: $X \rho Y \Leftrightarrow_{df} |X \cap Y| > |X - Y|$ (for any $X,Y \subseteq \mathscr{U}$).

Lindström's quantifiers are classified according to the number structure of their parameters. So, with any such quantifier a corresponding natural number sequence, said to be a *type*, is associated: $(n_1,n_2, \ldots ,n_k)$.

In accordance with the last considered examples, the first six quantifiers are of type (1) and $(Q_{most})_{\mathscr{U}}$ - of type (1,1). The Lindström's notions of a local and global quantifiers are presented as follows (Pogonowski J. and Smigerska J. 2008).

Definition 2.41 (*generalised local and global quantifiers*)

The *generalised local quantifier* defined on $\mathscr{U}$ of type $\{n_1,n_2, \ldots ,n_k)$ is introduced as an arbitrary k-ary relation between the subsets $\mathscr{U}^{n_1}, \ldots , \mathscr{U}^{n_k}$. The *generalised global quantifier* defined on $\mathscr{U}$ of type $(n_1,n_2, \ldots ,n_k)$ is considered as a functor Q assigning to each non-empty $\mathscr{U}$ a local quantifier $Q_{\mathscr{U}}$ of type $(n_1,n_2, \ldots ,n_k)$[‡].

Generalised quantifiers, in accordance with their *type*[§], can be either *monadic* (if any such quantifier is of type (1,1, ... ,1)) or *polyadic*. For convenience, the monadic quantifiers of type (1) , (1,1) , (1,1,1), etc. are also known as *unary*, *binary*, *ternary*, etc. quantifiers. In general, the following *lexicographic* (quantifier type) *order* was presented (Hella L. 1989): (1) < (1,1) < ... < (2) < (2,1) < (2,1,1) < … < (2,2) < … < (3) < …

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[*] Provided there is no ambiguity and for convenience, instead of the originally used ′ M ′, the space is here denoted by ′ $\mathscr{U}$ ′.

[†] Nicholas Rescher, born 1928 and Chen Chung Chang, born 1927.

[‡] A quantifier Q of type $(n_1,n_2, \ldots ,n_k)$ can be considered as a function associating with each $\mathscr{U}$ a quantifier $Q_{\mathscr{U}}$ on $\mathscr{U}$ of that type, i.e. a k-ary relation between relations over $\mathscr{U}$. For (any relation) $X_i \subseteq \mathscr{U}^{n_i}$ $Q_{\mathscr{U}} X_1, \ldots ,X_k$ means that relation $Q_{\mathscr{U}}$ holds for the arguments $X_1, \ldots , X_k$ (Westerståhl D. 2001). Provided there is no ambiguity and for convenience, instead of R_i, here the symbols X_i are used (without parentheses).

[§] *Type theory* is a branch of c*omputational logic* that studies types (informally object attributes). See: *The Free Encyclopaedia, The Wikimedia Foundation, Inc.* The above used notions are set-theoretic.Linguists often prefer lambda notation, from the simply *typed lambda calculus* (i.e. Alonzo Church's typed λ - calculus: Alonzo Church: 1903 – 1995). This is a functional framework, where everything except primitive objects like *individuals* (type e) and *truth values* (type t) is a function. Binary relations are of type (e, (e,t)), type (1) quantifiers now get the type ((e,t), t), type (1,1) quantifiers are of type ((e,t), ((e,t),t)), etc. (Westerståhl D. 2001).

Let $f: \mathcal{U} \to \mathcal{U}'$ be a bijection. In the case of Lindström's quantifiers, the ISOM condition is presented as follows*.

ISOM $\quad (\rho_1,\rho_2, \ldots ,\rho_k) \in Q_{\mathcal{U}} \Leftrightarrow (f(\rho_1),f(\rho_2), \ldots ,f(\rho_k)) \in Q_{\mathcal{U}'}$.

Some example Lindström's quantifiers are given below (Pogonowski J. and Smigerska J. 2008), see also (Westerståhl D. 2001). In accordance with the last work, e.g. $(Q_{all})_{\mathcal{U}}$ is equivalently defined in the following simplified form (for all $\mathcal{U}$ and all $X,Y \subseteq \mathcal{U}$): $all_{\mathcal{U}}(X,Y) \Leftrightarrow_{df} X \subseteq Y$ (another examples are also considered: left to the reader).

$(Q_{all})_{\mathcal{U}} =_{df} \{ (X,Y) \in \mathcal{U}^2 \ / \ X \subseteq Y\}$,

$(Q_{some})_{\mathcal{U}} =_{df} \{ (X,Y) \in \mathcal{U}^2 \ / \ X \cap Y \neq \varnothing\}$,

$(Q_{more})_{\mathcal{U}} =_{df} \{ (X,Y) \in \mathcal{U}^2 \ / \ |X| > |Y| \}$,

$(Q_{I})_{\mathcal{U}} =_{df} \{ (X,Y) \in \mathcal{U}^2 \ / \ / \ |X| = |Y| \}$. *(Härtig† quantifier)*

A generalisation of Rescher's, Chang's and Härtig's quantifiers is *Henkin's quantifier* (Henkin L‡. 1961). The last quantifier, denoted be Q_H , can be considered as a special case of a generalised quantifier (Badia A. 2009). A simplest form of this quantifier is given below.

$$((Q_H)_{\mathcal{U}}\, x,y,z,t)\ \varphi(x,y,z,t) \Leftrightarrow_{df} \begin{pmatrix} \forall_x \exists_z \\ \forall_y \exists_t \end{pmatrix} \varphi(x,y,z,t).$$

Let f and g be two functions defined on $\mathcal{U}$. The above Henkin's quantifier, of type (4), can be also equivalently represented as follows.

$$Q_H =_{df} \{ \rho \subseteq \mathcal{U}^4 \ / \ \exists_f \exists_g \forall_x \forall_y\ (x, f(x), y, g(y)) \in \rho \}.$$

Several properties of generalised quantifiers are given below (Pogonowski J. and Smigerska J. 2008, Westerståhl D. 2011).

CONSERV — *Conservativity* (preference for the first argument): for all $\mathcal{U}$ and all $X_1, \ldots ,X_k,Y \subseteq \mathcal{U}$: $Q_{\mathcal{U}}X_1, \ldots ,X_k,Y \Leftrightarrow Q_{\mathcal{U}}X_1, \ldots ,X_k,(X_1 \cup \ldots \cup X_k) \cap Y$.

EXT — *Extension* (universum independence): $X_1, \ldots ,X_k \subseteq \mathcal{U} \subseteq \mathcal{U}' \Rightarrow (Q_{\mathcal{U}}X_1, \ldots ,X_k \Leftrightarrow Q_{\mathcal{U}'}X_1, \ldots ,X_k)$.

UNIV — (*CONSERV* and *EXT*): $Q_{\mathcal{U}}X_1, \ldots ,X_k,Y \Leftrightarrow Q_{X_1 \cup \ldots \cup X_k} X_1, \ldots ,X_k,(X_1 \cup \ldots \cup X_k) \cap Y$.

QUANT — (Quantifier independence wrt object features): for all $\mathcal{U}$, $\mathcal{U}'$, all bijections $f: \mathcal{U} \to \mathcal{U}'$ and all $X_1, \ldots ,X_k \subseteq \mathcal{U}$: $Q_{\mathcal{U}}X_1, \ldots ,X_k \Leftrightarrow Q_{\mathcal{U}'} f(X_1), \ldots ,f(X_k)$.

The above conditions CONSERV, EXT and QUANT of the most simple quantifiers, of type (1,1),, i.e. *binary quantifiers* – considered as relations, are presented as follows.

CONSERV — For all $\mathcal{U}$, X and Y: $Q_{\mathcal{U}}X,Y \Leftrightarrow Q_{\mathcal{U}}X,X \cap Y$.

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* Instead of R, the letter ρ is used here to denote a relation.

† Klaus Härtig, born 1941.

‡ Leon Albert Henkin (1921 – 2006).

EXT $X,Y \subseteq \mathscr{U} \subseteq \mathscr{U}' \Rightarrow Q_{\mathscr{U}} X,Y \Leftrightarrow Q_{\mathscr{U}'} X,Y$.

QUANT For all $\mathscr{U}$, $\mathscr{U}'$, all bijections $f : \mathscr{U} \to \mathscr{U}'$ and all X,Y: $Q_{\mathscr{U}} X,Y \Leftrightarrow Q_{\mathscr{U}'} f(X), f(Y)$.

A quantifier satisfying, at the same time, the CONSERV, EXT and QUANT conditions is said to be a *logical* one. The following property is satisfied.

Theorem 2.2

A binary quantifier is logical iff for all $\mathscr{U}$ and $\mathscr{U}'$, $X,Y \subseteq \mathscr{U}$ and $X',Y' \subseteq \mathscr{U}'$: $|X - Y| = |X' - Y'|$ and $|X \cap Y| = |X' \cap Y'| \Rightarrow (Q_{\mathscr{U}} X,Y \Leftrightarrow Q_{\mathscr{U}'} X',Y')$.

Proof(if-implication):

By using QUANT we can obtain: $Q_{\mathscr{U}} X, X \cap Y = Q_{\mathscr{U}'} X', X' \cap Y'$. And next, by UNIV we have: $Q_{\mathscr{U}} X,Y \Leftrightarrow Q_{\mathscr{U}'} X',Y'$. □

Proof (only-if-implication):

In accordance with this implication, the above condition QUANT is satisfied. Consider $\mathscr{U}$. Let $X,Y \subseteq \mathscr{U}$. Assume that $\mathscr{U}' = X' = X$. Then $Q_{\mathscr{U}} X,Y \Leftrightarrow Q_X X, X \cap Y$. And hence, UNIV is satisfied. □

It was shown that the class of logical quantifiers is closed under the operations conjunction, disjunction and negation, i.e. if Q_1 and Q_2 are logical quantifiers, such ones are also: $Q_1 \wedge Q_2$, $Q_1 \vee Q_2$ and $\sim Q_1$. With any binary (k + 1) argument quantifier Q, the following operations are also associated (Pogonowski J. and Smigerska J. 2008).

Internal conjunction (two kinds)*: $(Q_{\wedge 1})_{\mathscr{U}} X_1, \dots, X_k, Y \Leftrightarrow_{df} Q_{\mathscr{U}} X_1 \cap \dots \cap X_k, Y$ and $(Q_{\wedge 2})_{\mathscr{U}} X_1, \dots, X_k, Y \Leftrightarrow_{df} Q_{\mathscr{U}} X_1, Y \wedge \dots \wedge Q_{\mathscr{U}} X_k, Y$.

Internal negation: $(Q_{\sim})_{\mathscr{U}} X_1, \dots, X_k, Y \Leftrightarrow_{df} Q_{\mathscr{U}} X_1, \dots, X_k, \mathscr{U} - Y$.

The *dual quantifier* Q^d of Q is defined as follows: $\sim (Q \sim) = (\sim Q) \sim$. The external negation (the set of sets that are not in Q) and the internal negation (the set of complements of the sets in Q) correspond to the notions of a proposition negation and a predicative phrase negation, respectively†. De Morgan´s laws are associated only with the external negation, e.g. $\sim (Q \wedge Q') \Leftrightarrow \sim Q \vee \sim Q'$ (in a similar way for disjunction,i.e. the rule NA). As an example, by using the internal negation we can obtain: $(Q \wedge Q') \sim \Leftrightarrow Q \sim \wedge Q' \sim$. In the case of dual quantifiers we have: $(Q \wedge Q')^d \Leftrightarrow Q^d \vee Q'^d$ (similarly for disjunction), see: (Westerståhl D. 2001).

The class of logical quantifiers is closed under (the two kinds of) internal conjunction and internal disjunction, internal negation, and quantifier 242ulfilment242.

An n-argument quantifier $Q_{\mathscr{U}}$ is said to be *trivial* if it is either an empty or a complete relation on $\mathbb{P}(\mathscr{U})$. The following condition is introduced.

NONTRIV There exist universums $\mathscr{U}$ in which $Q_{\mathscr{U}}$ is *nontrivial*.

---

* The two kinds of an *internal disjunction* can be introduced in a similar way: left to the reader.

† The internal and external negations are also known as: *inner negation* (or *post-complement*) and *outer negation* (or *pre-complement*), respectively. Moreover, these two negations and dual are *idempotent*, i.e. $Q = \sim\sim Q = Q\sim\sim = Q^{dd}$ (see: Westerståhl D. 2001).

The class of nontrivial quantifiers is not closed under Boolean operations. A more hard version of NONTRIV is *ACT*: $Q_{\mathscr{U}}$ is nontrivial (for any $\mathscr{U}$). However, a harder than the previous one is the following version (Benthem J.van[*]. 1986: generalised for (k + 1)-argument quantifiers by Westerståhl D[†]. 2001, 2011).

VAR[‡] For any $\mathscr{U}$ and all $X_1, \dots, X_k \subseteq \mathscr{U}$ with $X_1 \cap \dots \cap X_k \neq \varnothing$, there exist Y_1, Y_2 such that: $Q_{\mathscr{U}}X_1, \dots, X_k, Y_1$ and $\sim Q_{\mathscr{U}}X_1, \dots, X_k, Y_2$.

In fact, VAR implicates ACT, which implicates NONTRIV. The opposite implications are not satisfied (Pogonowski J. and Smigerska J. 2008). In the case of binary quantifiers, the last condition is presented as follows.

VAR For any $\mathscr{U}$ and $\varnothing \neq X \subseteq \mathscr{U}$, there exist Y_1, Y_2 such that: $Q_{\mathscr{U}}X, Y_1$ and $\sim Q_{\mathscr{U}}X, Y_2$.

The most important type in natural language contexts is (1,1), i.e. binary quantifiers (Westerståhl D. 2011). Some monotonicity properties related to these quantifiers are given below (Pogonowski J. and Smigerska J. 2008).

Definition 2.42 (*monotonicity properties*)

Let Q be a binary quantifier. Then Q is:

$\mathrm{MON}\uparrow \;\Leftrightarrow_{df}\; Q_{\mathscr{U}}X,Y \wedge Y \subseteq Y' \Rightarrow Q_{\mathscr{U}}X,Y'$,

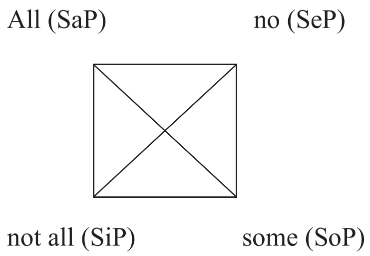
$\mathrm{MON}\downarrow \;\Leftrightarrow_{df}\; Q_{\mathscr{U}}X,Y \wedge Y' \subseteq Y \Rightarrow Q_{\mathscr{U}}X,Y'$,

$\uparrow\mathrm{MON} \;\Leftrightarrow_{df}\; Q_{\mathscr{U}}X,Y \wedge X \subseteq X' \Rightarrow Q_{\mathscr{U}}X',Y$,

$\downarrow\mathrm{MON} \;\Leftrightarrow_{df}\; Q_{\mathscr{U}}X,Y \wedge X' \subseteq X \Rightarrow Q_{\mathscr{U}}X',Y$.

We shall say that Q is *right monotone* (*RMON*) iff it is either MON ↑ or MON ↓ . Similarly, Q is *left monotone* (*LMON*) iff it is either ↑ MON or ↓ MON. Q is ↑ MON ↑ iff it is ↑ MON and MON ↑. In a similar way are introduced: ↓ MON ↓ , ↑ MON ↓ and ↓ MON ↑ .

According to the last definition, the above four double monotonicities correspond to the four vertices (i.e. *some*, *no*, *not all* and *all*, respectively) of a *logical square* (known also as: *square of opposition* or *logical quadrat*)[§], see Figure 3.2 below.



[*] Johan van Benthem, born 1949.

[†] Dag Westerståhl, born 1946.

[‡] Or: VARIETY.

[§] The difference between the terms '*contradiction*' and '*contrariety*' (or '*opposition*') was first studied by Aristoteles (384 – 322 b.c.). But, as a diagram, the first logical square was done after the works of Apuleius L.M. (c.e.124 – 170) and Boethius A.M.S. (c.e. 477/80 – 524): see: *The Free Encyclopaedia, The Wikimedia Foundation, Inc*.

More formally: SaP $\Leftrightarrow_{df} \underset{S}{\forall}$ (S is P), SeP $\Leftrightarrow_{d} \sim\underset{S}{\exists}$ (S is P), SiP $\Leftrightarrow_{d} \underset{S}{\exists}$ (S is +P), and SoP $\Leftrightarrow_{d} \underset{S}{\exists}$ (S is not P). In accordance with De Morgan´s laws (see: Subsection 3.3, Thesis 2.135): $\sim\underset{S}{\forall}$ (S is P) $\Leftrightarrow_{d} \underset{S}{\exists}$ (S is not P). And hence we have the following two contradictions: SaP – SoP and also SeP – SiP. Also, the following two implications are satisfied: SaP $\Rightarrow$ SiP and SeP $\Rightarrow$ SoP. Moreover, the last two implications are equivalent (according to rules CC, SR and N∀).

Figure 3.2 Logical ssquare[*]

The following property is satisfied (Pogonowski J. and Smigerska J. 2008).

Theorem 2.3

Let CONSERV and VAR be satisfied. Then the above four quantifiers, related to the logical square, are the unique ones of double monotonicities.

Proof:

Assume that Q is $\downarrow$MON$\downarrow$, $\mathscr{U}$ is an universum and X, Y $\subseteq \mathscr{U}$. We should show that Q is '*no*', i.e. $Q_{\mathscr{U}}X,Y \Leftrightarrow X \cap Y = \varnothing$.

Proof(only-if-implication)[†]*:*

Assume that $X \cap Y = \varnothing$. Let $X' \neq \varnothing$ and $X \subseteq X'$. According to VAR, there exists a set $Z \subseteq \mathscr{U}$ such that $Q_{\mathscr{U}}X',Z$. Since Q is $\downarrow$MON then $Q_{\mathscr{U}}X,Z$. Next, by using MON$\downarrow$, we can obtain: $Q_{\mathscr{U}}X,\varnothing$ (in accordance with Definition 2.42: $Q_{\mathscr{U}}X,Z \wedge \varnothing \subseteq Z \Rightarrow Q_{\mathscr{U}}X,\varnothing$). Since $X \cap Y = \varnothing$, then we have: $Q_{\mathscr{U}}X,\varnothing \Leftrightarrow Q_{\mathscr{U}}X, X \cap Y$. And hence, by CONSERV we can obtain: $Q_{\mathscr{U}}X,Y$. □

Proof(if-implication):

Assume that the relation $Q_{\mathscr{U}}X,Y$ is satisfied. Hence, by $\downarrow$MON$\downarrow$ we can obtain: $Q_{\mathscr{U}}X \cap Y, X \cap Y$ (since $X \cap Y \subseteq X, Y$). According to MON$\downarrow$, for any $Z \subseteq \mathscr{U}$ we have: $Q_{\mathscr{U}}X \cap Y, X \cap Y \cap Z$ (since $X \cap Y \cap Z \subseteq X \cap Y$). Then, by using CONSERV we have: $Q_{\mathscr{U}}X \cap Y, Z$ (let $A =_{df} X \cap Y$, then: $Q_{\mathscr{U}}A, A \cap Z \Leftrightarrow Q_{\mathscr{U}}A, Z$). Using VAR we have: $X \cap Y = \varnothing$ (more formally, the proof of implication: $Q_{\mathscr{U}}X \cap Y, Z \Rightarrow X \cap Y = \varnothing$ is given below, the proofs of the rest three cases are similar: left to the reader). □

It is used in the next proof a more formal definition of the condition VAR related to binary quantifiers, presented in the original work (Benthem J. van. 1986): $\underset{A}{\forall}(A \neq \varnothing \Rightarrow \underset{B}{\exists} Q_{\mathscr{U}}A, B \wedge \underset{C}{\exists} \sim Q_{\mathscr{U}}A, C)$, for any $\mathscr{U}$ and A, B, C $\subseteq \mathscr{U}$.

$Q_{\mathscr{U}}X \cap Y, Z \Rightarrow X \cap Y = \varnothing$

Proof:

(1)	$\underset{A}{\forall}(A \neq \varnothing \Rightarrow \underset{B}{\exists} Q_{\mathscr{U}}A, B \wedge \underset{C}{\exists} \sim Q_{\mathscr{U}}A, C)$	{VAR}
(2)	$A \neq \varnothing \Rightarrow \underset{B}{\exists} Q_{\mathscr{U}}A, B \wedge \underset{C}{\exists} \sim Q_{\mathscr{U}}A, C$	$\{-\underset{A}{\forall} : 1\}$
(3)	$\underset{B}{\forall} \sim Q_{\mathscr{U}}A, B \vee \underset{C}{\forall} Q_{\mathscr{U}}A, C \Rightarrow A = \varnothing$	{CC, NK, N∃, – N, SR : 2}

[*] A similar (*set-theoretic*) interpretation of this square can be obtained if we consider S and P as two sets. Then SaP, SeP, SiP and SoP (i.e. *all S are P*, *no S is P*, *some S are P*, and *some S are not P*, respectively) should correspond to: $S \subseteq P$, $S \cap P = \varnothing$, $S \cap P \neq \varnothing$, and $S \nsubseteq P$, respectively (in accordance with the *Venn diagrams*: John Venn 1834 – 1923): see: *The Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[†] It is convenient to show first the only-if-implication.

(4) $\underset{B}{\forall} \sim Q_{\mathscr{U}}X \cap Y, B \;\vee\; \underset{Z}{\forall} Q_{\mathscr{U}}X \cap Y, Z \;\Rightarrow\; X \cap Y \;=\; \varnothing$. $\{A =_{df} X \cap Y,\; C =_{df} Z : 3\}$

(5) $\underset{Z}{\forall} Q_{\mathscr{U}}X \cap Y, Z$ $\{+ \underset{Z}{\forall}$: Theorem 2.3: if-implication$\}$

(6) $\underset{B}{\forall} \sim Q_{\mathscr{U}}X \cap Y, B \;\vee\; \underset{Z}{\forall} Q_{\mathscr{U}}X \cap Y, Z$ $\{+ A : 5\}$

$X \cap Y = \varnothing$. □ $\{- C : 4{,}6\}$

In the next considerations it is assumed that all quantifiers are logical (i.e. they satisfy the conditions CONSERV, EXT and QUANT) and satisfy NONTRIV. In the case of natural languages, it is convenient to be assumed the following additional condition (Pogonowski J. and Smigerska J. 2008).

FIN Only finite universums are considered, i.e. $|\mathscr{U}| \in \mathbb{N}$, for any $\mathscr{U}$.

However, some determiners, e.g. such as '*infinitely many*' or the dual: '*all but finitely many*', etc. should require the use of non-finite models. According to the last work, with any binary quantifier Q (which is logical) are associated some properties. In particular, $Q_{\mathscr{U}}$ may be: *symmetric*, *antisymmetric*, *asymmetric*, *reflexive*, *quasi-reflexive*, *weak-reflexive*, *antireflexive*, *linear*, *transitive*, *cyclic*, *Euclidean* or *non-Euclidean*, e.g. Q is symmetric iff $Q_{\mathscr{U}}\, X,Y \Rightarrow Q_{\mathscr{U}}\, Y,X$, *antisymmetric* iff $Q_{\mathscr{U}}\, X,Y \wedge Q_{\mathscr{U}}\, Y,X \;\Rightarrow\; X = Y$ (e.g. *some* and *all*, respectively, etc.: left to the reader). In particular, there was shown lack (in natural languages) of quantifiers which are asymmetric, cyclic or Euclidean.

Any binary logical quantifier $Q_{\mathscr{U}}$ can be identified by a binary relation $\rho_{Q_{\mathscr{U}}}$ defined over a set of cardinal numbers as follows: $x\, \rho_{Q_{\mathscr{U}}}\, y \Leftrightarrow_{df} \underset{X}{\exists}\, \underset{Y}{\exists}\, ((|X - Y| = x) \wedge (|X \cap Y| = y) \wedge Q_{\mathscr{U}}X,Y)$. On the other hand, for any a priori given binary relation $\rho_{Q_{\mathscr{U}}}$ over a set of cardinal numbers, we can obtain (corresponding to $\rho_{Q_{\mathscr{U}}}$) a binary logical quantifier $Q_{\mathscr{U}}$ defined as: $Q_{\mathscr{U}}X,Y) \Leftrightarrow_{df} |X - Y|\, \rho_{Q_{\mathscr{U}}}\, |X \cap Y|$.

The above correspondence help us to consider quantifiers as some ordered pairs associated with the vertices of a *binary tree* having as a *root* (0,0) and the *set of out-incident vertices*[*] {(x + 1, y), (x, y + 1)}, associated with any vertex (x,y). And hence, there exists a possibility of studying various properties related to these quantifiers (Pogonowski J. and Smigerska J. 2008).

In the next parts of the last work are briefly considered: some methodological aspects, there are presented such quantifiers as:

$Q_0\, x\;\; A(x)$ '*there exist infinitely many* x *such that* A(x)',
$Q_1\, x\;\; A(x)$ '*there exist uncountably many* x *such that* A(x)',
$Q_C\, x\;\; A(x)$ '*there exist as much objects* x *as in the whole* $\mathscr{U}$ ' (Chang's quantifier),
Is nore X *than* Y,
Henkin's quantifier and
Vilenkin – Shreider quantifier .[†]

The last quantifier (originally denoted by $Q_m\, x\;\; A(x)$) is presented as follows (Vilenkin N. Ya. and Shreider Yu. A. 1977).

$Q_{V\text{-}S}\, x\;\; A(x)$ '*such objects* x *that* A(x) *form majority in* $\mathscr{U}$ ',

Let $X \neq \varnothing$ be a set and $\mathscr{B}(X)$ be a Boolean algebra (involving not necessary all subsets) such that $X \in \mathscr{B}(X)$. The family M(X) of elements in $\mathscr{B}(X)$ is said to be a *majority system* in X if the following three conditions are satisfied (it is assumed below that $\in$, $\subseteq$ and $\notin$ bind more strongly than the symbol of conjunction).

(1) $M(X) \neq \varnothing$,
(2) $A \in M(X) \wedge A \subseteq B \;\Rightarrow\; B \in M(X)$ and

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[*] See (Berge C. 1973), Claude Jacques Berge (1926 – 2002).

[†] Naum Yakovlevich Vilenkin (1920 – 1991), Yulii Anatol'evich Shreider (1927 – 1998).

(3) $A \in M(X) \Rightarrow$ (the set complement) $A' \notin M(X)$.

The majority space is defined as follows: $(X, M(X))$. If $A \in M(X)$ then A is said to be a *majority* in X. Obviously $\varnothing \notin X$, $X \in M(X)$. Moreover, if $A \in M(X)$ and $B \in M(X)$ then $A \cap B \neq \varnothing$.

The following semantics of $Q_{V\text{-}S}$ is presented (Pogonowski J. and Smigerska J. 2008): $\mathscr{U} \models Q_{V\text{-}S}\, x\ A(x) \Leftrightarrow_{df} \{a \in dom(\mathscr{U}) / \mathscr{U} \models A(a)\}$ is a majority in $dom(\mathscr{U})$, for some (majority) system $M(dom(\mathscr{U}))$.

In accordance with the last work, the following two topics are also considered: *expressive power of logic frameworks* and *infinitary logics*. Important results concerning this expressive power are related to *Lindström's limitation theorems*.

Infinitary logics are logics in which: the considered language allows formulae with non-finite length and/or infinitely long proofs or non-finitary (proof) rules. It was shown (Barwise[*] K.J. 1975) that the existence of some countable sets (a generalisation of heritable finite sets), known as *admissible sets*, on whom (interpreted as sets of formulae codes) becomes the possibility of studying *recursion theory* and *proof theory*. There are very many applications of Barwise's theorem, e.g. there exists a possibility to show that any countable transitive model for ZFC[†] has a proper finite extension. Barwise's work can be considered as an unification in the studies of model theory, recursion theory and set theory. Especially useful in this work was the axiomatic approach given by (Kripke S. 1964) and (Platek R.A. 1966), known as: *Kripke – Platek set theory* (see the next Chapter, Subsection 5.6). A more formal treatment is omitted here.

4. Non-classical calculus

There are briefly considered some other non-classical systems such as: fuzzy predicate calculus, modal and temporal predicate calculi and also the intuitionistic and paraconsistent predicate logic systems.

4.1. Fuzzy predicate calculus

The main results considered below are under (Cintula P., Fermüller C.G., and Noguera C. 2017), see also (Hájek P. 2005) and (Metcalfe G. et al. 2009). And so, consider a *fuzzy propositional logic* L. There exists an uniform way of introducing its *first-order predicate logic* (denoted here by) $L\forall$ in a *predicate language* $\mathscr{PL}$ (defined in a similar way as in the classical case, see Subsection 3.1). Here, there are assumed only t-norm based such logics. The considered *semantics* is presented by structures and predicate symbols are interpreted as functions (similarly as in the classical case, see Subsection 3.6). More formally, a *structure* $\mathbf{M} =_{df} (M, f_{\mathbf{M}}, P_{\mathbf{M}})$ consist of a *domain* $M \neq \varnothing$, a function $f_{\mathbf{M}} : M^n \rightarrow M$ and a function $P_{\mathbf{M}} : M^n \rightarrow [0,1]$ ($n \in \mathbb{N}$; $f_{\mathbf{M}}, P_{\mathbf{M}} \in \mathscr{PL}$).

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[*] Kenneth Jon Barwise (1942 – 2000).

[†] *Zermelo-Fraenkel axiomatics* (ZF or ZFC: ´C´ stands for ´Choice´ (the *axiom of Choice*): Ernst Friedrich Ferdinand Zermelo (1871 – 1953), Adolf Abraham Halevi Fraenkel (1891 – 1965). According to this system, the letters of the sets may appear on both sides of ´∈´, but those for elements may only appear on the left side (as in KPU axiomatic system: see Subsection 5.6). The following axioms were used in ZF: *axiom of extensionality*, *axiom of regularity* (or *foundation*), *axiom schema of specification* (or *separation* or of *restricted comprehension*), *axiom of pairing*, *axiom of union*, *axiom schema of replacement*, *axiom of infinity* and *axiom of power set*. Here, the used two *axiom schemes* can be considered as a standard way of introducing axioms having the same syntactic structure, e.g. the axiom A1 (the first *law of the hypothetical syllogism*: *law of Duns Scotus*, see Subsection 1.7), of *Łukasiewicz's implication-negation axiomatic system*: $p \Rightarrow (\sim p \Rightarrow q)$ can be generalised for any two formulae φ and ψ, as follows: $\varphi \Rightarrow (\sim \varphi \Rightarrow \psi)$. A more formal treatment is left to the reader.

Any *evaluation* v of object variables in M defines values of *terms* and truth values of *atomic formulae* (for f_M and P_M, respectively)[*]. The *quantifier terms* are defined as follows.

$$||(\underset{x}{\forall})\,\varphi||_v =_{df} \inf\{||\varphi||_{v[x:a]} \;/\; a \in M\} \quad \text{and} \quad ||(\underset{x}{\exists})\,\varphi||_v =_{df} \sup\{||\varphi||_{v[x:a]} \;/\; a \in M\},$$

where v[x : a] is the evaluation sending x to a (keeping the values of other variables unchanged). Any other formula is computed in accordance with the truth values of the corresponding propositional connectives in L.

The following axioms are accepted for L∀ (Cintula P., Fermüller C.G., and Noguera C. 2017).

(P) The (first-order) instances of the axioms of L[†] ∈

(∀1) $\underset{x}{\forall}\,\varphi(x) \Rightarrow \varphi(y)$

(∃1) $\varphi(y) \Rightarrow \underset{x}{\exists}\,\varphi(x)$

(∀2) $\underset{x}{\forall}\,(\chi \Rightarrow \varphi) \Rightarrow (\chi \Rightarrow \underset{x}{\forall}\,\varphi)$

(∃2) $\underset{x}{\forall}\,(\varphi \Rightarrow \chi) \Rightarrow (\underset{x}{\exists}\,\varphi \Rightarrow \chi\}$

(∀3) $\underset{x}{\forall}\,(\chi \vee \varphi) \Rightarrow \underset{x}{\forall}\,\chi \vee \varphi$

The deduction rules of L∀ are the same as in L plus the rule of generalisation, i.e. '+∀' (see Subsection 3.2).

The sequent calculus becomes elusive for some non-classical logics. A significant challenge was the introduction of analytic proof systems, i.e. a natural generalisation of Gentzen – style sequent systems suitable for such logics. *Hypersequent systems*, first presented in (Pottinger G. 1983) and (Avron A. 1987), and surveyed in (Avron A. 1996) and (Baaz M. et al. 2003) are one such generalisation. 'Hypersequent calculus do not alter the definition of a sequent at all, but just add an additional level of context of ordinary sequents. Just as in classical sequent calculus, hypersequent calculus consist in initial hypersequents (i.e., axioms) as well as logical and structural rules. The axioms and logical rules are essentially the same as in sequent calculus[‡]. The only difference is the presence of side hypersequents, denoted by H and H′, representing (possibly empty) hypersequents' (Baaz M. et al. 2003).

Hypersequents are sequences[§] of ordinary sequents (called *components*), e.g. $\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ldots \mid \Gamma_n \vdash \Delta_n$, where the *hypersequent bar* ′ | ′ (known also as *pipe operator*) is a *meta – level disjunction*[**]. It is here assumed that the consequent Δ_I consists at most one formula (I = 1, … ,n). If for all I, Δ_I consists of a single formula, the hypersequent is called *single – conclusioned* (Avron A. 1996).

The *generic interpretation* of a sequent $\Gamma \vdash B$, denoted by *Int* $(\Gamma \vdash B)$, is defined by $(\wedge\Gamma \Rightarrow B*)$ where ′$\wedge\Gamma$′ stands for the conjunction of the formulae in Γ or ′⊤′ if Γ is empty and ′B∗′ is B or ′⊥′ if B is empty. The *generic interpretation of a hypersequent* *Int* $(\Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid \ldots \mid \Gamma_n \vdash A_n) =_{df}$

---

[*] $||f_M(t_1, \ldots, t_n)||_v =_{df} f_M(||t_1||_v, \ldots, ||t_n||_v)$, similarly for P_M.

[†] The axioms of the following three basic fuzzy propositional logics: Łukasiewicz's BL, *Gödel's* BL, and product (logic) BL (Hájek P. 2002, 2005). As an illustration, Łukasiewicz's BL axiomatic system is given in Subsection 2.2 (the main problem here is the computational effectiveness: using only ′– C′ and ′RR′).

[‡] See Subsection 1.8.

[§] Provided there is no ambiguity and for convenience, instead of '*multiset*', the term '*sequence*' is used here.

[**] The standard interpretation of the pipe operator is usually disjunction. Intuitively, a hypersequent is true in a certain state iff one of its components is true in that (relative to some semantics which makes the last statement meaningful), see: (Avron A. 1996).

$\overset{n}{\underset{i=1}{\vee}}$ *Int* ($\Gamma_i \vdash A_I$) , where ´ ∨ ´ is a meta – level disjunction and ´⊤´ and ´⊥´ correspond to the logical constants *´true´* and *´false´*, respectively. The *structural rules* are here divided into *internal* and *external* ones. The internal rules deal with formulae withim components and are the same as in ordinary sequent calculus (e.g. see Subsection 2.4, linear logic: contraction and weakening). The external structural rules, i.e. the *external contraction* and *external weakening*, manipulate whole components of a hypersequent (Baaz M. et al. 2003).

A rule such as external contraction can be used to eliminate duplicate components, for instance. *Shuffling rules* are a class of external structural rules which combine or exchange information from multiple components, such as the following crucial *communication rule1* given in *Avron's calculus*[*] HG for *Gödel-Dummet[†]t logic* (Avron A. 1996), where ´ $?_i$ ´ are finite sequences of formulae, ´ G_i ´ and ´ H_i ´ are variables for (possible empty) hypersequents, I = 1,2.

$$Com : \frac{G_1 \mid ?_1 \vdash A_1 \mid H_1 \qquad G_2 \mid ?_2 \vdash A_2 \mid H_2}{G_1 \mid G_2 \mid ?_1 \vdash A_2 \mid ?_2 \vdash A_1 \mid H_1 \mid H_2}$$

In most hypersequent calculi, the only axioms are of the form $A \vdash A$ (or even $p \vdash p$, p is atomic), exactly as in the standard sequent calculus (Avron A. 1996).

In particular, the following simplified version of this rule was presented by Rothenberg R.[‡]

$$Com : \frac{G_1 \mid \Gamma_1, \Gamma_1' \vdash A_1 \qquad G_2 \mid \Gamma_2, \Gamma_2' \vdash A_2}{G_1 \mid G_2 \mid \Gamma_1, \Gamma_2' \vdash A_2 \mid \Gamma_2, \Gamma_1' \vdash A_1}$$

Let G_1 , Γ_1 , Γ_1', A_1 , G_2 , Γ_2 , Γ_2' and A_2 correspond to p, q, r, s, t, u, v and w, respectively. The proof of the last rule is then reduced to the proof of the following formula.

$$(p \vee (q \wedge r \Rightarrow s)) \wedge (t \vee (u \wedge v \Rightarrow w)) \Rightarrow p \vee t \vee (q \wedge v \Rightarrow w) \vee (u \wedge r \Rightarrow s)$$

Proof:

(1)	$p \vee (q \wedge r \Rightarrow s)$	{1,2 / a}
(2)	$t \vee (u \wedge v \Rightarrow w)$	
(3)	~ p	
(4)	~ t	
(5)	q	
(6)	v	{aip / NA, NC, – K}
(7)	~ w	
(8)	u	
(9)	r	
(10)	~ s	
(11)	$q \wedge r \Rightarrow s$	{ – A : 1,3}
(12)	$u \wedge v \Rightarrow w$	{ – A : 2,4}
(13)	$\sim q \vee \sim r$	{Toll, NK : 10,11}
(14)	~ r	{ – A : 5,13}
	contr. □	{9,14}[*]

[*] See: (Avron A. 1991).

[†] Kurt Gödel (1906 – 1978), Michael Anthony Eardley Dummet (1925 – 2011).

[‡] Rothenberg R., *An hypersequent calculus for Łukasiewicz logic without the merge rule*. Scotland's University of St. Andrews 2pp: file:///C:/Users/user/Documents/HYPERSEQUENT%20CALCULUS%20%202.pdf.

In general, hypersequents were used in some systems e.g. such as: HG (in Gentzen's sequential calculus), HIL (in intuitionistic logic), HIF (in intuitionistic fuzzy logic), HLC (HIL + *Com*), GŁ (in infinite-valued Łukasiewicz's logic), etc.

'I proof search with shuffling rules must deal with an exponential number of cases, which is impractical for hypersequents with many components or large components. The complexity of implementing root-first proof search on these rules for hypersequent calculi can outweigh advantages these calculi may provide over their sequent counterparts' Rothenberg R.[†] In particular, in the last work was presented a new simplified version of the hypersequent system GŁ, introduced in (Ciabattoni A. and Metcalfe G. 2003), called GŁ2. The two *weakening rules* given in GŁ are absorbed into the axioms of GŁ2. Moreover, the *merge rule* (a simplified shuffling rule, making proof search expensive) is omitted. The obtained system GŁ2′ is shown below.

(*Axiom*) $G \mid \Gamma', \underbrace{\bot, \dots, \bot}_{n}, \Delta' \vdash A_1, \dots, A_n, \Delta' \quad (n \geq 0)$

$$+C_a : \quad \frac{G \mid \Gamma, B \vdash A, \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma, A \Rightarrow B \vdash \Delta}$$

$$+C_c : \quad \frac{G \mid \Gamma, A \vdash B, \Delta \qquad G \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash A \Rightarrow B, \Delta}$$

$$EC : \quad \frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta}$$

$$S : \quad \frac{G \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G \mid \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2}$$

And so, in accordance with this work, it was shown that GŁ2′ is *sound* and *complete*[‡] for infinite – valued Łukasiewicz's logic. Moreover all rules in this system, except ′S′, are invertible.

The quantifier rules of hypersequent calculus for first-order logic are similar to these ones used in the classical sequent calculus (see Subsection 3.5). As an example, the rule ′$+\forall_a$′ of the classical calculus is now presented as follows (the rest three rules can be represented in a similar way: left to the reader).

$$+\forall_{a/h} : \quad \frac{G \mid A(t), \ \Gamma \vdash B}{G \mid \forall_x A(x), \ \Gamma \vdash B}$$

[*] Another contradiction can be obtained starting with lines (2) and (4): left to the reader.

[†] Rothenberg R., *An hypersequent calculus for Łukasiewicz logic without the merge rule*. Scotland's University of St. Andrews 2pp: file:///C:/Users/user/Documents/HYPERSEQUENT%20CALCULUS%20%202.pdf

[‡] A logical system is *sound* (i.e. has the *soundness property*) iff every formula that can be proved in this system is logically valid wrt the semantics of this system. More formally: $A_1, \dots, A_n \vdash B \Rightarrow A_1, \dots, A_n \models B$ (i.e. if B is derived by $A_1, \dots, A_n$ then B is a *tautological entailment* wrt $A_1, \dots, A_n$). A logical system is *complete* wrt some property iff every formula having this property can be derived by this system (for a more information see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*).

Let G , Γ and B correspond to p, q and r, respectively. The proof of the last rule is then reduced to the proof of the following formula.

$$P \vee (A(t) \wedge q \Rightarrow r) \;\Rightarrow\; p \vee ((\underset{x}{\forall} A(x)) \wedge q \Rightarrow r)$$

Proof(+ $\forall_{a/h}$):

(1)	$p \vee (A(t) \wedge q \Rightarrow r)$	{ a}
(2)	$\sim p$	
(3)	$\sim ((\underset{x}{\forall} A(x)) \wedge q \Rightarrow r)$	{aip / NA, – K}
(4)	$\underset{x}{\forall} A(x)$	
(5)	q	{NC, – K : 3}
(6)	$\sim r$	
(7)	$A(t) \wedge q \Rightarrow r$	{ – A : 1,2}
(8)	A(t)	{–∀ : 4}
(9)	$A(t) \wedge q$	{+ K : 8,5}
(10)	r	{– C : 7,9}
	contr. □	{6,10}

The proofs of the rest formulae, i.e. ´+ $\exists_{a/h}$´ , ´+ $\forall_{c/h}$´ and ´+ $\exists_{c/h}$´, are left to the reader. The considered here rules are a part of the hypersequent calculus HIF for first-order Gödel's logic given in (Baaz M. and Zach R. 2000). A more formal treatment is omitted here.

Fuzzy quantifiers and generalised fuzzy quantifiers

In classical predicate logic, the universal and existential quantifiers are introduced as two constants (see Subsections 3.1 and 4.1). Below is briefly presented the notion of a *fuzzy quantifier*, mainly under (Losada D.E. et al. 2006). Some information concerning the notion of a *generalised fuzzy quantifier* is also given.

Fuzzy quantifiers are very useful and important in areas such as: information retrieval, fuzzy queries, fuzzy constraints, fuzzy data mining applications, fuzzy dependencies, fuzzy intelligence systems, etc. More generally, fuzzy quantification is an important topic in fuzzy theory and its applications. 'Fuzzy or linguistic quantifiers allow us to express fuzzy quantities or proportions in order to provide an approximate idea of the number of elements of a subset fulfilling a certain condition or the proportion of this number in relation to the total number of possible elements' (Galindo J. et al. 2008). The considered in this work (two kinds of) fuzzy quantifiers may be useful in any fuzzy database. The following definition was given.

Definition 2.43 (*absolute and relative quantifiers*)

A fuzzy quantifier named Q is is represented as a function Q whose domain depends on whether it is absolute or relative:

$Q_{abs} : \mathbb{R} \rightarrow [0,1]$ and
$Q_{rel} : [0,1] \rightarrow [0,1]$.

In fact, in comparison with the original Zadeh's fuzzy quantifiers (Zadeh L.A. 1983)*, the set of all nonnegative real numbers $\mathbb{R}_+$ is now extended to $\mathbb{R}$. According to this definition, $dom(Q_{rel}) =_{df} [0,1]$ because the division $\frac{a}{b} \in [0,1]$, where ´a´ is the *number of elements fulfilling some condition* and ´b´ is the *total number of elements*. Let ´ɣ´† be the *value of quantification* defined as follows: if $Q =_{df} Q_{abs}$ then $ɣ =_{df}$ a else

* Lotfi Aliasker Zadeh (1921 – 2017).

† Provided there is noambiguity, this symbol is used instead the original one: ´φ´.

$\gamma =_{df} \frac{a}{b}$. And so, the *251ulfilment degree* is defined as Q(ɣ). If the function of the quantifier (absolute or relative) Q(ɣ) = 1 the Q is completely satisfied. The value ´0´ indicates that Q is not fulfilled at all. Any intermediate value indicates an intermediate 251ulfilment degree for Q,

However, 'given a certain linguistic expression, it is often difficult to achieve consensus on a) the most appropriate mathematical definition for a given quantifier and b) the adequacy of a particular numerical value as the evaluation result for a fuzzy quantified sentence. This is especially problematic when linguistic expressions involve several fuzzy properties. To overcome this problem, some authors have proposed indirect definitions of fuzzy quantifiers through semi-fuzzy quantifiers (Glöckner I. 1999), (Glöckner I. and Knoll A. 2001), (Glöckner I. 2004). A fuzzy quantifier can be defined from a semi-fuzzy quantifier through a so-called *quantifier fuzzification mechanism* (QFM). The motivation of this class of indirect definitions is that *semi-fuzzy quantifiers* (SFQ) are closer to the well-known crisp quantifiers and can be defined in a more natural and intuitive way' (Losada D.E. et al. 2006).

Fuzzy set theory allows definition of sets having not well defined boundaries. Initially, in accordance with the last work, some basic concepts of fuzzy set theory are briefly presented below.[*]

Let $\mathscr{U}$ be a universe and A be a *fuzzy set*. This set can be characterised by a *membership function* with the form: $\mu_A : \mathscr{U} \to [0,1]$, where $\mu_A(u)$ represents its *grade* (or *degree*) *of membership* to the fuzzy set A with 0 (1) corresponding to *no* (to *full*) membership in A. The *fuzzy set operations* can be introduced in several ways, e.g. the *complement* of a fuzzy set A and the *intersection and* the *union* of the fuzzy sets A and B are typically defined as follows: $\mu_{A'}(u) =_{df} 1 - \mu_A(u)$, $\mu_{A \cap B}(u) =_{df} \min\{\mu_A(u), \mu_B(u)\}$ and $\mu_{A \cup B}(u) =_{df} \max\{\mu_A(u), \mu_B(u)\}$, where ´min´ and ´max´ are the well-known *Zadeh's t-norm* and *t-conorm*, respectively (see Subsection 2.2). Next, by $\mathbb{P}(\mathscr{U})$ and $\widetilde{\mathbb{P}}(\mathscr{U})$ we shall denote the *crisp* and the *fuzzy power sets* of $\mathscr{U}$ ($\widetilde{\mathbb{P}}(\mathscr{U})$ is the set of all fuzzy sets that can be defined on $\mathscr{U}$). In particular, if $\mathscr{U}$ is finite, e.g. $\mathscr{U} =_{df} \{u_1, \dots, u_n\}$, a *discrete fuzzy set* A on $\mathscr{U}$ is usually denoted as: $A =_{df} \{\mu_A(u_1)/u_1, \dots, \mu_A(u_n)/u_n\}$. The notions of unary fuzzy and unary semi-fuzzy quantifiers[†] are given in the next two definitions (Losada D.E. et al. 2006).

Definition 2.44 (*unary fuzzy quantifier*)

A unary fuzzy quantifier $\widetilde{Q}$ on a base set $\mathscr{U} \neq \varnothing$ is a mapping $\widetilde{Q} : \widetilde{\mathbb{P}}(\mathscr{U}) \to [0,1]$.

Definition 2.45 (*unary semi-fuzzy quantifier*)

A unary semi-fuzzy quantifier Q on a base set $\mathscr{U} \neq \varnothing$ is a mapping $Q : \mathbb{P}(\mathscr{U}) \to [0,1]$.

The following example was given in (Losada D.E. et al. 2006).

Example 2.44(*relative semi-fuzzy quantifier*)

Let $(Q_{about\_half})_{\mathscr{U}} : \mathbb{P}(\mathscr{U}) \to [0,1]$ be defined as follows. It is assumed that $\mathscr{U}$ is finite and $d =_{df} |A|/|\mathscr{U}|$, where $A \subseteq \mathscr{U}$.

$$(Q_{about\_half})_{\mathscr{U}}(A) =_{df} \begin{cases} 0 & , d < 0.3 \\ 2(\frac{d-0.3}{0.2})^2 & , 0.3 \leq d < 0.4 \\ 1 - 2(\frac{d-0.5}{0.2})^2 & , 0.4 \leq d < 0.6 \\ 2(\frac{d-0.7}{0.2})^2 & , 0.6 \leq d < 0.7 \\ 0 & , \text{otherwise.} \end{cases}$$

---

[*] See also: Section 7 of Chapter III of this book. Provided there is no ambiguity and for convenience, instead of X and in accordance with Subsection 3.8, the universum is here denoted by $\mathscr{U}$.

[†] Or equivalently: *monadic fuzzy* and *monadic semi-fuzzy quantifiers of type* (1), see Subsection 3.8.

Let $\mathscr{U} =_{df} \{u_1, \ldots, u_{10}\}$ be a set of individuals. Assume that $A = \{u_1, u_4, u_8, u_{10}\}$ is a subset containing these individuals which are taller than 1.70 m. Hence, the evaluation of the expression '*about half of people are taller than 1.70m*', $(Q_{about\_half})_{\mathscr{U}}(A) = 1 - 2(\frac{0.4-0.5}{0.2})^2 = 0.5$. □

The (general concept of the) quantifier fuzzification mechanism is introduced in the next definition (here it is used the unary version of this mechanism).

Definition 2.46 (*quantifier fuzzification mechanism*)[*]

A QFM can be considered as a mapping, say χ_{QFM}, with *domain* in the universe of semi-fuzzy quantifier and *codomain* (or equivalently: *range*) in the universe of fuzzy quantifier, where $dom(\chi_{QFM}) \subseteq \mathbb{P}(\mathscr{U})$ and $cod(\chi_{QFM}) \subseteq \widetilde{\mathbb{P}}(\mathscr{U})$. If $dom(\chi_{QFM}) = \mathbb{P}(\mathscr{U})$ then $\chi_{QFM} : \mathbb{P}(\mathscr{U}) \rightarrow \widetilde{\mathbb{P}}(\mathscr{U})$.

There exist different vertions for QFM (Glöckner I. 1999, 2004). As an example, the notion of α-cut was used in (Losada D.E. et al. 2006)[†]. Here, e.g. the following QFM was used for (a unary semi-fuzzy quantifier) Q:

$\int_0^1 Q(A_{\geq \alpha})\, d\alpha$ or (if $\mathscr{U}$ is finite) as a corresponding finite sum, equivalent to this one given by the *OWA method*[‡] (Delgado M. et al. 2000).

The introduction of QFM's to generalised quantifier theory can be considered as an important step in the development of the linguistics and computer science. A more formal treatment is here omitted, see also (Glöckner I. 2006, 2009). A survey in this area is given in (Dvořák A. and Holčapek M. 2018): left to the reader.

4.2. Modal, deontic and temporal calculus

The logical rules used in modal, deontic and temporal predicate logics are an extension of these ones applied in propositional logic with corresponding rules concerning quantifiers, see Subsection 2.3. Unfortunately, the corresponding deontic version of *Gödel's axiom* G1, i.e. ´! φ ⇒ φ´ is not satisfied.[§] Without loss of generality and for simplicity, the considerations in this subsection are restricted only to the two basic types of quantifiers used in the classical first-order predicate calculus.

Modal predicate calculus

With each of the two basic (*universal* and *existential*) quantifiers can be associated a modal functor of *necessity* or also a modal functor of *possibility*. Moreover, any of the last two basic quantifiers may be bounded or not. And so, we can obtain a total of eight rules of omitting an universal and an existential quantifiers.

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[*] Provided there is no ambiguity, Definition 2.46 is a modification of this one given in (Losada D.E. et al. 2006) wrt to the following expression: '$F : (Q : \mathbb{P}(\mathscr{U}) \mapsto [0,1]) \mapsto (\widetilde{Q} : \widetilde{\mathbb{P}}(\mathscr{U}) \mapsto [0,1])$'.

[†] Given a fuzzy set $A \in \widetilde{\mathbb{P}}(\mathscr{U})$ and $\alpha \in [0,1]$, the *cut of level α* of A, in short: *α-cut* of A, is the crisp set $A_{\geq \alpha} =_{df} \{u \in \mathscr{U} / \mu_A(u) \geq \alpha\}$, see Section 7 of this book.

[‡] The Ordered Weighted Averaging method (Yager R.R. 1991, 1992).

[§] Consider the following formula: !p ⇒ p. We have:

(1)	!p	{ a}
(2)	~ p	{aip}
(3)	□ (~ p ⇒ ∗)	{df ´!´ ; 1}
(4)	~ p ⇒ ∗	{G1}
(5)	∗	{– C : 2,4} ?

Let first consider the following formula: ´$\Box \underset{x}{\exists} A(x)$´. Assume that this formula is satisfied. So, there exists some ´a´ such that ´$\Box A(a)$´ (i.e. some ´a´ having necessarily this property ´A´). And hence, the following implication is satisfied: $\Box \underset{x}{\exists} A(x) \Rightarrow \Box A(a)$[*] (in a similar way, considering the universal quantifier or the modal functor of possibility). As an example, the *rule of omitting a bounded existential quantifier*, given in Subsection 3.2 and denoted below by ´$- \Box \exists^*$´), is now represented as follows.

$$- \Box \exists^* : \quad \frac{\Box \underset{\varphi(x)}{\exists} \psi(x)}{\begin{array}{l}\Box\, \varphi(x / \xi_{\beta_1,\beta_2,\,\ldots\,,\,\beta_n}) \\ \Box\, \psi(x / \xi_{\beta_1,\beta_2,\,\ldots\,,\,\beta_n})\end{array}}$$

Proof($- \Box \exists^*$):

$$\begin{array}{lll}\Box \underset{\varphi(x)}{\exists} \psi(x) & \Leftrightarrow_{df} & \Box \underset{x}{\exists} (\varphi(x) \wedge \psi(x)) \\ & \Rightarrow & \Box (\varphi(x / \xi_{\beta_1,\beta_2,\,\ldots\,,\,\beta_n}) \wedge \psi(x / \xi_{\beta_1,\beta_2,\,\ldots\,,\,\beta_n})) \\ & \Rightarrow & \Box\, \varphi(x / \xi_{\beta_1,\beta_2,\,\ldots\,,\,\beta_n}) \wedge \Box\, \psi(x / \xi_{\beta_1,\beta_2,\,\ldots\,,\,\beta_n}). \ \Box \quad \{T\ 2.23 \text{ of Subsection 2.3, SR, } - K\}\end{array}$$

Some example theses are given below.

Thesis 2.149

$$\Box \underset{x}{\exists} \underset{y}{\forall} R(x,y) \Rightarrow \Diamond \underset{y}{\forall} \underset{x}{\exists} R(x,y)$$

Proof:

(1)	$\Box \underset{x}{\exists} \underset{y}{\forall} R(x,y)$	{a}
(2)	$\sim \Diamond \underset{y}{\forall} \underset{x}{\exists} R(x,y)$	{aip}
(3)	$\Box \underset{y}{\exists} \underset{x}{\forall} \sim R(x,y)$	{N◇,N∀,N∃,SR : 2}
(4)	$\underset{x}{\exists} \underset{y}{\forall} R(x,y)$	{- □ : 1}[†]
(5)	$\underset{y}{\exists} \underset{x}{\forall} \sim R(x,y)$	{- □ : 3}
(6)	$\underset{y}{\forall} R(a,y)$	{– ∃ : 4}
(7)	$\underset{x}{\forall} \sim R(x,b)$	{– ∃ : 5}
(8)	R(a,b)	{– ∀ : 6}
(9)	~ R(a,b)	{– ∀ : 7}
	contr. □	{8,9}

Thesis 2.150

$$\Diamond \underset{x}{\forall} A(x) \vee \Diamond \underset{x}{\forall} B(x) \Rightarrow \Diamond \underset{x}{\forall} (A(x) \vee B(x))$$

Proof:

(1)	$\Diamond \underset{x}{\forall} A(x) \vee \Diamond \underset{x}{\forall} B(x)$	{a}

[*] Concerns a set of ´*individuals*´ that all belong to the same universum.

[†] Corresponds to G1.

(2)	$\sim \Diamond \forall_x (A(x) \vee B(x))$	{aip}
(3)	$\Box \exists_x (\sim A(x) \wedge \sim B(x))$	$\{N\Diamond, N\forall, NA, SR : 2\}$
(4)	$\Box (\sim A(a) \wedge \sim B(a))$	$\{-\ \Box\exists : 3\}$
(5)	$\Box \sim A(a)$	{ T 2.23 of Subsection 2.3, SR, – K}
(6)	$\Box \sim B(a)$	
(7)	$\sim \Diamond A(a)$	$\{N\Diamond :\ 5,6\}$
(8)	$\sim \Diamond B(a)$	
(1.1)	$\Diamond \forall_x A(x)$	{ada}
(1.2)	$\Diamond A(a)$	$\{-\ \Diamond\forall :\ 1.1\}$
	contr.	{7,1.2}
(2.1)	$\Diamond \forall_x B(x)$	{ada}
(2.2)	$\Diamond B(a)$	$\{-\ \Diamond\forall :\ 2.1\}$
	contr. □	{8,2.2}

Thesis 2.151(*prenex form*)

$$\Diamond \forall_x A(x) \vee \Diamond \forall_x B(x) \Leftrightarrow \Diamond \forall_x \Diamond \forall_y (A(x) \vee B(y))$$

Proof(if-implication):

(1)	$\Diamond \forall_x A(x) \vee \Diamond \forall_x B(x)$	{a}
(2)	$\Box \exists_x \Box \exists_y (\sim A(x) \wedge \sim B(y))$	$\{aip/N\Diamond, N\forall, NA, SR\}$
(3)	$\Box \sim A(a)$	$\{-\ \Box\exists$, T 2.23 of Subsection 2.3, SR, – K : 2}
(4)	$\Box \sim B(b)$	
(5)	$\sim \Diamond A(a)$	{3,4}
(6)	$\sim \Diamond B(b)$	
(1.1)	$\Diamond \forall_x A(x)$	{ada}
(1.2)	$\Diamond A(a)$	$\{-\ \Diamond\forall :1.1\}$
	contr.	{5,1.2}
(2.1)	$\Diamond \forall_x B(x)$	{ada}
(2.2}	$\Diamond B(b)$	$\{-\ \Diamond\forall : 2.1\}$
	contr. □	{6,2.2}

Proof(only-if-implication):

(1)	$\Diamond \forall_x \Diamond \forall_y (A(x) \vee B(y))$	{a}
(2)	$\sim \Diamond \forall_x A(x)$	{aip}

(3) $\sim \Diamond \underset{x}{\forall} B(x)$

(4) $\Box \underset{x}{\exists} \sim A(x)$ {N◇,N∀,SR : 2}

(5) $\Box \underset{x}{\exists} \sim B(x)$ {N◇,N∀,SR : 3}

(6) $\Box \sim A(a)$ {- □∃ : 4}

(7) $\Box \sim B(b)$ {- □∃ : 5}

(8) $\Box \sim A(a) \wedge \Box \sim B(b)$ {+ K : 6,7}

(9) $\Box (\sim A(a) \wedge \sim B(b))$ {T 2.23 of Subsection 2.3 : 8}

(10) $\Box \sim (A(a) \vee B(b))$ {NA,SR : 9}

(11) $\sim \Diamond(A(a) \vee B(b))$ {N◇ : 10}

(12) $\Diamond(A(a) \vee B(b))$ {– ◇∀ : 1}

contr. □ {11,12}

The following thesis is a modal version of T 2.137, given in Subsection 3.3.

Thesis 2.152

$$\Box \underset{A(x)}{\exists} \underset{B(y)}{\forall} R(x,y) \Rightarrow \Diamond \underset{B(y)}{\forall} \underset{A(x)}{\exists} R(x,y)$$

Proof:

(1) $\Box \underset{A(x)}{\exists} \underset{B(y)}{\forall} R(x,y)$ {a}

(2) $\sim \Diamond \underset{B(y)}{\forall} \underset{A(x)}{\exists} R(x,y)$ {aip}

(3) $\Box \underset{B(y)}{\exists} \underset{A(x)}{\forall} \sim R(x,y)$ {N◇,N∀*,N∃*,SR : 2}

(4) $\Box A(a)$

(5) $\Box \underset{B(y)}{\forall} R(a,y)$ {– □ ∃* : 1}

(6) $\Box B(b)$

(7) $\Box \underset{A(x)}{\forall} \sim R(x,b)$ {– □ ∃* : 3}

(8) $\Box \underset{y}{\forall} (B(y) \Rightarrow R(a,y))$ {SR : 5}

(9) $\Box (B(b) \Rightarrow R(a,b))$ {– □ ∀ : 8}

(10) $\Box \underset{x}{\forall} (A(x) \Rightarrow \sim R(x,b))$ {SR : 7}

(11) $\Box (A(a) \Rightarrow \sim R(a,b))$ {– □ ∀ : 10}

(12) $\Box B(b) \Rightarrow \Box R(a,b)$ {G2, – C : 9}*

(13) $\Box A(a) \Rightarrow \Box \sim R(a,b)$ {G2, – C : 11}

(14) $\Box R(a,b)$ {– C : 6,12}

* It is used Gödel's axiom G2: see Subsection 2.3.

(15)	$\Box \sim R(a,b)$	{– C : 4,13}
(16)	R(a,b)	{- $\Box$: 14}
(17)	$\sim$ R(a,b)	{- $\Box$: 15}
	contr. $\Box$	{16,17}

Deontic predicate calculus

Some example theses related to deontic predicate logic are given below. Traditionally, as in the previous considerations, it is used assumptional proof style. In accordance with the correspondence between modal an deontic functors (i.e. ´$\Box$, $\Diamond$´ and ´!, δ´ , respectively), e.g. the following rule, related to ´– $\Box$ $\exists^*$ ´, can be introduced (here, instead of the modal T 2.23, it is used the deontic thesis T 2.50).

$$-!\ \exists^* : \quad \frac{!\ \underset{\varphi(x)}{\exists}\ \psi(x)}{\begin{array}{l} !\ \varphi(x / \xi_{\beta_1,\beta_2,\ \dots\ ,\ \beta_n}) \\ !\ \psi(x / \xi_{\beta_1,\beta_2,\ \dots\ ,\ \beta_n}) \end{array}}$$

Thesis 2.153

$! \underset{x}{\forall} A(x) \ \Rightarrow\ ! \underset{x}{\exists} A(x)$

Proof:

(1)	$! \underset{x}{\forall} A(x)$	{a}
(2)	$\sim ! \underset{x}{\exists} A(x)$	{aip}
(3)	$\delta \sim \underset{x}{\exists} A(x)$	{N! , see T 2.48 : 2}
(4)	$\delta \underset{x}{\forall} \sim A(x)$	{N∃,SR : 3}
(5)	$\delta \sim A(a)$	{– δ ∀ : 4}
(6)	$\sim ! A(a)$	{N! : 5}
(7)	$! A(a)$	{– ! ∀ : 1}
	contr. $\Box$	{6,7}

Thesis 2.154

$\delta \underset{x}{\forall} A(x) \ \Rightarrow\ \delta \underset{x}{\exists} A(x)$

Proof:

(1)	$\delta \underset{x}{\forall} A(x)$	{a}
(2)	$\sim \delta \underset{x}{\exists} A(x)$	{aip}
(3)	$! \underset{x}{\forall} \sim A(x)$	{Nδ,see T 2.49,N∃,SR : 2}
(4)	$\delta A(a)$	{– δ ∀ : 1}
(5)	$! \sim A(a)$	{– ! ∀ : 3}
(6)	$\sim \delta A(a)$	{ Nδ : 5}
	contr. $\Box$	{5,6}

Thesis 2.155

$! \underset{x}{\forall} A(x) \;\Rightarrow\; \delta \underset{x}{\exists} A(x)$

Proof:

(1)	$! \underset{x}{\forall} A(x)$	{a}
(2)	$\sim \delta \underset{x}{\exists} A(x)$	{aip}
(3)	$! \underset{x}{\forall} \sim A(x)$	{Nδ,see T 2.49,N∃,SR : 2}
(4)	$!\, A(a)$	{– ! ∀ : 1}
(5)	$! \sim A(a)$	{– ! ∀ : 3}
(6)	$\delta\, A(a)$	{T 2.54 of Subsection 2.3 : 4}
(7)	$\sim \delta\, A(a)$	{ Nδ : 5}
	contr. □	{6,7}

The following formula is a deontic version of T 2.152.

Thesis 2.156

$! \underset{A(x)}{\exists} \underset{B(y)}{\forall} R(x,y) \;\Rightarrow\; \delta \underset{B(y)}{\forall} \underset{A(x)}{\exists} R(x,y)$

Proof:

(1)	$! \underset{A(x)}{\exists} \underset{B(y)}{\forall} R(x,y)$	{a}
(2)	$\sim \delta \underset{B(y)}{\forall} \underset{A(x)}{\exists} R(x,y)$	{aip}
(3)	$! \underset{B(y)}{\exists} \underset{A(x)}{\forall} \sim R(x,y)$	{Nδ,N∀*,N∃*,SR : 2}
(4)	$!\, A(a)$	{–! ∃* : 1}
(5)	$! \underset{B(y)}{\forall} R(a,y)$	
(6)	$!\, B(b)$	{–! ∃* : 3}
(7)	$! \underset{A(x)}{\forall} \sim R(x,b)$	
(8)	$! (B(b) \;\Rightarrow\; R(a,b))$	{– ! ∀ : 5}
(9)	$! (A(a) \;\Rightarrow\; \sim R(a,b))$	{– ! ∀ : 7}
(10)	$!\, B(b) \;\Rightarrow\; !\, R(a,b)$	{T 2.55 of Subsection 2.3, – C : 8}
(11)	$!\, A(a) \;\Rightarrow\; ! \sim R(a,b)$	{T 2.55 of Subsection 2.3, – C : 9}
(12)	$!\, R(a,b)$	{– C : 6,10}
(13)	$! \sim R(a,b)$	{– C : 4,11}
(14)	$\delta\, R(a,b)$	{T 2.54 of Subsection 2.3, – C : 12}
(15)	$\sim \delta\, R(a,b)$	{T 2,49 of Subsection 2.3 : 13}
	contr. □	}14,15}

Temporal predicate calculus

The axioms and rules presented in Subsection 2.3 (see *Manna and Pnueli's temporal logic*) deal only with the propositional fragment of this logic. The considered deductive system based on these axioms and rules is complete for proving the validity of any propositional temporal formula (Manna Z. and Pnueli A. 1992)*. In accordance with the last work, there was also presented an extension of this system by additional axioms and rules, to deal with the first-order elements such as: variables, equality and quantification (and hence, handling a large number of cases). Unfortunately, this extension do not lead to a complete system.

The notion of state x-variant is generalised at the level of a model. Assume that σ: $s_0, s_1, s_2, \ldots$ and σ': $s_0', s_1', s_2', \ldots$ are two models over $V \subseteq \mathscr{V}$ (the vocabulary). We shall say that σ' is a *x-variant* of σ if s_j' is a x-variant of s_j (i.e. differs from s_j by at most the interpretation given to x: see Subsection 2.3). The following two definitions were used for an existentially and a universally quantified formulae, respectively (Manna Z. and Pnueli A. 1992).

$$(\sigma,j) \vDash \exists_x \varphi \;\Leftrightarrow_{df}\; (\sigma',j) \vDash \varphi \quad \text{for some } \sigma', \text{ a x-variant of } \sigma$$

$$(\sigma,j) \vDash \forall_x \varphi \;\Leftrightarrow_{df}\; (\sigma',j) \vDash \varphi \quad \text{for every } \sigma', \text{ a x-variant of } \sigma$$

According to the last work, the introduction of variables into formulae should require the consideration of schemes containing sentence symbols with parameters. In the case that the considered formula φ contains quantifications, it is required that the process of any such instantiation does not capture the occurrences of variables that are free in the replacing formula. In general, any such *replacement* $\alpha : p(x_1, \ldots, x_n) \leftarrow \chi(x_1, \ldots, x_n)$ should be *amissible* for φ ($n \geq 0$). If α is a replacement admissible for φ then $\varphi[\alpha]$ is referred as an *admissible instantiation* of φ.

Below it is assumed that V is partitioned into two subsets of variables: rigid and flexible ones. A *rigid variable* must have the same value in all states of a computation, while a *flexible variable* may assume different values in different states of computation. And hence, it is assumed that all occurrences of parametrised sentence symbols in φ must be rigid (Manna Z. and Pnueli A. 1992). In particular, the following example was given in this work.

Example 2.45(*admissible instantiation*)

Let $v \in V$ be rigid. Consider the following state-valid scheme.

$$\varphi : \forall_v ((x \geq v) \Rightarrow p(v)) \;\Rightarrow\; \forall_v ((x \geq v+1) \Rightarrow p(v+1))$$

Applying rule GEN followed by the instantiation: $\alpha : p(u) \leftarrow \diamond(x = u)$, we can obtain the following valid entailment $\Box\, \varphi[\alpha]$.

$$\forall_v ((x \geq v) \Rightarrow \diamond(x = v)) \;\Rightarrow\!>\; \forall_v ((x \geq v+1) \Rightarrow \diamond(x = v+1))^{\dagger}. \;\Box$$

In accordance with the above considerations, for instantiations of schemes that may contain quantifiers, the applied replacements must be admissible. And hence, the following *stipulation to rule INST* is introduced (Manna Z. and Pnueli A. 1992): in the use of INST for deriving $\varphi[\alpha]$ from φ, the replacement α must be

* Provided there is no ambiguity and for convenience, in the next considerations, we shall use basic notions related to this subsection.

† Since ′⇒′ and ′⇔′ denote the logical connectives of implication and equivalence, it is used here the same form of the corresponding abbreviations (Manna Z. and Pnueli A. 1992), as these used in Subsection 2.3: $\varphi \Rightarrow\!> \psi \Leftrightarrow_{df} \Box(\varphi \Rightarrow \psi)$ and $\varphi <\!\Leftrightarrow\!> \psi \Leftrightarrow_{df} \Box(\varphi \Leftrightarrow \psi)$, respectively.

admissible for φ. Some additional notions and/or axioms concerning variables, equality and quantifiers, related to this work, are also illustrated below.

The introduction of variables should require replacements of the form: ´$x \leftarrow e$´, which replace a variable x with an *expression* e. Any such expression e is said to be *rigid* if it does not refer to any flexible variable. This replacement is called *compatible* if either both x and e are rigid or x is flexible. Assume that $p(x)$ has one or more occurrences of x and that there is no quantification over x. We shall say that $x \leftarrow e$ is *admissible* for $p(x)$ if it is compatible and none of the variables appearing in e is quantified in $p(x)$. According to the last considerations, we shall also say that e is *admissible* for $p(x)$ and we shall write $p(e)$ for the instantiated formula $p(x)[x \leftarrow e]$. The following axioms concerning equality were introduced.

(REFL-E)	$\Box(e = e)$	{the axiom of *reflexivity for equality*}
(REPL-E)	$(e_1 = e_2) \Rightarrow> (p(e_1) \Leftrightarrow p(e_2))$	{the axiom of *replacement of equals by equals*}
(SUBS-E)	$\Box(e_1 = e_2) \Rightarrow (p(e_1) <\Leftrightarrow> p(e_2))$	{the axiom of *substitutivity of equality*}
(FRAME)	$p \Rightarrow> \circ p$	{the *FRAME axiom*}

It is assumed in the last axiom that the state formula p is rigid. In particular, in accordance with this axiom, there were derived some theses (Manna Z. and Pnueli A. 1992). As an example, the proof of the opposite implication (wrt FRAME), i.e. ´$\circ p \Rightarrow> p$´ is illustrated below. We shall first present the proof of the following rule (R1, described as a sequent): $p <\Leftrightarrow> q \vdash p \Rightarrow> q$.

Proof(R1):

(1)	$p <\Leftrightarrow> q$	{a}
(2)	$\Box(p \Leftrightarrow q)$	{df ´$<\Leftrightarrow>$´: 1}
(3)	$\Box((p \Rightarrow q) \wedge (q \Rightarrow p))$	{ df ´$\Leftrightarrow$´, SR : 2}
(4)	$\Box(p \Rightarrow q)$	{T 2.94, – K : 3}
	$p \Rightarrow> q$. □	{df ´$\Rightarrow>$´: 4}

Proof($\circ p \Rightarrow> p$):

(1)	$\sim p \Rightarrow> \circ \sim p$	{FRAME}
(2)	$\Box(\sim p \Rightarrow \circ \sim p)$	{df ´$\Rightarrow>$´: 1}
(3)	$\Box(\sim \circ \sim p \Rightarrow p)$	{CC, SR : 2}
(4)	$\sim \circ \sim p \Rightarrow> p$	{ df ´$\Rightarrow>$´: 3}
(5)	$\circ \sim p \Rightarrow> \sim \circ p$	{FA2, R1}
(6)	$\Box(\circ \sim p \Rightarrow \sim \circ p)$	{df ´$\Rightarrow>$´: 5}
(7)	$\Box(\circ p \Rightarrow \sim \circ \sim p)$	{CC, SR : 6}
(8)	$\circ p \Rightarrow> \sim \circ \sim p$	{df ´$\Rightarrow>$´: 7}
	$\circ p \Rightarrow> p$. □	{E-TRANS : 8,4}

In the next considerations, there are first presented two axioms characterising properties of the *next* and *previous* values of variables, denoted below by x^+ and x^-, respectively. Then there are also given several axioms and rules concerning quantifiers (Manna Z. and Pnueli A. 1992).

Let u be a rigid variable and x be either rigid or flexible one. By $\varphi(u,x)$ it is denoted a state formula in which the only free variables are u or x or both. The following two axioms are introduced.

(NXTV) $\varphi(u,x^+)$ <⇔> $\bigcirc\varphi(u,x)$

(PRVV) $\varphi(u,x^-)$ <⇔> (*first*[*] $\wedge$ $\varphi(u,x)$) $\vee$ $\ominus\varphi(u,x)$

The above two axioms can be generalised for state formulae of the form $\varphi(u_1, \dots ,u_m, x_1, \dots ,x_n)$, where $u_1, \dots , u_m$ are rigid. Moreover, in accordance with these two axioms, the following two properties can be shown: $\Box(u = u^+)$ and $\Box(u = u^-)$. By Thesis 2.94 and FA1 it follows that: $u = u^+ = u^-$.

The following *axioms for quantifiers* were introduced. The two formulae, presented in Q-DUAL (*quantifier duality*) are similar to theses T 2.135 and T 2.136 (see Subsection 3.3).

(Q-DUAL) $\sim \exists_x\, p(x)$ <⇔> $\forall_x \sim p(x)$

$\sim \forall_x\, p(x)$ <⇔> $\exists_x \sim p(x)$

The next two axioms (*quantifier instantiation* and *universal commutation*[†]) are given as follows (u, x denote variables and p(u), p(x) are formulae, e is an admissible expression for p(u)).

(∀-INS) $\forall_u\, p(u)$ ⇒> p(e)

(∀○-COM) $\forall_x \bigcirc p(x)$ <⇔> $\bigcirc \forall_x\, p(x)$

According to the last axiom, similar formulae related to ∀⊖-COM, ∃○-COM and ∃⊖-COM are also derived. Moreover, similar properties are satisfied for the weak version ´$\tilde{\ominus}$ ´ (left to the reader).

There is only one basic inference rule concerning quantifiers. Let u be a variable, p and q(u) – two formulae. Assume that u has no free occurrences in p. This rule, called *universal generalisation* and denoted by ´∀-GEN´, is represented as follows (Manna Z. and Pnueli A. 1992).

(∀-GEN) $\dfrac{p \Rightarrow> q(u)}{p \Rightarrow> \forall_u\, q(u)}$

Proof(∀-GEN)*:*

Let ´p ⇒> q(u) ´ be valid and σ be an arbitrary model. Assume that p holds at position j of σ[‡]. We should show that ´$\forall_u$ q(u) ´ also holds at that position. Consider any σ´, an u-variant of σ. Since p does not depend on u, p holds at (σ´,j). Then ´q(u) ´ also holds at (σ´,j)[§]. Thus, ´q(u) ´ holds at j of all u-variants of σ. And hence, ´$\forall_u$ q(u) ´ holds at (σ,j). □

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[*] ´~ ⊖T (or equivalently: $\tilde{\ominus}$ F)´: 'This formula states that there is no previous position that satisfies ´ T ´. Since all positions that are in the model satisfy ´ T ´, this is equivalent to the following: ´*there is no previous position*´. Note that this formula always holds at the initial position of every model, and nowhere else' (Manna Z. and Pnueli A. 1992).

[†] Known as one of *Barcam's axioms* (Ruth Barcan Marcus: 1921 – 2012).

[‡] In short: '*p holds at (σ,j)*'

[§] The proof is given below.

(1)	p ⇒> q(u)	{ a}
(2)	□(p ⇒ q(u))	{ df ´⇒>´: 1}
(3)	p	{p does not depend on u}
(4)	p ⇒ q(u)	{FA1}

In accordance with the above axioms and the last rule, many derived rules can be obtained, e.g. E-INST (expression instantiation), ∃-INTR (∃-introduction), QQ-INTR, where ´Q´ is either universal or existential quantifier, defined as ´∀∀−INTR´ and ´∃∃−INTR´, (four rules) respectively, etc. (Manna Z. and Pnueli A. 1992):

(E-INST) $p(u) \vdash p(e)$ (if ´e´ is admissible for p(u))

(∃-INTR) $p(u) \Rightarrow> q \vdash \exists_u (p(u) \Rightarrow> q)$ (if ´u´ has no free occurrences in ´q´)

(QQ-INTR) $p(x) \Rightarrow> q(x) \vdash Q_x\, p(x) \Rightarrow> Q_x\, q(x)$

$p(x) <\Leftrightarrow> q(x) \vdash Q_x\, p(x) <\Leftrightarrow> Q_x\, q(x)$

Here, as an illustration, is the proof of one of the rules ´∀∀−INTR´, presented in this work.

Proof($p(x) \Rightarrow> q(x) \vdash \forall_x\, p(x) \Rightarrow> \forall_x\, q(x)$)

(1) $p(x) \Rightarrow> q(x)$ {a}

(2) $\forall_x\, p(x) \Rightarrow> p(x)$ {∀-INS, $e =_{df} x$}

(3) $\forall_x\, p(x) \Rightarrow> q(x)$ {E-TRANS: 2,1}

$\forall_x\, p(x) \Rightarrow> \forall_x\, q(x)$. □ {∀-GEN : 3}

The above introductory notions are only an illustration of this excellent work. Manna and Pnueli's temporal logic was interpreted over general models. On the other hand, the considered system was also used for such specific models, corresponding to computations and program validity (left to the reader).

As stated earlier, the temporal predicate calculus cannot be axiomatised completely, but this system is sufficient to deduce a lot of useful theorems*. The presented here first-order linear temporal predicate logic includes four future functors, i.e. ´○, □, ◇, U´ and five axioms: *modus ponens*, *necessitation* (similarly to GR in modal logic: $\models \varphi \Rightarrow \models \Box\, \varphi$), *generalisation* (see: ´∀-GEN´), *reflexivity* (axiom for equality) and substitutivity (corresponds to ´E-INST´). In the next considerations, there are presented concepts needed for specification and verification of concurrent systems, in particular related to the *mutual exclusion problem*: as an example, Peterson's algorithm is considered (Peterson G.L. 1981). Some other works are cited below.

A temporal predicate calculus based on Łukasiewicz's ternary logic system $Ł_3$ with $W_3 =_{df} \{0, 1/2, 1\}$† was presented in (Chiriţă C. 2008). The proposed system *alphabet* consists of the following primitive symbols: a countable set V of variables (x, y, …), an arbitrary set of constant symbols, an arbitrary set of predicate symbols: with each such symbol is associated a natural number n (> 0, the arity of P), the propositional connectives ´~ ´and ´⇒´, Prior's tense logic functors ´G´ and ´H´ (see Subsection 2.3), the universal quantifier ´∀´ , parentheses ´(,) ´ and square brackets ´[,] ´. In addition to the axioms of three-valued Łukasiewicz's logic, there are used five axioms concerning ´G´,´H´ and ´P´. Here, by F(φ) it is denoted 'the set of free variables of φ. Next, there are given four axioms concerning the universal quantifier. The proposed system uses the following four inference rules: *modus ponens*, *generalisation* and two kinds *temporal generalisation* (wrt ´G´ and ´H´). It is shown that this system is complete. From a formal point of wiev, this system should be defined as a *tense predicate calculus*.

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q(u). □ {– C : 4,3}

* Yakar R., *Temporal logic.Theory and Applications*. Tel - Aviv University 58pp: file:///F:/Temporal%20predicate%20calculus/paper%202.pdf

† See: Subsection 2.1.

In the next work (Venkata S.R.P. 2015), there are studied some (dynamic) problems in *artificial intelligence* involving *time constraints* (e.g. such as: ′*before time*′, ′*after time*′, ′*in time*′, etc.). And hence, the considered problems may contain an incomplete information, i.e. *knowledge representation*. Any proposition containing an incomplete information can be considered as a fuzzy one. Moreover, any such proposition may contain some time constrains. So, it is proposed a *fuzzy temporal logic* system dealing with an incomplete information.

'Temporal specifications are often used when phenomena are modelled and dynamics play a main role. If simulation is one of the aims of modelling, usually a restricted, executable modelling language format is used, based on some form of past to future implications. In this paper a detailed transformation procedure is described that takes any temporal predicate logic specification and generates a specification in a past implies future normal format. The procedure works for temporal specifications in which the atoms either express time ordering relation or are state-related, i.e. include only one time variable' Treur J.[*] The considered approach applies to more than one specific logic. In particular, it is introduced the notion of *state - time partitioned formula* and it is assumed that any formula is state - time partitioned[†].

'The affinity of special and temporal phenomena has been recognised for a long time in the literature. This paper investigates temporal changes of topological relationships and thereby integrates two important research areas: first, two-dimensional topological relationships that have been investigated quite intensively, and second, the change of special information over time that has recently been identified as an important research topic'[‡]. In this work, there are investigated *spatio - temporal predicates* for describing developments of special topological relationships. In accordance with this work, the last predicates can be obtained by temporal aggregation of elementary spatial predicates and sequential composition. For this purpose, it is presented a new *spatio - temporal data model* (an extension of spatial data model to temporal concepts: a more integrated view of space and time, allowing the treatment of continuous special changes). The concept of *spatio - temporal data types*[§] is next introduced. A temporal version of an object of type α is then given by a function from time to α (motivated by the observation that anything that changes over time can be expressed as a function over time), see: (Erwig M. et al. 1998). And hence, *spatio-temporal objects* are considered as special instances of *temporal objects* where α is a spatial data type (like point or region). The universal and existential quantifiers are overloaded considering these two quantifiers as (temporal) *universal* and *existential aggregations*, respectively. Let p be a spatial predicate and S_1 and S_2 be two spatio - temporal objects. The *semantics of existential quantification* is presented as follows: $\exists$ p (S_1, S_2) is true iff p is true for the values of S_1 and S_2 at some time, where $\exists\, p =_{df} \lambda (S_1, S_2). \exists\, t: p (S_1(t), S_2(t))$[**] and ′t′ ranges over time. Let now t_1 and t_2 be the respective domains of S_1 and S_2. As it was shown in this work, the *semantics of universal quantification* $\forall p (S_1, S_2)$ should depend in general, on whether quantification ranges over: *time*, $t_1 \cup t_2$, t_1, t_2, $t_1 \cap t_2$. The most restrictive is the first case, i.e. ′*time*′. Let $\gamma \in \{\cup, \cap, \pi_1, \pi_2\}$, where $\pi_i(x_1, \ldots, x_i, \ldots, x_n) =_{df} x_i$. The obtained quantifiers are defined as follows: $\forall_\gamma\, p =_{df} \lambda (S_1, S_2). \forall t \in \gamma(dom(S_1), dom(S_2)): p (S_1(t), S_2(t))$. In the next considerations are introduced *basic spatio-temporal predicates*, defined by temporal lifting and aggregation (left to the reader).

'Computer systems continue to grow in complexity and the distinctions between hardware and software keep on blurring. Out of this has come an increasing awareness of the need for behavioral models suited for specifying and reasoning about both digital devices and programs. Contemporary hardware description languages are not

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[*] Treur J., *Past-future separation and normal forms in temporal predicate logic specifications*. Vrije Universiteit Amsterdam: 27pp: file:///F:/Temporal%20predicate%20calculus/paper%205.pdf.

[†] An atom that involves only one time variable or constant is called *state - related atom*. A formula is called *state - time partitioned* if every atom that occurs in it either is a state – related atom or *time - ordering atom* (e.g. ′$t_1 < t_2$′).

[‡] Erwig M. and Schneider M., *Spatio-temporal predicates*. FernUniversität Hagen. 38pp: file:///F:/Temporal%20predicate%20calculus/paper%204.pdf.

[§] 'Abstract data types whose values can be interpreted as complex entities into databases and whose definition and integration into databases is independent of a particular DBMS model', i.e. a particular data base model and structure.

[**] The used the *lambda notation*, in general: ′$\lambda(x_1, \ldots, x_n). e$′ denotes a function that takes arguments $x_1, \ldots, x_n$ and returns a value determined by the expression e.

sufficient because of various conceptual limitations' (Moszkowski B. 1982). This report describes 'a logical notation for reasoning about digital circuits. The formalism provides a rigorous and natural basis for device specification as well as for proving properties such as correctness of implementation. Conceptual levels of circuit operation ranging from detailed quantitative timing and signal propagation up to functional behavior are integrated in a unified way. A temporal predicate calculus serves as the forward core of the notation, resulting in a versatile tool that has more descriptive power than any conventional hardware specification language'. This notation is based on discrete time intervals and combines aspects of standard temporal logics with features of dynamic logic (Moszkowski B. 1982).

4.3. Other systems

Some considerations concerning the intuitionistic and paraconsistent predicate logic systems are briefly presented below.

Intuitionistic predicate logic

Intuitionistic predicate logic can be introduced as an extension of the classical intuitionistic propositional calculus by adding some axioms and rules related to the classical predicate logic. The used below notations are in accordance with (Hilbert D. and Bernays P. 1934, 1939)[*], where syntactic variables are used not only to denote rules, but also to denote theses related to the subject language (*The little encyclopaedia of logic* 1988).

Let φ be a thesis in intuitionistic propositional logic. Then φ is accepted as an axiom in intuitionistic predicate logic. As axioms, are also accepted formulae related to the following two schemes (denoted in the metalanguage).

$$\underset{x}{\forall} B(x) \Rightarrow B(a)$$

$$B(a) \Rightarrow \underset{x}{\exists} B(x)$$

The following two deduction rules are introduced: rules of *joining an universal* and *an existential quantifiers* (denoted in the same way, i.e. by ′+∀′ and ′+∃′, respectively).

If $B \Rightarrow A(x)$ then $B \Rightarrow \underset{x}{\forall} A(x)$

If $A(x) \Rightarrow B$ then $\underset{x}{\exists} A(x) \Rightarrow B$

In accordance with the last two rules, it is assumed that $A(x)$ is an arbitrary formula having x as a free variable, whereas B is a formula in which x is not free (*The little encyclopaedia of logic* 1988). Here, the following example was given.

Example 2.46(a *requirement for B*)

Let $B \Rightarrow A(x)$ be the following formula: $(x < 2) \Rightarrow (x < 3)$. Assume that x is free. According to ′+∀′, by assuming in the antecedent of the above implication that $x =_{df} 1$, we can obtain: $(1 < 2) \Rightarrow \underset{x}{\forall} (x < 3)$. Let now $A(x) \Rightarrow B$ be the above formula. By assuming in the consequent of the last implication that $x =_{df} 4$, we can obtain: $\underset{x}{\exists} (x < 2) \Rightarrow (4 < 3)$. □

The above two rules ′+∀′ and ′+∃′ can be accepted as *primitive* ones (since they were accepted without using any proofs at all). In accordance with these rules there exists, inter alia, a possibility for obtaining some *derived rules*. As an example, consider the following *rule of generalisation* (denoted below by ′RGEN′).

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[*] David Hilbert (1862 – 1943), Paul Bernays (1888 – 1977)

$$\text{RGEN}: \quad \frac{A(x)}{\forall_x A(x)}$$

The proof of this rule is given below (*The little encyclopaedia of logic* 1988).

Proof(RGEN):

(1)	$A(x)$	{a}
(2)	$A(x) \Rightarrow ((p \vee \sim p) \Rightarrow A(x))$	{law of simplification}
(3)	$(p \vee \sim p) \Rightarrow A(x)$	{– C : 2,1}
(4)	$(p \vee \sim p) \Rightarrow \forall_x A(x)$	{′+∀′: 3}
	$\forall_x A(x)$. □	{– C : 4, law of excluded middle}

The obtained in this way a set of theses should be de facto a proper subset of the classical predicate calculus. As an example, there are no intuitionistic theses the following ones of the classical predicate logic (e.g. wrt the use of the law: $\sim\sim p \Rightarrow p$):

$$\sim \forall_x A(x) \Rightarrow \exists_x \sim A(x)$$
$$\sim \exists_x \sim A(x) \Rightarrow \forall_x A(x)$$
$$\sim\sim \exists_x A(x) \Rightarrow \exists_x \sim\sim A(x)$$
$$\forall_x (A(x) \vee \sim A(x))$$
$$\forall_x (p \vee A(x)) \Rightarrow (p \vee \forall_x A(x))$$

The proof of the last thesis is given below (the rest proofs are omitted: left to the reader).

Proof(the last formula):

(1)	$\forall_x (p \vee A(x))$	{a}
(2)	$\sim (p \vee \forall_x A(x))$	{aip}
(3)	$\sim p$	{NA, – K : 2}
(4)	$\sim \forall_x A(x)$	
(5)	$\exists_x \sim A(x)$	{N∀ : 4}
(6)	$\sim A(a)$	{–∃ : 5}
(7)	$p \vee A(a)$	{–∀ : 1}
(8)	p	{– A : 6,7}
	contr. □	{3,8}

It was shown that the intuitionistic propositional calculus is *decidable* (as in the classical case (see Theorem 1.30 of Subsection 1.6). However, the intuitionistic predicate calculus is not decidable, even assuming that this calculus is one-argument: which is decidable in the classical case (Kripke S. 1965)[*]. On the other hand, the formulae of intuitionistic predicate calculus can be interpreted as open sets of a topological space. The validation V(-) of the calculus formulae into the open sets of a topological space is realised as follows: $(\phi_1, \ldots, \phi_{n-1} \models \phi_n)$

[*] *Semantical analysis of intuitionistic logic*: Saul Aaron Kripke, born 1940.

$\Rightarrow$ $V(\phi_1) \cap \ldots \cap V(\phi_{n-1}) \subseteq V(\phi_n)$. This way, it was presented a *completeness theorem* of this calculus, see: Valentini S.[*] Here, in particular, it was used *Rasiowa - Sikorski*[†] *- like theorem* for the countable *Heyting algebras*, see: (Rasiowa H. and Sikorski R. 1963).

There were given various approaches concerning the (first-order) intuitionistic predicate logic, e.g. (Schütte K. 1977), where the intuitionistic predicate calculus is introduced 'syntactically by means of axioms and basic inferences which are motivated in a natural way by logical inferences'.

Let φ be *derivable i*n the intuitionistic predicate calculus, in short: $\vdash_{IPC} \varphi$. Assume that φ is *valid*, i.e. thesis, in short: $\vDash \varphi$. We can interpreted logical implications as *oriented edges* of a *digraph*, e.g. the implication ′$\varphi \Rightarrow \psi$′ can be represented represented by the oriented edge: (φ,ψ). Next, by ′φ - IT′ we shall denote the fact that ′φ is *intuitively true*′.And hence, in accordance with the last considerations, the following '*oriented cycle*' can be obtained (Swart H.C.M. de. 1977): $(\vdash_{IPC} \varphi, \varphi$ - IT), (φ - IT, $\vDash \varphi$), ($\vDash \varphi, \vdash_{IPC} \varphi$). There were proposed various interpretations and/or suggestions related to this digraph. The models validity in the last work is defined by 'validity in the nodes of some partially ordered set'.

It is sometimes convenient to consider some (*proper*) *superset* of theses of Heyting's system of intuitionistic logic (see Subsection 2.4). Any such superset, closed wrt the primitive rules 'RR'[‡] and '− C', is said to be an *intermediate logic*. The last term was first introduced in (Umezawa T. 1959, 1960) and hence, there was initiated a systematic research in this area. In fact, such logic systems were introduced in an early time, i.e. during the search for semantics of intuitionistic logic (Heyting A. 1930). However, the considered here matrix model was not an adequate such semantical model. But, in fact, this model can be considered as the first intermediate logic (i.e. superintuitionistic logic). The axiomatics of the last system was presented in (Łukasiewicz J. 1938). Here, to Heyting's axiomatic system (of the *intuitionistic propositional logic*, see Subsection 2.4) there was added the following formula: $(\sim p \Rightarrow q) \Rightarrow (((q \Rightarrow p) \Rightarrow q) \Rightarrow q)$. And hence, the last logic is also known as Heyting's – Łukasiewicz's one.[§] In a similar way (adding new formulae to the axioms of intuitionistic logic) there were introduced such logics as: *weak excluded middle logic* (Ghilardi S. 1999), *Dummett's logic*, *Kreisel*[**] *- Putnam's logic*, *Scott's logic*, *Jankov*[††]*'s logic* (Jankov V.A. 1973): see '*Jankov formulas and intermediate logics'* (Bezhanishvili N. and Jongh D. de. 2006), see also (Rybakov V.V. 1992), and so on.

Paraconsistent predicate logic

A formal system (deductive system, deductive theory, . . .) is said to be *inconsistent* if there is a formula φ of the system such that φ and its negation, $\sim\varphi$, are both theorems of this system. In the opposite case, the system is called *consistent* (Da Costa N.C.A. 1974). The propositional logics introduced in (Da Costa N.C.A. and Wolf R.G. 1980) were extended to first-order predicate calculi (Da Costa N.C.A. and Wolf R.G. 1985). The aim of the last work was a formalisation of certain aspects of *dialectics*[‡‡] in accordance wich the interpretation given in (McGill V.J. and Parry W.T. 1948)[§§]. Some results concerning the above introduced first-order predicate calculi are given below.

Let DL (i.e. *dialectical logic*) be the propositional logic system given in (Da Costa N.C.A. and Wolf R.G. 1980). The obtained new predicate logic system, denoted by DL^Q, has the following *primitive symbols* (Da Costa N.C.A. and Wolf R.G. 1985):

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[*] Valentini S., A simple proof of the completeness theorem of the intuitionistic predicate calculus with respect to the topo logical semantics. Italy's University of Padova 10pp: file:///C:/Users/user/Documents/INT%20PRED%20Clc%20italia.pdf.

[†] Helena Rasiowa (1917 – 1994), Roman Sikorski (1920 – 1983)

[‡] *Rule of definitional replacement of one formula by another*, see Subsection 1.7.

[§] See: (*Formal logic. Encyclopedical outline with applications to informatics and linguistics* 1987).

[**] Georg Kreisel (1923 – 2015)

[††] Vadim Anatol'evich Jankov, born 1935.

[‡‡] In general, the dialectical principle of the *unity of opposites* (related to the *system-constructivist theory*) contributes to the understanding of the relationship between some external (e.g. social, …) and internal (e.g. individual, …) perspectives (Surikova S. 2007) .

[§§] Vivian Jerauld McGill (1897 – 1977), William Tuthil Parry (1908 – 1988)

(1) The *connectives*: $\wedge$, $\vee$, $\Rightarrow$ and $^\circ$, where the last symbol (known as '*stability operator*') is defined as follows: $\varphi^\circ \Leftrightarrow_{df} \sim(\varphi \wedge \sim\varphi)$,

(2) The *universal* and *existential quantifiers*, i.e. $\forall$ and $\exists$ (respectively),

(3) *Individual variables* (an infinitely denumerable set),

(4) Three disjoint sets of *individual constants*: $\underline{A}$, $\underline{B}$ and $\underline{C}$ such that $\underline{A} \cup \underline{B} \cup \underline{C} \neq \varnothing$,

(5) Three disjoint non-empty sets $\underline{A}'$, $\underline{B}'$ and $\underline{C}'$, containing constant predicate symbols of any rank $n \in \mathbb{N}$,

(6) For any $n \in \mathbb{N}$, containing an infinite denumerable set of predicate variables of rank n, and

(7) *Parentheses* (left: '(' and right ')').

The following syntactical notions are used below (Da Costa N.C.A. and Wolf R.G. 1985). The letters A, B and C, with or without subscripts, are employed as *metalinguistic variables* for formulae.* By x, y and z, with or without subscripts, there are denoted *individual variables*. The letters a, b and c are used as *syntactical variables* for individual constants and t - denotes arbitrary *term*. The above extension of DL to DL^Q was realised by using the following additional axioms.

(A1) $C \Rightarrow A(x) \,/\, C \Rightarrow \forall_x A(x)$

(A2) $\forall_x A(x) \Rightarrow A(t)$

(A3) $A(t) \Rightarrow \exists_x A(x)$

(A4) $A(x) \Rightarrow C \,/\, \exists_x A(x) \Rightarrow C$

(A5) $\forall_x (A(x))^\circ \Rightarrow (\forall_x A(x))^\circ$

(A6) $\forall_x (A(x))^\circ \Rightarrow (\exists_x A(x))^\circ$

($\diamond$A7) If A and B are *congruent* formulae, i.e. $A \triangleq B$ † or one is obtained from the other by suppression of vacuous quantifiers, then $'A \Leftrightarrow B'$ is an axiom.

In accordance with the last work, some properties concerning DL^Q were also shown, e.g. it was shown that all schemata and rules of classical *positive predicate logic*‡ are valid in DL^Q. Moreover DL^Q is *undecidable*, also it is a *conservative* extension of DL (i.e. schemata not valid in DL are not valid in DL^Q either). The presented system DL^Q is consistent and nontrivial, etc. In addition to some formulae in DL (e.g. $A \wedge \sim A \Rightarrow B$, $(A \Rightarrow B) \Rightarrow (\sim B \Rightarrow \sim A)$, $\sim\sim A \Leftrightarrow A$, etc.). In particular, the following two formulae are not satisfied in DL^Q (Theorem 8 of the above work). A more formal treatment is left to the reader.

$$\exists_x A(x) \Leftrightarrow \sim \forall_x \sim A(x)$$

$$\forall_x A(x) \Leftrightarrow \sim \exists_x \sim A(x)$$

'*Paraconsistency* is, roughly speaking, a property of negations (a negation-like operators, and is known to be useful for representing inconsistency-tolerant reasoning more appropriately. Examples of paraconsistent negations are De Morgan's type negations such as *strong negation* (Nelson D. 1949), *negations based on four-valued logic* (Belnap N.D. 1977) and *negations based on bilattice logics* (Arieli O. and Avron A. 1996)', see: (Kamide N. and Wansing H. 2010). The notion of a *co-implication* (first introduced in intuitionistic logic)

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* In accordance with (Hilbert D. and Bernays P. 1934, 1939), as in the previous considerations concerning *intuitionistic predicate logic*.

† Two formulae are said to be *congruent* (Kleene S.C. 1952) in the case that one can be transformed into the other only in accordance with the *commutative axiom*. And hence, congruent formulae may differ only in the order of enumeration of their terms and in the order of the literals associated with any term.

‡ A formula φ is a well-formed *positive predicate formula* iff it is formed in accordance with the usual rules of formation and neither $'\sim'$ nor the existensial quantifier $'\exists_x'$ occur in φ (Asenjo F.G. 1996).

combined with paraconsistent negations was first presented in (Wansing H. 2008). And hence, two new 'first-order paraconsistent logics with *De Morgan type negations* and *co-implication*[*], called *symmetric paraconsistent logic* (SPL) and *dual paraconsistent logic* (DPL)' were introduced by Kamide N. and Wansing H. (2010). Here, a Gentzen's type sequent calculi was used. 'The logic SPL is symmetric in the sense that the rule of contraposition is admissible in cut-free SPL. By using this symmetry property, a simpler cut-free sequent calculus for SPL is obtained. The logic DPL is not symmetric, but it has the duality principle. Simple semantics for SPL and DPL are introduced, and the completeness theorems with respect to these semantics are proved. The cut-elimination theorems for SPL and DPL are proved in two ways: One is a syntactical way which is based on the embedding theorems of SPL and DPL into Gentzen's LK[†] (Gentzen G.K.E. 1934, 1935), and the other is a semantical way which is based on the completeness theorems'. The main difference between the above two logics is related to the use of the corresponding four inference rules of adding a paraconsistent negation of an implication / co-implication to the antecedent / consequent of a sequent. The presented inference rules for quantifiers, related to SPL and DPL are the same. They are illustrated below. Here, the paraconsistent negation is denoted by ´¬ ´.

$$+\neg\forall_a: \quad \frac{\neg\,\varphi[z/x],\ \Gamma \vdash \Theta}{\neg \underset{x}{\forall}\, \varphi,\ \Gamma \vdash \Theta}$$

$$+\neg\exists_a: \quad \frac{\neg\,\varphi[t/x],\ \Gamma \vdash \Theta}{\neg \underset{x}{\exists}\, \varphi,\ \Gamma \vdash \Theta}$$

$$+\neg\forall_c: \quad \frac{\Gamma \vdash \Theta,\ \neg\,\varphi[t/x]}{\Gamma \vdash \Theta, \neg \underset{x}{\forall}\, \varphi}$$

$$+\neg\exists_c: \quad \frac{\Gamma \vdash \Theta,\ \neg\,\varphi[z/x]}{\Gamma \vdash \Theta, \neg \underset{x}{\exists}\, \varphi}$$

In accordance with the last formulae, ´t´ and ´z´ are used as a *term* and an *eigenvariable*, i.e. an individual variable with the eigenvariable condition, respectively. Moreover, $\varphi[t/x]$ denotes a formula obtained from φ by replacing all free occurrences of the individual variable x in φ by the term t, but avoiding a clash of variables (Kamide N. and Wansing H. 2010). Let ´¬ ´ and ´~ ´ be the paraconsistent and the classical negations, respectively. As an example, some properties given in the last work are illustrated below.

$\neg\neg\,\varphi \Leftrightarrow \varphi$	{SPL}
$\neg\sim\varphi \Leftrightarrow \sim\neg\,\varphi$	{SPL}
$\neg(\varphi \Rightarrow \psi) \Leftrightarrow (\neg\,\psi \Leftarrow \neg\,\varphi)$	{SPL}
$\neg(\varphi \Leftarrow \psi) \Leftrightarrow (\neg\,\psi \Rightarrow \neg\,\varphi)$	{SPL}
$\neg(\varphi \Rightarrow \psi) \Leftrightarrow (\varphi \Leftarrow \psi)$	{DPL}
$\neg(\varphi \Leftarrow \psi) \Leftrightarrow (\varphi \Rightarrow \psi)$	{DPL}

[*] *De Morgan's type negations* have the common characteristic axioms of De Morgan's laws. There are used in the last work two types of negation connective: *classical* (here denoted by ´¬ ´) and *paraconsistent* one, i.e. *De Morgan's type negation*, denoted by ´ ~ ´). Provided there is no ambiguity and for convenience, we shall use here the reverse version of these designations. Moreover, instead of $\alpha, \beta, \dots$, we shall denote formulae by $\varphi. \psi, \dots$, etc. The *co-implication* connective ´⇐´, known also as *subtraction* (or *difference*) *operator*, is defined as follows: $p \Leftarrow q \ \Leftrightarrow_{df}\ p \wedge \sim q$ (Kamide N. and Wansing H. 2010). Obviously, in classical logic: $p \Leftarrow q \ \Leftrightarrow\ \sim(p \Rightarrow q)$.

[†] It is used Gentzen's type sequent calculi by extending LK.

De Morgan's laws wrt the paraconsistent negation are satisfied in the same manner as in the classical predicate logic, e.g. $\neg \exists_x \varphi \Leftrightarrow \forall_x \neg \varphi$. In particular, it was shown that the propositional fragment of SPL is decidable, DPL is paraconsistent wrt ′¬ ′, etc.

In general, the notion of a paraconsistent negation involves some fundamental questions, mainly from philosophical point of view,in particular 'the question of the existence of paraconsistent logic is still an open problem', see: Béziau J -Y* (left to the reader).

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In this chapter were considered classical and also various non-standard predicate logic systems. In particular, first-order predicate logic is a commonly accepted standard for the formalisation of mny important notions used in mathematics, in particular in discrete mathematical structures (e.g. sets, relations, algebraic systems, etc). Sets are presented in the next chapter.

* 'To know if paraconsistent negations are negations is a fundamental issue: if they are not, paraconsistent logic does not properly exist. In a first part we present a philosophical discussion about the existence of paraconsistent logic and the surrounding confusion about the emergence of possible paraconsistent negations. In a second part we have a critical look at the main paraconsistent negations as they appear in the literature.' Béziau J -Y., *Are paraconsistent negations negations*? (Dedicated to Prof. Newton C.A. da Costa for his 70th birthday), Stanford University, Centre for the Study of Language and Information, SNSF, 22pp: file:///F:/Intuitionistic%20and%20paraco%20predic%20logics%20%20and%20paracons%20set%20theory/PARACONSISTENT%20NEGATION%20ARTYKULY/paraconsistent%20negations%20is%20negation.pdf.

III. Sets

Set theory is a basic tool in discrete mathematics and also in mathematical analysis (concerning infinite sets). Initially, there is given a historical outline related to the development of this theory (*The little encyclopaedia of logic* 1988). And next, starting with the axiomatic foundations, some well-known (set-algebraic) classical basic notions and definitions are introduced (Słupecki J. and Borkowski L. 1967). The most of considered proofs are from assumptions. Several applications are also given. In the next considerations Kripke- Platek set theory is briefly presented. Some comments concerning commonsense sets are also given. Next some elements of non-classical set theories are given, such as: multisets (or bags) and multirelations, fuzzy sets and fuzzy relations, rough sets or also non-standard approaches: fuzzy rough sets, interval type-2 fuzzy sets, near sets, forcing and non-wellfounded sets, and paraconsistent sets. Bunch theory is also briefly presented. It is shown that the generalised Łukasiewicz's fuzzy t-norm (introduced in Chapter I) may be considered as an adequate t-norm in the case of obtaining a distance function of the Minkowski class. Some other properties and examples are also given. In particular, a possibility of a generalisation and improving of the notions of lower and upper approximations used in fuzzy rough sets is also presented.

5. Classical set theory

Set theory was originated by G. Cantor, in accordance with his works for the time period 1871 – 1883: e.g. (Cantor G. 1878)[*]. And hence, were given basic properties concerning sets, other related notions and numerous theorems (being of fundamental importance until now). However, some parts of this theory, in particular set algebra, were formed in earlier time periods: as a beginning, in Leibniz's works and next developed by J. H. Lambert and L. Euler[†] (Euler's geometrical relationships between sets). A first attempt of formal introduction of the notion of relation was also given: W.S. Hamilton and A. De Morgan. There was also presented a formalised calculus of sets, now known as *Boolean algebra*: G. Boole.[‡] Relation calculus in combination with set algebra were developed in works given by Ch.S. Peirce and E. Schröder. Advanced studies of non-finite sets were presented in some works concerning mathematical analysis, e.g. R. Dedekind or P. du Bois-Reymond.[§] However, all these earlier developments were rather fragmentary than general, in comparison with the precision of Cantor's work. On the other hand, the notion of set, used in the last work, was considered rather intuitive. In consequence, on ground of Cantor's theory of sets there were upraised several antinomies, e.g. to the most known belong the following two ones: Russel's and Burali - Forti's antinomies.[**] Preventing antinomies to appear becomes possible by restricting the scope of the notion '*set*'. The first axiomatic system, satisfying this restriction, was introduced by E. Zermelo in 1904, see: (Zermelo E. 1908). Another such prevention (by distinguishing *logical types*) was proposed by B. Russell (Whitehead A.N. and Russell B. 1913). Zermelo's system was extended in succession by A. Fraenkl[††] and T. Skolem. An axiomatic system was also proposed by J. von. Neumann,

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[*] Georg Ferdinand Ludwig Philipp Cantor (1845 – 1918)

[†] Gottfried Wilhelm Leibniz (1646 – 1716), Johann Heinrich Lambert (1728 – 1777), Leonhard Paul Euler (1707 – 1783)

[‡] William Stirling Hamilton (1788 – 1856), Augustus De Morgan (1806 – 1871), George Boole (1815 – 1864)

[§] Charles Sanders Peirce (1839 – 1914), Friedrich Wilhelm Karl Ernst Schröder (1841 – 1902), Julius Wilhelm Richard Dedekind (1831 – 1916), Paul David Gustav du Bois-Reymond (1831 – 1889)

[**] Bertrand Russell (1872 – 1970), Cesare Burali-Forti (1861 – 1931)

[††] Adolf Abraham Halevi Fraenkel (1891 – 1965)

extended by K. Gödel and P. Bernays, Quine's set theory (a version on the basis of Zermelo's axiomatic system and Russell's type theory)[*], etc.

5.1. The axiomatic approach

The human intuition is not sufficient for deciding about various general notions used in set theory. The main idea presented in Zermelo's system is a precise characterisation of the following two fundamental notions: ´∈´ (e.g. ´x ∈ X´, similarly: ´x ∈ y´, if ´y´ is a set) and the notion of a set (e.g. ´ X ´, in a symbolic way denoted below as Z(X) , or Z(y) , if ´y´ is a set (Grzegorczyk A. 1969).[†] In accordance with the last work, the axiomatic system given by E. Zermelo (1908) is presented as follows.

A1 (*axiom of extensionality*)	$Z(X) \wedge Z(Y) \wedge \forall_{x} (x \in X \Leftrightarrow x \in Y) \Rightarrow X = Y$
A2 (*existence of a two-element set*)	$\forall_{x} \forall_{y} \exists_{W} (Z(W) \wedge \forall_{z} (z \in W \Leftrightarrow (z = x) \vee (z = y)))$
A3 (*existence of union of sets belonging to a family*)	$Z(X) \wedge \forall_{x \in X} Z(x) \Rightarrow \exists_{Y} (Z(Y) \wedge \forall_{x} (x \in Y \Leftrightarrow \exists_{z} (z \in X \wedge x \in z)))$
A4 (*powerset of a set*)	$Z(X) \Rightarrow \exists_{Y} (Z(Y) \wedge \forall_{x} (x \in Y \Leftrightarrow Z(x) \wedge x \subseteq X^{‡}))$
A5 (specification axiom)	$Z(X) \Rightarrow \exists_{Y} (Z(Y) \wedge \forall_{x} (x \in Y \Leftrightarrow x \in X \wedge \Phi(x)^{§}))$
A6 (*infinity axiom*)	$\exists_{X} (Z(X) \wedge \exists_{x} (x \in X) \wedge \forall_{x \in X} Z(x) \wedge \forall_{x \in X} \exists_{y \in X} (x \subsetneq y)^{**})$

The above system is usually extended by the following axiom (first introduced by E. Zermelo in 1904 and known as *axiom of choice*: AC): for any *disjoint union* (i.e. union of pairwise disjoint, non-empty sets) there exists a subset of this union containing exactly one member from each of the above sets. Next, the above presented axiomatics was extended by A. A. H. Fraenkel, introducing the following two additional axioms: *axiom of regularity* (or *foundation*) and *axiom of replacement* (left to the reader: see also Subsection 9.6: *non-well-founded set theories*). And hence, the obtained in this way system of axioms (known as ZFC, ´C´ from ´*choice*´) was accepted as the most widely used one.

5.2. Basic notions and definitions

Some well-known (set-algebraic) classical basic notions and definitions are first introduced (Słupecki J. and Borkowski L. 1967). The most of considered proofs are from assumptions. Several applications are also given. The following designations are used below.

x, y, z, …, a, b, c, … set's *elements* (*individuals*),

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[*] Ernst Friedrich Ferdinand Zermelo (1871 – 1953), Adolf Abraham Halevi Fraenkel (1891 – 1965), Thoralf Skolem (1887 – 1963), John von Neumann (1903 – 1957), Kurt Gödel (1906 – 1978), Paul Bernays (1888 – 1977), Willard Van Orman Quine (1908 – 2000)

[†] Andrzej Grzegorczyk (1922 – 2014)

[‡] $x \subseteq X \Leftrightarrow_{df} \forall_{z} (z \in x \Rightarrow z \in X)$

[§] ´ Φ(x) ´ denotes a mathematical expression, describing some property associated with x. Sometimes A5 is also known as an *axiom of separation* or also *restricted comprehension*. In accordance with A5, any definable subclass of a set is a set.

[**] $x \subsetneq y \Leftrightarrow_{df} x \subseteq y \wedge x \neq y$

X, Y, Z, …, A, B, C, … sets of individuals, and
X, **Y**, **Z**, …, **A**, **B**, **C**, … *families* of sets (i.e. sets which elements are also sets).

Let $\mathscr{U}$ be the *universum*. There are considered below only individuals belonging to $\mathscr{U}$. The following axiom is accepted.

(A5.1) $x \in \mathscr{U}$

Definition 5.1 (*range equality*)

$X \doteq Y \Leftrightarrow_{df} \forall_x (x \in X \Leftrightarrow x \in Y)$

Definition 5.2 (*subset*)

$X \subseteq Y \Leftrightarrow_{df} \forall_x (x \in X \Rightarrow x \in Y)$

In accordance with the last definition, we shall say that X is a *subset* of Y (or equivalently that Y is a *superset* of X. Some set-algebraic properties are illustrated below.

Thesis 5.1

$X \subseteq \mathscr{U}$

Proof:

(1)	$x \in \mathscr{U} \Rightarrow (x \in X \Rightarrow x \in \mathscr{U})$	$\{\models (p \Rightarrow (q \Rightarrow p))\}$
(2)	$x \in X \Rightarrow x \in \mathscr{U}$	{– C : 1, A 5.1}
(3)	$\forall_x (x \in X \Rightarrow x \in \mathscr{U})$	{+∀ : 2}
	$X \subseteq \mathscr{U}$. □	{Df. 5.2 : 3}

The following designation is also used: $X \nsubseteq Y \Leftrightarrow_{df} \sim (X \subseteq Y)$. In accordance with Definition 5.2, we can obtain.

$\sim (X \subseteq Y)$	$\Leftrightarrow \sim \forall_x (x \in X \Rightarrow x \in Y)$	{Df. 5.2}
	$\Leftrightarrow \exists_x (x \in X \wedge x \notin Y)$. □	{ N∀, NC, SR}

An indirect version of the proof of T 5.1 is given below.

(1)	$X \nsubseteq \mathscr{U}$	{aip}
(2)	$\exists_x (x \in X \wedge x \notin \mathscr{U})$.	{df. ′⊈′ : 1}
(3)	$a \notin \mathscr{U}$	{–∃, – K :2}
(4)	$a \in \mathscr{U}$	{–∀ : A 5.1}
	contr. □	{3,4}

The range equality is *reflexive*, *symmetric* and *transitive*, i.e. the following thesis is satisfied (Słupecki J. and Borkowski L. 1967)

Thesis 5.2

(a) $X \doteq Y$,
(b) $X \doteq Y \Rightarrow Y \doteq X$ and
(c) $X \doteq Y \wedge Y \doteq Z \Rightarrow X \doteq Z$. □

As an example, the following proof of T 5.2(c) was presented.

(1)	$X \doteq Y$	{a}
(2)	$Y \doteq Z$	
(3)	$\underset{x}{\forall}(x \in X \Leftrightarrow x \in Y)$	{Df. 5.1 : 1}
(4)	$\underset{x}{\forall}(x \in Y \Leftrightarrow x \in Z)$	{Df. 5.1 : 2}
(5)	$x \in X \Leftrightarrow x \in Y$	{−∀ : 3}
(6)	$x \in Y \Leftrightarrow x \in Z$	{−∀ : 4}
(7)	$x \in X \Leftrightarrow x \in Z$	{TE : 5, 6}
(8)	$\underset{x}{\forall}(x \in X \Leftrightarrow x \in Z)$	{+∀ : 7}
	$X \doteq Z$. □	{Df: 5.1 : 8}

In accordance with the last work, the following theses and lemmas were also presented (the corresponding proofs are left to the reader).

Thesis 5.3

(a) $X \doteq Y \Rightarrow X \subseteq Y$ and
(b) $X \subseteq Y \wedge Y \subseteq X \Rightarrow X \doteq Y$. □

Thesis 5.4 (*reflexivity and transitivity of set inclusion*)

(a) $X \subseteq X$ and
(b) $X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z$. □

Lemma 5.1

$x = y \Rightarrow \underset{X}{\forall}(x \in X \Leftrightarrow y \in X)$. □

Definition 5.3

(a) $x \in \{y\} \Leftrightarrow x = y$
(b) $x \in \{y_1, y_2, \dots, y_n\} \Leftrightarrow x = y_1 \vee x = y_2 \vee \dots \vee x = y_n$

Lemma 5.2

$\underset{X}{\forall}(x \in X \Leftrightarrow y \in X) \Rightarrow x = y$. □

In accordance with the last two lemmas, we have:

Thesis 5.5

$x = y \Rightarrow \underset{X}{\forall}(x \in X \Leftrightarrow y \in X)$. □

Let $X = Y$. According to T 5.2(a) and EI (the rule of *extensionality for identity*), the next thesis is obtained.

Thesis 5.6

$X = Y \Rightarrow X \doteq Y$. □

The following axiom (*extensionality for sets*) was introduced.

(A5.2) $X \doteq Y \;\Rightarrow\; X = Y$

Thesis 5.7

$X \doteq Y \;\Leftrightarrow\; X = Y$. □ {A 5.2, T 5.6}

In accordance with the above considerations, the *set equality* is introduced as follows. Some other definitions are also given below.

Definition 5.4 (*set equality*)

$X = Y \;\Leftrightarrow_{df}\; \underset{x}{\forall} (x \in X \;\Leftrightarrow\; x \in Y)$ {Df. 5.1, T 5.7}

Corollary 5.1

The set equality is *reflexive*, *symmetric* and *transitive*. □ {T 5.2, T 5.7}

Let $x \notin X \;\Leftrightarrow_{df}\; \sim(x \in X)$. The following definitions are also introduced.

Definition 5.5 (*set union*)

$x \in X \cup Y \;\Leftrightarrow_{df}\; x \in X \;\vee\; x \in Y$

Definition 5.6 (*set intersection*)

$x \in X \cap Y \;\Leftrightarrow_{df}\; x \in X \;\wedge\; x \in Y$

Definition 5.7 (*set difference*)

$x \in X - Y \;\Leftrightarrow_{df}\; x \in X \;\wedge\; x \notin Y$

Definition 5.8 (*set complement*)

$x \in X' \;\Leftrightarrow_{df}\; x \notin X$

In accordance with the last two definitions and A 5.1: $X' =_{df} \mathscr{U} - X$.

Definition 5.9 (*empty set*)

$x \in \varnothing \;\Leftrightarrow_{df}\; x \in \mathscr{U}'$

Definition 5.10 (*proper subset*)

$X \subsetneq Y \;\Leftrightarrow_{df}\; X \subseteq Y \;\wedge\; X \neq Y$

As an example, in accordance with the last definition, we have: $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ (the sets of *natural numbers*, *integer numbers*, *rational numbers*, *real numbers* and *complex numbers*, respectively)*.

Definition 5.11 (*symmetric set difference*)

$X \div Y \;\Leftrightarrow_{df}\; (X - Y) \cup (Y - X)$

Next we shall say that two sets X and Y are *disjoint* iff $X \cap Y = \varnothing$. And for simplicity, e.g. instead of ′$x \in X \wedge y \in X$′ we shall use: ′$x,y \in X$′.

* More formally: $(\mathbb{N} \subsetneq \mathbb{Z}) \wedge (\mathbb{Z} \subsetneq \mathbb{Q}) \wedge (\mathbb{Q} \subsetneq \mathbb{R}) \wedge (\mathbb{R} \subsetneq \mathbb{C})$.

There exists a graphical method of representing relationships between (a finite number of) sets. This method, introduced in 1880 by John Venn*, is known as *Venn diagrams* and any such diagram shows all possible relations between a finite collection of different sets†. Some generalisation of the last diagrams was proposed in (Łuszczewska - Rohmanowa S‡. 1953). Here, instead of circles, ellipses were used.

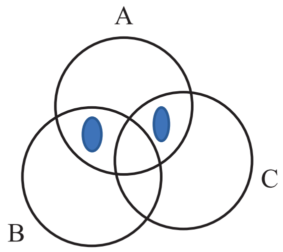


Figure 5.1 An example Venn diagram for $A \cap (B \div C)$

Set intersection is *left distributive* over symmetric set difference, i.e. the following equality is satisfied:

$A \cap (B \div C) = (A \cap B) \div (A \cap C)$. And hence, the left and the right side of this equality correspond to the same Venn diagram (see the above Figure 5.1: left to the reader). A more formal proof of the last set equality is given below (the left side is first considered).

$x \in A \cap (B \div C) \Leftrightarrow x \in A \wedge x \in (B \div C)$ {Df. 5.6}
$\Leftrightarrow x \in A \wedge x \in (B - C) \cup (C - B)$ {Df. 5.11, SR}
$\Leftrightarrow x \in A \wedge (x \in B \wedge x \notin C \vee x \in C \wedge x \notin B)$ {Df. 5.5,Df. 5.7,SR}
$\Leftrightarrow x \in A \wedge x \in B \wedge x \notin C \vee x \in A \wedge x \in C \wedge x \notin B$ ($\wedge$ is distributive over $\vee$)§.

$x \in (A \cap B) \div (A \cap C) \Leftrightarrow x \in (A \cap B) \wedge x \notin (A \cap C) \vee x \in (A \cap C) \wedge x \notin (A \cap B)$
$\Leftrightarrow x \in (A \cap B) \wedge \sim(x \in A \cap C) \vee x \in (A \cap C) \wedge \sim(x \in A \cap B)$
$\Leftrightarrow x \in (A \cap B) \wedge \sim(x \in A \wedge x \in C) \vee x \in (A \cap C) \wedge \sim(x \in A \wedge x \in B)$
$\Leftrightarrow x \in (A \cap B) \wedge (x \notin A \vee x \notin C) \vee x \in (A \cap C) \wedge (x \notin A \vee x \notin B)$**
$\Leftrightarrow x \in A \wedge x \in B \wedge x \notin A \vee x \in A \wedge x \in B \wedge x \notin C \vee x \in A \wedge x \in C \wedge x \notin A \vee x \in A \wedge x \in C \wedge x \notin B$
$\Leftrightarrow x \in A \wedge x \in B \wedge x \notin C \vee x \in A \wedge x \in C \wedge x \notin B$. □

Right distributivity is also satisfied (since ′∩′ is commutative: left to the reader). Moreover, it can be shown that set union and intersection are two *associative* and *mutually distributive* set operations, *De Morgan′s laws* are also satisfied (left to the reader), etc. In general, there exists some correspondence between the propositional calculus and set algebra (Słupecki J. and Borkowski L. 1967). And hence, the following properties hold (the corresponding proofs are left to the reader).

$A \cup (B \cup C) = (A \cup B) \cup C$

---

* John Venn (1834 – 1923)

† Unlike Venn diagrams, *Euler diagrams* (introduced in 1768: Leonhard Paul Euler 1707 – 1783) show only relevant relationships.

‡ Seweryna Łuszczewska - Rohmanowa (1904 –

§ In fact, conjunction and disjunction are commutative and mutually distributive logical operations.

** In accordance with NK (see T 1.8 of Subsection 1.3) and SR.

$A \cap (B \cap C) = (A \cap B) \cap C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

5.3. Generalised unions and intersections

Let $X_1, X_2, \ldots$ be a set sequence. The notions of a generalised union and intersection are defined as follows*.

Definition 5.12 (*generalised union*)

$$x \in \bigcup_i X_i \Leftrightarrow_{df} \exists_i (x \in X_i)$$

Definition 5.13 (*generalised intersection*)

$$x \in \bigcap_i X_i \Leftrightarrow_{df} \forall_i (x \in X_i)$$

As an illustration, some properties are presented below.

Thesis 5.8

$$\forall_i (X \subseteq X_i) \Rightarrow X \subseteq \bigcap_i X_i$$

Proof:

(1)	$\forall_i (X \subseteq X_i)$	{a}
(2)	$X \not\subseteq \bigcap_i X_i$	{aip}
(3)	$\exists_x (x \in X \land x \notin \bigcap_i X_i)$	{df. ′$\not\subseteq$′: 2}
(4)	$a \in X$	{–∃ : 3}
(5)	$a \notin \bigcap_i X_i$	
(6)	$\sim (a \in \bigcap_i X_i)$	{df. ′$\notin$′: 5}
(7)	$\sim \forall_i (a \in X_i)$	{Df.5.13, SR : 6}
(8)	$\exists_i (a \notin X_i)$	{N∀ : 7}
(9)	$a \notin X_{i_0}$	{–∃ : 8}
(10)	$a \in X \land a \notin X_{i_0}$	{+K : 4,9}

* A more general approach was presented in (Słupecki J. and Borkowski L. 1967). It was considered a function φ(X) , defined for any set X of this sequence and having as values sets.

(11) $\underset{x}{\exists}$ (x ∈ X ∧ x ∉ X_{i_0}) {+∃ : 10}

(12) X ⊈ X_{i_0} {df.´⊈ ´ : 11}

(13) X ⊆ X_{i_0} {−∀ : 1}

contr. □ {12,13}

The following theses are also satisfied (some proofs are left to the reader).

Thesis 5.9

$\underset{i}{\forall}$ (X_i ⊆ X) ⇒ $\bigcup_i X_i$ ⊆ X . □

Thesis 5.10

$\underset{i}{\forall}$ (X_i ⊆ Y_i) ⇒ $\bigcup_i X_i \subseteq \bigcup_i Y_i$. □

Thesis 5.11

$\underset{i}{\forall}$ (X_i ⊆ Y_i) ⇒ $\bigcap_i X_i \subseteq \bigcap_i Y_i$. □

Thesis 5.12

$X \cap \bigcup_i X_i = \bigcup_i (X \cap X_i)$. □

Thesis 5.13

$X \cup \bigcap_i X_i = \bigcap_i (X \cup X_i)$. □

The corresponding proofs of T 5.12 and T 5.13 should require the use of the following thesis (see Subsection 3.3) : ⊨ $\underset{x}{Q}$ (p • A(x)) ⇔ p • $\underset{x}{Q}$ A(x), where Q ∈ {∀, ∃}, • ∈ {∧ , ∨} (Q and • are interpreted in the same manner in a given predicate formula).

De Morgan´s laws are illustrated by the next two theses.

Thesis 5.14

$(\bigcup_i X_i)' = \bigcap_i X_i'$

Proof:

x ∈ $(\bigcup_i X_i)'$	⇔	~ (x ∈ $\bigcup_i X_i$)	{Df. 5.8}
	⇔	~ $\underset{i}{\exists}$ (x ∈ X_i)	{Df. 5.12, SR}
	⇔	$\underset{i}{\forall}$ (x ∉ X_i)	{N∃}
	⇔	$\underset{i}{\forall}$ (x ∈ X_i´)	{Df. 5.8, SR}
	⇔	x ∈ $\bigcap_i X_i'$. □	{Df. 5.13}

Thesis 5.15

$(\bigcap_i X_i)' = \bigcup_i X_i'$. □

Thesis 5.16

$\bigcup_i X_i - \bigcup_i Y_i \subseteq \bigcup_i (X_i - Y_i)$

Proof:

$x \in \bigcup_i X_i - \bigcup_i Y_i$	$\Leftrightarrow$	$x \in \bigcup_i X_i \wedge x \notin \bigcup_i Y_i$	{Df. 5.7}
	$\Leftrightarrow$	$\exists_i (x \in X_i) \wedge \sim \exists_i (x \in Y_i)$	{Df. 5.12, df.´∉´ , SR}
	$\Leftrightarrow$	$\exists_i (x \in X_i) \wedge \forall_i (x \notin Y_i)$	{ N∃, SR}
	$\Rightarrow$	$\exists_i (x \in X_i \wedge x \notin Y_i)$	{– C wrt $\models \exists_x A(x) \wedge \forall_x B(x) \Rightarrow \exists_x (A(x) \wedge B(x))$}
	$\Leftrightarrow$	$\exists_i (x \in X_i - Y_i)$	{Df. 5.7, SR}
	$\Leftrightarrow$	$x \in \bigcup_i (X_i - Y_i)$. □	{Df. 5.12}

5.4. Cartesian products and relations

Let (x,y) be an *ordered pair*[*] having as a *first element* x and as a *second element* y. Any such ordered pair is presented with a family of two sets: $(x,y) =_{df} \{\{x\}, \{x,y\}\}$ (Kuratowski K. 1921). It is used the following axiom.

(A5.3) $(x,y) = (z,t) \Leftrightarrow x = z \wedge y = t$

Definition 5.14

1. $x \in \{y\} \Leftrightarrow_{df} x = y$
2. $x \in \{y_1, y_2, \dots, y_n\} \Leftrightarrow_{df} x = y_1 \vee x = y_2 \vee \dots \vee x = y_n$

The proof of A5.3 is given below (Słupecki J. and Borkowski L. 1967)[†].

Proof A5.3b:

(1)	$x = z$	{a}
(2)	$y = t$	
(3)	$\{x\} = \{z\}$	{Df.5.14(1)}
(4)	$\{x,y\} = \{z,t\}$	
(5)	$(x,y) = \{\{x\},\{x,y\}\} = \{\{z\},\{z,t\}\} = (z,t)$. □	{df. ´ (x,y) ´, ´ (z,t) ´}

Proof A5.3a:

(1)	$(x,y) = (z,t)$	{a}
(1.1)	$x \neq y$	{ada}
(1.2)	$\{x\} = \{z\}$	{df. ´ (x,y) ´, ´ (z,t) ´}
(1.3)	$\{x,y\} = \{z,t\}$	
(1.4)	$x = z \wedge y = t$	{1.2, 1.3}

[*] Ordered pairs were introduced independently in 1914 by Norbert Wiener (1894 – 1964) and Felix Hausdorff (1868 – 1942). But the most simple and useful definition was this one given in 1921 by Kazimierz Kuratowski (1896 – 1980).

[†] Provided there is no ambiguity, the proof of A5.3b is first presented.

(2.1)	$x = y$	{ada}
(2.2)	$(x,y) = \{\{x\}\}$	{df. ′ (x,y) ′}
(2.3)	$(z,t) = \{\{x\}\}$	{1}
(2.4)	$x = y = z = t$	{2.2, 2.3}
(2.5)	$x = z \wedge y = t$	{2.4}
	$x = z \wedge y = t$. □	{1.4, 2.5}

Definition 5.15 (*Cartesian product*)[*]

For $X, Y \subseteq \mathscr{U}$: $X \times Y =_{df} \{ (x,y) \ / \ x \in X \wedge y \in Y\}$

Example 5.1 (*right distributivity*)[†]

(a) $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$
(b) $(X - Y) \times Z = (X \times Z) - (Y \times Z)$

Proof(a):

$(x,y) \in (X \cap Y) \times Z$	$\Leftrightarrow$	$x \in X \cap Y \wedge y \in Z$	{Df.5.15}
	$\Leftrightarrow$	$x \in X \wedge x \in Y \wedge y \in Z$	{df. ′∩′, SR}
	$\Leftrightarrow$	$(x \in X \wedge y \in Z) \wedge (x \in Y \wedge y \in Z)$	{$p \wedge p \Leftrightarrow p$, $p \wedge q \Leftrightarrow q \wedge p$, SR}[‡]
	$\Leftrightarrow$	$(x,y) \in X \times Z \wedge (x,y) \in Y \times Z$	{Df.5.15, SR}
	$\Leftrightarrow$	$(x,y) \in (X \times Z) \cap (Y \times Z)$. □	{df. ′∩′}

Proof(b):

$(x,y) \in (X \times Z) - (Y \times Z)$	$\Leftrightarrow$	$(x,y) \in X \times Z \wedge (x,y) \notin Y \times Z$	{df. ′−′}
	$\Leftrightarrow$	$(x,y) \in X \times Z \wedge \sim((x,y) \in Y \times Z)$	{df. ′∉′, SR}
	$\Leftrightarrow$	$(x,y) \in X \times Z \wedge \sim(x \in Y \wedge y \in Z)$	{Df.5.15, SR}
	$\Leftrightarrow$	$x \in X \wedge y \in Z \wedge (x \notin Y \vee y \notin Z)$	{Df.5.15, NK, SR}
	$\Leftrightarrow$	$x \in X \wedge y \in Z \wedge x \notin Y \vee x \in X \wedge y \in Z \wedge y \notin Z$	{distributivity of ′∧′ wrt ′∨′}
	$\Leftrightarrow$	$x \in X \wedge y \in Z \wedge x \notin Y$	{$y \in Z \wedge y \notin Z$}
	$\Leftrightarrow$	$(x \in X \wedge x \notin Y) \wedge y \in Z$	{associativity wrt ′∧′}
	$\Leftrightarrow$	$x \in X - Y \wedge y \in Z$	{df. ′−′}
	$\Leftrightarrow$	$(x,y) \in (X - Y) \times Z$. □	{Df.5.15}

It can be observed that, e.g. $(A \cup B) \times (C \cup D) = (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$. In general, the following property is satisfied.

Thesis 5.17

$$(\bigcup_i X_i) \times (\bigcup_i Y_i) = \bigcup_{i,j} (X_i \times Y_j)$$

Proof:

---

[*] Derived from Descartes' *analytic geometry* (René Descartes 1596 - 1650, also known by his Latin name Renatus Cartesius). Provided there is no ambiguity and for simplicity, the parentheses related to the used here conjunction will be omitted.

[†] Given in (Słupecki J. and Borkowski L. 1967)

[‡] Idempotence, commutativity and associativity of conjunction (i.e. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$).

$$
\begin{aligned}
(x,y) \in (\bigcup_i X_i) \times (\bigcup_i Y_i) &\Leftrightarrow x \in \bigcup_i X_i \wedge y \in \bigcup_i Y_i && \{\text{Df. 5.15}\} \\
&\Leftrightarrow \exists_i (x \in X_i) \wedge \exists_i (y \in Y_i) && \{\text{Df. 5.12,SR}\} \\
&\Leftrightarrow \exists_i \exists_j (x \in X_i \wedge y \in Y_j) && \{\models \exists_x A(x) \wedge \exists_x B(x) \Leftrightarrow \exists_x \exists_y (A(x) \wedge B(y))\} \\
&\Leftrightarrow \exists_{i,j} ((x,y) \in X_i \times Y_j)^* && \{\text{Df. 5.15, SR}\} \\
&\Leftrightarrow (x,y) \in \bigcup_{i,j} (X_i \times Y_j). \square && \{\text{Df. 5.12}\}
\end{aligned}
$$

Sometimes it is useful to consider sets satisfying some conditions. And so, let $\Phi(x)$ be an arbitrary condition associated with the free variable x. By $\Phi(a)$ it is denoted an expression obtained from $\Phi(x)$ substituting x by a. The following definition is introduced.

Definition 5.16 (*sets with conditions*)†

$$x \in \{ a / \Phi(a) \} \Leftrightarrow_{df} \Phi(x)$$

Some properties are illustrated below (Słupecki J. and Borkowski L. 1967). The proofs of T 5.19 and T 5.21 are left to the reader.

Thesis 5.18

$$\{ a / \Phi(a) \vee \Psi(a) \} = \{ a / \Phi(a) \} \cup \{ a / \Psi(a) \}$$

Proof:

$$
\begin{aligned}
x \in \{ a / \Phi(a) \vee \Psi(a) \} &\Leftrightarrow \Phi(x) \vee \Psi(x) && \\
&\Leftrightarrow x \in \{ a / \Phi(a) \} \vee x \in \{ a / \Psi(a) \} && \{\text{Df. 5.16, SR}\} \\
&\Leftrightarrow x \in \{ a / \Phi(a) \} \cup \{ a / \Psi(a) \}. \square && \{\text{df. } ´\cup´\}
\end{aligned}
$$

Thesis 5.19

$$\{ a / \Phi(a) \wedge \Psi(a) \} = \{ a / \Phi(a) \} \cap \{ a / \Psi(a) \}. \square$$

Thesis 5.20

$$\forall_x (\Phi(x) \Leftrightarrow \Psi(x)) \Leftrightarrow \{ a / \Phi(a) \} = \{ a / \Psi(a) \}$$

Proof(a):

(1) $\forall_x (\Phi(x) \Leftrightarrow \Psi(x))$ {a}

(2) $\sim (\{ a / \Phi(a) \} = \{ a / \Psi(a) \})$ {aip}

(3) $\sim \forall_x (x \in \{ a / \Phi(a) \} \Leftrightarrow x \in \{ a / \Psi(a) \})$ {df. ´=´, SR}

(4) $\exists_x \sim ((x \in \{ a / \Phi(a) \} \Rightarrow x \in \{ a / \Psi(a) \}) \wedge (x \in \{ a / \Psi(a) \}$ $\Rightarrow x \in \{ a / \Phi (a) \}))$ {N∀, – E, SR}

---

* $\exists_{i,j} =_{df} \exists_i \exists_j$ is an abbreviation.

† The set of all a satisfying condition $\Phi(a)$, another designation: $\mathop{E}_a \Phi(a)$ ('E' from French word '*ensemble*'). $\mathop{E}_{a,b} \Phi(a,b)$ can be introduced in a similar way.

(5)	$\exists_{a} (x \in \{ a / \Phi(a) \} \wedge x \notin \{ a / \Psi(a) \} \vee x \in \{ a / \Psi (a) \} \wedge x \notin \{ a / \Phi (a) \})$	{NK, NC, SR}
(6)	$x_0 \in \{ a / \Phi(a) \} \wedge x_0 \notin \{ a / \Psi(a) \} \vee x_0 \in \{ a / \Psi (a) \} \wedge a_0 \notin \{ a / \Phi (a)\}$	{$-\exists$ }
(1.1)	$x_0 \in \{ a / \Phi(a) \}$	{ada}
(1.2)	$x_0 \notin \{ a / \Psi(a) \}$	
(1.3)	$\sim (x_0 \in \{ a / \Psi(a)\}$	{df. ´$\notin$´}
(1.4)	$\sim \Psi(x_0)$	{Df. 5.16}
(1.5)	$\Phi(x_0)$	{Df. 5.16 : 1.1}
(1.6)	$\Phi(x_0) \Rightarrow \Psi(x_0)$	{$-\forall$, – E : 1}
(1.7)	$\Psi(x_0)$	{– C : 1.5, 1.6}
	contr.	{1.4, 1.7}
(2.1)	$x_1 \in \{ a / \Psi(a) \}$	{ada}
(2.2)	$x_1 \notin \{ a / \Phi(a) \}$	
(2.3)	$\Psi(x_1)$	{Df. 5.16: 2.1}
(2.4)	$\sim \Phi(x_1)$	{df. ´$\notin$´,Df.5.16}
(2.5)	$\Psi(x_1) \Rightarrow \Phi(x_1)$	{$-\forall$, – E : 1}
(2.6)	$\Phi(x_1)$	{– C : 2.3, 2.5}
	contr. □	{2.4, 2.6}

A simplified proof can be obtained by using the notion of an exclusive disjunction. This is illustrated below.

Proof(b):

(1)	$\{ a / \Phi(a) \} = \{ a / \Psi(a) \}$	{a}
(2)	$\sim \forall_{x} (\Phi(x) \Leftrightarrow \Psi(x))$	{aip}
(3)	$\exists_{x} (\Phi(x) \not\Leftrightarrow \Psi(x))$	{N$\forall$: 2}
(4)	$\Phi(x_0) \not\Leftrightarrow \Psi(x_0)$	{$-\exists$: 3}
(5)	$\Phi(x_0) \wedge \sim \Psi(x_0) \vee \sim \Phi(x_0) \wedge \Psi(x_0)$	{df.´$\not\Leftrightarrow$´: 4}
(1.1)	$\Phi(x_0)$	{ada}
(1.2)	$\sim \Psi(x_0)$	
(1.3)	$x_0 \in \{ a / \Phi(a) \}$	{Df. 5.16}
(1.4)	$\sim (x_0 \in \{ a / \Psi(a) \})$	{Df.5.16, SR}
(1.5)	$x_0 \notin \{ a / \Psi(a) \})$	{df. ´$\notin$´: 1.4}
(1.6)	$x_0 \in\{ a / \Psi(a) \})$	{1}

	contr.	{1.5, 1.6}
(2.1)	$\sim \Phi(x_0)$	
(2.2)	$\Psi(x_0)$	{ada}
(2.3)	$x_0 \notin \{ a \ / \ \Phi(a) \})$	{df. ′∉′: 2.1}
(2.4)	$x_0 \in \{ a \ / \ \Psi(a) \})$	{Df. 5.16 : 2.2}
(2.5)	$x_0 \in \{ a \ / \ \Phi(a) \})$	{1}
	contr. □	{2.3, 2.5}

Thesis 5.21

$\underset{x}{\forall} (\Phi(x) \Rightarrow \Psi(x)) \ \Leftrightarrow \ \{ a \ / \ \Phi(a) \} \ \subseteq \ \{ a \ / \ \Psi(a) \}$. □

Relations are one of the most important topics in mathematical logic. Below are only considered relations which are binary (commonly used e.g. in abstract algebraic systems).

Definition 5.17 (*binary relation*)

An arbitrary subset $\rho \subseteq X \times Y$ of Cartesian product of two sets X and Y.

According to the last definition, ρ is a set of ordered pairs (x,y) such that $x \in X$ and $y \in Y$. For simplicity, instead of $(x,y) \in \rho$ we shall use: $x \rho y$ (*elements* x *and* y *are in* ρ). Similarly, $x \rho' y$ iff $(x,y) \notin \rho$ (*elements* x *and* y *are not in* ρ). We shall say that $\rho = X \times Y$ is *complete*. Similarly, $\rho = \varnothing$ is said to be *empty*.

Definition 5.18 (ρ is *function*)[*]

A binary relation ρ is *function* iff $\underset{x}{\forall} \underset{y}{\forall} \underset{z}{\forall} (x \rho y \wedge x \rho z \Rightarrow y = z)$

Let $\rho \subseteq X \times Y$. The *domain* and *codomain* (called also: *range* or *image*) of ρ are defined as follows: $dom(\rho) =_{df} \{x \in X \ / \ \underset{y \in Y}{\exists} (x \rho y)\}$ and $cod(\rho) =_{df} \{y \in Y \ / \ \underset{x \in X}{\exists} (x \rho y)\} = dom(\rho^{-1})$, where by ′$\rho^{-1}$′ it is denoted the *opposite* (or *inverse*) *relation* to ρ, i.e. $\rho^{-1} \subseteq Y \times X$ such that: $y \rho^{-1} x \Leftrightarrow_{df} x \rho y$.

Definition 5.19 (*composition of two binary relations*)

Let $\rho \subseteq X \times Y$ and $\sigma \subseteq Y \times Z$ be two binary relations. The *composition* (called also *superposition* or *relative product*) $\rho \circ \sigma =_{df} \{(x,z) \in X \times Z \ / \ \underset{y \in Y}{\exists} (x \rho y \wedge y \sigma z)\} \subseteq X \times Z$.

Example 5.2 (*composition*)

---

[*] Definition 5.18, given in (Słupecki J. and Borkowski L. 1967), seems to be the most simple one. In general, it is required the use of the quantifier $\exists_{\leq 1}$ ("*there exists at most one*"), see Subsection 3.4. And hence, a particular case is the following expression: $\underset{x \in X}{\forall} \ \underset{y \in Y}{\exists!} (x \rho y)$.
We shall say that $\rho \subseteq X \times Y$ is a *mapping on* X *into* Y iff $\underset{x \in X}{\forall} \ \underset{y \in Y}{\exists} (x \rho y) \wedge \underset{x \in X}{\forall} (x \rho y \wedge x \rho y' \Rightarrow y = y')$. Any such mapping is *function* if $X, Y \subseteq \mathbb{R}$, or *transformation*: in the case of sets of points (Kerntopf P. 1967). It can be observed that the last expression is logically equivalent to the following more simpler form (left to the reader): $\underset{x \in X}{\forall} ((\underset{y \in Y}{\exists} x \rho y) \wedge (x \rho y \wedge x \rho y' \Rightarrow y = y'))$. A more information concerning mappings will be presented in Part II.

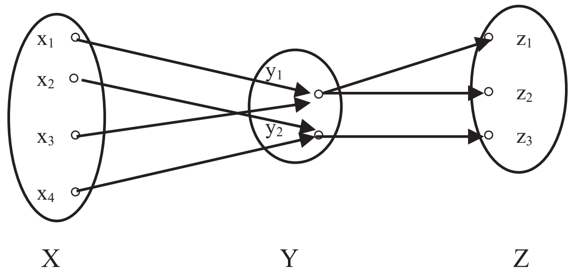


Figure 5.2 An example composition of two binary relations

Consider the following two binary relations: $\rho =_{df} \{(x_1,y_1), (x_2,y_2), (x_3,y_1), (x_4,y_2)\}$ and $\sigma =_{df} \{(y_1,z_1), (y_1,z_2), (y_2,z_3)\}$. Hence, $\rho \circ \sigma = \{(x_1,z_1), (x_1,z_2), (x_2,z_3), (x_3,z_1),(x_3,z_2), (x_4,z_3)\}$. It can be observed that $\rho \cup \sigma = \{(x_1,y_1), (x_2,y_2), (x_3,y_1), (x_4,y_2), (y_1,z_1), (y_1,z_2), (y_2,z_3)\} \subseteq \{(x_1,y_1), (x_1,y_2), (x_1,z_1), (x_1,z_2), (x_1,z_3), (x_2,y_1), (x_2,y_2), (x_2,z_1), (x_2,z_2), (x_2,z_3), (x_3,y_1), (x_3,y_2), (x_3,z_1), (x_3,z_2), (x_3,z_3), (x_4,y_1), (x_4,y_2), (x_4,z_1), (x_4,z_2), (x_4,z_3), (y_1,y_1), (y_1,y_2), (y_1,z_1), (y_1,z_2), (y_1,z_3), (y_2,y_1), (y_2,y_2), (y_2,z_1), (y_2,z_2), (y_2,z_3)\} = (X \cup Y) \times (Y \cup Z)$. □

Some properties, concerning the notions of domain and codomain are illustrated below.

Thesis 5.22

$dom(\rho \cup \sigma) = dom(\rho) \cup dom(\sigma)$

Proof:

For any $x \in X$:

$x \in dom(\rho \cup \sigma)$	$\Leftrightarrow$	$\exists_{y \in Y} x (\rho \cup \sigma) y$	{df. ´ *dom* ´}
	$\Leftrightarrow$	$\exists_{y \in Y} (x \rho y \vee x \sigma y)$	{df. ´$\cup$´, SR}
	$\Leftrightarrow$	$\exists_{y} (y \in Y \wedge (x \rho y \vee x \sigma y))$	{df. ´ $\exists_{\varphi(x)} \psi(x)$ ´}
	$\Leftrightarrow$	$\exists_{y} ((y \in Y \wedge x \rho y) \vee (y \in Y \wedge x \sigma y))$	{distributivity of ´$\wedge$´}
	$\Leftrightarrow$	$\exists_{y} (y \in Y \wedge x \rho y) \vee \exists_{y} (y \in Y \wedge x \sigma y)$	{logical equivalency of the last two lines}*
	$\Leftrightarrow$	$\exists_{y \in Y} (x \rho y) \vee \exists_{y \in Y} (x \sigma y)$	{df. ´ $\exists_{\varphi(x)} \psi(x)$ ´, SR}
	$\Leftrightarrow$	$x \in dom(\rho) \vee x \in dom(\sigma)$	{df. ´ *dom* ´}
	$\Leftrightarrow$	$x \in dom(\rho) \cup dom(\sigma)$. □	{df. ´$\cup$´}

Some other properties are presented below (the corresponding proofs of T 5.23 - T 5.26 are left to the reader).

Thesis 5.23

---

* This logical equivalence is similar as in the case of classical one, i.e. $\exists_x (A(x) \vee B(x)) \Leftrightarrow \exists_x A(x) \vee \exists_x B(x)$. However, the use of the above two variables should require more complicated proof. In fact, the corresponding proofs (a and b) are ramified indirect ones. They are left to the reader: see Subsection 3.3.

$cod(\rho \cup \sigma) = cod(\rho) \cup cod(\sigma)$. □

Thesis 5.24

$dom(\rho \cap \sigma) \subseteq dom(\rho) \cap dom(\sigma)$. □

Thesis 5.25

$cod(\rho \cap \sigma) \subseteq cod(\rho) \cap cod(\sigma)$. □

Thesis 5.26

$dom(\rho - \sigma) \subseteq dom(\rho) - dom(\sigma)$. □

Thesis 5.27

$cod(\rho - \sigma) \subseteq cod(\rho) - cod(\sigma)$

Proof:

For any $y \in Y$:

$y \in cod(\rho - \sigma)$	$\Leftrightarrow$	$\underset{x \in X}{\exists}\ x\,(\rho - \sigma)\,y$	{df. ′ *cod* ′}
	$\Leftrightarrow$	$\underset{x \in X}{\exists}\ (x\,\rho\,y \wedge x\,\sigma' y)$	{df. ′ – ′}
	$\Rightarrow$	$\underset{x \in X}{\exists}\ (x\,\rho\,y) \wedge \underset{x \in X}{\exists}\ (x\,\sigma' y)$	{this implication is satisfied}[*]
	$\Leftrightarrow$	$y \in cod(\rho) \wedge y \in cod(\sigma')$	{df. ′ *cod* ′}
	$\Leftrightarrow$	$y \in cod(\rho) \wedge y \notin cod(\sigma)$	{Df. 5.8, SR}
	$\Leftrightarrow$	$y \in cod(\rho) - cod(\sigma)$. □	{Df. 5.7, SR, Df. 5.2}

Thesis 5.28

$dom(\rho)' \subseteq dom(\rho')$

Proof:

For any $x \in X$:

(1)	$x \in dom(\rho)'$	{a}
(2)	$x \notin dom(\rho')$	{aip}
(3)	$x \notin dom(\rho)$	{Df. 5.8 : 1}
(4)	$\sim (x \in dom(\rho))$	{df. ′∉′: 3}
(5)	$\sim \underset{y \in Y}{\exists}\ (x\,\rho\,y)$	{df. ′ *dom* ′, SR : 4}
(6)	$\underset{y \in Y}{\forall}\ (x\,\rho' y)$	{ N∃[*] : 5}
(7)	$\sim (x \in dom(\rho'))$	{df. ′∉′: 2}

[*] The last implication is satisfied, as in the classical case: $\underset{x}{\exists}(A(x) \wedge B(x)) \Rightarrow \underset{x}{\exists}A(x) \wedge \underset{x}{\exists}B(x)$. The corresponding ramified indirect proof of implication $\underset{x \in X}{\exists}\ (x\,\rho\,y \wedge x\,\sigma' y) \Rightarrow \underset{x \in X}{\exists}\ (x\,\rho\,y) \wedge \underset{x \in X}{\exists}\ (x\,\sigma' y)$ is left to the reader: see Subsection 3.3 .

(8)	$\sim \exists_{y \in Y} (x \rho' y)$	{df. ′ *dom* ′, SR : 7}
(9)	$\forall_{y \in Y} (x \rho y)$	{ N∃*: 8}
(10)	$\forall_{y \in Y} (x \rho' y) \wedge \forall_{y \in Y} (x \rho y)$	{ + K : 6,9}
(11)	$\forall_{y \in Y} (x \rho' y \wedge x \rho y)$	{logical equivalency of the last two lines}
	contr. □	{11}

The proof of the next thesis is very similar to the proof of the previous one (left to the reader).

Thesis 5.29

$cod(\rho)' \subseteq cod(\rho')$. □

Definition 5.20 (*ρ defined in X*)

Let $\rho \subseteq X \times X$. Then we shall say that ρ is a *binary relation defined in X* (or equivalently: a *binary relation on X*).

A particular case are the following two binary relations: $0_X =_{df} \{(x,x) / x \in X\}$ and $1_X =_{df} X \times X$ (Kerntopf P. 1967).

Let ρ be a binary relation on X. We shall say that:

ρ is *reflexive*	$\Leftrightarrow_{df}$	$\forall_{x \in X} (x \rho x)$
ρ is *antireflexive**	$\Leftrightarrow_{df}$	$\forall_{x \in X} (x \rho' x)$
ρ is *symmetric*	$\Leftrightarrow_{df}$	$\forall_{x,y \in X} (x \rho y \Leftrightarrow y \rho x)$
ρ is *antisymmetric*†	$\Leftrightarrow_{df}$	$\forall_{x,y \in X} (x \rho y \Rightarrow y \rho' x)$
ρ is *weak antisymmetric*	$\Leftrightarrow_{df}$	$\forall_{x,y \in X} ((x \rho y) \wedge (y \rho x) \Rightarrow (x = y))$
ρ is *transitive*	$\Leftrightarrow_{df}$	$\forall_{x,y,z \in X} ((x \rho y) \wedge (y \rho z) \Rightarrow (x \rho z))$
ρ is *connected*	$\Leftrightarrow_{df}$	$\forall_{x,y \in X} ((x \neq y) \Rightarrow ((x \rho y) \vee (y \rho x)))$

It can be observed that a relation ρ which is not reflexive, not necessarily should be antireflexive (and vice versa). The following implication is also satisfied: ρ is *antisymmetric* $\Rightarrow$ ρ is *weak antisymmetric* (the proof of this implication is given below). The only one symmetric and weak antisymmetric relation is 0_X (Kerntopf P. 1967).

Thesis 5.30

$$\forall_{x,y \in X} (x \rho y \Rightarrow y \rho' x) \Rightarrow \forall_{x,y \in X} ((x \rho y) \wedge (y \rho x) \Rightarrow (x = y))$$

---

* Called also: *irreflexive*.

† Known also as: *strong antisymmetric* or *asymmetric*. The above notions are here considered as central. In general, some other notions were also introduced, e.g. binary relations which are: *quasi-reflexive*, or *Euclidean* (*right* or *left* ones), or *wheely*, or *serial*, or *trichotomous* (for any x,y ∈ X: exactly one of xρy, yρx, *or* $x = y$ *holds*: *The Free Encyclopaedia, The Wikimedia Foundation, Inc*), etc.: left to the reader.

Proof:

(1) $\forall_{x,y \in X} (x\ \rho\ y \Rightarrow y\ \rho'\ x)$ {a}

(2) $\sim \forall_{x,y \in X} ((x\ \rho\ y) \wedge (y\ \rho\ x) \Rightarrow (x = y))$ {aip}

(3) $\exists_{x,y \in X} ((x\ \rho\ y) \wedge (y\ \rho\ x) \wedge (x \neq y))$ { N∀*, NC, SR : 2}

(4) a, b ∈ X

(5) a ρ b

(6) b ρ a {– ∃*, – K : 3}

(7) a ≠ b

(8) a, b ∈ X ⇒ (a ρ b ⇒ b ρ′ a) {– ∀* : 1}

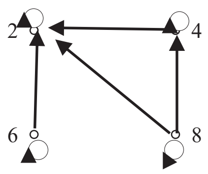
(9) a ρ b ⇒ b ρ′ a {– C : 4,8}

(10) b ρ′ a {– C : 5,9}

contr. □ {6,10}

Example 5.3 (*binary relation*)

Figure 5.3(a,b) below is an example of binary relation ρ represented by (directed) *graph* or also as a *Boolean* (*vertex – vertex incidence*) *matrix*, respectively. Provided there is no ambiguity and for convenience, graph vertices are denoted by the (first four even) natural numbers, i.e. X = {2, 4, 6, 8}. For any x,y ∈ X : x ρ y $\Leftrightarrow_{df}$ the *remainder* of the *Euclidean division*[*] of x by y is equal to 0, i.e. x *mod* y = 0. It can be observed that ρ is reflexive and weak antisymmetric. The corresponding analysis related to ρ′ and ρ^{-1} is left to the reader.



(a) The graph G_ρ of ρ

$$M_\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) The matrix M_ρ of ρ

Figure 5.3 An example binary relation ρ

Let γ , ρ , σ ⊆ X × X. We have: γ ∘ (ρ ∪ σ) = (γ ∘ ρ) ∪ (γ ∘ σ), i.e. composition is is *left distributive* over set union. And so, the following thesis is satisfied.

Thesis 5.31

$$\forall_{x,y \in X} (x\,(\gamma \circ (\rho \cup \sigma))\,y \Leftrightarrow x\,((\gamma \circ \rho) \cup (\gamma \circ \sigma))y)$$

Proof(a):

[*] Known also as: *division with remainder* (Euclides 430 – c.360 b.c.).

(1)	$\sim \forall_{x,y \in X} (x\,(\gamma \circ (\rho \cup \sigma))\,y \Rightarrow x\,((\gamma \circ \rho) \cup (\gamma \circ \sigma))y)$	{a}
(2)	$\exists_{x,y \in X} (x\,(\gamma \circ (\rho \cup \sigma))\,y \wedge x\,((\gamma \circ \rho) \cup (\gamma \circ \sigma))'y)$	{N∀*, NC, SR : 1}
(3)	$a, b \in X$	
(4)	$a\,(\gamma \circ (\rho \cup \sigma))\,b$	{– ∃*, – K : 2}
(5)	$a\,((\gamma \circ \rho) \cup (\gamma \circ \sigma))'\,b$	
(6)	$\exists_{z \in X} (a\,\gamma\,z \wedge z\,(\rho \cup \sigma)\,b)$	{df. ´∘´: 4}
(7)	$a\,(\gamma \circ \rho)'\,b$	{df. ´ ′ ´, df. ´∪´, NA, SR, – K : 5}
(8)	$a\,(\gamma \circ \sigma)'\,b$	
(9)	$\sim \exists_{u \in X} (a\,\gamma\,u \wedge u\,\rho\,b)$	{df. ´ ′, ∘ ´: 7}
(10)	$\sim \exists_{w \in X} (a\,\gamma\,w \wedge w\,\sigma\,b)$	{df. ´ ′, ∘ ´: 8}
(11)	$\forall_{u \in X} (a\,\gamma'\,u \vee u\,\rho'\,b)$	{N∃*, NK, SR : 9}
(12)	$\forall_{w \in X} (a\,\gamma'\,w \vee w\,\sigma'\,b)$	{N∃*, NK, SR : 10}
(13)	$c \in X$	
(14)	$a\,\gamma\,c$	{– ∃*, – K : 6}
(15)	$c\,(\rho \cup \sigma)\,b$	
(16)	$u =_{df} c$	{– ∀* : 11}
(17)	$c \in X \Rightarrow a\,\gamma'\,c \vee c\,\rho'\,b$	
(18)	$w =_{df} c$	{– ∀* : 12}
(19)	$c \in X \Rightarrow a\,\gamma'\,c \vee c\,\sigma'\,b$	
(20)	$c\,\rho\,b \vee c\,\sigma\,b$	{df. ´ ∪ ´: 15}
(21)	$a\,\gamma'\,c \vee c\,\rho'\,b$	{– C : 13, 17}
(22)	$a\,\gamma'\,c \vee c\,\sigma'\,b$	{– C : 13, 19}
(23)	$c\,\rho'\,b$	{– A : 14, 21}
(24)	$c\,\sigma'\,b$	{– A : 14, 22}
(25)	$c\,\sigma\,b$	{– A : 20, 23}
	contr. □	

Proof(b):

(1)	$\sim \forall_{x,y \in X} (x\,((\gamma \circ \rho) \cup (\gamma \circ \sigma))y \Rightarrow x\,(\gamma \circ (\rho \cup \sigma))\,y)$	{a}

(2)	$\exists_{x,y \in X}$ (x ((γ ∘ ρ) ∪ (γ ∘ σ))y ∧ x (γ ∘ (ρ ∪ σ)) ′ y)	{N∀*, NC, SR : 1}
(3)	a,b ∈ X	
(4)	a ((γ ∘ ρ) ∪ (γ ∘ σ)) b	{− ∃*, − K : 2}
(5)	a (γ ∘ (ρ ∪ σ))′ b	
(6)	a (γ ∘ ρ) b ∨ a (γ ∘ σ) b	{ df. ′ ∪ ′ : 4}
(7)	~ $\exists_{z \in X}$ (a γ z ∧ z (ρ ∪ σ) b)	{ df. ′ ′′, ∘ ′: 5}
(8)	$\forall_{z \in X}$ (a γ′ z ∨ z (ρ ∪ σ)′ b)	{N∃*, NK, SR : 7}
(9)	$\exists_{u \in X}$ (a γ u ∧ u ρ b) ∨ $\exists_{w \in X}$ (a γ w ∧ w σ b)	{ df. ′ ∘ ′,SR : 6}
(1.1)	$\exists_{u \in X}$ (a γ u ∧ u ρ b)	{ada}
(1.2)	c ∈ X	
(1.3)	a γ c	{− ∃* , − K : 1.1}
(1.4)	c ρ b	
(1.5)	c ∈ X ⇒ a γ′ c ∨ c (ρ ∪ σ)′ b	{− ∀* : 8}
(1.6)	a γ′ c ∨ c (ρ ∪ σ)′ b	{− C : 1.2, 1.5}
(1.7)	c (ρ ∪ σ)′ b	{− A : 1.3, 1.6}
(1.8)	c ρ′ b	{− K : 1.7}*
	contr.	{1.4, 1.8}
(2.1)	$\exists_{w \in X}$ (a γ w ∧ w σ b)	{ada}
(2.2)	d ∈ X	
(2.3)	a γ d	{− ∃* , − K : 2.1}
(2.4)	d σ b	
(2.5)	d ∈ X ⇒ a γ′ d ∨ d (ρ ∪ σ)′ b	{− ∀* : 8}
(2.6)	a γ′ d ∨ d (ρ ∪ σ)′ b	{− C : 2.2, 2.5}
(2.7)	d (ρ ∪ σ)′ b	{− A : 2.3, 2.6}
(2.8)	d σ′ b	{− K : 2.7}
	contr. □	{2.4, 2.8}

Right distributivity or use of *set intersection* in T 5.31 (instead of set union) are left to the reader. The notions of power set and partition of a set are presented below.

Definition 5.21 (*power set*)

The *power set* (or the *powerset*) of a set X, ℙ(X) $=_{df}$ { Y / Y ⊆ X }*.

---

*c (ρ ∪ σ)′ b ⇔ ~ ((c,b) ∈ (ρ ∪ σ))

⇔ ~ (((c,b) ∈ ρ) ∨ ((c,b) ∈ σ))

⇔ ((c,b) ∉ ρ) ∧ ((c,b) ∉ σ) (see Subsection 1.2: Rule of omitting a conjunction).

Let X be a finite n - element set. The number of all different non-empty subsets of X containing i elements from n (i = 1, 2, …, n) is $\sum_{i=1}^{n}\binom{n}{i}$. Since $\varnothing \in \mathbb{P}(X)$, we have: $\sum_{i=1}^{n}\binom{n}{i} + 1 = \sum_{i=1}^{n}\binom{n}{i} + \binom{n}{0} = \sum_{i=0}^{n}\binom{n}{i}$ $= 2^n$ [†].

Example 5.4 (*power set*)

Consider the set $X =_{df} \{x_1, x_2, x_3, x_4\}$, n = 4. According to Definition 5.21, the obtained power set $\mathbb{P}(X)$ will contain the following $2^4 = 16$ subsets. As an illustration, we observe that non-empty subsets are related to the *vertices*, *undirected* (i.e. *non-oriented*) *edges* or *graph cycles*, as it is shown below (see Figure 5.4).

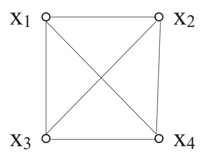


Figure 5.4 A nonoriented graph

$\binom{4}{0} = 1$: $\varnothing$

$\binom{4}{1} = 4$: $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$

$\binom{4}{2} = 6$: $\{x_1,x_2\}, \{x_1,x_3\}, \{x_1,x_4\}, \{x_2,x_3\}, \{x_2,x_4\}, \{x_3,x_4\}$

$\binom{4}{3} = 4$: $\{x_1,x_2,x_3\}, \{x_1,x_2,x_4\}, \{x_1,x_3,x_4\}, \{x_2,x_3,x_4\}$

$\binom{4}{4} = 1$: $\{x_1,x_2,x_3,x_4\}$ $(= X)$

$1 + 4 + 6 + 4 + 1 = 16 = 2^4$. □

Definition 5.22 (*partition*)

Partition of a set X is a family (or equivalently: collection) $\Pi(X) =_{df} \{ X_i / X_i \subseteq X,\ i \in I\}$ satisfying the following three conditions.

(1) $\underset{i \in I}{\forall} (X_i \neq \varnothing)$,

---

[*] Obviously, the sets $\varnothing, X \in \mathbb{P}(X)$. We observe that the number of elements of X do not depend on their ordering and also subsets of X are not multisets (see Subsection 6.1). The powerset $\mathbb{P}(X)$ is also equivalently denoted by 2^X.

[†] The *combination* of i elements from n: $\binom{n}{i} =_{df} \frac{n!}{i!(n-i)!} = \frac{n(n-1)...(n-i+1)}{i!}$. It is assumed that: $\binom{n}{n} = \binom{n}{0} = 1$.

(2) $\bigcup_{i \in I} X_i = X$ and

(3) $\forall_{i,j \in I} \; (i \neq j \Rightarrow X_i \cap X_j = \varnothing)$.

Let X be a finite n - element set. Hence, the number of all possible partitions is solely determined by n. We shall denote this number by d_n[*] . The following proposition was shown (Grinshpan A., *The number of partitions of a set*. Drexel University, Philadelphia, 1pp., http://www.math.drexel.edu/~tolya/partitions.pdf).

Proposition 5.1

For any integer $n \geq 0$, $d_{n+1} = \sum_{i=0}^{n} d_i \binom{n}{i}$, where $d_0 =_{df} 1$. □

Example 5.5 (*partition*)

Let X be the set from the previous example. We have: $d_4 = \sum_{i=0}^{3} d_i \binom{3}{i} = 1 + 3 + 6 + 5 = 15$. The obtained values for d_i ($i = 0,1,2,3$) correspond to the first four Bell's numbers 1, 1, 2 and 5, respectively, e.g. $d_3 =_{df} d_0 \binom{2}{0} + d_1 \binom{2}{1} + d_2 \binom{2}{2} = 1 \times 1 + 1 \times 2 + 2 \times 1 = 5$. And hence, $d_3 \binom{3}{3} = 5$.

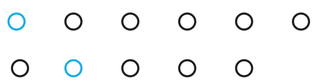
The following partitions are obtained.

$\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$	$\{x_2, x_3\}, \{x_1\}, \{x_4\}$
$\{x_1, x_2\}, \{x_3\}, \{x_4\}$	$\{x_2, x_4\}, \{x_1\}, \{x_3\}$
$\{x_1, x_3\}, \{x_2\}, \{x_4\}$	$\{x_3, x_4\}, \{x_1\}, \{x_2\}$
$\{x_1, x_4\}, \{x_2\}, \{x_3\}$	$\{x_2, x_3\}, \{x_1, x_4\}$
$\{x_1, x_2, x_3\}, \{x_4\}$	$\{x_2, x_4\}, \{x_1, x_3\}$
$\{x_1, x_2, x_4\}, \{x_3\}$	$\{x_1, x_2\}, \{x_3, x_4\}$
$\{x_1, x_3, x_4\}, \{x_2\}$	$\{x_1, x_3\}, \{x_2, x_4\}$
$\{x_1, x_2, x_3, x_4\}$. □	

A given partition $\Pi(X)$ can be represented by *dots* in *Ferrers diagrams* or also by *boxes* (or squares) in *Young diagrams*. The last diagram (often also called Ferrers diagram) is useful in the study of symmetric functions and group representation theory[†].

Example 5.6 (*Ferrers diagram*)

According to the previous example, the number $d_4 = 1 + 3 + 6 + 5 = 15$ ($= 6 + 5 + 3 + 1$) is represented as follows.



[*] The n-th Bell's number, i.e. the number of non-empty disjoint subsets a set of size n, e.g. the first ones are 1, 1, 2, 5, 15, 52, 203, 877, 4140, etc. (Eric Temple Bell 1883 – 1960). An expression for dn may be given also in terms of Stirling's numbers (James Stirling 1692 – 1770): S{n,k) / k! is the number of partitions of an n-element set into k non-empty disjoint subsets, where Sn,k) is the number of *surjective* (or *onto*) functions from an n – element set onto a k – element set (see the above cited work).

[†] Norman Macleod Ferrers (1829 – 1903), Alfred Young (1873 – 1940): some function is said to be *symmetric* iff the function value is independent of the argument ordering, e.g. f(x,y) is *symmetric* iff f(x,y) = f(y,x) for all pairs $(x,y) \in dom(f)$, see: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

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○

By turning this diagram (around the main diagonal: circles in blue color) we can obtain another partition of 15, i.e. 4 + 3 + 3 + 2 + 2 + 1, said to be *conjugate*[*]. □

The next considerations (in particular: Definitions 5.23 - 5.28 and Theorem 5.1) are under (Kerntopf P. 1967).

Definition 5.23 (*equivalence relation*)

A binary relation ρ on X is *equivalence* iff it is at the same time reflexive, symmetric and transitive on X.

Example 5.7 (*equivalence relation*)

Let ρ be a binary relation on $\mathbb{Z}$ (the set of integer numbers) defined as follows: $x\, \rho\, y$ iff $x - y$ is even. For any $x, y, z \in \mathbb{Z}$ we have:

(1) $x\, \rho\, x$, since: $x - x = 0$

(2) $x\, \rho\, y \Leftrightarrow y\, \rho\, x$, since: $x - y$ is even iff $y - x =_{df} -(x - y)$ is even and

(3) $(x\, \rho\, y) \wedge (y\, \rho\, z) \Rightarrow x\, \rho\, z$, since: $x - z = (x - y) + (y - z)$. □

As an example, according to Definition 5.23, the following binary relation is also an equivalence.

$$\forall_{N_1, N_2 \in \text{BIPN}} \left(N_1 \ b\!\approx\ N_2 \ \Leftrightarrow_{df} \ SM(N_1) \approx SM(N_2) \right)$$

Here, BIPN denotes the class of *Boolean interpreted Petri nets*, $SM(N_i)$ is the finite-state sequential machine corresponding to N_i ($i = 1,2$). The above *behavioral equivalence* and *state machine equivalence* relations are denoted by ' b≈ ' and '≈' , respectively (Tabakow I.G. 1989) .

Definition 5.24 (*equivalence class*[†] *and quotient set*)

Let ρ be an equvalence on X. The *equivalence class* for $x \in X$, denoted by $[x]_\rho =_{df} \{y \in X / y\, \rho\, x\} \subseteq X$. The *quotient set* for ρ , denoted by $X / \rho =_{df} \{[x]_\rho / x \in X\}$.

In particular, according to Example 5.5 we can obtain: $X / 0_X = \{\{x_1, x_2, x_3, x_4\}\}$ and $X / 1_X = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}$. We observe that the number of subsets in $X / 0_X$ is minimal (equal to 1) and this number is maximal in $X / 1_X$ (equal to the number of elements in X). The last property is always satisfied if X is finite.

The following properties hold (the corresponding proofs are left to the reader)

Corollary 5.2

For any $x, y, z \in X$:

(1) $x \in [x]_\rho$

(2) $(y \in [x]_\rho) \wedge (z \in [x]_\rho) \Rightarrow y\, \rho\, z$

(3) $y\, \rho\, x \Rightarrow [y]_\rho = [x]_\rho$

(4) $x\, \rho' y \Rightarrow [x]_\rho \cap [y]_\rho = \varnothing$. □

Theorem 5.1

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[*] A partition having itself as conjugate is said to be self-conjugate, e.g. the partition of 9: 3 + 3 + 3

[†] Not be confused with the corresponding music term.

Any equivalence ρ on X generates some partition $\Pi_\rho(X)$ and vice versa, i.e. any partition $\Pi(X)$ generates an equivalence relation ρ_Π on X.

Proof(a):

Consider the family X / ρ. In accordance with Coroll. 5.2 (1), any $x \in X$ belongs to some subset of X, more exactly to $[x]_\rho$. Assume the existence of an element $u \in X$ such that $u \in [y]_\rho$ and $u \in [z]_\rho$. We can obtain: $(u \rho y) \wedge (u \rho z)$. Since ρ is symmetric: $u \rho y \Leftrightarrow y \rho u$. By using SR we can obtain: $(y \rho u) \wedge (u \rho z)$. From transitivity of ρ it follows that $y \rho z$. According to Coroll. 5.2 (3) and ′− C′ we have: $[y]_\rho = [z]_\rho$. And so, $\Pi_\rho(X) =_{df} X / \rho$.

Proof(b):

Let $\Pi(X)$ be given. The following relation ρ_Π on X is defined: $\forall_{x,y \in X} (x \rho_\Pi y \Leftrightarrow_{df} \exists_{i \in I} ((x \in X_i) \wedge (y \in X_i)))$. This relation is reflexive, symmetric and transitive and hence an equivalence on X (left to the reader). □

Example 5.8 (*generated partition*)

In accordance with the previous example, the following partition is generated: $\Pi_\rho(X) = X / \rho = \{E, O\}$, where ′E′ and ′O′ denote the subsets of event and odd integer numbers. □

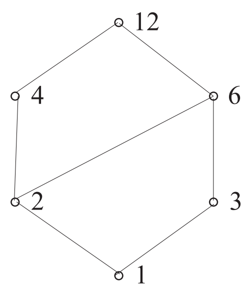
Definition 5.25 (*partial order*)

A binary relation ρ on X is *partial order* iff it is at the same time reflexive, weak antisymmetric and transitive on X. If ρ is a partial order on X then X is said to be a *partially ordered set* (or: *poset*).

Any finite partially ordered set can be graphically represented by using *Hasse's diagrams*[*].

Example 5.9 (*Hasse's diagram*)

Any natural number $n \in \mathbb{N}$ can be represented as follows: $n = \prod_{i=1}^{r} p_i^{a_i}$, where any p_i is a *prime number*[†] and the *number of divisors* of n, denoted by $\tau(n) =_{df} \prod_{i=1}^{r}(a_i + 1)$ (Euclides 430 – c.360 b.c.), e.g. for $n = 12$, we can obtain: $12 = 2^2 \times 3^1$. In a similar way, e.g. $28 = 2^2 \times 7^1$ and $45 = 3^2 \times 5^1$. In this particular case we have: $\tau(12) = \tau(28) = \tau(45) = 6$. We have the following set of divisors for $n = 12$: {1, 2, 3, 4, 6, 12}. The remaining two sets are presented as follows: {1, 2, 4, 7, 14, 28} and {1, 3, 5, 9, 15, 45}. Since the obtained diagrams (considered as nonoriented graphs) are of the same structure, Hasse's diagram associated with $n = 12$ is only presented (see Figure 5.5 given below: the remaining two diagrams are left to the reader). □



[*] Helmut Hasse (1898 – 1979)

[†] A natural number having as divisors only 1 and p_i, itself: e.g. 2, 3, 5, 7, etc. Nonprime natural numbers ($\neq 1$) are said to be *composite*.

Figure 5.5 An example Hasse's diagram*

It can be observed that the opposite of a partial order relation, i.e. ρ^{-1} is also a partial order (left to the reader). The *partial order relation*, the *opposite partial order relation* and their *strong* versions are denoted below as follows: $'\succcurlyeq'$, $'\preccurlyeq'$, $'\succ'$, and $'\prec'$, respectively. For example: $(x \succ y) \Leftrightarrow_{df} (x \succcurlyeq y) \wedge (x \neq y)$.

Definition 5.26 (*linear order*)

Let ρ be a partial order on X. If ρ is also connected then X is said to be an *ordered set* (or *linearly ordered set*).

As an example, any subset of $\mathbb{R}$ is a linearly ordered set wrt $'\geq'$. The following two definitions are also introduced (Kerntopf P. 1967).

Definition 5.27 (*minimal and maximal elements*)

Let X be a partially ordered set. Assume that there exists an element $\wedge \in X$ such that: $\forall_{x \in X} (\wedge \preccurlyeq x)$. We shall say that $'\wedge'$ is the *minimal element* in X. Similarly, assume now that there exists an element $\vee \in X$ such that: $\forall_{x \in X} (\vee \succcurlyeq x)$. We shall say that $'\vee'$ is the *maximal element* in X.

Definition 5.28 (*infimum and supremum*†)

Let X be a partially ordered set. We shall say that $x \in X$ is a *lower bound* for $Y \subseteq X$ if $\forall_{y \in Y} (x \preccurlyeq y)$. The lower bound x is said to be *greatest lower bound* for Y (i.e. *infimum* for Y) if for any other lower bound x' for Y we have: $x' \preccurlyeq x$. In a similar way, $x \in X$ is an *upper bound* for $Y \subseteq X$ if $\forall_{y \in Y} (x \succcurlyeq y)$. The upper bound x is said to be *least upper bound* for Y (i.e. *supremum* for Y) if for any other upper bound x' for Y we have: $x' \succcurlyeq x$.

It can be observed that infimum and supremum (if they exist) are the only one for any partially ordered set (some other considerations are omitted below)‡.

According to Definition 5.23, any binary relation ρ on X is equivalence iff it is reflexive, symmetric and transitive. Assume now that ρ is reflexive and symmetric but not transitive. In classical set theory any such relation can be made transitive, and hence equivalence. The corresponding process is known as *transitive closure* of ρ. In nonclassical set theory (e.g. fuzzy set theory) a similar process becomes more complex.

Let ρ be a binary relation on X, $\rho^i =_{df} \rho \circ \rho \circ ... \circ \rho$ (i times, $i \in I$) and $\rho^+ =_{df} \bigcup_i \rho^i$. Provided there is no ambiguity and for convenience, $\rho^0 =_{df} \{(x,x) / x \in X\}$. The following two theses are satisfied§

Thesis 5.32

A binary relation ρ on X is transitive iff $\rho^2 \subseteq \rho$.

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* In particular, such diagrams are used in *lattice theory* (will be presented in Part II of this book).

† from the Latin *'infimus'* and *'supremus'*: lowest and greatest. The last two notions are related to any partially ordered set, but mainly used in the case of numerical sets.

‡ As an example, any *lattice* is associated with some partial order relation, and vice versa. *Lattice theory* will be presented in Part II of this work.

§ Friedrich Wilhelm Karl Ernst Schröder (1841 – 1902)

Thesis 5.33

A binary relation ρ on X is transitive iff $\rho^+ = \rho$.

More formally, T 5.32 corresponds to the following expression.

$$\forall_x \forall_y \forall_z (x \rho y \wedge y \rho z \Rightarrow x \rho z) \Leftrightarrow \forall_x \forall_y (\exists_z ((x \rho z \wedge z \rho y) \Rightarrow x \rho y)$$

Proof(T 5.32a):

(1)	$\forall_x \forall_y \forall_z (x \rho y \wedge y \rho z \Rightarrow x \rho z)$	{a}
(2)	$\sim \forall_x \forall_y (\exists_z ((x \rho z \wedge z \rho y) \Rightarrow x \rho y)$	{aip}
(3)	$\exists_x \exists_y (\exists_z (x \rho z \wedge z \rho y) \wedge x \rho' y)$	{N∀, NC, SR : 2}
(4)	$\exists_z (a \rho z \wedge z \rho b)$	
(5)	$a \rho' b$	{– ∃, – K : 3}
(6)	$a \rho c \wedge c \rho b$	{– ∃ : 4}
(7)	$a \rho c \wedge c \rho b \Rightarrow a \rho b$	{– ∀ : 1}
(8)	$a \rho b$	{– C : 6,7}
	contr.	{5,8}

Proof(T 5.32b):

(1)	$\forall_x \forall_y (\exists_z (x \rho z \wedge z \rho y) \Rightarrow x \rho y)$	{a}
(2)	$\sim \forall_x \forall_y \forall_z (x \rho y \wedge y \rho z \Rightarrow x \rho z)$	{aip}
(3)	$\exists_x \exists_y \exists_z (x \rho y \wedge y \rho z \wedge x \rho' z)$	{ N∀, NC, SR : 2}
(4)	$a \rho b$	
(5)	$b \rho c$	{– ∃, – K : 3}
(6)	$a \rho' c$	
(7)	$\exists_z (a \rho z \wedge z \rho c) \Rightarrow a \rho c$	{– ∀ : 1}
(8)	$\sim \exists_z (a \rho z \wedge z \rho c)$	{Toll : 6,7}
(9)	$\forall_z (a \rho' z \vee z \rho' c)$	{N∃, NK, SR : 8}
(10)	$a \rho' b \vee b \rho' c$	{– ∀ : 9}
(1.1)	$a \rho' b$	{ada}
	contr.	{4, 1.1}
(2.1)	$b \rho' c$	{ada}
	contr. □	{5, 2.1}

Assume now that $\rho^+ = \rho$. Then $\rho^2 \subseteq \rho$. In accordance with T 5.32 ρ is transitive (using rule TC). And hence, the opposite implication, i.e. T 5.33b is satisfied.

Proof(T 5.33a):

(1)	ρ is transitive	{a}
(2)	$\rho^+ \neq \rho$	{aip}
(3)	$\sim(\rho^+ \subseteq \rho \;\wedge\; \rho \subseteq \rho^+)$	{df. ´≠´ : 2}
(4)	$\rho^+ \nsubseteq \rho \;\vee\; \rho \nsubseteq \rho^+$	{NK, SR : 3}
(5)	$\rho \subseteq \rho^+$	{df. ´ρ^+´}
(6)	$\rho^+ \nsubseteq \rho$	{– A : 4,5}
(7)	$\sim \underset{x}{\forall}\,\underset{y}{\forall}\,(x\,\rho^+ y \;\Rightarrow\; x\,\rho\,y)$	{df. ´ ⊈ ´ : 6}
(8)	$\underset{x}{\exists}\,\underset{y}{\exists}\,(x\,\rho^+ y \;\wedge\; x\,\rho' y)$	{N∀, NC, SR : 7}
(9)	$a\,\rho^+ b$	
(10)	$a\,\rho' b$	{– ∃, – K : 8}
(11)	$(a,b) \in \bigcup_i \rho^i$	{df. ´ρ^+´}
(12)	$\underset{i}{\exists}\,(a\,\rho^i b)$	{df. ´ $\bigcup_i \rho^i$ ´}
(13)	$a\,\rho^{i_0}\,b$	{– ∃ : 12}
(14)	$a\,(\rho^{i_0-1} \circ \rho)\,b$	{df. ´ ρ^{i_0} ´ : 13}
(15)	$\underset{x}{\exists}\,(a\,\rho^{i_0-1} x \;\wedge\; x\,\rho\,b)$	{Df. 5.19 : 14}
(16)	$a\,\rho^{i_0-1} c_1$	
(17)	$c_1\,\rho\,b$	{– ∃, – K : 15}
(18)	$a\,(\rho^{i_0-2} \circ \rho)\,c_1$	{df. ´ ρ^{i_0-1} ´ : 16}
(19)	$\underset{x}{\exists}\,(a\,\rho^{i_0-2} x \;\wedge\; x\,\rho\,c_1)$	{Df. 5.19 : 18}
(20)	$a\,\rho^{i_0-2} c_2$	
(21)	$c_2\,\rho\,c_1$	{– ∃, – K : 19}

Finally, the following two lines can be obtained (after $4(i_0 - 1)$ additional steps):

$a\,\rho^{i_0-(i_0-1)}\,c_{i_0-1}$ ($= a\,\rho\,c_{i_0-1}$) and $c_{i_0-1}\,\rho\,c_{i_0-2}$.

And hence, using ´+ K´, we can obtain: $a\,\rho\,c_{i_0-1} \;\wedge\; c_{i_0-1}\,\rho\,c_{i_0-2} \;\wedge\; \ldots \;\wedge\; c_3\,\rho\,c_2 \;\wedge\; c_2\,\rho\,c_1 \;\wedge\; c_1\,\rho\,b$*. By Definition 5.19 and the *associativity property* of conjunction, finally we ca obtain: $a\,\rho\,b$. But this is a contradiction wrt line (10) of the above proof. □

It is assumed in the last thesis that the composition operation is an *associative* one. The proof of this property is given below.

Thesis 5.34 (*associativity of composition*)

* The i_0 leaves of the corresponding binary tree with a root node: $a\,\rho^{i_0}\,b$.

Let ρ, γ and σ be defined on X. The following property is satisfied (for any x,y $\in$ X): $(x,y) \in \rho \circ (\gamma \circ \sigma) \Leftrightarrow (x,y) \in (\rho \circ \gamma) \circ \sigma$. And hence, the following formula is satisfied.

$$\exists_z (x\rho z \wedge \exists_u (z\gamma u \wedge u\sigma y)) \Leftrightarrow \exists_z (\exists_u (x\rho u \wedge u\gamma z) \wedge z\sigma y)$$

Proof(T 5.34a):

(1)	$\exists_z (x\rho z \wedge \exists_u (z\gamma u \wedge u\sigma y))$	{a}
(2)	$\sim \exists_z (\exists_u (x\rho u \wedge u\gamma z) \wedge z\sigma y)$	{aip}
(3)	$\forall_z (\forall_u (x\rho' u \vee u\gamma' z) \vee z\sigma' y)$	{N∃, NK, SR : 2}
(4)	$x\rho a$	{− ∃, − K : 1}
(5)	$\exists_u (a\gamma u \wedge u\sigma y)$	
(6)	$a\gamma b$	{− ∃ : 5}
(7)	$b\sigma y$	
(8)	$\forall_u (x\rho' u \vee u\gamma' b) \vee b\sigma' y$	{− ∀ : 3}
(1.1)	$\forall_u (x\rho' u \vee u\gamma' b)$	{ada}
(1.2)	$x\rho' a \vee a\gamma' b$	{− ∀ : 1.1}
(1.3)	$a\gamma' b$	{− A : 1.2, 4}
	contr.	{6, 1.3}
(2.1)	$b\sigma' y$	{7, 2.1}
	contr. □	

Proof(T 5.34b):

(1)	$\exists_z (\exists_u (x\rho u \wedge u\gamma z) \wedge z\sigma y)$	{a}
(2)	$\sim \exists_z (x\rho z \wedge \exists_u (z\gamma u \wedge u\sigma y))$	{aip}
(3)	$\forall_z (x\rho' z \vee \forall_u (z\gamma' u \vee u\sigma' y))$	{N∃, NK, SR : 2}
(4)	$\exists_u (x\rho u \wedge u\gamma c) \wedge c\sigma y$	{− ∃ : 1}
(5)	$\exists_u (x\rho u \wedge u\gamma c)$	{− K : 4}
(6)	$c\sigma y$	
(7)	$x\rho d$	{− ∃ : 5}
(8)	$d\gamma c$	
(9)	$x\rho' d \vee \forall_u (d\gamma' u \vee u\sigma' y)$	{− ∀ : 3}
(1.1)	$\forall_u (d\gamma' u \vee u\sigma' y)$	{ada}
(1.2)	$d\gamma' c \vee c\sigma' y$	{− ∀ : 1.1}

(1.3)	$c\,\sigma' y$	{– A : 8, 1.2}
	contr.	{6, 1.3}
(2.1)	$x\,\rho' d$	{ada}
	contr. □	{7, 2.1}

The notion of *reflexive and transitive closure* $\rho^* =_{df} \rho^+ \cup \rho^0$. Since X is finite, we have: $(\rho^*)^* = \rho^*$.

Thesis 5.35

$\rho^* \cup \sigma^* \subseteq (\rho \cup \sigma)^*$

The following proposition is used in the proof of T 5.35.

Proposition 5.2

Let ρ and σ be defined on X and $n \in \mathbb{N}$: $(x,y) \in \rho^n \Rightarrow (x,y) \in (\rho \cup \sigma)^n$ (for any $(x,y) \in X$)*. Assume that the last implication is satisfied for $n =_{df} k$. The proof for $n =_{df} k + 1$ is given below.

Proof:

(1)	$(x,y) \in \rho^{k+1}$	{a}
(2)	$(x,y) \notin (\rho \cup \sigma)^{k+1}$	{aip}
(3)	$(x,y) \in \rho^k \circ \rho$	{SR : 1}
(4)	$\exists_z (x\,\rho^k z \wedge z\,\rho\,y)$	{df. ′∘′ : 3}
(5)	$x\,\rho^k b$	
(6)	$b\,\rho\,y$	{– ∃, – K : 4}
(7)	$(x,y) \notin (\rho \cup \sigma)^k \circ (\rho \cup \sigma)$	{df. ′∘′, SR : 2}
(8)	$\sim \exists_z (x\,(\rho \cup \sigma)^k z \wedge z(\rho \cup \sigma)y)$	{df. ′∘′, SR : 7}
(9)	$\forall_z (x((\rho \cup \sigma)^k)' z \vee z(\rho \cup \sigma)' y)$	{NE, NK, SR : 8}
(10)	$x((\rho \cup \sigma)^k)' b \vee b(\rho \cup \sigma)' y$	{– ∀ : 9}
(1.1)	$x((\rho \cup \sigma)^k)' b$	{ada}
(1.2)	$x((\rho \cup \sigma)^k) b$	{– C : 5, ind. assup: $x\,\rho^k b \Rightarrow x(\rho \cup \sigma)^k b$}
	contr.	{1.1, 1.2}
(2.1)	$b(\rho \cup \sigma)' y$	{ada}
(2.2)	$\sim (b\,\rho\,y \vee b\,\sigma\,y)$	{df. ′∪′: 2.1}

* This proof is inductive wrt k, e.g. for k = 2 is left to the reader. The proof of implication: $(x,y) \in \sigma^n \Rightarrow (x,y) \in (\rho \cup \sigma)^n$ (for any $(x,y) \in X$) is similar as the case of the antecedent: $(x,y) \in \rho^n$ (see Proposition 5.2: left to the reader).

(2.3)	b ρ′y	{NA, – K : 2.2}
	contr. □	{6, 2.3}

According to T 5.35, the following implication should be satisfied.

$(x,y) \in \rho^* \cup \sigma^* \Rightarrow (x,y) \in (\rho \cup \sigma)^*$ (for any $x,y \in X$)

Proof(T 5.35):

(1)	$(x,y) \in \rho^* \cup \sigma^*$	{a}
(2)	$\sim ((x,y) \in (\rho \cup \sigma)^*)$	{aip}
(3)	$\sim ((x,y) \in (\rho \cup \sigma)^+ \cup (\rho \cup \sigma)^0)$	{df. ′ $(\rho \cup \sigma)^*$ ′, SR : 2}
(4)	$\sim ((x,y) \in (\rho \cup \sigma)^+ \vee (x,y) \in (\rho \cup \sigma)^0)$	{df. ′∪′, SR : 3}
(5)	$(x,y) \notin (\rho \cup \sigma)^+$	{NA, – K : 4}
(6)	$(x,y) \notin (\rho \cup \sigma)^0$	
(7)	$\sim ((x,y) \in (\rho \cup \sigma)^+)$	{df. ′∉′ : 5,6}
(8)	$\sim ((x,y) \in (\rho \cup \sigma)^0)$	
(9)	$\sim ((x,y) \in \bigcup_i (\rho \cup \sigma)^i)$	{df. ′$(\rho \cup \sigma)^+$ ′, SR : 7}
(10)	$\sim ((x,y) \in \{(x,x) / (x,x) \in X \times X\}$	{df. ′ $(\rho \cup \sigma)^0$ ′ : 8}
(11)	$(x,y) \notin \rho^0$	{10}
(12)	$(x,y) \notin \sigma^0$	
(13)	$\sim \exists_i ((x,y) \in (\rho \cup \sigma)^i)$	{df. ′ $\bigcup_i (\rho \cup \sigma)^i$ ′ : 9}
(14)	$\forall_i ((x,y) \notin (\rho \cup \sigma)^i)$	{N∃, SR : 13}
(15)	$(x,y) \in \rho^* \vee (x,y) \in \sigma^*$	{df. ′∪′ : 1}
(1.1)	$(x,y) \in \rho^*$	{ada}
(1.2)	$(x,y) \in \rho^+ \cup \rho^0$	{df. ′ρ^* ′ : 1.1}
(1.3)	$(x,y) \in \rho^+ \vee (x,y) \in \rho^0$	{df. ′∪′ : 1.2}
(1.4)	$(x,y) \in \rho^+$	{– A : 11, 1.3}
(1.5)	$(x,y) \in \bigcup_i \rho^i$	{df. ′ρ^+ ′ : 1.4}
(1.6)	$\exists_i ((x,y) \in \rho^i)$	{df. ′ $\bigcup_i \rho^i$ ′ : 1.5}
(1.7)	$(x,y) \in \rho^{i_0}$	{– ∃ : 1.6}
(1.8)	$(x,y) \notin (\rho \cup \sigma)^{i_0}$	{– ∀ : 14}
(1.9)	$(x,y) \in (\rho \cup \sigma)^{i_0}$	{– C : Prop. 5.2, 1.7, $n =_{df} i^0$}

	contr.	{1.8, 1.9}
(2.1)	$(x,y) \in \sigma^*$	{ada}
(2.2)	$(x,y) \in \sigma^+ \cup \sigma^0$	{df. $'\sigma^*'$: 2.1}
(2.3)	$(x,y) \in \sigma^+ \vee (x,y) \in \sigma^0$	{df. $'\cup'$: 2.2}
(2.4)	$(x,y) \in \sigma^+$	{– A : 12, 2.3}
(2.5)	$(x,y) \in \bigcup_i \sigma^i$	{df. $'\sigma^+'$: 2.4}
(2.6)	$\exists_i ((x,y) \in \sigma^i)$	{df. $'\bigcup_i \sigma^i'$: 2.5}
(2.7)	$(x,y) \in \sigma^{i_0}$	{– ∃ : 2.6}
(2.8)	$(x,y) \notin (\rho \cup \sigma)^{i_0}$	{– ∀ : 14}
(2.9)	$(x,y) \in (\rho \cup \sigma)^{i_0}$	{– C : Prop. 5.2*, 2.7, $n =_{df} i_0$}
	contr. □	{2.8, 2.9}

In accordance with T 5.32, the following *transitive closure algorithm* (in short: *TC algorithm*) is used.

TC algorithm

(1) $\hat{\rho} =_{df} \rho \cup \rho^2$;
(2) if $\hat{\rho} = \rho$ then ρ is transitive : end ;
(3) $\rho =_{df} \hat{\rho}$: go to (1). □

Example 5.10 (*TC algorithm*)

Figure 5.6 Two binary relations on $X =_{df} \{x_1, x_2, x_3\}$

Consider ρ and σ of Figure 5.6. We have: $\rho = \{(x_1,x_2), (x_2,x_3), (x_1,x_3)\}$ and $\rho^2 = \{(x_1,x_3)\}$. And hence, ρ is transitive {TC algorithm, step (2)}. Let now consider σ. We have: $\sigma = \{(x_1,x_2), (x_2,x_3), (x_3,x_1)\}$, $\sigma^2 = \{(x_1,x_3), (x_2,x_1), (x_3,x_2)\}$, $\sigma^3 = \{(x_1,x_1), (x_2,x_2), (x_3,x_3)\}$, $\sigma^4 = \sigma$, $\sigma^5 = \sigma^2$, $\sigma^6 = \sigma^3$, etc. And finally, $\sigma^+ = \sigma \cup \sigma^2 \cup \sigma^3 \cup \ldots = \sigma \cup \sigma^2 \cup \sigma^3 = \{(x_1,x_2), (x_2,x_3), (x_3,x_1), (x_1,x_3), (x_2,x_1), (x_3,x_2), (x_1,x_1), (x_2,x_2), (x_3,x_3)\}$ (see Figure 5.7 given below). Here, any σ^i is related to some *directed path of length* i (i = 1.2, …). We observe that the obtained closure is also reflexive, i.e. $\sigma^+ = \sigma^*$. □

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* Assuming the antecedent $'(x,y) \in \sigma^n'$, i.e. $(x,y) \in \sigma^n \Rightarrow (x,y) \in (\rho \cup \sigma)^n$ / $n =_{df} i_0$.

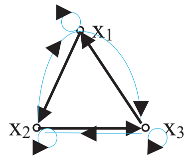

Figure 5.7 The transitive closure for σ

The above method becomes inefficient for a large graph size. Some simplification can be obtained by using multiplication over graph incidence matrices. This is illustrated in the next example.

Example 5.11 (*TC algorithm*)

Let σ be the binary relation of the previous example. The graph incidence matrix is represented by the following square matrix of order 3: $M_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. And hence, $M_{\sigma^2} = M_\sigma \times M_\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$[*]. The following steps are realised.

(1) $M_{\hat\sigma} = M_\sigma \cup M_{\sigma^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ {TC algorithm: step (1)}

(2) $M_{\hat\sigma} \neq M_\sigma$ {TC algorithm: step (2)}

(3) $M_\sigma =_{df} M_{\hat\sigma}$ {TC algorithm: step (3)}

(4) $M_{\hat\sigma} = M_\sigma \cup M_{\sigma^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cup \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right)$ {TC algorithm: step (1)}

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(5) $M_{\hat\sigma} \neq M_\sigma$ {TC algorithm: step (2)}

(6) $M_\sigma =_{df} M_{\hat\sigma}$ {TC algorithm: step (3)}

(7) $M_{\hat\sigma} = M_\sigma \cup M_{\sigma^2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ {TC algorithm: step (1)}

(8) $M_{\hat\sigma} = M_\sigma$: σ is transitive : end. □ {TC algorithm: step (2)}

The concept of an occurrence (Petri) net, i.e. a directed acyclic graph that represents causality[†] and concurrency information about a single execution of a system, was introduced in (Best, E. and Devillers R. 1987).

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[*] The matrix multiplication operation is similar to the classical one, i.e. $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ (i,j = 1,2, … ,n), for any A,B,C (square matrices of order n, where C = A × B). But now, instead of *sum* and *multiplication*, the operations *maximum* and *minimum* are used, respectively.

[†] And hence, occurrence nets are known also as *causal* ones, e.g. (Kummer O. and Stehr M - O. 1997).

Some research concerning: *structural properties*, e.g. (Haar S. 1999), *orthomodular lattices* related to these nets (Bernardinello L. et al. 2009) or *timed structured occurrence nets* (Bhattacharyya A. et al. 2016), was also presented. In accordance with the last works, some information, concerning the two fundamental binary relations *li* and *co* is given below. The following definition is firs presented (P and T denote the *sets of places* and *transitions*, F is said to be a *flow relation*).

Definition 5.29 (*finite net*)[*]

A *finite net* is a triple $N =_{df} (P, T, F)$ where:

(i) $P \cap T = \varnothing$,

(ii) $P \cup T \neq \varnothing$,

(iii) $F \subseteq (P \times T) \cup (T \times P)$ and

(iv) $dom(F) \cup cod(F) = P \cup T$.

Let $x \in P \cup T$. The *pre-set* and *post-set* associated with x are defined as follows: ${}^{\bullet}x =_{df} \{y \in P \cup T \,/\, (y,x) \in F\}$ and $x^{\bullet} =_{df} \{y \in P \cup T \,/\, (x,y) \in F\}$, respectively. Obviously: $y \in {}^{\bullet}x \Leftrightarrow x \in y^{\bullet}$.

Let P and T be countable. The following definition was used in (Bernardinello L. et al. 2009) .

Definition 5.30 (*occurrence net*)

A net $N = (P, T, F)$ is an *occurrence net* iff

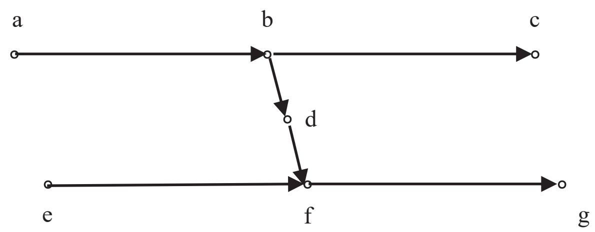
(i) $\underset{p \in P}{\forall} ((|{}^{\bullet}p| \leq 1) \wedge (|p^{\bullet}| \leq 1))$[†] and

(ii) $\underset{x,y \in P \cup T}{\forall} ((x,y) \in F^{+} \Rightarrow (y,x) \notin F^{+})$.

According to the last definition, any occurrence net N is *conflict-free* (i) and *acyclic* (ii). The obtained structure $(X, \sqsubseteq)$ derived from N as follows: $X =_{df} P \cup T$ and $\sqsubseteq \;=_{df} F^{*}$ is a partially ordered set.

Let $'\leq'$ be a partial order relation on a set A. The following two relations can be derived (for any $a,b \in A$): a *li* $b \Leftrightarrow_{df} (a < b) \vee (b < a) \vee (a = b)$ and a *co* $b \Leftrightarrow_{df} a$ *li*$'$ b. The last two relations are symmetric and not transitive. Moreover, *li* is reflexive, while *co* is antireflexive. Let now ρ be defined on A and $\rho_{/B}$ be ρ restricted to $B \subseteq A$. The notion of region is introduced as follows: B is a *region* iff (i) $\rho_{/B} = B \times B$ and (ii) $\underset{a \in A-B}{\forall} \; \underset{b \in B}{\exists} (a \rho' b)$. A frequently used example for *li* and *co* is the following one.

Example 5.12 (*li- and co-graphs,regions*)



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[*] If places and transitions are interpreted as conditions and events (respectively), we can obtain a Petri net, consisting of conditions and events, having tokens in some conditions (the *initial case*). In a similar way, there are obtained a more complex models e.g. such as: place-transition nets or individual-token sets (Reisig W. 1985, 1992) . Some time it is assumed (in a more general context) that P and T are *countable* (see the next subsection), e.g. assuming occurrence nets, see: (Bernardinello L. et al. 2009). In the last case, instead of '*finite net*', it is used the term '*net*'.

[†] Here $|{}^{\bullet}p|$ and $|p^{\bullet}|$ denote the *cardinalities* of ${}^{\bullet}p$ and $p^{\bullet}$, respectively: see the next subsection.

Figure 5.8 An example partially ordered set

In accordance with Figure 5.8, the obtained graphs for relations <u>*li*</u> and <u>*co*</u> are shown below.

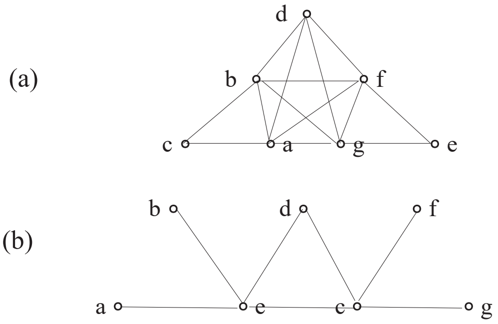


Figure 5.9 The <u>*li*</u> - and <u>*co*</u> - graphs

The regions of <u>*li*</u> and <u>*co*</u>, denoted by L's and C's, are called *lines* and *cuts*, respectively. According to Figure 5.9 (a) we have the following three lines: L_1 = {a, b, d, f, g}, L_2 = {a, b, c} and L_3 = {f, g, e}. In a similar way, by Figure 5.9 (b) we can obtain the following five cuts: C_1 = {c, d, e}, C_2 = {a, e}, C_3 = {b, e}, C_4 = {c, f} and C_5 = {c, g}. □

Let $|A|$ be the *cardinality* of A (the number of elements if the set A is finite: see the next subsection). The following notion was introduced.

Definition 5.31 (*k-dense set*)

The poset X is *k-dense* iff $|L \cap C| = 1$ (for any L and C).

Example 5.13

In accordance with the last definition, the poset A of Figure 5.8 is a k-dense set. However, the poset shown in Figure 5.10 below is not k-dense. We have: L_1 = {a,b}, L_2 = {c,b}, L_3 = {c,d} and C_1 = {a,c}, C_2 = {b,d}, C_3 = {a,d}.And hence, $L_2 \cap C_3 = \varnothing$. This poset becomes k-dense by using the additional edge (a,d): left to the reader. □

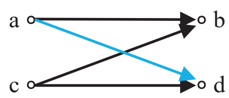


Figure 5.10

Proposition 5.3

$|L \cap C| \leq 1$ (for any L and C)

Proof:

Let a, b $\in$ L $\cap$ C. Assume that a $\neq$ b. We have: a <u>*li*</u> b and a <u>*co*</u> b. And hence: a = b (a contr). □

The study of such relations as <u>*li*</u> and <u>*co*</u> derived from the partially ordered set (X, ⊑) was presented in (Bernardinello L. et al. 2009). A more formal treatment is omitted: left to the reader.

5.5. Equinumerosity and countability

The following definition is first introduced.

Definition 5.32 (*function*)

Let $\rho \subseteq X \times Y$. We shall say that ρ is a *function* iff $\underset{x\in X}{\forall} \; \underset{y\in Y}{\exists!} \,(x \rho y)$.

Non-finite sets are usually interpreted as non-empty sets having cardinality different from any natural number. In the next definition, the notion of a natural number is eliminated.[*]

Definition 5.33 (*non-finite set*)

X is *a non-finite set in Dedekind's sense* $\Leftrightarrow \underset{Y\subseteq X}{\exists} (Y \neq X \wedge Y \sim X)$. {see Df. 5.34, below}

In the next considerations we shal assume that X and Y are two numerical sets. Moreover, instead of *relational notation* ´$x \rho y$´ sometimes, for convenience, we shall use the *functional notation* ´ $y = f(x)$ ´[†]. The notions of injection, surjection and bijection are introduced as follows[‡].

Let $f : X \to Y$ (*arrow notation*). We shall say that f is *injection* (*into function* or *one-to-one function*, in short: $f \in 1-1$) iff $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for any $x_1, x_2 \in X$. This function is *surjection* (*onto function*) iff *cod*(f) = Y. And finally, f is *bijection* iff it is injection and surjection (or equivalently: ρ and ρ^{-1} are functions). The equinumerosity of two sets is presented as follows (Słupecki J. and Borkowski L. 1967).

Definition 5.34 (*set equinumerosity*)

Let $\rho \subseteq X \times Y$. We shall say that ρ *establishes the equinumerosity* of X and Y, in short: $X \sim_\rho Y$ iff $\rho \in 1-1 \wedge X = dom(\rho) \wedge Y = cod(\rho)$. The last two sets are *equinumerous*, in short: $X \sim Y$ iff $\underset{\rho}{\exists} (X \sim_\rho Y)$.

According to the last definition, the *equinumerosity relation* ´~´ is reflexive, symmetric and transitive, i.e a relation of type equivalence (the corresponding proofs are omitted: see the last cited work)[§]. The *cardinality* of a set X is below denoted by $|X|$. We shall assume that: $|\varnothing| = 0$ and $|X| \in \mathbb{N}$ (the set of *natural numbers*), if X is a non-empty finite set.

(A5.4) $|X| = |Y| \Leftrightarrow X \sim Y$

Example 5.14

Let $\mathbb{N}$, $\mathbb{N}_E$ and $\mathbb{N}_O$ be the set of *natural numbers*, i.e. $\mathbb{N} =_{df} \{1, 2, ...\}$ and the subsets of *even* and *odd natural numbers*, respectively. We have: $\mathbb{N} \sim \mathbb{N}_E$ and $\mathbb{N} \sim \mathbb{N}_O$ (see below). □

$\mathbb{N}$:	1	2	3	4	5 …
$n \mapsto 2n$[**] :	2	4	6	8	10 …
$n \mapsto 2n-1$:	1	3	5	7	9 …

---

[*] Julius Wilhelm Richard Dedekind (1831 – 1916)

[†] There exist various notations related to this notion, e.g. *arrow notation*, *index notation*, *dot notatio*n, etc. Here, it is used Euler's *functional notation* (Leonhard Euler 1707 – 1783).

[‡] Terms used by the French mathematicians group *Bourbaki*.

[§] $X \sim_\rho X$, $X \sim_\rho Y \Rightarrow Y \sim_{\rho-1} X$, $X \sim_\rho Y \wedge Y \sim_\sigma Z \Rightarrow X \sim_{\rho\circ\sigma} Z$

[**] Known by Galileo Galilei (1564 – 1642)

Definition 5.35 (*cardinal number*)

m is a *cardinal number* (or *cardinal*) $\Leftrightarrow_{df} \underset{X}{\exists}(Z(X) \wedge m =_{df} |X|)$.

In particular, in accordance with Definition 5.35, it follows that *zero* and *natural numbers* are cardinal numbers. Next, we shall consider infinite sets (i.e. sets that are not finite). The notion of set countability is introduced as follows.

Definition 5.36 (*countable set*)

Let X be an infinite set. We shall say that X is a countable set iff $X \sim \mathbb{N}$.

The countability property (for numerical sets) is discussed below. It is first considered the set $\mathbb{Z}$ of integer numbers.

Proposition 5.4

$\mathbb{Z} =_{df} \{0, \pm 1, \pm 2, \ldots\}$ is a countable set.

Proof:

For any $n \in \mathbb{Z}$, let $f : \mathbb{Z} \rightarrow \mathbb{N}$ such that: $f(n) =_{df}$ *if* $n \geq 0$ *then* $2n + 1$ *else* $-2n$. We have: $f(0) = 1$, $f(-1) = 2$, $f(1) = 3$, $f(-2) = 4$, $f(2) = 5$, $f(-3) = 6$, $f(3) = 7$, etc. □

Thesis 5.36

Let X and Y be two countable sets. Then $X \cup Y$ is a countable set.

Proof:

Assume that X and Y are countable. And hence, their elements can be represented by the following two sequences.

X : $x_1\ x_2\ x_3\ \ldots$

Y : $y_1\ y_2\ y_3\ \ldots$

For any $n \in \mathbb{N}$, let $f : \mathbb{N} \rightarrow X \cup Y$ such that: $f(n) =_{df}$ if n is *odd* then $x_{(n+1)/2}$ else $y_{n/2}$ (see below). □

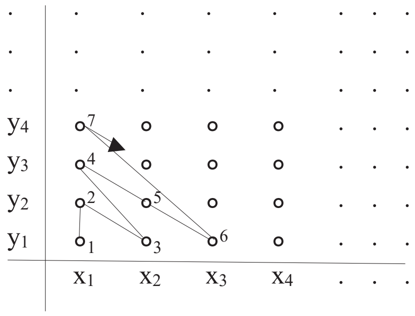
$\mathbb{N}$:	1	2	3	4	5 …
$X \cup Y$:	x_1	y_1	x_2	y_2	x_3 …

Thesis 5.37

Let X and Y be two countable sets. Then $X \times Y$ is a countable set.

Proof:

Assume that X and Y are countable. Let $X \times Y =_{df} \{ (x,y) \ / \ x \in X \wedge y \in Y\}$. For any two different pairs pairs $(x_i,y_j), (x_m,y_n) \in X \times Y$, the binary relation $\rho \subseteq (X \times Y) \times (X \times Y)$ can be defined as follows: $(x_i,y_j)\ \rho\ (x_m,y_n) \Leftrightarrow_{df} (i + j < m + n) \vee ((i + j = m + n) \wedge (i < m))$: see below. □



Corollary 5.3

The set of *rational numbers* $\mathbb{Q} =_{df} \{ \frac{m}{n} \ / \ (m,n) \in \mathbb{Z} \times \mathbb{N} \}$ is a countable set. □ {Prop. 5.4, T 5.37}

Obviously, every subset of a countable set is also a countable one. In particular, the above thesis T 5.36 can be extended for any finite union of countable sets.

In general, there exist infinite sets that are not countable. In particular, it was shown that the set of real numbers $\mathbb{R}$ is *uncountable* (Cantor G. 1874). 'Cantor's discovery of uncountable sets in 1874 was one of the most unexpected events in the history of mathematics. Before 1874, infinity was not even considered a legitimate mathematical subject by most people, so the need to distinguish between countable and uncountable infinities could not have been imagined' (Stillwell J. 2010). The uncountability of $\mathbb{R}$ is presented in the next thesis (Kuratowski K. 1966). The following function *tangens* was used: $y =_{df} tg(\pi(x - 1/2))$ (in fact: $y = tg(-\pi / 2)$, for $x = 0$ and $y = tg(\pi / 2)$, for $x = 1$, $\pm\pi / 2$ are two breakpoints)*. The last map is a bijection. And so, $(0,1) \sim \mathbb{R}$.

Thesis 5.38

The set of real numbers $\mathbb{R}$ is uncountable.

Proof:

Assume that $\mathbb{R}$ is a countable set. And hence, the elements of $\mathbb{R}$ can be represented as a *sequence*: (r_n). Let $r_1, r_2, \ldots r_n, \ldots$ be the corresponding elements belonging to $pq =_{df} [0,1]$. The first element of this sequence can be eliminated from pq by dividing this interval into three parts of length equal to 1/3. If r_1 belongs to one of these three parts, then, as a next subinterval is selected one of the remaining two parts. If r_1 belongs to the borderline of two parts, then as a next subinterval is selected the remaining one. And hence, the next selected subinterval, denoted by p_1q_1 of length 1/3 is such one that $r_1 \notin p_1q_1$. In a similar way, after selecting p_2q_2 of length $1/3^2$ from p_1q_1 we can eliminate r_2, i.e. $r_2 \notin p_2q_2$, … , from p_nq_n of length $1/3^n$ we can eliminate r_n, etc. And finaly, we have: $\bigcap_n p_n q_n = \{c\}$, such that $c \notin (r_n)$ and $c = \lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n$. □

The next example is under (Ross K.A. and Wright C.R.B. 1999).

Example 5.15

The set $F =_{df} \{f / f: \mathbb{N} \to \{0,1\}\}$ is uncountable. In fact, assume that F is a countable set. And so, the elements of F can be represented as a *sequence*: (f_n). Let now $f^*(n) =_{df}$ if $f_n(n) = 1$ then 0 else 1. And hence, $f^* \in F$ and $f^* \notin (f_n)$. □

---

* The *tangens* $y = tg(x)$ has a *period* of π and *asymptotes* $x = (k + 1/2)\pi$, $k \in \mathbb{Z}$. This function is *monotonic increasing* for $x \in [-\pi/2, +\pi/2]$ and having values from $-\infty$ to $+\infty$. The function's behaviour is repeated with period π.

Consider now the following non-empty set Σ, named *alphabet*. The elements of this set are named *letters*. A *word* is a finite sequence of letters belonging to Σ. The *set of all finite words* in this alphabet is denoted by $\overline{\Sigma}$. The *empty word* is denoted by λ. Assume now that $\Sigma =_{df} \{0,1\}$ and $\Sigma^* =_{df} \overline{\Sigma} \cup \{\lambda\}$. The last set is countable as it is shown in the next example.

Example 5.16

Consider the set Σ^*. By ′nb′ we shall denote the ′*number of bits*′ of a non-empty word $x \in \Sigma^*$, e.g. nb(011) = 3, for x = 011. The *decimal value* of any binary word (or *string*) is defined as follows: $\perp x =_{df} \sum_{j=1}^{n} a_j . 2^{n-j}$, where $n =_{df}$ nb, e.g. $\perp(0101) = 0 \times 2^{4-1} + 1 \times 2^{4-2} + 0 \times 2^{4-3} + 1 \times 2^{4-4} = 5$*. The above set Σ^* is countable. The elements of this set can be ordered as follows (see below). □

For any $x, y \in \Sigma^*$:

(1) $\lambda \prec x$,

(2) $x \prec y$ iff $nb(x) < nb(y) \vee nb(x) = nb(y) \wedge \perp x < \perp y$.

Since $\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma^i$, where $\Sigma^0 = \{\lambda\}$, $\Sigma^1 = \{0,1\} = \Sigma$, $\Sigma^2 = \{00, 01, 10, 11\}$, etc., we can obtain:

Σ^* :	λ	0	1	00	01	10	11	000	001	010 …
ℕ :	1	2	3	4	5	6	7	8	9	10 …

Let T be the set of *transcendental* (or *irrational*) *numbers* (Liouville J. 1851)†, i.e. the set of real or complex numbers that are not *algebraic* ones (not be solutions of polynomial equations with integer coeficients: e.g. π, e, 2π, e – 1, etc.). It was shown by Cantor that T is an uncountable set‡. A more formal treatment is left to the reader.

In accordance with (A5.4) and Definition 5.35, the cardinality of X, i.e. $|X|$ can be considered as an object associated with any Y such that $Y \sim X$. However, we have no any rule related to the map: $X \to |X|$ and at the same time satisfying (A5.4)§. And so, the last problem can be eliminated by assuming the following Zermelo's axiom, given in 1904 and known as the *axiom of choice* (another possibility is the use of Zermelo-Fraenkel's *axiom of regularity* or *foundation*). In general, the introduction of cardinal numbers will not be possible without using one of the last two axioms. In fact, some other approaches were also given, e.g. use of *relation type axiom* (a weakly version) instead of the regularity one: (Kuratowski K. and Mostowski A. 1966). Let **Z** denotes a *family of sets*. *Zermelo's axiom of choice* can be presented as as follows.

(A5.5) $\mathbf{Z} \neq \varnothing \wedge \forall_X (X \in \mathbf{Z} \Rightarrow X \neq \varnothing) \wedge \forall_{P,Q} (P,Q \in \mathbf{Z} \wedge P \neq Q \Rightarrow P \cap Q = \varnothing) \Rightarrow$
$\exists_Y \forall_X (X \in \mathbf{Z} \Rightarrow \exists!_x (x \in X \cap Y))$

---

* The above considered formula can be generalised for a radix m as follows: $\sum_{j=1}^{n} a_j . m^{n-j}$, where n and m correspond to the p*osition number* and used *radix*, respectively. For example, assuming n = 3 and m = 4, we can obtain: $\perp(232) = 2 \times 4^{3-1} + 3 \times 4^{3-2} + 2 \times 4^{3-3} = 46$.

† Joseph Liouville (1809 – 1882). The existence of *irrational numbers* was well-known even in Ancient Greece, e.g. Pithagoras (c.570 – c.495 b.c.).

‡ Cantor's *'diagonal' proof* concerning the existence of *irrational numbers* (1873): http://www.mathpages.com/home/kmath371.htm.

§ *Formal logic. Encyclopedical outline with applications to informatics and linguistics* (1987).

Let Y^X be a set of maps from X to Y, i.e. $Y^X =_{df} \{f / f : X \to Y\}$. The following two definitions are introduced (Słupecki J. and Borkowski L. 1967: for convenience, instead of the original X^Y, the set Y^X is used in the second definition).

Definition 5.37 (*set of maps*)

Let $\rho \subseteq X \times Y$. We shall say that $\rho \in Y^X$ (*the function* ρ *is a map from* X *to* Y) iff $dom(\rho) = X$ and $\rho(X) \subseteq Y$.

Definition 5.38 (*cardinality of a set of maps*)

$$|Y|^{|X|} = |Y^X| .$$

In the next considerations, the cardinalities of $\mathbb{N}$ and $\mathbb{R}$, i.e. $|\mathbb{N}|$ and $|\mathbb{R}|$, we shall denote by the following two cardinal numbers: $\aleph_0$ and c, known as *alef-zero*[*] and *continuum*, respectively. Some properties of these two numbers are presented below (Słupecki J. and Borkowski L. 1967).

Definition 5.39 (*sequence stock*)

The set X is *in stock* for the sequence $\{x_n\}$ $\Leftrightarrow \underset{x}{\forall} (x \in X \Leftrightarrow \underset{i}{\exists} (i \in \mathbb{N} \wedge x = x_i))$.

The following proposition was shown.

Proposition 5.5

$$|X| = \aleph_0 \Leftrightarrow \underset{\{x_n\}}{\exists} (\underset{i,j}{\forall} (i \neq j \Rightarrow x_i \neq x_j) \wedge X \text{ is in stock for } \{x_n\}). \square$$

Definition 5.40

$$|X| \leq |Y| \Leftrightarrow \underset{Z \subseteq Y}{\exists} (X \sim Z) .$$

According to the last definition and reflexivity of equinumerosity, the following corollary was given.

Corollary 5.4

$$X \subseteq Y \Rightarrow |X| \leq |Y| . \square$$

Definition 5.41

$$|X| < |Y| \Leftrightarrow |X| \leq |Y| \wedge |X| \neq |Y| .$$

Thesis 5.39

$\aleph_0 \leq c . \square$ {Coroll. 5.4, Df. 5.41}

It was shown that the right side of Definition 5.33 is logicaly equivalent to the following expression (see the next thesis): $\underset{P \subseteq X}{\exists} (|P| = \aleph_0)$[†].

---

[*] The used by Cantor symbol ´ℵ´ denotes the first letter of Hebraic alphaet.

[†] The following two implications are satisfied (Słupecki J. and Borkowski L. 1967). The proof of these two implications is left to the reader.

(1) $P \subseteq X \wedge |P| = \aleph_0 \Rightarrow \underset{Y \subseteq X}{\exists} (Y \neq X \wedge X \sim Y). \square$

Thesis 5.40

X is *a non-finite set in Dedekind's sense* $\Leftrightarrow \underset{P \subseteq X}{\exists} (|P| = \aleph_0)$. □

According to the last thesis, the existence of a non-finite set in Dedekind's sense is logically equivalent to the existence of a countable subset of this set.

Definition 5.42 (*transfinite cardinality*)

The cardinality of a non-finite set in Dedekind's sense is said to be *transfinite* one.

Corollary 5.5

A cardinal number m is transfinite iff $\aleph_0 \leq m$. □ {T 5.40, Df.5.42, Coroll. 5.4}

Corollary 5.6

The cardinal numbers $\aleph_0$ and c are transfinite. □ {T 5.40, Df.5.42, Coroll. 5.4}

The following thesis is satisfied (Słupecki J. and Borkowski L. 1967).

Thesis 5.41

$\aleph_0 \neq$ c

Proof:

Assume that $\aleph_0 =$ c {aip}. Let X $=_{df}$ $\mathbb{R}$. And hence, in accordance with Proposition 5.5, there exists a sequence $\{x_n\}$ such that $\mathbb{R}$ is in stock for $\{x_n\}$. On the other hand, it is known that for any $x \in \mathbb{R}$, there exists exactly one *proper representation* of x as non-finite decimal* (obviously, starting from a given position, all next digits may be equal to zero). We can obtain the following proper representations (any c_i denotes the integer part of the real number x_i and any y_{ij} – the j^{th} digit of x_i):

$x_1 = c_1, y_{11}y_{12}y_{13} \dots y_{1n} \dots$
$x_2 = c_2, y_{21}y_{22}y_{23} \dots y_{2n} \dots$
....
$x_n = c_n, y_{n1}y_{n2}y_{n3} \dots y_{nn} \dots$
....

Consider now the sequence $\{z_n\}$ such that: $z_n =_{df}$ if $y_{nn} \neq 1$ then 1 else 2. We have a proper representation of some real number $z = 0, z_1z_2z_3 \dots$ and $z \notin \{x_n\}$. □ {contr.}

Corollary 5.7

$\aleph_0 <$ c. □ {T 5.39, T 5.41, Df. 5.41}

In particular, the following two theses were also shown (the corresponding proofs are omitted).

Thesis 5.42

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(2) $\underset{Y \subseteq X}{\exists} (Y \neq X \wedge X \sim Y) \Rightarrow \underset{P \subseteq X}{\exists} (|P| = \aleph_0)$. □

* See also: (Knuth D.E. 1997: Donald Ervin Knuth, born 1938).

$X \subseteq \mathbb{R} \;\wedge\; |X| = \aleph_0 \;\Rightarrow\; |\mathbb{R} - X| = c.$ □

According to the last thesis, the cardinality of $\mathbb{R}$ is invariant under removing a countable subset of elements.

Thesis 5.43

$n^{\aleph_0} = c$, for any $n \in \mathbb{N} - \{1\}$. □

Example 5.17

Let F be the set considered in Example 5.15. Since $F =_{df} \{0,1\}^{\mathbb{N}}$ then $|F| = |\{0,1\}^{\mathbb{N}}| = |\{0,1\}|^{|\mathbb{N}|} = 2^{\aleph_0} = c.$ □ {Df. 5.38, T 5.43}

The above introductory notions related to classical set theory are an illustration of this fundamental work. A more formal treatment is omitted: left to the reader. In the next considerations Kripke - Platek and other set theories are cited. A brief presentation of Kripke - Platek axiomatic set theory is only given.

5.6. Kripke - Platek and other set theories

Zermelo's axiomatic set theory was extended by adding new axioms, e.g. the *axioms of replacement*[*] and *regularity* (or *foundation*) introduced in Zermelo-Fraenkel set theory.[†] On the other hand, there were also presented some alternative approaches, e.g. such as (Holmes M.R. 2017): *typed theories of sets* (Whitehead A.N. and Russell B. 1913), *Ackermann's set theory* (theory of classes in which some classes are sets), *Quine's new foundations* (as a simplification of the theory of types), *positive set theories* (with topological motivation), *constructive* (i.e. intuitionistic) *set theories*, *set theory for nonstandard analysis*, etc.[‡] The classical *Kripke - Platek set theory* is an axiomatic system weaker than Zermelo - Fraenkel set theory. *Kripke - Platek set theory with urelements* (i.e. with basic elements: allowing large or high-complexity objects) is an axiomatic system based on the classical Kripke-Platek set theory.[§] The introduction of urelements was of interest for technical reasons in model theory (Barwise K. J. 1975). The technical advantage of this theory is that all of its constructions are 'absolute' in a suitable sense. This makes the theory suitable for the development of an extension of recursion theory to sets (Holmes M.R. 2017). A brief presentation of the last system (denoted in short by KPU) is given below.[**]

It is used in KPU a *two-sorted first-order predicate logic language* with a single binary relation symbol ´∈´, denoted by L^* and defined as follows. Letters of the *sort*: p, q, r, … denote *urelements* (if they are used) and letters of the *sort*: a, b, c, … denote sets (instead of the traditional capital letters: A, B, C, …). The letters: x, y, z, … may denote urelements or sets, e.g. $x =_{df} p$ or $x =_{df} a$. The letters of the sets may appear on both sides of ´∈´ (e.g. $a \in b$), but those for urelements may only appear on the left (e.g. $p \in b$). The presented below axioms (in particular: *Δ_0-separation* and *Δ_0-collection*) will require reference to a certain collection of formulae called *Δ_0-formulae*, consisting only formulae built by the constants: ´∈´, ´~´, ´∧´ , ´∨´ and bounded quantification. The following *KPU axiomatic system* was presented (e.g. in accordance with ZFC, since a and b may be arbitrary, $A1\,/_{ZFCstyle}$ can be represented as follows:

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[*] Published independently by Fraenkel A. A. H. and Skolem A. T. (in 1922)

[†] This axiomatic system was proposed in order to formulate a theory of sets free of paradoxes such as Russell's paradox: 1901 (Bertrand Russell 1872 – 1970). The same paradox was discovered in 1899 by Zermelo E. F. F. (but not published). Let $Y =_{df} \{X / X \notin X\}$. The following antinomy can be obtained: $Y \in Y \Leftrightarrow Y \notin Y$. Another paradoxes were also observed, e.g. such as: Burali-Forti's paradox (1897), concerning *ordinal numbers* (a generalisation of natural numbers: Cesare Burali-Forti 1861 – 1931) or Cantor's paradox (1895 - 9) concerning natural numbers (*The Free Encyclopaedia, The Wikimedia Foundation, Inc.).*

[‡] As an example: *small set theory* or also *double extension set theory*.

[§] Saul Aaron Kripke, born 1940 ; Richard Alan Platek, born 1940

[**] *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

A1 (*axiom of extensionality*) $\forall_x (x \in a \Leftrightarrow x \in b) \Rightarrow a = b$

A2 (*axiom of foundation*) For every formula $\varphi(x)$: $\exists_a \varphi(a) \Rightarrow \exists_a (\varphi(a) \wedge \forall_{x \in a} \sim \varphi(x))$

A3 (*axiom of pairing*) $\exists_a ((x \in a) \wedge (y \in a))$

A4 (*axiom of union*) $\exists_a \forall_{c \in b} \forall_{y \in c} (y \in a)$[*]

A5 (*axiom of Δ_0-separation*) $\exists_a \forall_x (x \in a \Leftrightarrow x \in b \wedge \varphi(x))$ (for every Δ_0-formula $\varphi(x)$)

A6 (*axiom of Δ_0-collection*) $\forall_{x \in a} \exists_y \varphi(x,y) \Rightarrow \exists_b \forall_{x \in a} \exists_{y \in b} \varphi(x,y)$

A7 (*axiom of set existence*) $\exists_a (a = a)$

For example, in accordance with ZFC, since a and b may be arbitrary, A1 / $_{\text{ZFCstyle}}$ can be also represented as follows:

A1 (*axiom of extensionality*) $\forall_a \forall_b (\forall_x (x \in a \Leftrightarrow x \in b) \Rightarrow a = b)$

The following two additional assumptions were used (describing the partition of objects into sets and urelements).

Assuumption 1 $\forall_p \forall_a (p \neq a)$

Assuumption 2 $\forall_p \forall_x (x \notin p)$

It can be observed that the used KPU axioms are *universal quantifier* closures[†] of the corresponding formulae. In particular, the above axioms A2, A5 and A6 are *axiom schemes*.[‡] This axiomatic system can be used in model theory of infinitary languages (assuming infinitary long statements or proofs). The transitive such models (inside a maximal universe) are called admissible sets. A more formal treatment is omitted: left to the reader.

5.7. Set theory: some aplications

Set theory (because of its general nature) has very many applications not only in *classical mathematics* (e.g. in *differential and integral calculus*) but also in *theoretical computer science* (in general, e.g. recursion theory) and *discrete mathematics* (in particular, e.g. the algebraic treatment of set operations). Set theory provides the basis of *general topology* (e.g *Polish spaces*), etc. Kolmogorov's axioms[§] (1933) are the foundations of probability theory (Kolmogorov A.N. 1956). 'These axioms remain central and have direct contributions to mathematics, the physical sciences, and real-world probability cases'(Aldous D.J.[**]). This system is presented below. Some introductory notions are first presented (Fisz M. 1969).

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[*] Or equivalently: $\exists_a \forall_{c \in b} (c \subseteq a)$

[†] The *universal quantifier closure* of a formula φ is the formula whith no free variables obtained by adding a universal quantifier for every free variable in φ, e.g. $\forall_x \forall_z (R(x) \vee \exists_y S(y,z))$ is an *universal quantifier closure* of the formula $\varphi =_{df} R(x) \vee \exists_y S(y,z)$.

[‡] Any *axiom scheme* can be considered as a standard way of introducing axioms having the same syntactic structure, e.g. the axiom A1 (the first *law of the hypothetical syllogism*: *law of Duns Scotus*, see Subsection 1.7) , of *Łukasiewicz's implication-negation axiomatic system*: $p \Rightarrow (\sim p \Rightarrow q)$ can be generalised for any two formulae φ and ψ, as follows: $\varphi \Rightarrow (\sim \varphi \Rightarrow \psi)$.

[§] Andrey Nikolaevich Kolmogorov (1903 – 1987)

[**] What is the significance of the Kolmogorov axioms? Retrieved November 19, 2019 (*The Free Encyclopaedia, The Wikimedia Foundation, Inc.*), https://www.stat.berkeley.edu/~aldous/Real_World/kolmogorov.html: David John Aldous, born 1952

Let E be the set of *elementary events* (called also '*atomic*') . Any *event* which is not elementary can be considered as a *subset of elementary events*, e.g. $A \subseteq E$. The set of all such subsets of elementary events is denoted by Z. According to Definition 5.21, for $|E| = n \in \mathbb{N}$[*] we have: $|Z| = 2^n$.

An event including all elements of E is said to be a *sure event*. An event which no includes any element of E is said to be *impossible* one. *Set inclusion* and *set equality* are introduced in the same manner as in classical set theory, assuming now that set elements are elementary events. In accordance with the last properties Z is defined as a *Borel's field set of events*.[†]

Kolmogorov's axiomatic system is presented as follows.

A1 (*axiom of non-negativity*)	The probability $p(A) \in \mathbb{R}$ of an event $A \in Z$ is a non-negative number, i.e $p(A) \geq 0$
A2 (*axiom of unit measure*)	$p(E) = 1$
A3 (*axiom of σ-additivity*)	$p(\bigcup_i A_i) = \sum_i p(A_i)$, for any countable sequence of disjoint sets $\{A_i\}$

The *event complement* is defined as follows: $A' =_{df} E - A$ (in a similar way as set complement: see Definition 5.8). Some consequences of the last axioms are illustrated below (Fisz M. 1969).

Proposition 5.6

$p(A') = 1 - p(A)$

Proof:

Since A and A′ are disjoint sets of elementary events, we have: $p(A \cup A') = p(A) + p(A') = 1$. And hence: $p(A') = 1 - p(A)$. □ {A2, A3}

Corollary 5.8

Since $p(A') \geq 0$ then $1 - p(A) \geq 0$. Hence $p(A) \leq 1$. We have: $0 \leq p(A) \leq 1$. □ {A1,Prop. 5.6}

Proposition 5.7

$p(\varnothing) = 0$

Proof:

For any A: $A \cup E = E$. Let $A =_{df} \varnothing$. Then A and E are disjoint and hence $p(A \cup E) = p(A) + p(E) = p(E) = 1$. Hence $p(A) = 0$. □ {A2,A3,Coroll. 5.8}

Proposition 5.8

$A \subseteq B \Rightarrow p(A) \leq p(B)$

Proof:

Assume that $A \subseteq B$. Then $B = A \cup (B - A)$. And hence: $p(B) = p(A) + p(B - A)$. Since $p(B - A) \geq 0$ then $p(B) \geq P(A)$. □ {A1,A3}

Proposition 5.9

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[*] In general, the set E may also be of cardinality $\aleph_0$ or c.

[†] The probability p(A) satisfying Kolmogorov's axiomatic system can be considered as a *normalised nonnegative* and *countably additive measure* in Borel's field set of events Z (Félix Édouard Justin Émile Borel: 1871 – 1956).

Let $A, B \in Z$ be two not necessarily disjoint sets of elementary events. Then: $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Proof:

Since $A \cup B = (A - B) \cup B$ and $A = (A - B) \cup (A \cap B)$ then: $p(A \cup B) = p(A - B) + p(B)$ and $p(A) = p(A - B) + p(A \cap B)$.

From the last equation we have: $p(A - B) = p(A) - p(A \cap B)$. And hence: $p(A \cup B) = (p(A) - p(A \cap B)) + p(B) = p(A) + p(B) - p(A \cap B)$. □ {A3}

Since ′ ∩ ′ and ′ ∪ ′ are two *commutative*, *associative* and *mutually distributive* binary operations, in accordance with the last proposition, for any three events $A, B, C \in Z$ we can obtain:

$p(A \cup B \cup C)$	$=$	$p((A \cup B) \cup C)$	{assoc.}
	$=$	$p(A \cup B) + p(C) - p((A \cup B) \cap C)$	{Prop. 5.9}
	$=$	$(p(A) + p(B) - p(A \cap B)) + p(C) - p((A \cap C) \cup (B \cap C))$	{Prop. 5.9, comm. and distrib}
	$=$	$p(A) + p(B) + p(C) - p(A \cap B) - (p(A \cap C) + p(B \cap C) - p((A \cap C) \cap (B \cap C))$	{Prop. 5.9}
	$=$	$p(A) + p(B) + p(C) - p(A \cap B) - p(A \cap C) - p(B \cap C) + p(A \cap B \cap C)$. □	{comm., $C \cap C = C$}

Let now the above sets A, B and C be denoted by A_1, A_2 and A_3, respectively. The following expression can be obtained (the used here abbreviation ′ lt ′ denotes ′*less than*′, i.e. ′<′).

$$p(\bigcup_{i=1}^{3} A_i) = \sum_{i=1}^{3} p(A_i) - \sum_{i_1, i_2 = 1 / i_1 \text{ lt } i_2}^{3} p(A_{i_1} \cap A_{i_2}) + (-1)^{3+1} \cdot p(A_1 \cap A_2 \cap A_3)$$

In general, for $A_1, A_2, \dots, A_n \in Z$ $(n \geq 3)$ the following expression can be obtained (known as *Poincaré's formula*.[*]

$$p(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} p(A_i) - \sum_{i_1, i_2 = 1 / i_1 \text{ lt } i_2}^{n} p(A_{i_1} \cap A_{i_2}) + \sum_{i_1, i_2, i_3 = 1 / i_1 \text{ lt } i_2 \text{ lt } i_3}^{n} p(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \dots + (-1)^{n+1} \cdot p(A_1 \cap \dots \cap A_n)$$

Let A be an event. Consider the sequence: $B_1, B_2, \dots, B_n$ of mutually exclusive events forming a *complete system*. Assume now that A can be satisfied iff there exists B_i such that B_i is satisfied ($i \in \{1, n\}$). And hence, the event A can be decomposed into the subcases $A \cap B_1, A \cap B_2, \dots, A \cap B_n$, i.e. $A = \bigcup_{i=1}^{n} (A \cap B_i)$. And so, we can obtain: $p(A) = p(\bigcup_{i=1}^{n} (A \cap B_i)) = \sum_{i=1}^{n} p(A \cap B_i)$. Since $p(A \cap B_i) = p(B_i) \cdot p(A / B_i)$ we have (the *complete probability* formula[†]): $p(A) = \sum_{i=1}^{n} p(B_i) \cdot p(A / B_i)$.

[*] Jules Henri Poincaré (1854 – 1912)

[†] Known also as: '*total probability theorem*'.

On the other hand: $p(A) \cdot p(B_i / A) = p(B_i) \cdot p(A / B_i)$. Hence: $p(B_i / A) = \dfrac{p(B_i) \cdot p(A/B_i)}{p(A)}$. And finally we have: $p(B_i / A) = \dfrac{p(B_i) \cdot p(A/B_i)}{\sum_{i=1}^{n} p(B_i) \cdot p(A/B_i)}$. The last formula is known as *Bayes' theorem*, given in the mid-eighteenth century by Thomas Bayes (1701 – 1761).*

'*Discrete event systems* (known also as: *event-driven systems*) are systems whose dynamic behaviour is driven by asynchronous occurrences of discrete events'. *Petri nets*† are fundamental models of such systems (Reisig W. 1985, 1992). Set theory provides a foundation for Petri net theory and its applications. As an illustration (at the end of this subsection), the notions of *D-partition* and *net k-distinguishability* are introduced below (Tabakow I.T. 2008)‡. At first, some basic notions are given.

In general any *place-transition net* $N =_{df} (T, P, A, ,K, M_0, W)$, where (T, P, A) is a finite net containing sets of *transitions* , *places*, and *arcs* called also *edges*, $K : P \rightarrow (IN_\omega - \{0\})$ and $W : A \rightarrow \mathbb{N}$ are the corresponding *place capacity* and *edge multiplicity* (called also *weight*) functions, respectively. In particular, N is *ordinary* iff $W(a) = 1$ (for any $a \in A$). The *initial marking vector* $M_0 : P \rightarrow IN_\omega$, where $\mathbb{N}$ denotes the set of all natural numbers, $IN =_{df} \mathbb{N} \cup \{0\}$, $IN_\omega =_{df} IN \cup \{\omega\}$, and ω is an infinite number such that: $\omega + k = \omega$ and $k < \omega$ (for any $k \in IN$) (Murata T. 1989, Reisig W. 1985, 1992). The *forward marking class* of N, i.e. $[M_0 > =_{df} \{M \in IN_\omega{}^P \ / \ \exists \tau \in T^* (M_0 [\tau > M) \}$.

In the next considerations we shall assume N is a *live* and *bounded* net. In the case of manufacturing systems the *net reversibility property* is also required. Moreover, for simplicity it is assumed below that N is *pure* (i.e. it has no self-loops) and *simple* (there are no different vertices in N having the same pre- and post-sets). The net P-*invariants* (*T-invariants*) are computed using $\underline{N} \cdot \underline{i} = \underline{0}$ ($\underline{N}^T \cdot \underline{x} = \underline{0}$), where $\underline{N}$ is the *PN-connectivity matrix* of N (having $|T|$ rows and $|P|$ columns). We have: $\underline{N} = \underline{N}^+ - \underline{N}^-$, where $\underline{N}^+$ and $\underline{N}^-$ are the corresponding *output* and *input matrices* for N. The *support* of any P-invariant $\underline{i}$ wrt N is defined as follows: $supp(\underline{i}) =_{df} \{p \in P / \underline{i}(p) \neq 0 \} \subseteq P$. Let I be the set of all (positive) P-invariants of N and $J \subseteq I$ is a subset. The *P-invariant matrix* of N wrt J is introduced as follows: $\underline{J} : J \times P \rightarrow IN$, where $\underline{J}(\underline{i},p) =_{df} \underline{i}(p) \in IN$.

For convenience only, we shall assume below that the *P-cover* J of N is a set of all positive and minimal P-invariants. Any such set is assumed to be a set of linearly independent P-invariants. And in fact, any P-invariant matrix $\underline{J}$ can be considered as an information system. Hence, we shall assume that the set of P-invariants J (i.e. 'attributes' of this information system) is a reduced set (Murata T. 1989). Also we shall use the notion of the *revised P-invariant matrix* of N, defined as: $\underline{\rho} : J \times P \rightarrow \{0.1\}$, where $\underline{\rho}(\underline{i},p) =_{df} 1$ iff $\underline{i}(p) \neq 0$ (Immanuel B. and Rangarajan K. 2001). Let $x \in T \cup P$. The *pre-set* (*post-set*) associated with x is defined as follows: ${}^\bullet x =_{df} \{ y \in T \cup P / (y,x) \in A \}$ ($x^\bullet =_{df} \{ y \in T \cup P / (x,y) \in A \}$). For any nonempty $X \subseteq T \cup P$: ${}^*X =_{df} \bigcup_{x \in X} {}^\bullet x$ and $X^* =_{df} \bigcup_{x \in X} x^\bullet$. Any ordinary net is a *marked graph* iff $| {}^\bullet p | = | p^\bullet | = 1$ (for any $p \in P$). For simplicity, it is assumed below N have a P-cover. Otherwise, this method is also applicable. In the last case some additional test points is necessary to be introduced. Moreover, it can be observed that a selection of a minimal (or in general: optimal) P-cover may not guarantee a better net fault distinguishability. The net's places are interpreted below as representing resource states or operations and the

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* (Bradistilov G. 1961): Georgi Bradistilov (1904 – 1977), (R0berts F.S. 1976): Fred Stefen Roberts, born: 1943.

† Carl Adam Petri (1926 – 2010)

‡ A fragment of this paper is here presented. The paper describes a method of diagnosis-time assessment in discrete event systems. Any such system is modelled by a *live*, *bounded*, and *reversible place-transition Petri net* N. Also there are assumed some deterministically given delays associated with the transitions of N and hence N is assumed to be deterministic timed. For this purpose two different type of (single) fault models are used, i.e. *place fault* and *transition fault* models and the corresponding diagnosis process is assumed to be *sequential*.

transitions as representing start or completion of the corresponding discrete event (Zhou M.C. and DiCesare F. 1993).

Let $[M_0>_\alpha =_{df} [M_0> \cup \{M_\alpha\}$, where M_0 is the initial marking and M_α be a marking of N such that $M_\alpha \notin [M_0>$. We shall say M_α is a *faulty marking*. Since $M \cdot \underline{i} = M_0 \cdot \underline{i}$, for any $M \in [M_0>$ and $\underline{i} \in J$ (Reisig W. 1985), then $\Delta M \cdot \underline{i} = 0$, where $\Delta M =_{df} M - M_0$. The last property is satisfied for any P-invariant $\underline{i} \in J$. Hence we can obtain $\underline{J} \cdot \Delta M^T = \underline{0}$. Therefore for $M \in [M_0>_\alpha$ the above equation may be violated. Thus we have: $\underline{J} \cdot \Delta M^T = \underline{a} \in \{0,1\}^{|J|}$ (for any $M \in [M_0>_\alpha$, obviously $\underline{a} = \underline{0}$ iff $M \in [M_0>$). Without losing any generality, below $(\underline{a})_s \neq 0$ are interpreted as $(\underline{a})_s = 1$ ($s \in \{1,...,|J|\}$). And hence, in accordance with (Murata T. 1983), any $(\underline{a})_s = 1$ will correspond to some subset of places $supp(\underline{i}_s) \subseteq P$ having a (potentially) faulty behaviour. Let $\Omega(\underline{a}) =_{df} \bigcap_{(\underline{a})_s=1} supp(\underline{i}_s) \cap \bigcap_{(\underline{a})_s=0} supp(\underline{i}_s)' \subseteq P$, where $supp(\underline{i}_s)' =_{df} P - supp(\underline{i}_s)$ is the corresponding set complement operation (provided there is no ambiguity we shall use below the same designation " ′ " as an index, e.g. to denote M′, i.e. the marking M for N′, where N′ is the net simulator corresponding to N, in a similar manner Ω′ is used for Ω of N′). The notions of D-partition and net k-distinguishability are given below (Tabakow I.G. 2000, 2005, 2006 / fault distinguishability …, 2007).

Definition 5.41

By a *D-partition of the set of places* P of a given place-transition net N wrt the P-cover J of N, denoted by $\Omega(N,J)$, or Ω if N and J are understood, we shall mean the (multi) family $\Omega =_{df} \{\Omega(\underline{a}) / \underline{a} \in \{0,1\}^{|J|}\}$.

Proposition 5.10

(a) $\Omega(\underline{0}) = \varnothing$,
(b) $\forall \underline{a},\underline{b} \neq \underline{0}$ $(\underline{a} \neq \underline{b} \Rightarrow \Omega(\underline{a}) \cap \Omega(\underline{b}) = \varnothing)$ and
(c) $\bigcup_{\underline{a} \in \{0,1\}^{|J|}} \Omega(\underline{a}) = P$. ▫

Definition 5.43

The Petri net N is a *k-distinguishable net under* Ω iff

(a) $\exists\, \Omega(\underline{a}) \in \Omega$ $(|\Omega(\underline{a})| = k)$ and
(b) $\forall\, \Omega(\underline{a}) \in \Omega$ $(|\Omega(\underline{a})| \leq k)$.

5.8. Commonsense sets: some comments

'Set theory has long been known for its phenomenal success in providing a basis for virtually all of mathematics, both in the philosophical sense of a precise foundation, and in the more prosaic sense of naturalness for defining other mathematical concepts with minimum pain. Here we explore the idea that also in commonsense reasoning there is a natural role for sets, although perhaps not as a basis for all such reasoning. Despite the wide variety of knowledge representation schemes that have been proposed, set theory seems not to have been explored as a vehicle for representing commonsense knowledge'.* In accordance with this work, 'it is argued that set theory provides a powerful addition to commonsense reasoning, facilitating expression of meta-knowledge, names, and self-reference. Difficulties in establishing a suitable language to include sets for such purposes are discussed, as well as what appear to be promising solutions. Ackermann's set theory as well as a more recent theory involving universal sets are discussed in terms of their relevance to commonsense'.†

* Perlis D., *Commonsense Set Theory*. University of Maryland Department of Computer Science College Park, Maryland 20742: file:///C:/Users/user/Documents/COMMONSENSE%20SETS%20Maryland%20to%20samo.pdf, 10pp. See also: (Geldenhuys A.E. et al. 1999).

† (Ackermann W.F. / Zur axiomatic …, 1956): Wilhelm Friedrich Ackermann (1896 – 1962)

‘The philosophical logician’s evaluation of a theory should be regulated by meta-theoretical norms which are neither rigid laws written on our genes not cultural artifacts embedded in ordinary discourse, but rather reflectively constructed expressions of our desire to be rational’(Poliard S. 2015). Some considerations, given in the last work, are cited below.

‘As I have already indicated, we shall find that a mere grasp of commonsense notions of set will not supply us with an even remotely adequate appreciation for mathematical sets. In fact, I doubt whether any ´commonsensical´ clarification of mathematical set theories is either possible or needed. My point is simply that we can be sure of this only after an investigation of ordinary language. And such an investigation is warranted even if its results are wholly negative’.

Some considerations related to Black’s ‘*The Elusiveness of Sets*’[*] were also given (Poliard S. 2015): ‘Black argues that commonsense sets (constructed in a certain way) are objects of the same type as mathematical sets. As a convenient shorthand we might say that Black ´identifies mathematical and commonsense sets´ or ´claims that mathematical sets are commonsense objects´’. ‘I gather that Black is actually making the much less dubious claim that one can plausibly assert both that sets are commonsense objects and they are apparently uncommonsensical ´abstract entities´ can be connected up with ordinary language. If this connection cannot be made, then sets may indeed be ´abstract entities´, but they are not commonsense objects’.

6. Multiset theory

‘A multiset is a collection of objects (called elements) in which elements may occur more than once. The number of times an element occurs in a multiset is called its multiplicity. The cardinality of a multiset is the sum of the multiplicities of its elements. Multisets are of interest in certain areas of mathematics, computer science, physics, and philosophy’(Blizard W.D. 1989). Many aspects related to multiset theory can be found in mathematical literature, e.g. works given by: Leibniz, Weierstrass, Dedekind, Cantor, Frege, Peano[†], etc., see (Blizard W.D. 1989). Since multisets (called also *bags* or *msets*) are generalisation of sets, in accordance with the last work, it is developed ‘a first-order two-sorted theory for multisets that ´contains´ classical set theory. The intended interpretation of the atomic formula $x \in^n y$ is ´x is an element of y with multiplicity n´.’ Next, it is presented a model of multiset theory using ZFC (i.e. Zermelo-Fraenkel axiomatics with the axiom of choice)[‡]. Here, the above atomic formula ´$x \in^n y$´ is interpreted as $y(x) = n$ (multisets are modelled by positive integer - valued functions). As an example, the *axiom of extensionality* in this theory is presented as follows.

$$\underset{x}{\forall}\ \underset{y}{\forall}\ (\underset{z}{\forall}\ \underset{n}{\forall}\ (z \in^n x \Leftrightarrow z \in^n y) \Rightarrow x = y)$$

Example 5.18 (*multiset*)

For any natural number $n > 1$, there exists exactly one such representation (called *canonical*}: see Example 5.9 of Subsection 5.4). Assume, e.g. $n = 12$ $(2^2 \times 3^1)$. Hence n is related to the following multiset: {2,2,3}. In a similar way: $n = 2700 = 2^2 \times 3^3 \times 5^2$ corresponds to {2,2,3,3,3,5,5}, etc.□

6.1. Basic notions and definitions

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[*] (Black M. 1971: Max Black 1909 – 1988)

[†] Gottfried Wilhelm Leibniz (1646 – 1716), Karl Theodor Wilhelm Weierstrass (1815 – 1897), Julius Wilhelm Richard Dedekind (1831 – 1916), Georg Ferdinand Ludwig Philipp Cantor (1845 – 1918), Friedrich Ludwig Gottlob Frege (1848 – 1925), Giuseppe Peano (1858 – 1932)

[‡] According to this system, the letters of the sets may appear on both sides of ´∈´, but those for elements may only appear on the left side (as in KPU axiomatic system: see Subsection 5.6).

Let $f_B : X \to \mathbb{N} \cup \{0\}$, where $|X| \in \mathbb{N}$ and $f_{B_i} =_{df} \#(x_i,B)$ be *the number* (or the *multiplicity*) *of* x_i's *in* B.[*] The following set is said to be a *basis*: $\{f_x / x \in X\}$.[†] And hence, $f_B = (f_{B1}, \ldots , f_{B|X|}) = \sum_{x \in X} \#(x,B) \cdot f_x$. Here, f_B is known as *Parikh's vector* (Parikh R.J. 1966).[‡]

Example 5.19 (*Parikh's vector*)

Let $X =_{df} \{x_1, x_2, x_3, x_4\}$ and $f_B =_{df} (3,1,2,0)$. Since $f_{x1} = (1, 0, 0, 0)$, $f_{x2} = (0, 1, 0, 0)$, $f_{x3} = (0, 0, 1, 0)$ and $f_{x4} = (0, 0, 0, 1)$, then $f_B = 3 \cdot f_{x1} + 1 \cdot f_{x2} + 2 \cdot f_{x3} + 0 \cdot f_{x4} = (3, 0, 0, 0) + (0, 1, 0, 0) + (0, 0, 2, 0) + (0, 0, 0, 0)$. □

Let A and B be two multisets on X. The *distance function* $d(A,B) =_{df} \sum_{x \in X} | f_A(x) - f_B(x) |$. In fact, the following properties are satisfied (for any A, B and C on X)..

$d(A,A) = 0$,

$d(A,B) = d(B,A)$ and

$d(A,B) + d(B,C) \geq d(A,C)$[§]

Let $\alpha, \beta \in \mathbb{R}$ be two arbitrary real numbers. The following property is satisfied: $|\alpha| + |\beta| \geq |\alpha + \beta|$. In particular, for $\alpha =_{df} a - c$ and $\beta =_{df} c - b$ we can obtain: $|a - c| + |c - b| \geq |a - b|$. And hence the last inequality is also satisfied (in accordance with the above notion of a distance function: a more formal treatment is left to the reader).

The *support* of a multiset B is defined as follows: $supp(B) =_{df} \{x \in X / \#(x,B) > 0\} \subseteq X$. The *cardinality* of B, i.e. $|B| =_{df} \sum_{x \in supp(B)} \#(x, B)$. Let $n \in \mathbb{N} \cup \{0\}$ and $B_n =_{df} \{x \in X / \#(x,B) \geq n\}$. We have: $B_0 = supp(B)$ and $B_n \subseteq supp(B)$.

6.2. Multiset operations

Some definitions concerning multisets are given below (Peterson J.L. 1981). The *multiset inclusion* relation and *multiset equation* (denoted by $'\subseteq_b'$ and $'=_b'$) are first introduced.

$$A \subseteq_b B \Leftrightarrow_{df} \forall_{x \in X} (\#(x,A) \leq \#(x,B))$$

$$A =_b B \Leftrightarrow_{df} \forall_{x \in X} (\#(x,A) = \#(x,B))$$

According to the last bag equation, we can obtain:

$$\forall_{x \in X} (\#(x,A) = \#(x,B)) \Leftrightarrow \forall_{x \in X} ((\#(x,A) \leq \#(x,B)) \wedge (\#(x,B) \leq \#(x,A)))$$

$$\Leftrightarrow \forall_{x \in X} (\#(x,A) \leq \#(x,B)) \wedge \forall_{x \in X} (\#(x,B) \leq \#(x,A)) \quad \{\text{Subsection 3.3}\}$$

---

[*] Because of the one-to-one correspondence between f_B and B and hence, for convenience, some times f_B is said to be multiset.

[†] Provided there is no ambiguity and for simplicity, f_{x_i} is used instead of $f_{\{x_i\}}$ (for any $x_i \in X$).

[‡] Rohit Jivanlal Parikh, born 1936.

[§] Corresponds to the *triangle inequality* in a metric space.

$\Leftrightarrow \quad (A \subseteq_b B) \wedge (B \subseteq_b A)$. □

The following *multiset operations* are also considered.

$A \cup B: \quad \#(x, A \cup B) =_{df} \max\{\#(x,A), \#(x,B)\}$

$A \cap B: \quad \#(x, A \cap B) =_{df} \min\{\#(x,A), \#(x,B)\}$

$A - B: \quad \#(x, A - B) =_{df} \#(x,A) - \#(x, A \cap B)$

$A \div B: \quad \#(x, A \div B) =_{df} \max\{\#(x, A - B), \#(x, B - A)\}$

$A + B: \quad \#(x, A + B) =_{df} \#(x,A) + \#(x,B)$

Example 5.20 (*set union*)

As in the previous example, let $X =_{df} \{x_1, x_2, x_3, x_4\}$ and A, B be two multisets on X, e.g. $f_A =_{df} (1,3,0,2)$ and $f_B =_{df} \{4,1,2,0)$, i.e. $A = \{x_1, x_2, x_2, x_2, x_4, x_4\}$ and $B = \{x_1, x_1, x_1, x_1, x_2, x_3, x_3\}$. We have: $f_{A \cup B} = \sum_{x \in X} \#(x, A \cup B) \cdot f_x = \sum_{x \in X} \max\{\#(x, A), \#(x, B)\} \cdot f_x = 4 \cdot f_{x1} + 3 \cdot f_{x2} + 2 \cdot f_{x3} + 2 \cdot f_{x4} = (4, 0, 0, 0) + (0, 3, 0, 0) + (0, 0, 2, 0) + (0, 0, 0, 2) = (4, 3, 2, 2) = \max\{f_A, f_B\}$. And hence: $A \cup B = \{x_1, x_1, x_1, x_1, x_2, x_2, x_2, x_3, x_3, x_4, x_4\}$. □

It can be shown that multiset union and multiset intersection are two *mutually distributive binary operations*, i.e. the following proposition is satisfied.

Proposition 5.11

For any multisets A, B and C:

$A \cap (B \cup C) =_b (A \cap B) \cup (A \cap C)$

and

$A \cup (B \cap C) =_b (A \cup B) \cap (A \cup C)$.

Proof:

In accordance with the first equality and the above definitions, we have:

$(L =)\ \min\{f_A(x), \max\{f_B(x), f_C(x)\}\} = \max\{\min\{f_A(x), f_B(x)\}, \min\{f_A(x), f_C(x)\}\}\ (= R)$

The proof of the first equality will require the consideration of the following $3^2 = 9$ cases (for any $f_A(x)$, $f_B(x)$, $f_C(x)$ and $x \in X$) and also two times subcases: $f_A(x) \geq f_C(x)$ and $f_A(x) < f_C(x)$, as it is shown below.

(1) $f_A(x) > f_B(x) > f_C(x)$
(2) $f_A(x) > f_B(x) = f_C(x)$
(3) $f_A(x) > f_B(x) < f_C(x)$ (plus subcases: $f_A(x) \geq f_C(x)$ and $f_A(x) < f_C(x)$)
(4) $f_A(x) = f_B(x) > f_C(x)$
(5) $f_A(x) = f_B(x) = f_C(x)$
(6) $f_A(x) = f_B(x) < f_C(x)$
(7) $f_A(x) < f_B(x) > f_C(x)$ (plus subcases: $f_A(x) \geq f_C(x)$ and $f_A(x) < f_C(x)$)
(8) $f_A(x) < f_B(x) = f_C(x)$
(9) $f_A(x) < f_B(x) < f_C(x)$

As an illustration, by assuming case (7) we can obtain:

$$\begin{aligned} L &= \min\{f_A(x), \max\{f_B(x), f_C(x)\}\} \\ &= \min\{f_A(x), f_B(x)\} \\ &= f_A(x) \end{aligned}$$

$R = \max\{\min\{f_A(x), f_B(x)\}, \min\{f_A(x), f_C(x)\}\}$
$= \max\{f_A(x), \min\{f_A(x), f_C(x)\}\}$
$= f_A(x) \quad \{\min\{f_A(x), f_C(x)\}\} = \text{if } f_A(x) \le f_C(x) \text{ then } f_A(x) \text{ else } f_C(x)\}$

The remaining cases are considered in a similar way. The proof of the second equality is also left to the reader. □

Example 5.21

The multiset intersection is not distributive over symmetric multiset difference, e.g. $A \cap (B \div C) \neq_b (A \cap B) \div (A \cap C)$. The left and the right sides of the last inequality are given below.

$L = \min\{f_A(x), \max\{f_B(x) - \min\{f_B(x), f_C(x)\}, f_C(x) - \min\{f_B(x), f_C(x)\}\}\}$ and
$R = \max\{\min\{f_A(x), f_B(x)\} - \min\{f_A(x), f_B(x), f_C(x)\}, \min\{f_A(x), f_C(x)\} - \min\{f_A(x), f_B(x), f_C(x)\}\}$

Assuming 3, 4 and 5, for $f_A(x)$, $f_B(x)$ and $f_C(x)$ respectively, we can obtain: $L = 1 \neq 0 = R$. □

In general, multiset complement is not a definable operation. However, in some particular cases this operation may be used. Let A,B and C be multisets. Assume that $A, B \subseteq_b C$. The last multiset can be considered as a multiset space wrt A and B. Hence, the *C – multiset complement* wrt A and B is introduced as follows: $A' =_{df} C - A$ and $B' =_{df} C - B$. {Df. 5.8}

Proposition 5.12

Let $A, B \subseteq_b C$ and A', B' be two C – multiset complements. The following *restricted De Morgan's laws* are satisfied: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

Proof:

Let $x \in X$ be arbitrary. The proof of the first equality is given below (the proof of the second equality is similar: left to the reader).

$L = f_{(A \cup B)'}(x) =_{df} f_C(x) - \max\{f_A(x), f_B(x)\}$
$R = f_{A' \cap B'}(x) =_{df} \min\{f_C(x) - f_A(x), f_C(x) - f_B(x)\}$

Let $f_A(x) \ge f_B(x)$ then: $L = f_C(x) - f_A(x) = R$. Assume now that $f_A(x) < f_B(x)$ we have: $L = f_C(x) - f_B(x) = R$. □

6.3. Multirelations

We shall consider below binary multirelations. Any such relation ρ can be represented as a subset of ordered pairs belonging to $X \times X$ and having some multiplicity, e.g. $\#((x_i,x_j), M_\rho)$, where M_ρ is the matrix associated with ρ*. Unfortunately, in the case of computer science theory and applications (e.g. logical and algebraic methods in programming or theory and application of relational structures) another approach is used, first presented in (Parikh R.J. 1983) and (Rewitzky I. 2003). Some basic notions and properties used below are under (Berghammer R. and Guttmann W. 2015)†. The notion of multirelation is first presented. In accordance with the last work, the *Z notation* is used below (Spivey J.M. 1998), e.g. $'R \subseteq A \times B'$ is denoted as: $'R : A \leftrightarrow B'$ (with *source* A and *target* B). Similarly, it is used matrix notation $'R_{x,y}'$ instead of $'(x,y) \in R'$

* This approach is useful in Petri net theory wrt the fundamental notion of *unfolding* (see the next subsection).

† See also the following extended version: Berghammer R. and Guttmann W., *An algebraic approach to multirelations and their properties*. 24pp: file:///C:/Users/user/Documents/ALGEBRAIC%20PROPERTIES%20TO%20MULTIRELATIONS.pdf.

or ′xRy′, etc.[*] The following three special relations are introduced: O, T and I (the *empty*, *universal* and *identity* relations, i.e. $O =_{df} \varnothing \subseteq X \times X$, $T =_{df} \mathscr{U} = X \times X$ and $I =_{df} \{(x,x) / x \in X\}$, respectively).

A binary multirelation can be considered as a relation in the usual sense, i.e. $R : A \leftrightarrow B$ (see Definition 5.17), assuming that (the targed) B is a powerset (see Definition 5.21)

Definition 5.44 (*multirelation*)

Let A, B be two sets and R be a binary relation under Definition 5.17 (here A and B correspond to X and Y, respectively) with the additional property that the targed (B) is a powerset (2^B). Then we shall say that R is a *multirelation*.

According to the last definition, $R : A \leftrightarrow 2^B$ is a multirelation, B is a superset.[†] In particular, the *empty*, the *universal* and the *membership multirelations* are denoted by: $\mathbf{O} : A \leftrightarrow 2^B$, $\mathbf{T} : A \leftrightarrow 2^B$ and $\mathbf{E} : A \leftrightarrow 2^A$, where $\mathbf{O} = \varnothing$, $\mathbf{T} = A \times 2^B$ and $\mathbf{E}_{x,Y} \Leftrightarrow_{df} x \in Y$ (for any $x \in A$ and $Y \in 2^A$).

Let $Q : A \times 2^B$ and $R : B \times 2^C$ be two multirelations. The *composition* of Q and R, denoted as $Q ; R : A \times 2^C$, is defined as follows.

$(Q ; R)_{x,Z} \Leftrightarrow_{df} \exists_{Y \in 2^B} Q_{x,Y} \wedge \forall_{y \in Y} R_{y,Z}$, for any $x \in A$ and $Z \in 2^C$.

The *transposition* of a multirelation R is not multirelation (the transposed relation, i.e. converse, is denoted by R^c, obviously: $(R^c)^c = R$). Instead, the dual operation is used (Berghammer R. and Guttmann W. 2015). The *dual* of R, denoted by $R^d : A \leftrightarrow 2^B$ is defined as below (here $\overline{Y}$ denotes the complement of Y, i.e. Y′ wrt the superset B).

$R^d_{x,Y} \Leftrightarrow_{df} \sim R_{x,\overline{Y}}$, for any $x \in A$ and $Y \in 2^B$.

The following operation precedence was introduced: the highest precedence have the unary operations *complement* and *dual*, then multirelation *composition* and next the multirelation operations of *union* and *intersection* (the definitions of the last two multirelation operations and multirelation *complement* are the same as in general relations, so they are omitted).

Let $R : A \leftrightarrow 2^B$. The following property is satisfied (for any $x \in A$ and $Y, Z \in 2^B$).

$R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z}$.

In accordance with the last property, if $x \in A$ is related to a set $Y \in 2^B$ then x also has to be related to all supersets of Y.

The notions of a contact multirelations and topological contact multirelations, concerning multirelations of type $R : A \leftrightarrow 2^A$ and first introduced in (Aumann G. 1970), are given below: *Aumann's axiomatic system* is first presented (Berghammer R. and Guttmann W. 2015)

A1 $\quad \forall_{x \in A} \sim R_{x,\varnothing}$

A2 $\quad \forall_{x \in A} R_{x,\{x\}}$

A3 $\quad \forall_{x \in A} \forall_{Y,Z \in 2^A} R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z}$

A4 $\quad \forall_{x \in A} \forall_{Y,Z \in 2^A} R_{x,Y} \wedge (\forall_{y \in Y} R_{y,Z}) \Rightarrow R_{x,Z}$

A5 $\quad \forall_{x \in A} \forall_{Y,Z \in 2^A} R_{x,Y \cup Z} \Leftrightarrow R_{x,Y} \vee R_{x,Z}$

---

[*] According to the used here Z notation and for convenience with the next considerations, the above binary relation will be denoted in this subsection by ′R′ instead of ′ρ′.

[†] Equivalently: $\rho \subseteq A \times \mathbb{P}(B)$.

Multirelations satisfying A1 - A3 are said to be *contact* ones. Any contact multirelation is called a *topological contact* if , in addition, A4 and A5 are also satisfied. According to A3, R is *up-closed multirelation* from A to 2^A.* Some properties (given in the extended vesion of this work: Section 3) are illustrated below.

In accordance with the last work, being relations, the multirelations of type $A \leftrightarrow 2^B$ form a *bounded distributive lattice* under the operations of union and intersection (which are mutually distributive).† There were studied various aspects summarised in twenty-one theorems. As an example, the following two theorems were presented.

Thesis 5.44

Let P, Q and R be arbitrary multirelations of the same type ´$A \leftrightarrow 2^B$´. We have:

(1) $\mathbf{O} ; R = \mathbf{O}$

(2) $\mathbf{E} ; R = R$

(3) $\mathbf{T} ; R = \mathbf{T}$

(4) $R \subseteq R ; \mathbf{E}$ {holds if R is up-closed}

(5) $(P \cup Q) ; R = P ; R \cup Q ; R$

(6) $(P \cap Q) ; R \subseteq P ; R \cap Q ; R$ {holds if P, Q are up-closed}

(7) $(P ; Q) ; R \subseteq P ; (Q ; R)$ {holds if Q is up-closed}

(8) $P ; Q \cup P ; R \subseteq P ; (Q \cup R)$

(9) $P ; (Q \cap R) \subseteq P ; Q \cap P ; R$. □

Thesis 5.45

Let Q and R be arbitrary multirelations of the same type ´$A \leftrightarrow 2^B$´. We have:

(1) $\mathbf{O}^d = \mathbf{T}$

(2) $\mathbf{E}^d = \mathbf{E}$

(3) $\mathbf{T}^d = \mathbf{O}$

(4) $(R^d)^d = R$

(5) $(Q \cup R)^d = Q^d \cap R^d$

(6) $(Q \cap R)^d = Q^d \cup R^d$

(7) $(Q ; R)^d \subseteq Q^d ; R^d$ {holds if Q is up-closed}

(8) $(Q ; R)^d = (Q ; \mathbf{E})^d ; R^d$. □

6.4. Applications

Multisets and multirelations arise naturally in modelling. Some interesting applications are cited below. '*Fibonacci‡ numbers* have been studied for a long time and have been generalized in several ways'(Munarini E.

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* In general, a multirelation $\rho \subseteq A \times 2^B$ is called *up-closed* if the following implication is satisfied: $a \rho X \wedge X \subseteq Y \Rightarrow a \rho Y$ (for any $a \in A$ and $X, Y \subseteq B$). The above notion can be introduced by using Parikh's vectors, e.g. $f_\rho : A \times 2^B \rightarrow \mathbb{N} \cup \{0\}$, left to the reader.

† *Lattice theory* will be considered in Part II of this book.

‡ Leonardo Fibonacci (c.1170 – c.1250)

2005). Some notions introduced in the last work and related to multisets are given below (we shall use the original designations, except for $\mathbb{N}$) [*].

A *multiset* on a set S is a function $\mu : S \to \mathbb{N} \cup \{0\}$, where $|S| \in \mathbb{N}$. By $\mu(x)$ it is denoted the *multiplicity* of $x \in S$ in μ. The *order* of μ is introduced as follows: $ord(\mu) =_{df} \sum_{x \in S} \mu(x)$. By M_n it is denoted the family of all multisets on $\langle n\rangle =_{df} \{1, 2, \dots , n\}$. It is said that μ on S is *m-filtering* if $\mu(x) < m$, for any $x \in S$. The families of *all m-filtering multisets on* $\langle n\rangle$ and *all m-filtering multisets of order* k on S are denoted by $M_n^{[m]}$ and $\binom{S;m}{k}$, respectively. The cardinality of the last family is the *André* (called also: *polynomial*) *coefficient* (André D. 1881)[†]: $\binom{|S|;m}{k}$. The following property holds:

$$(1 + x + x^2 + \dots + x^{m-1})^n = \sum_{k \geq 0} \binom{n;m}{k} \cdot x^k.$$

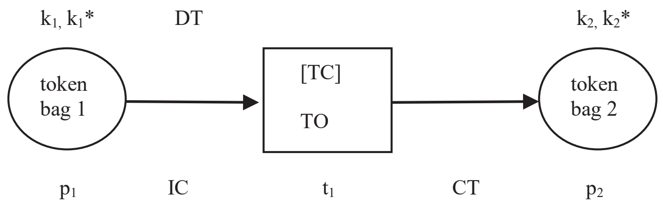
The *σ-stastistic* on M_n and the *conjugate* $\bar{\mu}$ of μ on $\mathbb{N} \cup \{0\}$ are defined as follows (Munarini E. 2005): $\sigma(\mu) =_{df} \sum_{x \in \langle n\rangle} x \cdot \mu(x)$ and $\bar{\mu}(x) =_{df} \mu(n + 1 - x)$, respectively. In particular: $ord(\bar{\mu}) = ord(\mu)$ and $\sigma(\bar{\mu}) = (n + 1)ord(\mu) - \sigma(\mu)$.

In accordance with the above illustrated properties (and also other ones, introduced in the previous considerations of this work: in particular, the definition of two kinds *Fibonacci permutations* on $\langle n\rangle$ having cardinalities that are *generalised Fibonacci numbers*) and using the notion of a multiset (e.g. in the proof of Theorem 11), there were introduced the *generalised q-Fibonacci polynomials* and two kinds *generalised q-Fibonacci numbers*. A more formal treatment is omitted: left to the reader.

Multisets are used as *token bags* in high level Petri nets, e.g. such as *numerical Petri nets* (Symons F.J.W. 1978, 1980) or c*oloured Petri nets* (Jensen K. 1987, Jensen K. and Kristensen L.M. 2009). A generic fragment of a typical numerical Petri net is shown in the next example (here, the used abbreviations DT, IC, TC, TO and CT denote as follows: *Destroyed Tokens*, *Input Condition*, *Transition Condition*, *Transition Operation* and *Created Tokens*, respectively).[‡]

Example 5.22

(a)



p-variables	initial values
x_1	a_1
…	…
x_n	a_n

[*] Provided there is no ambiguity and for convenience, here multiset is interpreted equivalently as a vector: ´μ´, the multiplicity of x in μ is denoted by $\mu(x)$, instead of $\#(x, \mu)$, the set S is used, instead of X, etc., see Subsection 6.1. Moreover, in the above original work, by $\mathbb{N}$ it is denoted the set of natural numbers extended by 0.

[†] See also: (Cmtet L. 1974).

[‡] There are two different types of place capacity: k and k^*. The first one sets a bound on the number of tokens of a particular value that can be resident in a place (as in *place/transition nets*, i.e. P/T nets). The second one, i.e. k^*, set a bound on the total number of tokens allowed in a place

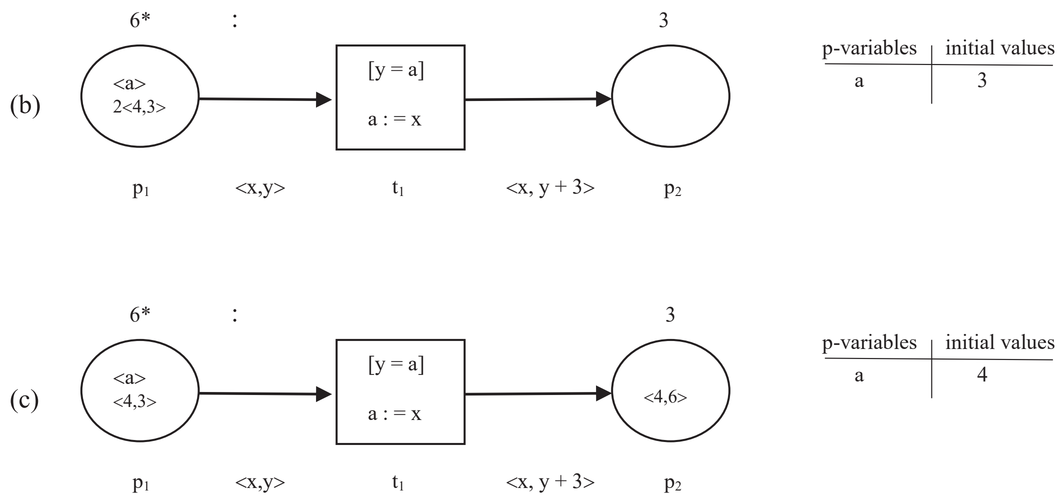


Figure 5.11 (a) A generic fragment, (b) Before firing t_1 and (c) After firing t_1

PN states are represented by *markings*. We have: $M_0 \rightarrow M_1$, where $M_0 =_{df}$ ({<a>, 2<4,3> }, ∅ / 3) denotes the *initial marking* and M_1 - the *next marking*: M_1 = ({<a>, <4,3> }, <4,6> / 4). □

A *categorical approach* for modelling of Petri nets was proposed in (Brown C. and Gurr D. 1990). An extension of the last work was proposed in (De Paiva V. 1991). Here, the main purpose was 'to deal with multirelations, as Petri nets are usually modelled using multirelations ´*pre*´ and ´*post*´'.[*] It is also presented a model of intuitionistic linear logic[†] (without modalities).

Another interesting study was given in (Furusawa H. et al. 2008). There are studied basic properties of up-closed multirelations. It is shown that the set of finitary total up-closed multirelations over a set forms a probabilistic Kleene algebra[‡].

'*Unfolding*, originally introduced in (McMillan K.L. 1993), is a method for *reachability analysis* which exploits and preserves concurrency information in the Petri net. It generates all possible firing sequences of the net, from the initial marking, whilst maintaining a partial order of events based on the causal relation induced by the net'.[§] The notion of unfolding 'plays a major role in the so called non sequential semantics of Petri nets as well as in model checking of concurrent and distributed systems or in control theory' (Pinna G.M. 2011). In accordance with the last work, the following two approaches (concerning the above method) have been proposed: *individual token philosophy* (dependences among events are represented either taking into account the whole history of the event) and *collection token philosophy* (considering the whole history irrelevant). And hence, there are proposed two kinds of unfoldings 'where the history is partially kept. These notions are based on *unravelling* a net rather than unfolding it'. And hence, there were introduced *unravel nets*: '*Causal nets* structurally capture dependencies (and conflict) whereas *occurrence nets* structurally capture the unique occurrence property of each transition'.[**] And so, the *unravel nets* are 'in between occurrence nets and causal nets'. Causal nets are structurally *safe nets*, i.e. *1-bounded* (or bounded for $n = 1$: see Example 2.33 of Subsection 3.1). The corresponding multirelations, denoted in the last work by F_{pre} and F_{post} are considered as a flow relation $F \subseteq (S \times T) \cup (T \times S)$,

[*] Corresponding to the notions of *precondition* and *postcondition* associated with some transition t (and related to the corresponding *preset* •t and *postset* t• of places).

[†] See Subsection 2.4.

[‡] The Kleene algebra of regular events will be presented in the next part of this work.

[§] Bonet B., Haslum P., Hickmott S. and Thiébaux S., *Directed unfolding of Petri nets*. file:///C:/Users/user/Documents/UNFOLDINGS%20IN%20PETRI%20NETS.pdf

[**] Any causal net is an occurrence net, but not vice versa.

where: $s\,F\,t$ iff $F_{pre}(s,t)$ and $t\,F\,s$ iff $F_{post}(t,s)$.* The used in this work notions were compared with the classical ones. Some definitions given in (Pinna G.M. 2011) are illustrated below (we shall use the original designations, except for $\mathbb{N}$).

Let A be a set. Any *multiset* of A is a function $m : A \to \mathbb{N} \cup \{0\}$. Here, the *family of all multisets* on A is denoted by μA. The usual operations on multisets are here used. The following definition is used: $m \leq m'$ if $m(a) \leq m'(a)$, for any $a \in A$. Also, it is introduced the multiset $[m]$ such that: $[m](a) =_{df}$ if $m(a) > 0$ then 1 else 0. A *multirelation* f from A to B (here denoted by $f : A \to B$) is a multiset of $A \times B$. In accordance with the main purpose of this work, there are considered only *finitary multirelations*, i.e. multirelations f such that the set $\{b \in B / f(a,b) > 0\}$ is finite.

Let $n_a =_{df} m(a)$. The above multirelation f induces the function $\mu f : \mu A \to \mu B$ (possibly partial, since infinite coefficients are disallowed) such that: $\mu f\left(\sum_{a \in A} n_a \cdot a \right) = \sum_{b \in B} \sum_{a \in A} (n_a \cdot f(a,b)) \cdot b$. A more formal treatment is omitted: left to the reader.

Multirelations introduced in (Rewitzky I. 2003) were considered as an 'alternative to predicate transformers for reasoning about programs' (Martin C.E. et al. 2007). Some notions related to these two works are given below,

'*Angelic nondeterminism* occurs when the choice is made by an 'angel': it is assumed that the angel will choose the *best possible outcome*. *Demonic nondeterminism* occurs when the choice is made by a 'demon': no assumption can be made about the choice made by the demon, so one must be prepared for the *worst possible outcome*. It is well known that both these kinds of behaviour can be described in the domain of *monotonic predicate transformers* (Back R.J.R. and Wright J. von. 1998), (Morgan C.C. 1998), but this is usually associated with the derivation of imperative, rather than functional programs' (Martin C.E. et al. 2007). According to the last work, the types of all up-closed multirelations with source A and targed B are denoted by $'A \rightrightarrows B'$. And hence, the angelic and demonic choices of two multirelations, e.g. Q and R are defined as follows: $Q \cup R : A \rightrightarrows B$ and $Q \cap R : A \rightrightarrows B$, respectively†.

Let α be a *program* allowing both angelic and demonic choice. By S it is denoted the set of all states 'regarded as infinite vectors of values of a countable collection of program variables'. And hence, the program α can be represented by the following binary multirelation (Rewitzky I. 2003): $R_\alpha \subseteq S \times 2^S$ and $s\,R_\alpha\,Q \Leftrightarrow_{df}$ program α, 'when started in state s, is guaranted to terminate in a state in which Q holds and every state in Q is a possible outcome of α'. For any $s \in S$, the set $R_\alpha(s) =_{df} \{Q \in 2^S \,/\, s\,R_\alpha\,Q\}$ 'captures the angelic choices available to the user' and for any $Q \in R_\alpha(s)$, the set $\{s' \in S / s' \in Q\}$ 'captures the demonic choices available of the machine'.

An ilustration of the above considerations are the two examples, given in (Martin C.E. et al. 2007) and 'demonstrating how multirelations can be used to specify and manipulate some *games* and *resource-sharing protocols*': left to the reader.

Let now $X =_{df} \{x_1, x_2, \dots, x_n\}$ be a set of elements called *nodes* (or *vertices*) and $U =_{df} \{u_1, u_2, \dots, u_m\} \subseteq X \times X = \{(x,y) / x,y \in X\}$ be a set of elements called *edges* (or *arcs*). The obtained pair $G =_{df} (X, U)$ is said to be a *graph*‡. A graph having all edges oriented is said to be *directed* (or *oriented*) one, e.g. see Figure 5.7. A graph without edge orientation is said to be an *undirected* (or *non-oriented*) one: e.g. see Figure 5.4.

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* ′S′ corresponds to the set of places, usually denoted by ′P′ (see Definition 5.29 (iii) of Subsection 5.4.).

† According to (Martin C.E. et al. 2007), there are specified systems 'that contain both angelic and demonic choice. One way to think of such specifications is as a contract (Back R.J.R. and Wright J. von. 1998) between two *agents*, both of whom are free to make various choices. One of the agents represents our interests, but the other may not. In this context, the angelic choices are interpreted as those made by our agent, since we assume that he will always make the choice that results in the best outcome for us. Conversely, the demonic choices are those made by the other agent, and since we have no control over them we must be prepared for all possibilities, including the worst'.

‡ Some elements concerning graph theory will be presented in the next part of this work.

Any edge may appear more than one time, but no more than p times if G is a *p-graph*. *Multigraphs* are considered as undirected graphs having multiple edges, associated with the same pairs of nodes (not necessarily different) (Berge C. 1973)[*]. As an example, *Euler's Königsberg's bridges* (L. Euler 1736)[†] and the obtained multigraph are shown in the next figure below (Mallion R.B. 2007).

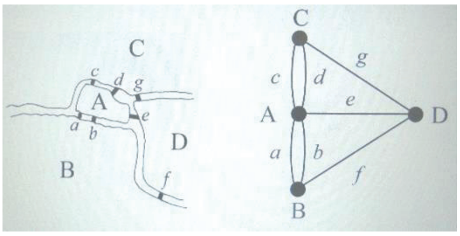

Figure 5.12 Euler's Königsberg's bridges

We observe that any (non-empty) multirelation, i.e. a multiset of ordered pairs, can be represented by some multigraph. In particular, *labelled directed multigraphs* (or *multidigraphs*) may be usable for describing various flight connections (by assigning to each edge its source and targed nodes, respectively)[‡].

7. Fuzzy set theory

Fuzzy sets (also known as *uncertain sets*) are an extension of classical ones by assigning membership degrees to elements of the considered set. Fuzzy sets were introduced introduced independently in (Zadeh L.A. 1965)[§] and (Klaua D. 1965}[**], 'as extension of the classical notion of set'. A more general (abstract algebraic) approach was given in (Salii V.N. 1965)[††]. In accordance with the last work, the following more general notion, called *L- relation*, was defined as follows, e.g. (Bělohlávek R. 2002): let X and Y be two non-empty sets and $R : X \times Y \rightarrow L$, where L is the *support set* of the *complete residuated* lattice $\mathcal{L}$.[‡‡] For $x \in X$ and $y \in Y$, $R(x,y) \in L$ is the *truth degree* to wich x and y are in the relation R.

7.1. Basic notions, definitions and fuzzy set operations

Fuzzy sets are an important scientific direction also having very many applications. 'More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of

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[*] Claude Jacques Berge (1926 – 2002)

[†] Leonhard Paul Euler (1707 – 1783)

[‡] T*he Free Encyclopaedia, The Wikimedia Foundation, Inc.*

[§] Lotfi Aliasker Zadeh (1921 – 2017)

[**] Dieter Klaua (1930 – 2014)

[††] Viacheslav Nikolaevich Salii, born 1939.

[‡‡] The algebraic system $(L, \wedge, \vee, \circ, e)$ is called a *residuated lattice – ordered monoid* or *residuated lattice* in short. It is assumed that $\mathcal{L}$ is *complete*: $\wedge, \vee \in L$ are the *infimum* and the *supremum* of the set L , respectively, see Definition 5.28. And finally, ´e´ is the unity element of this system: algebraic systems will be given in the second part of this book. See: Jipsen P. and C. Tsinakis C., *A Survey of residuated lattices*, 37pp: file:///C:/Users/user/Documents/RESIDIUATED%20LATTICES.pdf

membership' (Zadeh L.A. 1965). Some elements concerning fuzzy set theory are given below. Zadeh's definition of a fuzzy set is first presented (below X is said to be a *space point set*).[*]

Definition 5.45 (*fuzzy set*)

A *fuzzy set* A in X is characterised by a *membership function* $\mu_A(x)$ which associates with each point in X a real number in the interval [0,1], with the value of $\mu_A(x)$ of x representing the *grade* (or *degree*) *of membership* of x in A.

In accordance with the last definition, a fuzzy set is *empty*, i.e. $A = \varnothing$ iff $\mu_A(x) = 0$ (for all $x \in X$). Some basic fuzzy set operations are given below (Zadeh L.A. 1965).

Definition 5.46 (*fuzzy set equality*)[†]

$$A = B \Leftrightarrow_{df} \underset{x \in X}{\forall} (\mu_A(x) = \mu_B(x))$$

Definition 5.47 (*fuzzy set complement*)[‡]

$$\mu_{A'} =_{df} 1 - \mu_A \quad (\text{i.e. } \underset{x \in X}{\forall} (\mu_{A'}(x) =_{df} 1 - \mu_A(x))$$

Definition 5.48 (*fuzzy set inclusion*)

$$A \subseteq B \Leftrightarrow_{df} \mu_A \le \mu_B$$

Definition 5.49 (*fuzzy set union*)

$$\mu_{A \cup B} =_{df} \max\{\mu_A, \mu_B\}$$[§]

It was shown that the union of the fuzzy sets A and B is the *smallest fuzzy set* containing both A and B (Zadeh L.A. 1965). Let ´C´ be an arbitrary fuzzy set. The following implication is satisfied.

Corollary 5.9

$$A, B \subseteq C \Rightarrow A \cup B \subseteq C$$

Proof:

(1)	$A, B \subseteq C$	{a}
(2)	$\mu_A \le \mu_C$	{Df. 5.48}
(3)	$\mu_B \le \mu_C$	
(4)	$\mu_{A \cup B} =_{df} \max\{\mu_A, \mu_B\}$	{Df. 5.49}
(5)	$\mu_{A \cup B} \le \mu_C$	{ 2,3,4}
	$A \cup B \subseteq C$. □	{Df. 5.48}

Definition 5.50 (*fuzzy set intersection*)

$$\mu_{A \cap B} =_{df} \min\{\mu_A, \mu_B\}$$[**]

[*] Provided there is no ambiguity and for convenience, instead of ´$f_A(x)$´ (Zadeh L.A. 1965), we shall use equivalently: ´$\mu_A(x)$ ´. The universum is here denoted by X (a *space point set*).

[†] Instead of writing ´$\mu_A(x) = \mu_B(x)$ for all x in X´, we shall write more simply: ´$\mu_A = \mu_B$´, as in (Zadeh L.A. 1965).

[‡] Corresponds to: $X' =_{df} \mathscr{U} - X$ (see Definition 5.8).

[§] Or equivalently: $\mu_{A \cup B} =_{df} \mu_A \vee \mu_B$ (Zadeh L.A. 1965).

[**] Or equivalently: $\mu_{A \cap B} =_{df} \mu_A \wedge \mu_B$ (Zadeh L.A. 1965).

The intersection of the fuzzy sets A and B is the *largest fuzzy set* which is contained in both A and B (Zadeh L.A. 1965).

In accordance with the last two definitions, as a *t-norm* and a *t-conorm* there were used Zadeh's such ones. The use of another *t-norms*, *t-conorms* and *fuzzy negations*, e.g. Sugeno's or Yager's ones, should be also possible but more complicated (see Subsection 2.2).

Let ´C´ be an arbitrary fuzzy set. The following implication is satisfied (the proof is left to the reader).

Corollary 5.10

$C \subseteq A, B \;\Rightarrow\; C \subseteq A \cap B.$ □

It can be observed that two fuzzy sets A and B are *disjoint* iff $A \cap B = \varnothing$ (as in classical sets). Moreover the fuzzy set union and fuzzy set intersection are two *associative* and *mutually distributive* operations, *De Morgan´s laws* are also satisfied (as in classical set theory: see Subsection 5.2). And so, the following properties hold.

(1) $A \cup (B \cup C) = (A \cup B) \cup C$

(2) $A \cap (B \cap C) = (A \cap B) \cap C$

(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(5) $(A \cup B)' = A' \cap B'$

(6) $(A \cap B)' = A' \cup B'$

As an example, (3) corresponds to the proof of the following equality (for any $x \in X$).

$$\min\{\mu_A(x), \max\{\mu_B(x), \mu_C(x)\}\} = \max\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \mu_C(x)\}\}.$$

In a similar way, e.g. according to (6) the obtained equality is as follows.

$$(L =)\ 1 - \min\{\mu_A(x), \mu_B(x)\} = \max\{1 - \mu_A(x), 1 - \mu_B(x)\}\ (= R)$$

In particular, the proof of the first equality is very similar to the proof given in Proposition 5.11: left to the reader. The proof of the second equality is given below (it is sufficient to consider the following two cases: $\mu_A(x) > \mu_B(x)$ and $\mu_A(x) < \mu_B(x)$).

Proof(6):

(1.1)	$\mu_A(x) > \mu_B(x)$	{ada}
(1.2)	$1 - \mu_B(x) = 1 - \mu_B(x)$	$\{-\mu_A(x) < -\mu_B(x) : 1.1\}$
(2.1)	$\mu_A(x) < \mu_B(x)$	{ada}
(2.2)	$1 - \mu_A(x) = 1 - \mu_A(x)$	$\{-\mu_A(x) > -\mu_B(x) : 2.1\}$
	$L = R.$ □	{1.2, 2.2}

Some basic notions concerning fuzzy sets are also given below: see also (Bronstein I.N. et al. 2001).

(1) $\mu_{A^*}(x) =_{df}$ if $\mu_A(x) \geq 0.5$ then 1 else 0 {The *classical* (i.e. *ordinary*) set A^* corresponding to the fuzzy set A}

(2) $B =_{df} \bigcup_i A_i$ and $C =_{df} \bigcap_i A_i \;\Rightarrow\; \mu_B(x) =_{df} \sup_i \{\mu_{A_i}(x)\}$ and $\mu_C(x) =_{df} \inf_i \{\mu_{A_i}(x)\}$ {see Subsection 5.3}

(3) supp(A) $=_{df}$ $\{x \in X / \mu_A(x) > 0\}$ {fuzzy set *support*}

(4) tol(A) $=_{df}$ $\{x \in X / \mu_A(x) = 1\} \subseteq$ supp(A) {fuzzy set *tolerancy*)

(5) $A_{>\alpha} =_{df} \{x \in X / \mu_A(x) > \alpha\}$ {*α-cut*, $\alpha \in [0,1]$}

(6) $A_{\geq\alpha} =_{df} \{x \in X / \mu_A(x) \geq \alpha\}$ {*strict α-cut*, $\alpha \in [0,1]$}

(7) $\mu_{A_{norm}}(x) =_{df} \dfrac{\mu_A(x)}{\sup_{x \in X}\{\mu_A(x)\}}$ (if $A \neq \varnothing$) {fuzzy set *normalisation*}*

It can be observed that the above two α-cuts are classical sets. Moreover, supp(A) $= A_{>0}$. And finally, the strict 1-cut $A_{\geq 1}$ = tol(A). The following property is satisfied (Bronstein I.N. et al. 2001).

Proposition 5.13 (*fuzzy set representation*)

For any fuzzy set A on X, there exists exactly one *monotonic family of α-cuts* $\{A_{>\alpha} / \alpha \in (0,1]\}$ in X (exactly one *monotonic family of strict α-cuts* $\{A_{\geq\alpha} / \alpha \in (0,1]\}$ in X) such that: $\alpha < \beta \Rightarrow A_{>\alpha} \supseteq A_{>\beta}$ ($A_{\geq\alpha} \supseteq A_{\geq\beta}$).

And vice versa, if there exists a *monotonic family of subsets* $\{U_\alpha / \alpha \in (0,1]\}$ in X (a *monotonic family of subsets* $\{V_\alpha / \alpha \in (0,1]\}$ in X) then there exists exactly one fuzzy set U in X (exactly one fuzzy set V in X) such that $U_{>\alpha} = U_\alpha$ ($V_{\geq\alpha} = V_\alpha$) and $\mu_U(x) = \sup_{x \in U_\alpha}\{\alpha \in [0,1)\}$ ($\mu_V(x) = \sup_{x \in V_\alpha}\{\alpha \in (0,1]\}$). □

Let X be finite, A be a fuzzy set defined in X and $\mu_{A\alpha}(x) =_{df}$ if $\mu_A(x) \geq \alpha$ then 1 else 0 (corresponding to the notion of a *strict α-cut*). The fuzzy set A can be represented as follows: $\mu_A(x) = \bigcup_\alpha \alpha \cdot \mu_{A_\alpha}(x)$, $x \in X$. This is illustrated in the next example.

Example 5.23

Let A be a fuzzy set in $X =_{df} \{x_1, x_2, x_3, x_4\}$, defined by $\mu_A = (0.2, 0.5, 0.3, 0.9)$. We can obtain, e.g.

$$\begin{aligned} \mu_A(x_2) &= \max\{0.2 \times \mu_{A_{0.2}}(x_2), 0.3 \times \mu_{A_{0.3}}(x_2), 0.5 \times \mu_{A_{0.5}}(x_2), 0.9 \times \mu_{A_{0.9}}(x_2)\} \\ &= \max\{0.2 \times 1, 0.3 \times 1, 0.5 \times 1, 0.9 \times 0\} \\ &= 0.5. \end{aligned}$$ □

The following definition is introduced (Bronstein I.N. et al. 2001).

Definition 5.51 (*fuzzy set similarity*)†

Two fuzzy sets A and B are *similar* iff $\forall_{\alpha \in (0,1]} \exists_{\alpha_1,\alpha_2 \leq 1} (\alpha < \min\{\alpha_1,\alpha_2\}$, $\text{supp}(\alpha_1\mu_A)_\alpha \subseteq \text{supp}(\mu_B)_\alpha$ and $\text{supp}(\alpha_2\mu_B)_\alpha \subseteq \text{supp}(\mu_A)_\alpha$).

Proposition 5.14

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* The fuzzy set A is said to be *normalised* iff the *height* of A, i.e. $\sup_{x \in X}\{\mu_A(x)\} = 1$ (otherwise A is said to be *subnormal*).

† Let A, B be two fuzzy sets. The *similarity measure* $S(A,B) =_{df} (A \cap B) / (|A| + |B| - |A \cap B|)$, assuming that: $A \neq B \Rightarrow S(A,B) \leq \sigma$ (is the *threshold*). The considered also *possibility measure* 'quantifies the extent to which A and B overlap': $\Pi(A,B) =_{df} \sup_{x \in X}\{\min\{\mu_A(x), \mu_B(x)\}\}$: Mencar C., Castellano G., Fanelli A.M. and Bargiela A., Similarity vs. possibility in measuring fuzzy sets distinguishability, 7pp: file:///C:/Users/user/Documents/SIMILARITY%20AND%20POSSIBILITY%20IN%20FUZZY%20SETS.pdf.

Two fuzzy sets A and B are similar iff tol(A) = tol(B). □

Definition 5.52 (*strictly similarity*)

Two similar fuzzy sets A and B are *strictly similar* iff supp(A) = supp(B).

Sometimes there are required additional interpolating operators between t-norms and t-conorms: called *compensative* ones, e.g. *lambda-* and *gamma-operators* given below (Bronstein I.N. et al. 2001).

Let A and B be two fuzzy sets. The *lambda-* and *gamma-operators* (in short: *λ-operator* and *γ-operator*) are introduced as follows.

$\mu_{A\lambda B}(x) =_{df} \lambda(\mu_A(x)\mu_B(x)) + (1 - \lambda)(\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x))$

$\mu_{A\gamma B}(x) =_{df} (\mu_A(x)\mu_B(x))^{1-\gamma}((1 - (1 - \mu_A(x)))(1 - (1 - \mu_B(x))))^{\gamma}$

It can be observed that $\lambda = 0$ ($\gamma = 1$) and $\lambda = 1$ ($\gamma = 0$) correspond to the notions of *algebraic sum:* $x \oplus y =_{df} x + y - xy$ (*algebraic product:* $x \otimes y =_{df} xy$), respectively: see Subsection 2.2.

In accordance with the last work, the above considered gamma-operator was extended as follows (to more than two but a finite number) arguments): $\mu(x) =_{df} [\prod_{i=1}^{n} \mu_i(x)^{\delta_i}]^{1-\gamma} * [1 - \prod_{i=1}^{n}(1 - \mu_i(x))^{\delta_i}]^{\gamma}$, where $\sum_{i=1}^{n} \delta_i = 1$, $x \in X$ and $\gamma \in [0,1]$ (in particular, a similar operator was also given without using the *weights* δ_i).

Let A be a fuzzy set. For convenience, sometimes the following scaling operations are also used (Petry F.E. 1996): CON, DIL and INT (i.e. *concentration*, *dilation* and i*ntensification*, respectively), defined as follows.

CON(A): $\mu_{CON(A)}(x) =_{df} (\mu_A(x))^a$ (usually $a =_{df} 2$, $a \in \mathbb{N} - \{1\}$). This operation 'concentrates fuzzy elements by reducing the membership grade proportionally more for elements that have smaller membership grades'.

DIL(A): $\mu_{DIL(A)}(x) =_{df} (\mu_A(x))^{1/a}$ (usually $1/a =_{df} 0.5$). This operation 'dilates fuzzy elements by increasing membership grade more for the elements with smaller membership grades'.

INT(A) $\mu_{INT(A)}(x) =_{df}$ if $0 \leq \mu_A(x) \leq 0.5$ then $2(\mu_A(x))^2$ else $1 - 2(1 - \mu_A(x))^2$. This operation is 'like contrast of a picture. It raises the membership grade of those elements within the crossover points and reduces the membership grade of those outside the crossover points'.

In the scalar approaches (De Luca A. and S Termini S. 1972), the '*power of a finite fuzzy set*' A (or: the *scalar cardinality* of A or also: *sigma-count* of A) is defined as the sum of the membership degrees associated with A, i.e. $|A| =_{df} \sum_{x \in X} \mu_A(x)$. The last definition (convenient in applications) was generalised in (Dhar M. 2013) by using (in addition to the membership function) also a *reference function*. This approach was related to the notion of a *fuzzy number*, introduced in (Baruah H. K. 1999): A more formal treatment is omitted: left to the reader.

7.2. Fuzzy relations

Let X and Y be two *universal sets* and $G =_{df} X \times Y$ be the obtained *universal domain*. By F(X), F(Y) and F(G) we shall denote below the sets of all fuzzy subsets defined in X, Y and G, respectively. In general, fuzzy relations are considered as fuzzy sets of ordered tuples (or pairs: in the binary case). And hence, the most of the above considered properties are here also satisfied. Fuzzy relations can be introduced in a similar way as in the case of classical ones. Some basic notions, definitions or other considerations are given below (Bronstein I.N. et al. 2001).

Definition 5.53 (*binary fuzzy relation*)

A *binary fuzzy relation* ρ defined in $F(G)$* is a fuzzy subset of ordered pairs that belong to G.

According to the last definition, ρ can be described by the membership function $\mu_\rho : G \to [0,1]$ such that $\mu_\rho(x,y) \in [0,1]$ (for any $(x,y) \in G$). As fuzzy relations are particular fuzzy sets, we can use the same fuzzy set operations as in the previous subsection, e.g. the *union* $\rho \cup \sigma$ (assuming that $\rho, \sigma \in F(G)$) is defined as follows: $\mu_{\rho \cup \sigma}(x,y) =_{df} \max\{\mu_\rho(x,y), \mu_\sigma(x,y)\}$, for any $(x,y) \in G$.

Let $\rho \in F(G)$ be a fuzzy relation. The *opposite fuzzy relation* ρ^{-1} is defined as follows: $\mu_{\rho-1}(x,y) =_{df} \mu_\rho(y,x)$, for any $(x,y) \in G$.

In the next considerations there are cited such notions as: *projections*, *cylindric extensions* and *cylindric closure* and finaly the notion of a *fuzzy equivalence relation*. The notion of Cartesian product is first given.

Definition 5.54 (*Cartesian product of fuzzy sets*)†

Let A_i be fuzzy sets defined in the universal sets X_i $(1 \le i \le n)$. The *Cartesian product* $A_1 \times \ldots \times A_n$ is a fuzzy relation in the product space $X_1 \times \ldots \times X_n$, defined by its membership function $\mu_{A1 \times \ldots \times An}(x_1, \ldots ,x_n) =_{df} \min\{\mu_{Ai}(x_i) / x_i \in X_i, 1 \le i \le n\}$.

In accordance with the last definition, instead of the logical operation ′*minimum*′ sometimes it is used the algebraic operation *product:* ′$\times$′ , i.e. $\mu_{A1 \times \ldots \times An}(x_1, \ldots ,x_n)$ is defined as: $\mu_{A1}(x_1) \times \ldots \times \mu_{An}(x_n)$.

Corollary 5.11

Let $A \in F(X)$ and $B \in F(Y)$. The Cartesian product $A \times B \in F(G)$ is a fuzzy relation defined by its membership function: $\mu_{A \times B}(x,y) =_{df} \min\{\mu_A(x), \mu_B(y)\}$, for any $(x,y) \in G$. □ {Df. 5.54}

Consider the Cartesian product of all sets in the family $\mathcal{X} =_{df} \{X_1, X_2, \ldots , X_n\}$. Let $\rho \subseteq X_1 \times \ldots \times X_n$ be an *n-ary relation*. The *projection* of ρ on $\mathcal{Y} \subseteq \mathcal{X}$, denoted by $P(\rho\downarrow\mathcal{Y})$, disregards all sets in $\mathcal{X}$ except those in the family $\mathcal{Y}$.

Another operation on relations called *cylindric extension* was also introduced. This operation ‘produces largest fuzzy relation that is compatible with projection’, ‘is the least specific’ of all other such ones and guarantees that included information is complete. However, as it was observed, such reconstruction is limited. On the other hand, any such relation can be exactly reconstructed by taking set union of several of its projections or also by taking set intersection of their cylindric extensions. ‘The resulting relation is usually called *cylindric closure*’: left to the reader.

Binary fuzzy relations play a significant role among n-dimentional relations. Some additional notions concerning these relations are given below (see the last work cited below).

Let $\rho \in F(G)$ be a binary fuzzy relation. The *domain* and *codomain* (called also: *range* or *image*) of ρ are defined as follows: $\mu_{dom(\rho)}(x) =_{df} \max\{\mu_\rho(x,y) / y \in Y\}$ and $\mu_{cod(\rho)}(y) =_{df} \max\{\mu_\rho(x,y) / x \in X\}$, for any $x \in X$ and $y \in Y$, respectively.

The standard composition of two binary relations ρ and σ (see Definition 5.19) is generalised as follows.

Definition 5.55 (*composition of two binary fuzzy relations*)

Let $\rho(X,Y)$ and $\sigma(Y,Z)$ be two binary fuzzy relations. The obtained *composition* is defined as follows.

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* Provided there is no ambiguity and for convenience, sometimes instead of ′ρ defined in $F(G)$′ equivalently we shall use: ′$\rho(X,Y)$′. In a similar way, by ′$\sigma(Y \times Z)$′ we shall denote a binary fuzzy relation σ defined in $F(Y \times Z)$, etc.

† Kruse R. and Moewes C., *Fuzzy systems: fuzzy relations.* University of Magdeburg: 34pp. file:///C:/Users/user/Documents/FUZZY%20RELATIONS%20DEFINITION%20ETC.pdf.

$[\rho \circ \sigma](x,z) =_{df} \sup_{y \in Y}\{ \min\{\rho(x, y), \sigma(y, z)\} \}$*, for any $x \in X$ and $z \in Z$.

In accordance with the last definition, instead of *supremum* it is used the logical operation *maximum* if Y is finite. And hence, the obtained composition (or *superposition*) is called *max-min* one. The inverse of the max-min composition is defined by the following equality: $[\rho(X,Y) \circ \sigma(Y,Z)]^{-1} =_{df} \sigma^{-1}(Z,Y) \circ \rho^{-1}(Y,X)$.

It can be observed that the composition of two binary fuzzy relations is an associative binary operation. But this operation is not commutative. The matrix representation of this composition is realised similarly as in the classical case (see Example 5.11). And hence, the corresponding value of any matrix element, instead of {0,1}, will belong to the interval [0,1]. And so, we have multiplication over fuzzy graph incidence matrices: left to the reader.

According to this work (see below: the work marked by star), the above *max-min composition* can be also realised into two steps. The following definition is first introduced.

Definition 5.56 (*relational join*)

Let $\rho(X,Y)$ and $\sigma(Y,Z)$ be two binary fuzzy relations. The *relational join* of ρ and σ is defined as follows: $[\rho * \sigma](x,y,z) =_{df} \min\{\rho(x,y), \sigma(y,z)\}$†, for any $x \in X$, $y \in Y$ and $z \in Z$.

Corollary 5.12

$[\rho \circ \sigma](x,z) = \max_{y \in Y} \{[\rho * \sigma](x, y, z)\}$, for any $x \in X$ and $z \in Z$. □ {Df. 5.56}

The properties of *crisp binary equivalence relations* (see Definition 5.23) are extended for *binary fuzzy relations* as follows.

Definition 5.57 (*binary fuzzy equivalence relation*)

Let $\rho(X,X)$ be a fuzzy relation on X. We shall say that ρ is a *binary fuzzy equivalence relation* iff ρ is at the same time *reflexive*. i.e. $\rho(x,x) = 1$ (for any $x \in X$), *symmetric*, i.e. $\rho(x,y) = \rho(y,x)$ (for any $x,y \in X$) and *transitive*, i.e. if it satisfies: $\rho(x,z) \geq \max_{y \in Y}\{ \min\{\rho(x, y), \rho(y, z)\} \}$, for any $x, z \in X$‡.

The introduced in Subsection 2.2 t-norms and their t-conorms may be used for introducing new fuzzy relations. This is illustrated below.

Example 5.25

Let $X =_{df} \{0, 1/4, 2/4, 3/4, 1\}$. Consider the following binary relation $\overline{\otimes} \subseteq (X \times X)^2$ defined as follows: $(x, y)\overline{\otimes}(z, t) =_{df} (x\hat{\otimes}z, y\hat{\oplus}t)$, where $'\hat{\otimes}'$ and $'\hat{\oplus}'$ are the corresponding *generalised Łukasiewicz's fuzzy t-norm* and *t-conorm*, respectively. Assume that $\alpha =_{df} 2$. And hence, we can obtain.

$(x, y)\overline{\otimes}(z, t) = (\max\{0, x^2 + z^2 - 1\}^{1/2}, \min\{1, y^2 + t^2\}^{1/2})$.

The following implication is satisfied (for any pairs $\alpha, \beta, \gamma \in L =_{df} \{(x,y) / x + y \leq 1\}$).

$\beta \geq \alpha \Rightarrow \sigma(\beta, \gamma) \leq \sigma(\alpha, \gamma)$, where: $\sigma(\beta, \gamma) =_{df} \sup_{\delta \in L} \{\beta\overline{\otimes}\delta \leq \gamma\}$ and $\sigma(\alpha, \gamma) =_{df} \sup_{\delta \in L} \{\alpha\overline{\otimes}\delta \leq \gamma\}$.

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* Here: $[\rho \circ \sigma](x,z) =_{df} \mu_{\rho\circ\sigma}(x,z)$, $\rho(x,y) =_{df} \mu_\rho(x,y)$ and $\sigma(y,z) =_{df} \mu_\sigma(y,z)$. Another possible *compositions* may be the following ones: $\inf_{y \in Y}\{ \max\{\rho(x, y), \sigma(y, z)\}\}$ either $\sup_{y \in Y}\{ \rho(x, y) * \sigma(y, z)\}$.

† Here: $[\rho \circ \sigma](x,z) =_{df} \mu_{\rho\circ\sigma}(x,z)$, $\rho(x,y) =_{df} \mu_\rho(x,y)$ and $\sigma(y,z) =_{df} \mu_\sigma(y,z)$

‡ Obviously, the *transitivity property* may be introduced by using another *t-conorm* and *t-norm*, respectively: see Subsection 2.2. However, in applications, the above ones are widely used.

As an example: (2/4,1/4) $\geq$ (0, 2/4) $\Rightarrow$ (0, 3/4) $\leq$ (1/4, 3/4/). In general, σ is decreasing in the first argument. Here, 0_L = (0,1) and 1_L = (1,0) correspond to the *minimal* and *maximal* elements, respectively: a more formal treatment is left to the reader. □

7.3. Fuzzy distances, measures and transformations

The notion of a *fuzzy distance* can be introduced either as a 'generalisation of the classic distance between subsets of a metric space or as a distance between membership functions of fuzzy sets or also as a fuzzy metric by generalising a metric space to a fuzzy-metric one' (Bednář J. 2005). In general, there exist various definitions of distances between fuzzy sets.Obviously, it is assumed below that X is finite and non-empty. The presented distances are mainly concerned about the *image similarity*, but other applications are also possible.

Let A and B be two fuzzy sets. Some most frequently used definitions are illustrated below, e.g. (Janiˇs V. and Montes S 2007). The most simple ones are *Chebyshev*[*] *distance* and *Hamming*[†] *distance* (known also as: '*city block*' or '*Manhattan distance*').

Chebyshev distance: $$d(A,B) =_{df} \max_{x \in X}\{|\mu_A(x) - \mu_B(x)|\}$$

Hamming distance: $$d(A,B) =_{df} \sum_{x \in X} |\mu_A(x) - \mu_B(x)|$$

Minkowski[‡] *distance:* $$d(A,B) =_{df} \Big(\sum_{x \in X} |\mu_A(x) - \mu_B(x)|^n\Big)^{1/n}, \ n \in \mathbb{N}$$

Bray – Curtis / Sorensen distance[§]*:* $$d(A,B) =_{df} \sum_{x \in X} \frac{|\mu_A(x) - \mu_B(x)|}{\mu_A(x) + \mu_B(x)}$$

Squared chord distance $$d(A,B) =_{df} \sum_{x \in X} (\sqrt{\mu_A(x)} - \sqrt{\mu_B(x)})^2$$

Squared Chi – squared distance $$d(A,B) =_{df} \sum_{x \in X} \frac{(\mu_A(x) - \mu_B(x))^2}{\mu_A(x) + \mu_B(x)}$$

In particular, *Euclidean distance* can be obtained from Minkowski distance assuming $\alpha = 2$. The above nosion of a distance between two fuzzy sets A and B can be also used in some definitions concerning measures of fuzziness. Some such measures, denoted by $\varphi(A)$, are given below. Here, instead of B, there is used the *classical* (i.e. *ordinary*) set A^* corresponding to the fuzzy set A , i.e. B $=_{df}$ A^* (except for *entropy measure*, where it is used the *fuzzy set complement* of A, i.e. A´: see Subsection 7.1).

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[*] Pafnuty Lvovich Chebyshev (1821 – 1894).

[†] Richard Hamming (1915 – 1998).

[‡] Herman Minkowski (1864 – 1909)

[§] John Roger Bray, born 1929, John Thomas Curtis (1913 – 1961), Thorvald Julius Sorensen (1902 – 1973). This distance can be considered as a particular case (of the most generalised form, called) *Camberra distance*: $d(\underline{x},\underline{y}) =_{df} \sum_{i=1}^{n} \frac{|x_i - y_i|}{|x_i| + |y_i|}$, where: $\underline{x} =_{df} (x_1, \dots, x_n)$, $\underline{y} =_{df} (y_1, \dots, y_n) \in \mathbb{R}^n$: it was observed that the last distance (along with Bray – Curtis one) can perform very well in CBIR (Context Based Image Retrieval), significantly better than (mostly used) Manhattan or Euclidean ones.

Chebyshev measure: $\varphi(A) =_{df} 2d(A, A^*)$

Hamming measure: $\varphi(A) =_{df} \dfrac{2d(A, A^*)}{|X|}$

Minkowski measure $\varphi(A) =_{df} \dfrac{2d(A, A^*)}{\sqrt[n]{|X|}}$, $n \in \mathbb{N}$

*Entropy measure of fuzziness** $\varphi(A) =_{df} (-\sum_{x \in X}(\mu_A(x) \cdot \lg_2 \mu_A(x) + \mu_{A'}(x) \cdot \lg_2 \mu_{A'}(x))) / |X|$,

$\mu_A(x) \in (0,1)$, for any $x \in X$

Cosine similarity measure† $\varphi(A) =_{df} \dfrac{\sum_{x \in X} \mu_A(x) \cdot \mu_{A^*}(x)}{\sqrt{\sum_{x \in X} \mu_A(x)^2} \cdot \sqrt{\sum_{x \in X} \mu_{A^*}(x)^2}}$

The *generalised Łukasiewicz's fuzzy t-norm* and *t-conorm*, i.e. $x \hat{\otimes} y = \max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha}$ and $x \hat{\oplus} y = \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$, introduced in Subsection 2.2, can be defined for more than two (a finite number) arguments. The following *generalised t-norm* and *generalised t-conorm* are obtained.

$$\hat{\otimes}_{i=1}^{n} x_i = \max\{0, \sum_{i=1}^{n} x_i^\alpha - n + 1\}^{1/\alpha}$$

and

$$\hat{\oplus}_{i=1}^{n} x_i = \min\{1, \sum_{i=1}^{n} x_i^\alpha\}^{1/\alpha}.$$

The proofs of the last two formulae are inductive wrt n. As an example, the following proof for the second one can be obtained.

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* According to *Shannon's entropy* (Claude Elwood Shannon: 1916 – 2001). Here, $\mu_{A'}(x)$ corresponds to the notion of *fuzzy set complement*: in accordance with Definition 5.47, here it is implicitly assumed the *standard complement* (related to Łukasiewicz's negation: see Subsection 2.2).

† Corresponds to the notion of a *scalar product of two vectors*: $\underline{a} \cdot \underline{b} =_{df} |\underline{a}| \cdot |\underline{b}| \cdot \cos(\underline{a} \hat{\ } \underline{b})$. And hence: $\cos(\underline{a} \hat{\ } \underline{b}) = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}| \cdot |\underline{b}|} =$

$= \dfrac{\sum_{i=1}^{n} a_i \cdot b_i}{\sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}}$. Obviously: $\underline{a} \cdot \underline{b} \in \mathbb{R}$. It is assumed that $\underline{a}, \underline{b} \neq \underline{0}$. This measure i salso known as *Ochini coefficient* or *Otuska – Ochiai coefficient* (Janosuke Otuska: 1903 – 1950): *The Free Encyclopaedia, The Wikimedia Foundation, Inc*.

Proof:

Let $n =_{df} k$ and $F(k) =_{df} \min\{1, \sum_{i=1}^{k} x_i^{\alpha}\}^{1/\alpha}$. Since this t-conorm is associative , we can obtain:

$$F(k+1) = (\hat{\oplus}_{i=1}^{k} x_i) \hat{\oplus} x_{k+1}$$

$$= \min\{1, \{\min\{1, \sum_{i=1}^{k} x_i^{\alpha}\}^{1/\alpha}\}^{\alpha} + x_{k+1}^{\alpha}\}^{1/\alpha}$$

$$= \min\{1, \min\{1, \sum_{i=1}^{k} x_i^{\alpha}\} + x_{k+1}^{\alpha}\}^{1/\alpha}.$$

And so, it is necessarily to be shown that the left side (L) and the right side (R) of the following equality are equal.

(L) $\min\{1, \min\{1, \sum_{i=1}^{k} x_i^{\alpha}\} + x_{k+1}^{\alpha}\}^{1/\alpha} \stackrel{?}{=} \min\{1, \sum_{i=1}^{k+1} x_i^{\alpha}\}^{1/\alpha}$ (R).

Assume first that $\sum_{i=1}^{k} x_i^{\alpha} \geq 1$. Then: L = 1 = R. Let now $\sum_{i=1}^{k} x_i^{\alpha} < 1$. We can obtain: L = $\min\{1, \sum_{i=1}^{k} x_i^{\alpha} + x_{k+1}^{\alpha}\}^{1/\alpha} = \min\{1, \sum_{i=1}^{k+1} x_i^{\alpha}\}^{1/\alpha} = R$. □

In accordance with the above generalised t-conorm, the following fuzzy set distance and measure of fuzziness can be obtained (it is assumed that $\alpha =_{df} n \in \mathbb{N}$).

Generalised Łukasiewicz's distance $\quad d(A,B) =_{df} \min\{1, \sum_{x \in X} |\mu_A(x) - \mu_B(x)|^n\}^{1/n}, \; n \in \mathbb{N}$

Generalised Łukasiewicz's measure $\quad \varphi(A) =_{df} \dfrac{2d(A, A^*)}{|X|}$

Example 5.24 (*measures of fuzziness*)

Some computations, related to the above given five measures of fuzziness are illustrated below. As an example, assume that we have the following input data.

$x \in X$	x_1	x_2	x_3	x_4
$\mu_A(x)$	0.3	0.8	0.2	0.9
$\mu_{A^*}(x)$	0	1	0	1
$\mu_{A'}(x)$	0.7	0.2	0.8	0.1

The obtained fuzziness measures are given below.

Chebyshev measure $\quad \varphi(A) = 0.60$

Hamming measure $\quad \varphi(A) = 0.20$

Minkowski measure $\quad \varphi(A) = 0.4243\ldots$ / $n = 2$, $\varphi(A) = 0.22243\ldots$ / $n = 3$,
≈ 0.42 (*Euclidean measure*) $\quad \approx 0.22$

Entropy measure of fuzziness[*]	$\varphi(A) = 2.794249\ldots \approx 2.7942$, and hence: $2.7942 / 4 = 0.69855 \approx 0.70$
Cosine similarity measure	$\varphi(A) = 0.9563247\ldots \approx 0.96$ (or e.g. ≈ 0.24, dividing by $\|X\| = 4$)
Generalised Łukasiewicz's measure	$\varphi(A) = 0.2121320\ldots \approx 0.21$ / $n = 2$, $\varphi(A) = 0.1765$ / $n = 3$, ≈ 0.18. □

Let $F(X)$ be the set of all fuzzy subsets defined in X, $A \in F(X)$ with $\mu_A(x)$, defined for any $x \in X$ and having a measure of fuzziness $\varphi(A)$. In accordance with (De Luca A. and Termini S.A. 1972), the following three conditions should be satisfied by $\varphi(A)$.

(c_1) $\varphi(A) = 0$ if A is a *crisp set*, i.e. $\mu_A(x) \in \{0,1\}$, for any $x \in X$[†].

(c_2) $\varphi(A)$ should attain mximum value for $\mu_A(x) = 0.5$, for all $x \in X$.

(c_3) $\varphi(A) \geq \varphi(B)$ if $A, B \in F(X)$ and

$\mu_B(x) \geq \mu_A(x)$ if $x \in \{y / \mu_A(y) \geq 0.5\} \subseteq X$

$\mu_B(x) \leq \mu_A(x)$ if $x \in \{y / \mu_A(y) \leq 0.5\} \subseteq X$[‡].

As it was observed, the above three conditions are not complete. And so, they can not be considered as axioms for measure of fuzziness (Wang Z.-X. 1984), In fact, by (c_2) it follows that $\varphi(A)$ attains maximum at $\mu_A(x) = 0.5$ (for all $x \in X$). However, 'there never exists any fuzzy set B which is differend from A, such that sets A and B satisfy (c_3)'. And hence, (c_3) was revised in (Wang Z.-X. 1984). The following generalised version of (c_3) was introduced.

(c_3') $\varphi(A) \geq \varphi(B)$ if $A, B \in F(X)$ and

$\mu_B(x) \geq \max\{\mu_A(x), \mu_{A'}(x)\}$[§] if $x \in \{y / \mu_A(y) > 0.5\} \cup \{y / \mu_B(y) > 0.5\} \subseteq X$

$\mu_B(x) \leq \mu_A(x)$ if $x \in \{y / \mu_B(y) \leq 0.5\} \subseteq X$.

Let $\mathfrak{B}$ be a *Borel's field*[**] consisting of ordinary subsets of X. The notion of a *fuzzy measure space* is introduced as follows (Wang Z.-X. 1984).

Definition 5.58 (*fuzzy measure space*)

Let $\mu(\cdot) : \mathfrak{B} \to [0, 1]$. We shall say that $\mu(\cdot)$ is a *fuzzy measure* on $\mathfrak{B}$ if the following three conditions are satisfied.

(1) $\mu(\varnothing) = 0$, $\mu(X) = 1$

(2) $(A, B \in \mathfrak{B}) \wedge (A \cap B = \varnothing) \Rightarrow \mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ {*fuzzy additivity*}

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[*] $\log_a n = \dfrac{\log_b n}{\log_b a}$, e.g. $a \in \{:2, e\}$. Here it is assumed: $a =_{df} 2$ and $b =_{df} 10$.

[†] Or equivalently: if $A = A^*$.

[‡] The last condition (c_3) corresponds to the notion of a '*sharpness relation*' defined in [0, 1], introduced in (De Luca A. and Termini S.A. 1972). And hence, the above condition is a 'sharpened' version of this notion.

[§] Corresponds to the original text: '$\mu_B(x) \geq \mu_A(x) \vee (1 - \mu_{A'}(x))$'.

[**] Félix Édouard Justin Émile Borel (1871 – 1956).

(3) $\{A_n / n \in \mathbb{N}\} \subseteq \mathfrak{B}$ is a monotonic sequence $\Rightarrow$ $\mu(\lim A_n) = \lim \mu(A_n)$ {fuzzy continuity}

The ordered triple $(X, \mathfrak{B}, \mu)$ is said to be a *fuzzy measure space*.

Fuzzy measure theory was developed independently by Choquet G. (1953) and by Sugeno M. (1974)*: in the context of fuzzy integrals. By assuming an additive fuzzy measure, Choquet's integral is related to Lebesgue's† integral.

Some notions related to the fuzzy integrals are given below.‡ The last work concerns the theory of *multicriteria decision making* wrt many new approaches (or *paradigms*), 'introduced in the second half of the twentieth century, especially in Europe'. As it was stressed 'a common feature of all these approaches … is that we need somewhere a fundamental operation which is aggregation'. In accordance with this work, besides the *classical aggregation operators* (quasi-arithmetic means, median, weighted minimum and maximum and ordered weighted averaging operators) there were also considered some fuzzy measures related to the above two integrals (restricting definitions to the strict minimum). Some requirements (concerning mathematical or behavioral properties) were also presented.

Let $X =_{df} \{x_1, x_2, \dots, x_n\}$ be a finite set of criteria. The following definition was introduced.

Definition 5.59 (*fuzzy measure on the set X of criteria*)

A *fuzzy measure* on X is a set function $\mu : \mathbb{P}(X) \rightarrow [0,1]$ satisfying the following two axioms.

(A1) $\mu(\varnothing) = 0, \ \mu(X) = 1,$

(A2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$, for any $A,B \in \mathbb{P}(X)$.

This measure μ is *additive* (resp. *superadditive* / resp. *subadditive*) if $\mu(A \cup B) =_{df} \mu(A) + \mu(B)$ (resp. $\mu(A \cup B) \geq \mu(A) + \mu(B)$ / resp. $\mu(A \cup B) \leq \mu(A) + \mu(B)$).

According to this work, *fuzzy integrals* are considered as operators on $[0,1]^n$ and hence the obtained definitions are restricted to [0,1]-valued functions.

Definition 5.60 (*Sugeno's integral*)

Let μ be a fuzzy measure on X. *Sugeno's integral* of a function $f : X \rightarrow [0,1]$ wrt μ is defined as follows: $S_\mu(f(x_1), \dots, f(x_n)) =_{df} \bigvee_{i=1}^{n} (f(x_{(i)}) \wedge \mu(A_{(i)}))$, where $'\cdot_{(i)}'$ denotes a permutation of indexes as follows: $0 \leq f(x_{(1)}) \leq \dots \leq f(x_{(n)}) \leq 1$ and $A_{(i)} =_{df} \{x_{(i)}, \dots, x_{(n)}\}$.

Definition 5.61 (*Choquet's integral*)

Let μ be a fuzzy measure on X. *Choquet's integral* of a function $f : X \rightarrow [0,1]$ wrt μ is defined as follows: $C_\mu(f(x_1), \dots, f(x_n)) =_{df} \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)})) \cdot \mu(A_{(i)})$, where $f(x_{(0)}) =_{df} 0$.

Some properties of the above two integrals (concerning aggregation) were also presented (wrt 'their suitability for multicriteria decision problems').A more formal treatment is omitted here, see: (Grabisch M. 1995) and (Grabisch M. et al. 1995).

The following definition is introduced (Ruspini, E.H. 1969).

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* Gustave Choquet (1915 – 2006), Muchio Sugeno, born 1940.

† Henri Léon Lebesgue (1875 – 1941).

‡ Grabisch M., *The application of fuzzy integrals in Multicriteria Decision Making*.Thomson-CSF, Central Research Laboratory Domaine de Corbeville, 91404 Orsay cedex, France 13pp: file:///C:/Users/user/Documents/FUZZY%20INTEGRALS%202020%20ETC.pdf

Definition 5.62 (*Ruspini's partition*)

Ruspini's partition[*] is a finite family of fuzzy sets $\{f_1, f_2, \dots, f_n\}$, where $f_i : [0,1] \to [0,1]$, such that: $\sum_{i=1}^{n} f_i(x) = 1$ for all $x \in [0,1]$.

The notions of *F-transform* (equivalently: *F-transformation* or *fuzzy transformation*) and *inverse F-transform* can be introduced as follows.[†]

Definition 5.63 (*discrete F-transform*)

Assume that $f(x)$ is defined for $x_1, \dots, x_n \in [a,b]$ and $A =_{df} \{A_i(x) / i = 1, \dots, n\}$ is a Ruspini's partition. The vector of real numbers $(F_1, \dots, F_n)$ is a *discrete F-transform* of $f(x)$ if $F_i =_{df} \dfrac{\sum_{j=1}^{l} f(x_j) \cdot A_i(x_j)}{\sum_{j=1}^{l} A_i(x_j)}$ (for $i = 1, \dots, n$ and $n < l$).

Definition 5.64 (*inverse F-transform*)

Assume now that $(F_1, \dots, F_n)$ is the F-transform of $f(x)$ wrt A. The *inverse F-transform* is introduced as follows: $T_{f,n}(x) =_{df} \sum_{i=1}^{n} F_i \cdot A_i(x)$.

In accordance with the last work, the above two notions were used in *image noise reduction* and also for 'description of *fusion*[‡] using fuzzy transform'. A more formal treatment is omitted: left to the reader.[§]

7.4. Fuzzy-probabilistic models

The *fuzzy-probabilistic models* were introduced in such areas as, e.g. *fuzzy probability theory* (initiated by Kwakernaak H.[**]), *fuzzy probabilistic logic*, *fuzzy probabilistic expert systems*, *fuzzy probabilistic approach for determining safety*, and so on.

Some considerations related to the notion of *uncertainty* are first presented below (Möller B. and Beer M. 2004). Unfortunately, 'terminology related to measurement uncertainty is not used consistently among experts', e.g. see: *Definitions of Measurement Uncertainty Terms*[††].

In general, the term '*uncertainty*'[‡‡] is associated with something which can not be described exactly, e.g. 'each measurement is more or less uncertain' (Möller B. and Beer M. 2004). In accordance with the last

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[*] Enrique Hector Ruspini (1942 – 2019).

[†] Daňková M. and Perfilieva I., *Fuzzy transforms – a new bases for image fusion*. University of Ostrava. Institute for Research and Applications of Fuzzy Modeling, Ostrava, Czech Republic,21pp: file:///C:/Users/user/Documents/fuzzy%20trans%20image%20vision%20aplic%20Czesi.pdf.

[‡] In general, '*image fusion* is a process of combining relevant information from two or more images into a single image': see T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[§] See also: (Štepnička M. 2007).

[**] Huibert Kwakernaak, born 1937.

[††] https://users.physics.unc.edu/~deardorf/uncertainty/definitions.html

[‡‡] From: '*uncertain*': changeable, not reliable, not certainly knowing or known': Hornby A.S. with Cowie A.P. and Gimson A.C., *Oxford advanced learner's dictionary of current english*. Oxford University Press (1974) 1041pp: Science and Art, Sofia 1989.

work: 'Mathematical statistics offers methods for describing data samples with the aid of random variables. A common approach for this purpose is to specify a probability distribution function in order to obtain a stochastic data model … This uncertainty can neither be accounted for by the deterministic nor the stochastic data model'. Some other works, concerning *fuzzy probabilistics* and/or *structural reliability* are cited below.

The notions of '*interval of confidence*' and '*time dependent fuzzy set theory*' were introduced in (Aliev I.M. and Kara Z. 2004) and hence a 'general procedure to construct the membership function of the fuzzy reliability, when the failure rate is fuzzy'.

In particular, 'two types of uncertainties can be generally recognised in structural reliability: natural randomness of basic variables and vagueness of performance requirements. While the randomness of basic variables is handled by common methods of the probability theory, the vagueness of the performance requirements is described by the basic tools of the theory of fuzzy sets. Both the types of uncertainties are combined in the newly defined fuzzy probabilistic measures of structural reliability, the damage function and the fuzzy probability of failure' (Holický M. 2006).

'The process industry has always been faced with the difficult task of determining the required integrity of safeguarding systems such as Safety Instrumented Systems (SISs). The ANSI/ISA S84.01-1996 and IEC 61508 safety standards provide guidelines for the design, installation, operation, maintenance, and test of SIS. However, in the field, there is a considerable lack of understanding of how to apply these standards to both determine and achieve the required safety integrity level (SIL) for SIS. Moreover, in certain situations, the SIL evaluation is further complicated due to the uncertainty on reliability parameters of SIS components. This paper proposes a new approach to evaluate the "confidence" of the SIL determination when there is an uncertainty about failure rates of SIS components. This approach is based on the use of failure rates and fuzzy probabilities to evaluate the SIS failure probability on demand and the SIL of the SIS. Furthermore, we provide guidance on reducing the SIL uncertainty based on fuzzy probabilistic importance measures' (Sallak M. et al. 2008). According to this work, the SIS is 'a system composed of *sensors*, *logic solver* and *final elements* for the purpose of taking the process to a *safe state* when predetermined conditions are violated'. Here, by PFD it is denoted the probability that 'the SIS will fail such that it cannot respond to a potentially dangerous condition' (the *average probability* is denoted by PFD_{avg}). In particular, it was observed that 'the probabilistic approaches combined with *Monte Carlo simulation*[*] which evaluate the PFD of SIS from the failure probabilities of its components might be inappropriate (most of the available failure rates data are point values without information concerning the corresponding probability distributions)'.

'To address the fuzzy random uncertainty in structural reliability analysis, a novel method for obtaining the *membership function of fuzzy reliability* is proposed on the two orders four central moments (TOFM) method based on *envelope distribution*[†]. At each cut level, the envelope distribution is first constructed, which is a new expression to describe the bound of the fuzzy random variable distribution. The central moments of the bound distribution are determined by generating samples from the envelope distribution, and they are used to calculate the central moments of the limit state function based on the first two orders of the Taylor expansion. Thereafter, the modern approximation method is used to approximate the polynomial expression for the limit state function probability density function (PDF) by considering the central moments as constraint conditions. Thus, the reliability boundaries can be calculated under the considered cut level, and the membership function of the fuzzy reliability is subsequently obtained. Three examples are evaluated to demonstrate the efficiency and accuracy of the proposed method. Moreover, a comparison is made between the proposed method, Monte Carlo simulation (MCS) method, and fuzzy first-order reliability method (FFORM). The results show the superiority of the proposed method, which is feasible for the analysis of structural reliability with fuzzy randomness' (You L. et al 2019).

The notion of *imprecise reliability* was also studied by Utkin L.V. and Coolen F.P.A.[‡] The main aim of this work was to define what imprecise reliability is. And hence, there were discussed 'a variety of problems that can

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[*] Computational algorithms with repeated random sampling for obtaining numerical results (Enrico Fermi 1901 – 1954). See also: *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

[†] 'Suppose that we want to sample from a distribution f(x) that is difficult or impossible to sample from directly, but instead have a simpler distribution q(x) from which sampling is easy. The idea behind *rejection sampling* (aka *acceptance-rejection sampling*) is to sample from q(x) and apply some rejection/acceptance criterion such that the samples that are accepted are distributed according to f(x)': https://theclevermachine.wordpress.com/tag/envelope-distribution/.

[‡] Utkin L.V. and Coolen F.P.A., Imprecise reliability: An introductory overview, 46pp:

be solved by means of a framework of imprecise probabilities. From this point of view, various branches of reliability analysis were considered, including analysis of monotone systems, repairable systems, multi-state systems, structural reliability, software reliability, human reliability, fault tree analysis. Various types of initial information used in imprecise reliability were considered. Some open problems were briefly discussed in the concluding section'.

And so, according to the introductory considerations of the last work: 'A lot of methods and models in classical reliability theory assume that all probabilities are precise, that is, that every probability involved is perfectly determinable. Moreover, it is usually assumed that there exists some complete probabilistic information about the system and component reliability behavior. The completeness of the probabilistic information means that two conditions must be fulfilled: 1) all probabilities or probability distributions are known or perfectly determinable; 2) the system components are independent, i.e., all random variables, describing the component reliability behavior, are independent, or, alternatively, their dependence is precisely known. The precise system reliability measures can always (at least theoretically) be computed if both these conditions are satisfied (it is assumed here that the system structure is precisely defined and that there is a known function linking the system time to failure (TTF) and TTFs of components or some logical system function[*]). If at least one of these conditions is violated, then only interval reliability measures can be obtained. In reality, it is difficult to expect that the first condition is fulfilled. If the information we have about the functioning of components and systems is based on a statistical analysis, then a probabilistic uncertainty model should be used in order to mathematically represent and manipulate that information. However, the reliability assessments that are combined to describe systems and components may come from various sources. Some of them may be objective measures based on relative frequencies or on well-established statistical models. A part of the reliability assessments may be supplied by experts. If a system is new or exists only as a project, then there are often not sufficient statistical data on which to base precise probability distributions. Even if such data exist, we do not always observe their stability from the statistical point of view. Moreover, failure times may not be accurately observed or may even be missed. Sometimes, failures do not occur at all or occur partially, leading to censored observations of failure times, and the censoring mechanisms themselves may be complex and not precisely known. As a result, only partial information about reliability of system components may be available, for example, the mean time to failure (MTTF) or bounds for the probability of failure at a time. Of course, one can always assume that the TTF has a certain probability distribution, where, for example, exponential, Weibull and lognormal are popular choices. However, how should we trust the obtained results of reliability analyses if our assumptions are only based on our experiences or on those of experts. One can reply that if an expert provides an interval for the MTTF on the basis of his experience, why should we reject his assumptions concerning the probability distribution of TTFs? The fact is that judgements elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments. Therefore, any assumption concerning a certain probability distribution in combination with imprecision of expert judgements may lead to incorrect results which often cannot be validateddue to lack of (experimental) data. In many situations, it is unrealistic to assume that components of systems are independent'.

7.5. Applications

Theory of fuzzy sets (continually being developed) have a lot of applications in various areas, e.g. such as: safety, industrial engineering, reliability analysis, social sciences, production management, psychology, artificial intelligence, medicine (e.g. intuitionistic fuzzy sets, fuzziness in medical measurement: in general, the use of fuzzy technology), geography (e.g. fuzzy classifications in large geographical databases), the use of nonconventional methods in *fault diagnosis*, e.g. based upon some statistical and fuzzy concepts to vibrations (Stefanoiu D. and Ionescu F. 2006, Ştefanoiu D. et al. 2009) and so on. In particular, such or similar, areas are important topics in the annual International Conference FSTA (*Fuzzy Set Theory and Applications*). Some fuzzy set applications are given below. The next introductory definitions are first presented (Zhang X.L. and Xu Z.S. 2014).

Definition 5.65 (*Pythagorean fuzzy set*)

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file:///F:/FUZZY%20PROBABILISTIC%20St.Petrsburg.pdf.

[*] (Barlow R.E. and Proschan F. 1975).

A *Pythagorean fuzzy set* $\tilde{P}$ in X, introduced by Yager*, is defined as follows: $\tilde{P} =_{df} \{< x, \mu_{\tilde{P}}(x), \nu_{\tilde{P}}(x) > / x \in X\}$, where the functions $\mu_{\tilde{P}} : X \to [0,1]$ and $\nu_{\tilde{P}} : X \to [0,1]$ represent the *degrees of membership* and *nonmembership* of x in $\tilde{P}$, respectively. Here, the following *condition* should be satisfied: $0 \leq \mu_{\tilde{P}}^2(x) + \nu_{\tilde{P}}^2(x) \leq 1$ (for any $x \in X$).

In accordance with the last work, the following notions were also given: *Pythagorean index* associated with any x in $\tilde{P}$: $\pi_{\tilde{P}}(x) =_{df} (1 - \mu_{\tilde{P}}^2(x) - \nu_{\tilde{P}}^2(x))^{1/2}$ and this index is said to be a *completely Pythagorean* if $\mu_{\tilde{P}}^2(x) = \nu_{\tilde{P}}^2(x) = 0$, for all $x \in X$. The notions of *concentration* and *dilation* were introduced as follows: $CON(\tilde{P}) =_{df} \{< x, \mu_{CON(\tilde{P})}(x), \nu_{CON(\tilde{P})}(x) > / x \in X\}$, where: $\mu_{CON(\tilde{P})}(x) =_{df} (\mu_{\tilde{P}}(x))^2$ and $\nu_{CON(\tilde{P})}(x) =_{df} (1 - (1 - \nu_{\tilde{P}}^2(x))^2)^{1/2}$ and $DIL(\tilde{P}) =_{df} \{< x, \mu_{DIL(\tilde{P})}(x), \nu_{DIL(\tilde{P})}(x) > / x \in X\}$, where: $\mu_{DIL(\tilde{P})}(x) =_{df} (\mu_{\tilde{P}}(x))^{1/2}$ and $\nu_{DIL(\tilde{P})}(x) =_{df} (1 - (1 - \nu_{\tilde{P}}^2(x))^{1/2})^{1/2}$.

In accordance with the last considerations, the concept of Pythagorean fuzzy sets 'provides a novel way to model uncertainty and vagueness with high precision and accuracy compared to intuitionistic fuzzy sets' (Yang M. - S. and Hussain Z. 2018). Here, there were studied 'both *probabilistic* and *nonprobabilistic* types to calculate *fuzzy entropy* of Pythagorean fuzzy sets'. In particular, there were constructed new entropy measures based on 'probability-type, entropy induced by distance, Pythagorean index, and max-min operation'. There was also given an axiomatic definition of entropy for Pythagorean fuzzy sets, and so on (with application to *multicriterion decision making*).

In general, *Pythagorean fuzzy set theory* is a generalisation of *intuitionistic fuzzy set theory*. '*Pythagorean fuzzy sets*, proposed by Yager, have advantages in handling vagueness in the real world and possess good symmetry. The entropy measure is the most widespread form of uncertainty measure' (Han Q. et al. 2019). In the last work, there was given an 'improved *technique for order preference by similarity to an ideal solution* (TOPSIS) method to better deal with *multiple-attribute group decision making* (MAGDM) problems based on Pythagorean fuzzy soft sets. To better determine the weights of attributes, there was firstly defined a novel, more reasonable and valid *Pythagorean fuzzy soft entropy*': left to the reader.

The notions of *image segmentation* and *fuzzy clusterisation* (or: *clustering*) are briefly presented below. Some related expected values, such as: FEV, WFEV or FEI are also cited. And finaly, some considerations related to the *fuzzy control systems* are also given.

Fuzzy clustering (also referred as: *fuzzy c – means*, *fuzzy k – means* or *soft clustering*) was developed by Dunn J.C. (1973)† and improved by Bezdek J.C. (1978)‡. Some introductory notions concerning the image segmentation and fuzzy clustering are given below.

Image segmentation is 'the operation of partitioning an image into a collection of: *regions* (which usually cover the whole image), *linear structures* (such as *line segments*, *curve segments*), into *2D shapes* (such as circles, ellipses, ribbons – long, symmetric regions)'. *Clustering* is 'a more general term than *image segmentation*. We 'can cluster all sorts of data (usually represented as feature vectors, not just image pixels) or e.g. web pages, financial records, etc. Clustering is a large area of *machine learning* (not supervised, i.e. labels of feature vectors are not known)'§. Here, there were presented basic notions concerning the so-called *hard* (or *non-fuzzy*) *clustering*. A review work on fuzzy clustering is briefly presented below.

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* See: (Yager R.R. 2013) and (Yager R.R. and Abbasov A.M. 2013): Ronald R. Yager, born 1941.

† The used here term '*ISODATA*' denotes: ' *Iterative Self-Organizing DATA*' analysis technique. https://www.tandfonline.com/doi/abs/10.1080/01969727308546046

‡ *The Free Encyclopaedia, The Wikimedia Foundation, Inc*.See also: (Bezdek J.C. 1981).

§ Veksler O., Artificial Intelligence II, Lecture 15, Computer vision. Image segmentation: file:///C:/Users/user/Downloads/Image%20Segmentation%20or%20clustering.pdf.

'*Fuzzy clustering* is useful clustering technique which partitions the data set in fuzzy partitions and this technique is applicable in many technical applications like crime hot spot detection, tissue differentiation in medical images, software quality prediction etc. In this review paper, we have done a comprehensive study and experimental analysis of the performance of all major fuzzy clustering algorithms named: FCM, PCM, PFCM, FCM-σ, T2FCM, KT2FCM, IFCM, KIFCM, IFCM-σ, KIFCM-σ, NC, CFCM, DOFCM. To better analysis their performance we experimented with standard data points in the presents of noise and outlier' (Gosaina A. and Dahiyab S. 2016).

'*Clustering* is an important unsupervised form of classification technique of data mining [boows]. It divides the data elements in a number of groups such that elements within a group possess high similarity while they differ from the elements of other groups. Clustering can of two types: *hard clustering* and *fuzzy clustering* [boows]. When each element is solely dedicated to one group, that type of clustering is called *hard clustering*. In hard clustering, clusters have crisp sets for representing element's membership, i.e. the membership of elements in a cluster is assessed in binary terms according to a bivalent condition that an element either belongs or does not belong to the set. In contrast, when the elements are not solely belonging to any one group, instead they share some fraction of membership in a number of groups, that type of clustering is called *fuzzy clustering*. So, fuzzy clustering permits the gradual assessment of the membership of elements in a set which is described by a membership function valued in the real unit interval [0, 1]. Thus membership functions are represented as a fuzzy set which can be either *Type-I*, *Type-II* or *Intuitionistic*'[*](Gosaina A. and Dahiyab S. 2016). According to this work, the following algorithms are of Type-I : FCM, PCM, PFCM, NC, FCM-σ, CFCM and DOFCM. In a similar way we have: T2FCM, KT2FCM (of Type-II) and IFCM, IFCM-σ, KIFCM and KIFCM-σ (are intuitionistic ones).

The above cited algorithms have a different '*affinity*' wrt noise or also used data structures (e.g. the use of a *centroid,* i.e. the arithmetic mean of all figure points, instead of the center of the cluster), etc. The classical Bezdek's *fuzzy c – means algorithm*, i.e. FCM[†] ('one of the most renowned fuzzy clustering algorithms'), is presented below (Gosaina A. and Dahiyab S. 2016).

The last algorithm works under assumption that the *number of clusters* 'c' associated with the considered data set, denoted by $X =_{df} \{x_1, \dots, x_n\}$, is known and minimizes the following *objective function:* $J_{FCM} =_{df} \sum_{k=1}^{c} \sum_{i=1}^{n} u_{ik}^{m} \cdot d_{ik}^{2}$, where '$u_{ik}$' is the membership of datapoint 'x_i' in cluster 'k' (the process of assigning initial values to all u_{ik} is defined as *initial fuzzy pseudo - partition*[‡]) and $d_{ik} =_{df} | x_i - v_k |$ is the *Euclidean distance*[§] between x_i and cluster center v_k. According to 'u_{ik}', the following condition should be also satisfied: $\sum_{k=1}^{c} u_{ik} = 1$ (for i = 1, 2, … , n). It is assumed here that the *constant* $m =_{df} 2$, known as '*fuzzifier* or *fuzziness index*: as it controls the fuzziness of the resulting clusters'.

The *minimisation* of J_{FCM} is realised as follows (Bezdek J.C. 1981). The membership of every point is updated as follows.

$$u_{ik} =_{df} \frac{1}{\sum_{j=1}^{c} (\frac{d_{ki}}{d_{ji}})^{2/(m-1)}} \quad \text{(for } i = 1, 2, \dots, n \text{ and } k = 1, 2, \dots, c\text{)} \quad \text{and} \quad v_k =_{df} \frac{\sum_{i=1}^{n} u_{ik}^{m} \cdot x_i}{\sum_{i=1}^{n} u_{ik}^{m}}.$$

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[*] *Type-2 fuzzy sets* as a generalisation of ordinary fuzzy sets (i.e. *Type-1 fuzzy sets*) were introduced by Zadeh L.A. (1975): Lotfi Aliascer Zadeh (1921 – 2017). *Intuitionistic fuzzy sets* were introduced by Atanassov K.T. (1986). See also: (Dan S. et al. 2019).

[†] Known also as: *Euclidean fuzzy c – means algorithm* (Bezdek J.C. 1981).

[‡] The requirement of a crisp partition of X is replaced with the weaker requirement of a *fuzzy pseudopartition* on X (known also as a *fuzzy c-partition*).

[§] See Subsection 7.3.

Unfortunately, the above algorithm FCM 'failed to detect noise and outliers, so it treats the noisy points and outliers same as the actual data points. Thus, it's centroid is attracted towards outliers instead of the center of the cluster'. And hence, a problem may be the selection of (the number of clusters) 'c' (which are not sensitive wrt initial assignment of centroids).

Traditional clustering is an *unsupervised* one wrt the obtained *outcomes*, i.e. groupings of pixels having common characteristics. '*Supervised clustering* is based on the idea that a user can select sample pixels in an image that are representative of specific classes and then direct the image processing software to use these training sites as references for the classification of all other pixels in the image'[*].

In general, fuzzy clustering has many applications, e.g. such as (Chaudhan A. 2013): *medical imaging* (X-ray computer tomography / CT, magnetic resonance imaging / MRI, position emission tomography / PET), *image and speech enhancement*, *edge detection*, *video shot change detection* and so on. Some other considerations are omitted: left to the reader.

Instead of the use of *classical mean values* (e.g. such as: *arithmetic*, *geometric*, *square*, *harmonic*, *weighted arithmetic* or *weighted square* means)[†], in accordance with the fuzzy set theory,in particular there are used such notions as e.g. FEV (*fuzzy expected value*[‡]), WFEV (*weighted* FEV), GWFEV (*generalised* WFEV) or FEI (FE*interval*): (Schneider M. and Kandel A. 1993). The use of the first two notions are presented in the next example below.

Example 5.26 (FEV and WFEV)

n	μ_i	n_i	$\|\mu_T\|$	$\varphi(\mu) =_{df} \frac{\|\mu_T\|}{\|X\|}$	$\min\{\mu_i, \varphi(\mu)\}$
1	0.3	3	10	1.0	0.3
2	0.8 ←	2	7	0.7	(0.7)
3	0.5	1	5	0.5	0.5
4	0.9	4	4	0.4	0.4

Figure 5.13 FEV computation

Here: $\mu_i =_{df} \mu(x_i)$, where $x_i \in X$ and $|X| = 10$. If $n = 1$ then $|\mu_T| = 10$. The next value of $|\mu_T| = |X| - n_1 = 10 - 3 = 7$, etc. Since $\max\{\min\{\mu_i, \varphi(\mu)\}\} = 0.7$ corresponds to $\mu_i = 0.8$ (for $n = 2$) then we can obtain: FEV = 0.8. As an example, the corresponding arithmetic mean, equivalently represented as the following weighted one:

$$\frac{\sum_{i=1}^{n} \mu_i \cdot n_i}{\sum_{i=1}^{n} n_i} =_{df} \frac{0.3 \cdot 3 + 0.8 \cdot 2 + 0.5 \cdot 1 + 0.9 \cdot 4}{1 + 2 + 3 + 4} = 0.66 < 0.80 = 0.8 = \text{FEV}.$$

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[*] *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

[†] $(a_1 + \ldots + a_n) / n$, $(a_1 \cdot \ldots \cdot a_n)^{1/n}$, $((a_1^2 + \ldots + a_n^2) / n)^{1/2}$, $n / (1/a_1 + \ldots + 1/a_n)$, $(w_1 a_1 + \ldots + w_n a_n) / (w_1 + \ldots + w_n)$ and $((w_1 a_1^2 + \ldots + w_n a_n^2) / (w_1 + \ldots + w_n))^{1/2}$, respectively.

[‡] The following two notions are different: '*fuzzy expected value*' and '*expected value of a function* f(x) *having* x *as a fuzzy variable*'.

Let $s_0 =_{df}$ FEV and $\hat{s} =_{df} \dfrac{\sum_{i=1}^{n} \mu_i \cdot e^{-\beta \cdot |\mu_i - s|} \cdot n_i^{\lambda}}{\sum_{i=1}^{n} e^{-\beta \cdot |\mu_i - s|} \cdot n_i^{\lambda}}$. In accordance with the last definition, the WFEV is computed iteratively, starting with s_0, until $|\hat{s} - s| \leq \varepsilon$ (e.g. about 5 – 6 / $\varepsilon_1 = 0.01$ to 17 – 18 / $\varepsilon_2 = 0.001$ iterations. Practically, it is assumed: $\beta = 0.2$ and $\lambda = 1.0$).

Let now $F(i) =_{df} e^{-\beta \cdot |\mu_i - s|} \cdot n_i^{\lambda}$. And so, the following more simplified form can be obtained: $\hat{s} = \dfrac{\sum_{i=1}^{n} \mu_i \cdot F(i)}{\sum_{i=1}^{n} F(i)}$. As an example (in accordance with the above Figure 5.13 , to compute s_1), we can obtain: $F(1) = 3e^{-0.1}$, $F(2) = 2$, $F(3) = e^{-0.06}$ and $F(4) = 4e^{-0.02}$. Hence $s_1 \approx 0.6697$ and $|s_1 - s_0| = 0.1303$. □

Automatic control is one of the application areas of fuzzy set theory. The first successful application of fuzzy logic to the control (of a laboratory – scale process) was given in (Mamdani E.H. and Assilian S. 1975). Some industrial application (may be the first one) was given in (Holmblad L.P. and Ostergaard J.J. 1982). '*A fuzzy control system* is a control system based on fuzzy logic - a mathematical system that analyses analog input values in terms of logical variables that take on continuous values between 0 and 1, in contrast to *classical* or *digital logic*, which operates on discrete values of either 1 or 0 (*true* or *false*, respectively)'*. Some considerations are briefly presented below (Babuska R. and Mamdani E.H. 2008).

According to the last work: 'Many processes controlled by human operators in industry cannot be automated using conventional control techniques, since the performance of these controllers is often inferior to that of the operators. One of the reasons is that linear controllers, which are commonly used in conventional control, are not appropriate for nonlinear plants. Another reason is that humans aggregate various kinds of information and combine control strategies, that cannot be integrated into a single analytic control law. The underlying principle of *knowledge-based (expert) control* is to capture and implement experience and knowledge available from experts (e.g., process operators). A specific type of knowledge-based control is the fuzzy rule-based control, where the control actions corresponding to particular conditions of the system are described in terms of *fuzzy if-then rules*'.

In most cases fuzzy controllers are used for *direct feedback control*, e.g. *Mamdani (linguistic) controller*. 'However, it can also be used on the *supervisory level* as, e.g., a self-tuning device in a *conventional PID* (*Proportional-Integral-Differential*) *controller*', e.g. *Takagi-Sugeno controller* (Babuska R. and Mamdani E.H. 2008).

'Since the first use of fuzzy logic in the field of control engineering, it has been extensively employed in controlling a wide range of applications. The human knowledge on controlling complex and non-linear processes can be incorporated into a controller in the form of linguistic terms. However, with the lack of analytical design study it is becoming more difficult to auto-tune controller parameters. Fuzzy logic controller has several parameters that can be adjusted, such as: membership functions, rule-base and scaling gains. Furthermore, it is not always easy to find the relation between the type of membership functions or rule-base and the controller performance' (Saeed B. I. and Mehrdadi B. 2012). It is proposed in the last study 'a new systematic auto-tuning algorithm to fine tune fuzzy logic controller gains. A *fuzzy PID controller* is proposed and applied to several second order systems (initially, the controller gains are fixed and then automatically tuned to achieve the best possible performance). The relationship between the closed-loop response and the controller parameters is analysed to devise an auto-tuning method. The results show that the proposed method is highly effective and produces zero overshoot with enhanced transient response. In addition, the robustness of the controller is investigated in the case of parameter changes and the results show a satisfactory performance'.

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* *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

8. Rough set theory

Rough set theory was developed by Pawlak Z. in the early 1980's and first published in (Pawlak Z. 1982, 1991). 'The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of artificial intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful one is, no doubt, the fuzzy set theory proposed by Zadeh L.A. (1965). The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data - like probability in statistics, or basic probability assignment in *Dempster-Shafer theory*[*], grade of membership or the value of possibility in fuzzy set theory. …Rough set theory is another approach to vagueness. Similarly to fuzzy set theory it is not alternative to classical set theory but it is embedded in it. Rough set theory can be viewed as a specific implementation of Frege's idea of vagueness i.e. imprecision in this approach is expressed by a boundary region of a set, and not by partial membership, like in fuzzy set theory '.[†] Some basic notions concerning rough sets are given below.

8.1. The notion of approximation space

Let U be a set of objects and $\rho \subseteq U \times U$ be an *undiscernibility relation* 'representing our lack of knowledge about elements of U'. For simplicity, we shall assume below that ρ is an equivalence. Let also $X \subseteq U$. The last subset X can be characterised wrt ρ. 'To this end we will need the basic concepts of rough set theory', i.e. the following notions: *lover approximation*, *upper approximation* and *boundary region*, associated with X.

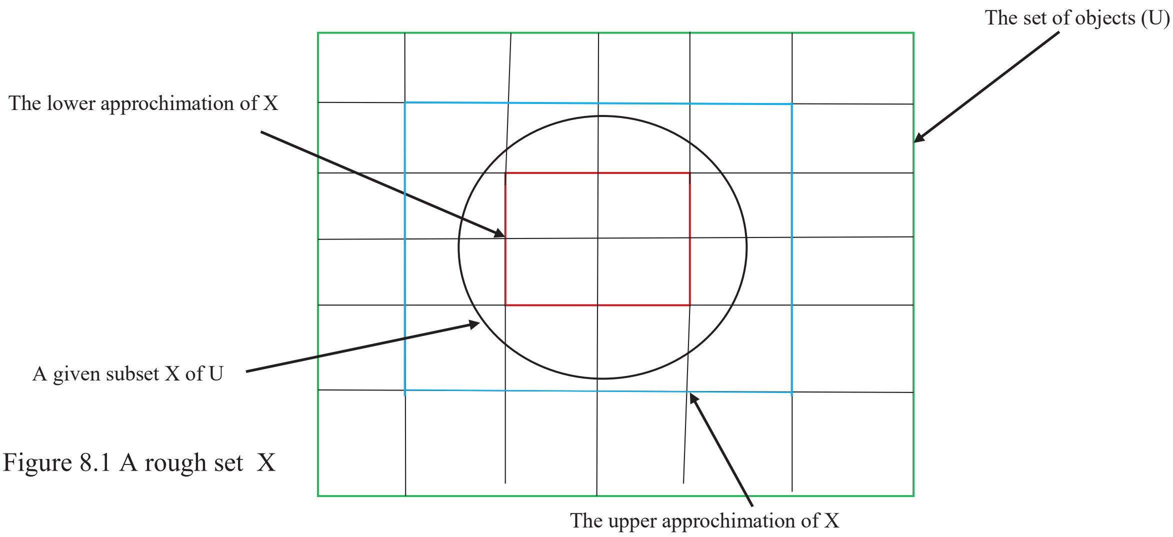

Figure 8.1 A rough set X

According to Figure 8.1, the *lower approchimation* of X, denoted by $\underline{A}$ (X) $=_{df}$ $\{x \in U / [x]_\rho \subseteq X\}$. In a similar way, the *upper approchimation* of X, denoted by $\bar{A}$ (X) $=_{df}$ $\{x \in U / [x]_\rho \cap X \neq \varnothing\}$. The *boundary region* of X, denoted here by BR(X) $=_{df}$ $\bar{A}$(X) − $\underline{A}$ (X).

Corollary 8.1

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[*] 'The *theory of belief functions*, also referred to as *evidence theory* or *Dempster–Shafer theory* (Arthur P. Dempster, born 1929 and Glenn Shafer, born 1946), is a general framework for reasoning with uncertainty, with understood connections to other frameworks such as probability, possibility and imprecise probability theories': *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

[†] Pawlak Z., *Rough sets*. 51pp: file:///C:/Users/user/Downloads/ZPAWLAK%20ROUGH!.pdf (Zdzislaw Pawlak 1926 – 2006).

$$\underline{A}(X) = \bigcup\{[x]_\rho \in U/\rho \;/\; [x]_\rho \subseteq X\} \quad \text{and} \quad \bar{A}(X) = \bigcup\{[x]_\rho \in U/\rho \;/\; [x]_\rho \cap X \neq \varnothing\}. \;\square$$

Corollary 8.2

Let $y \in U$. Then: $y \in \underline{A}(X) \Leftrightarrow \forall_{x \in U}(x\rho y \Rightarrow x \in X)$ and $y \in \bar{A}(X) \Leftrightarrow \exists_{x \in U}(x\rho y \wedge x \in X)$

Proof:

$y \in \underline{A}(X)$	$\Leftrightarrow$	$[y]_\rho \subseteq X$	{df. $\underline{A}(X)$}
	$\Leftrightarrow$	$\forall_{x \in U}(x \in [y]_\rho \Rightarrow x \in X)$	{df. $\subseteq$}
	$\Leftrightarrow$	$\forall_{x \in U}(x\rho y \Rightarrow x \in X)$.	{df. $[y]_\rho$}
$y \in \bar{A}(X)$	$\Leftrightarrow$	$[y]_\rho \cap X \neq \varnothing$	{df. $\bar{A}(X)$}
	$\Leftrightarrow$	$\exists_{x \in U}(x\rho y \wedge x \in X)$. $\square$	{df. $\cap$}

8.2. Rough set operations and approximation accuracy

Some properties related to the above two approchimations are cited below.

1. $\underline{A}(X) \subseteq X \subseteq \bar{A}(X)$
2. $\underline{A}(\varnothing) = \bar{A}(\varnothing) = \varnothing$* and $\underline{A}(U) = \bar{A}(U) = U$
3. $\bar{A}(X \cup Y) = \bar{A}(X) \cup \bar{A}(Y)$
4. $\underline{A}(X \cup Y) \supseteq \underline{A}(X) \cup \underline{A}(Y)$
5. $\bar{A}(X \cap Y) \subseteq \bar{A}(X) \cap \bar{A}(Y)$
6. $\underline{A}(X \cap Y) = \underline{A}(X) \cap \underline{A}(Y)$
7. $X \subseteq Y \Rightarrow (\underline{A}(X) \subseteq \underline{A}(Y)) \wedge (\bar{A}(X) \subseteq \bar{A}(Y))$ {See: T 1.5 of Subsection 1.3}
8. $\underline{A}(X') = \bar{A}(X)'$
9. $\bar{A}(X') = \underline{A}(X)'$
10. $\underline{A}(\underline{A}(X)) = \bar{A}(\underline{A}(X)) = \underline{A}(X)$
11. $\bar{A}(\bar{A}(X)) = \underline{A}(\bar{A}(X)) = \bar{A}(X)$

As an illustration, the proof of inclusion 5 is given below. And so, according to Corollary 8.2, the following implication should be shown.†

$$\exists_{x \in U}(x\rho y \wedge x \in X \cap Y) \Rightarrow \exists_{x \in U}(x\rho y \wedge x \in X) \wedge \exists_{x \in U}(x\rho y \wedge x \in Y)$$

---

* $|\underline{A}(\varnothing)| = |\bar{A}(\varnothing)| = |\varnothing| = 0$ (see Subsection 5.5).

† $\exists_x(A(x) \wedge B(x)) \Rightarrow \exists_x A(x) \wedge \exists_x B(x)$ is also satisfied using bounded quantifiers (see Subsection 3.3).

Proof

(1)	$\exists_{x \in U} (x\rho y \wedge x \in X \cap Y)$	{a}
(2)	$\sim (\exists_{x \in U} (x\rho y \wedge x \in X) \wedge \exists_{x \in U} (x\rho y \wedge x \in Y))$	{aip}
(3)	$\forall_{x \in U} (x\rho' y \vee x \notin X) \vee \forall_{x \in U} (x\rho' y \vee x \notin Y)$	{NK, N∃,SR : 2}
(4)	$a\rho y$	
(5)	$a \in X$	{− ∃, − K, df.'∩': 1}
(6)	$a \in Y$	
(1.1)	$\forall_{x \in U} (x\rho' y \vee x \notin X)$	{ada: 3}
(1.2)	$a\rho' y \vee a \notin X$	{−∀ : 1.1}
(1.3)	$a \notin X$	{− A : 4, 1.2}
	contr.	{5, 1.3}
(2.1)	$\forall_{x \in U} (x\rho' y \vee x \notin Y)$	{ada: 3}
(2.2)	$a\rho' y \vee a \notin Y$	{−∀ : 2.1}
(2.3)	$a \notin Y$	{− A : 4, 2.2}
	contr. □	{6, 2.3}

Another example is the proof of the above equality 9. This proof is given below.

$x \in \underline{A}(X)'$	$\Leftrightarrow$	$x \notin \underline{A}(X)$	{df. ' ′ '}
	$\Leftrightarrow$	$[x]_\rho \nsubseteq X$	{df. $\underline{A}(X)$}
	$\Leftrightarrow$	$[x]_\rho \cap X' \neq \varnothing$	{df. X′}
	$\Leftrightarrow$	$x \in \bar{A}(X')$. □	{df. $\bar{A}(X)$}

Let $\mu_X^\rho : U \to [0,1]$. Rough sets can be also introduced by using (instead of approximation) the last *rough membership function*, where: $\mu_X^\rho(x) =_{df} \dfrac{|X \cap [x]_\rho|}{|[x]_\rho|}$. We have: $\underline{A}(X) = \{x \in U / \mu_X^\rho(x) = 1\}$, $\bar{A}(X) = \{x \in U / \mu_X^\rho(x) > 0\}$ and $BR(X) = \{x \in U / 0 < \mu_X^\rho(x) < 1\}$. Several properties related to this function were also given.

In accordance with the last work: if $BR(X) = \varnothing$ then X is a *crisp set* else X is a *rough* one. Moreover, the following two definitions of rough sets were given: X is *rough* wrt ρ if $\underline{A}(X) \neq \bar{A}(X)$ or equivalently if $\exists_{x \in X} (0 < \mu_X^\rho(x) < 1\}$. The *accuracy of approximation*, denoted by $0 \leq \alpha_\rho(X) =_{df} \dfrac{|\underline{A}(X)|}{|\bar{A}(X)|} \leq 1$.

Obviously. If $\alpha_\rho(X) = 1$ then X is crisp else X is rough.

As an example, according to the above Figure 8.1, we have: $\alpha_\rho(X) = \frac{4}{16} = \frac{1}{4}$ (= 25%). And hence, X is rough.

Example 8.1 (*approximation accuracy*)

Let $\mathbb{R}_+ =_{df} [0, \infty)$ be the set of nonnegative real numbers and ρ be a binary relatin defined in $\mathbb{R}_+$ such that: $x\rho y \Leftrightarrow_{df} \lfloor x \rfloor = \lfloor y \rfloor$, where $\lfloor x \rfloor =_{df} \max\{n \leq x \ / \ n \in \mathbb{Z}\}$.

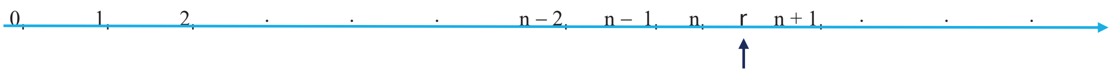

Figure 8.2

According to Figure 8.2, we can obtain: $\underline{A}\,([0,r)) = \bigcup_{i=0}^{n-1} [i, i+1)$ and $\bar{A}([0,r)) = \bigcup_{i=0}^{n} [i, i+1)$. And hence: $\alpha_\rho(X) = \dfrac{n}{n+1}$. □

8.3. Information systems, reducts and decision tables

The notion of an information system is introduced as follows.*

Definition 8.1 (*information system*)

An *information system* $S =_{df} (U, A)$† , where: $U \neq \varnothing$ is a finite set of objects, called *universe*, $A \neq \varnothing$ is a finite set of *attributes*, $a : U \rightarrow V_a$ (is the *value set associated with* $a \in A$) and $V =_{df} \bigcup_{a \in A} V_a$ is the *domain* of A.

Let $S = (U, A)$ and $A \subseteq B$. The obtained information system $S^* =_{df} (U, B)$ is said to be a *B - extension* of S, and S is said to be a *subsystem* of S^*. In general, there is no possibility of distinguishing all single objects using A of S. And so, differend objects may have the same values of attributes. Consequently, any set of attributes will establish a partition of U (Pawlak Z. 1991).

In accordance with (Pawlak Z., *Rough sets*: Chapter 2 {p.14: see below), the above considerations have been presented more precisely as follows.‡

Let $S = (U, A)$. Assume now that $B \subseteq A$. Any such subset is associated with a binary relation $\underline{\text{ind}}(B) \subseteq U \times U$, called *indiscernibility relation* and defined as follows.

$\underline{\text{ind}}(B) =_{df} \{(x,y) \in U \times U \ / \ \forall_{a \in B} (a(x) = a(y))\}$.

We have: $x \ \underline{\text{ind}}(B)\ y \Leftrightarrow \forall_{a \in B} (a(x) = a(y))$, for any $x, y \in U$. Obviously, this relation is an equivalence one. The family of all equivalence classes of ind(B) are denoted by U/ind(B) or in short: U/B (i.e. the partition determined by B). A *block* of the last partition containing x is denoted by B(x). The obtained blocks are referred as *B-elementary sets*. The last sets are considered as 'basic building blocks (concepts) of our knowledge about reality'.

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* Pawlak Z., *Rough sets*. 51pp: file:///C:/Users/user/Downloads/ZPAWLAK%20ROUGH!.pdf: Chapter 2 {p.14: *Rough sets and reasoning from data*). See also: Skowron A. and Suraj Z., ICS Report PAS: 18 / 1993 and (Skowron A. and Rauszer C. 1991).

† This information system is some times equivalently introduced as follows: $S =_{df} (U, A, V, f)$, where $f : U \times A \rightarrow V$, V is the union of all $V_a / a \in A$ and $a(x) =_{df} f(x,a)$.

‡ According to the used designations in Subsection 8.1: $\rho_A =_{df} \rho$. And hence $\rho_B =_{df} \underline{\text{ind}}(B)$.

If $\underline{\text{ind}}$ (B) = $\underline{\text{ind}}$ (B – {a}) then a ∈ B is *superfluous* else a is *independent*. The obtained lower, upper approximations and the corresponding boundary region are said to be: *B-lower approximation*, *B-upper approximation* and *B-boundary region*, defined as follows:

$\underline{B}(X) =_{df} \{x \in U / B(x) \subseteq X\}$,

$\bar{B}(X) =_{df} \{x \in U / B(x) \cap X \neq \varnothing\}$ and

$BR(X) =_{df} \bar{B}(X) - \underline{B}(X)$, respectively.

In a similar way as in the previous considerations, if BR(X) = ∅ then X is a *crisp set* else X is a *rough* one. The above lower and upper approximations of a set can be considered 'as *interior* and *closure* operations in a *topology* generated by the indiscernibility relation'. *

There were introduced the following basic classes of rough sets (said also '*categories of vagueness*').

$\underline{B}(X) \neq \varnothing, \bar{B}(X) \neq U$ ⇔ X is *roughly B-definable*

$\underline{B}(X) = \varnothing, \bar{B}(X) \neq U$ ⇔ X is *internally B-indefinable*

$\underline{B}(X) \neq \varnothing, \bar{B}(X) = U$ ⇔ X is *externally B-definable*

$\underline{B}(X) = \varnothing, \bar{B}(X) = U$ ⇔ X is *totally B-indefinable*

The obtained *accuracy of approximation* is introduced in a similar way as in the previous subsection, but wrt $B \subseteq A$.

Definition 8.2 (*reduct*)

We shall say that $B^* \subseteq B$ is *reduct* iff B^* is a set of independent attributes and B^* preserves classification, i.e. $\underline{\text{ind}}(B^*) = \underline{\text{ind}}(B)$. The set of all reducts of B, denoted by $RED(B) =_{df} \{ B^* \subseteq B / B^*$ is reduct of $B\} \subseteq 2^B$.

The following property is satisfied (connecting the notion of a core and reducts).

Proposition 8.1

$$CORE(B) = \bigcap_{B^* \in RED(B)} B^* . \square$$

Let consider the following example (Pawlak Z., *Rough sets*: Chapter 2: Table 1 / p.15 + p.17).

Example 8.2 (*information system*)

	a	b	c	d
x_1	0	1	2	1
x_2	1	0	2	1
x_3	1	1	3	1
x_4	0	1	4	0
x_5	1	0	2	0
x_6	0	1	3	1

Figure 8.3

In accordance with the last figure and for convenience, here *patients* p_i (i = 1, … ,6) are denoted by x_i . The set of attributes '*Headache*, *Muscle - pain*, *Temperature* and *Flu*' are denoted by a, b, c and d, respectively. And

* *Interior* of a subset X of U: the union of all subsets of X that are open in U. *Closure* of a subset X of U: all points in X together with all limit points. The *boundary* $bd(X) =_{df} cl(X) - int(X)$. E.g. see: *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

hence $A =_{df} \{a, b. c, d\}$. The integers 0,1,2,3 and 4 are here used to denote: '*no*', '*yes*', '*high*', '*very high*' and '*normal*', respectively.

We have: $U =_{df} \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Consider the concept 'Flu': $X =_{df} \{x_1, x_2, x_3, x_6\} \subseteq U$ and $B =_{df} \{a, b, c\} \subseteq A$. Hence: $U/\underline{ind}(B) = \{\{x_1\}, \{x_2, x_5\}, \{x_3\}, \{x_4\}, \{x_6\}\}$. Since $\underline{B}(X) = \{x_1\} \cup \{x_3\} \cup \{x_6\} = \{x_1, x_3, x_6\} \neq \varnothing$ and $\overline{B}(X) = \{x_1\} \cup \{x_2, x_5\} \cup \{x_3\} \cup \{x_6\} = \{x_1, x_2, x_3, x_5, x_6\} \neq U$ this concept is roughly B-definable. The obtained accuracy of approximation $\alpha_{\underline{ind}(B)}(X) = \frac{3}{5}$. □

Let $B \subseteq A$. The *discernibility matrix* M(B) is defined as follows: $[c_{ij}] =_{df} \{a \in B / a(x_i) \neq a(x_j)\}$ $\{i,j = 1, 2, \ldots, n)$, where c_{ij} is the set of all attributes discerning x_i and x_j. The calculation of reducts and core requires the use of only the following two laws:* *idempotence* $(x \cdot x = x = x + x)$ and *absorption* $(x + x \cdot y = x = x \cdot (x + y))$. It was shown that the core is the set of all single element entries of the discernibility matrix M(B), i.e. $CORE(B) = \{a \in B / c_{ij} = \{a\}$, for some i,j} (Pawlak Z., *Rough sets*: Chapter 2 { p.24).

Example 8.3 (*reducts and core*)

	a	b	c	d
x_1	1	2	3	2
x_2	1	3	2	1
x_3	0	1	2	0
x_4	1	2	2	1

(a)

	x_1	x_2	x_3	x_4
x_1	∅			
x_2	bcd	∅		
x_3	abcd	abd	∅	
x_4	cd	b	abd	∅

(b)

Figure 8.4 A simple information system (a) and its discernibility matrix† (b)

According to Figure 8.4(b) we can obtain: $F(A) =_{df} (b + c + d)(a + b + c + d)(a + b + d)(c + d)(b)(a + b + d) = (a + b + d)(c + d)b = (ac + ad + bc + bd + dc + d)b = acb + bc + db$.‡ And hence, the set of all reducts $RED(A) = \{\{a,b,c\}, \{b,c\}, \{b,d\}\}$. $CORE(A) = \{b\}$. □

Any finite information system can be interpreted as a *decision table* by partitioning the set of attributes A of this system into two classes of attributes: *condition* and *decision* ones (see Definition 8.3 given below§). We shall assume below that the obtained decision table is *consistent*, i.e. the same conditions should not implicate different decisions (Pawlak Z., *Rough sets*: Chapter 2 { p.18 - 19). As an example, by assuming a, b and c as *condition attributes* and d as *decision attribute* we can obtain: the information system of Figure 8.3 is *inconsistent* (the same conditions and different decision) and this one shown in Figure 8.4 is consistent. Some well known notions concerning decision tables are cited below.

Definition 8.3 (*decision table*)**

A *decision table* is any information system of the form $S_d =_{df} (U, A \cup \{d\})$, where $d \notin A$ is a distinguished attribute called *decision attribute* (in short: *decision*). The elements of A are called *conditions*. By $A \times V$ it is denoted the *set of all descriptors*, i.e. ordered pairs of the form (a,v) over A and V, where V is the set of all admissible values associated with any $a \in A$.

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* *Lattice theory* will be presented in the next part of this work.

† Provided there is no ambiguity and for simplicity, the subsets included in the discernibility matrix are presented as strings, e.g. $bcd =_{df} \{b,c,d\} \subseteq A \cup \{d\}$.

‡ By starting with leaving parentheses we can obtain $1^1 \cdot 2^1 \cdot 3^3 \cdot 4^1 = 216$ *Cauchy products* (Augustin-Louis Cauchy, 1759 – 1857) and $1^3 \cdot 2^5 \cdot 3^5 \cdot 4^2 = 124416$, in accordance with the previous Example 8.2.

§ Usually it is assumed only one decision attribute.

** See also: Pawlak Z., *Decision tables and decision spaces*, Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, file:///C:/Users/user/Downloads/Pawlak%20dec%20tables.pdf, 6pp.

Let $V_a \subseteq V$ be the subset of all admissible values associated with some $a \in A$. The *set of terms* is introduced as the least set containing descriptors (over A and V) and closed wrt the following three basic logical connectives: $\sim$, $\wedge$, $\vee$.* The *term* τ in S_d, in short $\|\tau\|_{S_d}$ (or $\|\tau\|$ if S_d is known), was inductively defined as follows.

$$\|(a,v)\| =_{df} \{x \in U / a(x) = v\} \quad (a \in A,\ v \in V_a)$$

$$\|\tau_1 \vee \tau_2\| = \|\tau_1\| \cup \|\tau_2\|$$

$$\|\tau_1 \wedge \tau_2\| = \|\tau_1\| \cap \|\tau_2\|$$

$$\|\sim\tau\| = U - \|\tau\|$$

Consider the above decision table S_d. Let $B \subseteq A$, $a \notin B$ and $d_a^B : U \rightarrow 2^{V_a}$ such that $d_a^B(x) =_{df} \{v \in V_a / \exists_{y \in U} (x\underline{ind}(B)y \wedge a(y) = v)\} \subseteq V_a$.

Definition 8.4 (*decision rule*)

The following implication: $(a_1 = v_1) \circ (a_2 = v_2) \circ \ldots \circ (a_n = v_n) \Rightarrow (a = v)$† is said to be a *decision rule*, where $\circ \in \{\wedge, \vee\}$. This rule is *true in* S_d if $\|(a_1 = v_1) \circ (a_2 = v_2) \circ \ldots \circ (a_n = v_n)\| \subseteq \|a = v\|$ and *has an example in* S_d if $\|(a_1 = v_1) \circ (a_2 = v_2) \circ \ldots \circ (a_n = v_n)\| \neq \varnothing$.

The set of all decision rules related to S_d can be obtained from the corresponding matrix $M(S_d)$ by using first the following algorithm. By M_k it is denoted below the matrix $M(x_k)$ corresponding to the k^{th} row of the discernibility matrix associated with $M(S_d)$, where $x_k \in U$ and $|U| = n$.

Algorithm 8.1 (*M_k generation*)

Input: The discernibility matrix of S_d, $B \subseteq A$

Output: $\{M_k \ / \ k = 1 \ldots n\}$

For $k = 1$ to n

(1) if $i \neq k$ then $c_{ij} =_{df} \varnothing$ (for any i,j; $i \neq j$, $i \neq k$)

(2) if $c_{kj} \neq \varnothing$ and $d_a^B(x_j) \neq \{v_k\}$ then $\hat{c}_{kj} =_{df} c_{kj} \cap B$ else $\hat{c}_{kj} =_{df} \varnothing$.

End. □

Example 8.4 (*decision rules*)

Let $B =_{df} A$. Consider the information system and its discernibility matrix shown in the above Figure 8.4 (a,b). The obtained decision table S_d and the matrix M_1 related to the first row x_1 are shown in Figure 8.5(a,b) below. Since '$\underline{ind}$' is an equivalence (in particular reflexive) and all rows are different, the obtained sets corresponding to any x_i, i.e. $d_a^B(x_i)$ are one-element sets, as it is shown in Figure 8.5(a) below

* See Subsection 1.1.

† The following three designations are equivalent: (a,v), a = v and a_v.

	a	b	c	d	$d_a^B(x)$
x_1	1	2	3	2	{2}
x_2	1	3	2	1	{1}
x_3	0	1	2	0	{0}
x_4	1	2	2	1	{1}

(a)

	x_1	x_2	x_3	x_4
x_1	∅	bc	abc	c
x_2				
x_3		∅		
x_4				

(b)

Figure 8.5 A decision table S_d (a) and the matrix M_1 related to the row x_1 (b)

As an example, consider the first row (corresponding to x_1) of Figure 8.4(b): (∅ , bcd, abcd, cd). Since $d_a^B(x_2)$, $d_a^B(x_3)$ and $d_a^B(x_4)$ are different from $d_a^B(x_1) = \{2\}$, in accordance with step (2) of the above Algorithm 8.1, we can obtain: $\{b,c,d\} \cap A = \{b,c\}$, $\{a,b,c,d\} \cap A = \{a,b,c\}$ and $\{c,d\} \cap A = \{c\}$: as it is shown in the above Figure 8.5(b).

Since $x_1 \Rightarrow d_2$ and $(b + c)(a + b + c)c = c$ then $c_1 \Rightarrow d_2$ is the first obtained decision rule. In a similar way we can obtain: $b_2 \Rightarrow d_1$, $a_3 \vee b_3 \Rightarrow d_0$ and $c_4 \wedge b_4 \Rightarrow d_1$ (corresponding to x_2, x_3 and x_4 , respectively). In accordance with T 1.9 of Subsection 1.3 (*law of addition of antecedents*: AA), the above two implications having the same consequent d_1 can be represented as follows: $b_2 \vee c_4 \wedge b_4 \Rightarrow d_1$. □

8.4. Other applications

The applications of rough sets are related to the following their advantages:[*] '(1) It does not need any preliminary or additional information about data - like probability in statistics, grade of membership in the fuzzy set theory, (2) It provides efficient methods, algorithms and tools for finding hidden patterns of data, (3) It allows to evaluate the significance of data, (4) It allows to generate in automatic way the sets of decision rules from data. (5) It is easy to understand, (6) It offers straightforward interpretation of obtained results, (7) It is suited for concurrent (parallel) distributed processing.' As an illustration, some applications of rough sets are cited below.

A comprehensive study of the literature on applications of rough set theory in civil engineering was presented in (Wang W. et al. 2010). 'The relationships between rough set theory and other mathematical methods, such as conventional statistical methods, fuzzy sets, and evidence theory, were briefly introduced. The applications of rough set theory in civil engineering were discussed in structure engineering, pavement engineering, traffic engineering, transportation engineering, land management, and water resources management. The great majority of civil engineering applications of rough set theory were based on the rule induction. Recent works on integration of rough set theory with other computing paradigms such as genetic algorithm, fuzzy logic, evidence theory were also presented'.

The use of rough set theory in deriving information from large data sets was considered in (Das R. et al. 2017). 'Rough Set theory is a very handy tool for imprecise and vague pattern of data . This paper shows how the concept of RST being used in deriving information from hidden pattern of data . From large data base software industries are the object of interest for applying Rough Set concept on the collected data. The set of rules which have been derived will be helpful in the development of software industries. This paper has used two types of

[*] Novak-Brzezinska A., *Rough set theory in decision support systems*. 104pp: file:///C:/Users/user/Downloads/ROUGH%20SETS.pdf

techniques in finding the reduct, first one uses *cluster* in finding different dissimilar groups, the other one is the application of quick reduct algorithm in deriving the rules verifying them by using *strength verification*'.*

An use of gene selection based on rough set applications was studied in (Anitha K. 2012). 'Gene selection is a main procedure of discriminate analysis of microarray data which is the process of selecting most informative genes from the whole gene data base. This paper approach a method for selecting informative genes by using rough set theory. Rough set theory is a effective mathematical tool for selecting informative genes. This paper describes basics of rough set theory and rough set attribute reduction by *quick-reduct based genetic algorithm*'.

Rough modelling as an approach to model construction was proposed in (Løken T. and Komorowski J. 2001). 'Traditional data mining methods based on rough set theory focus on extracting models which are good at classifying unseen objects. If one wants to uncover new knowledge from the data, the model must have a high descriptive quality— it must describe the data set in a clear and concise manner, without sacrificing classification performance. Rough modeling, introduced by Kowalczyk W. (1998), is an approach which aims at providing models with good predictive and descriptive qualities, in addition to being computationally simple enough to handle large data sets. As rough models are flexible in nature and simple to generate, it is possible to generate a large number of models and search through them for the best model. Initial experiments confirm that the drop in performance of rough models compared to models induced using traditional rough set methods is slight at worst, and the gain in descriptive quality is very large.'

An approach of finding reducts of composed information systems was presented in (Kryszkiewicz M. and Rybinski H. 1994).' A set-theoretical approach to finding reducts of composed information systems is presented. It is shown how the search space can be represented in form of a pair of boundaries. It is also shown, how reducts of composing information systems can be used to reduce the search space of the composed system. Presented solutions are implied directly from the properties of composed monotonic Boolean functions.'

'Rough set theory offers new insight into Bayes' theorem†. The look on Bayes' theorem offered by rough set theory is completely different to that used in the Bayesian data analysis philosophy. It does not refer either to prior or posterior probabilities, inherently associated with Bayesian reasoning, but it reveals some probabilistic structure of the data being analysed. It states that any data set (decision table) satisfies total probability theorem and Bayes' theorem … the rough set approach to Bayes' theorem shows close relationship between logic of implications and probability, first studied by Łukasiewicz'.‡ (Pawlak Z., *Rough sets*: Chapter 3: p. 29).

The proposed in (Łukasiewicz J. 1913) use of logic as a mathematical foundation of probability considers the replacement of probability by truth values of indefinite propositions, i.e. *propositional functions* (Pawlak Z., *Rough sets*: Chapter 4: p. 37). And so, let U be a non-empty finite set and $\phi(x)$ be a *propositional function*. 'The meaning of $\phi(x)$ in U, denoted by $|\phi(x)|$, is the set of all elements of U, that satisfies $\phi(x)$ in U. The truth value of $\phi(x)$ is defined as card $|\phi(x)|$ / card U. For example, if $U =_{df} \{1, 2, 3, 4, 5, 6\}$ and $\phi(x)$ is the propositional function $x > 4$, then the truth value of $\phi(x) = 2/6 = 1/3$. If the truth value of $\phi(x)$ is 1, then the propositional function is true, and if it is 0, then the function is false. Thus the truth value of any propositional function is a number between 0 and 1. Further, it is shown that the truth values can be treated as probability and that all laws of probability can be obtained by means of logical calculus'. In accordance with the last work, Łukasiewicz's approach is represented differently, i.e. by using flow graphs (optimal flow analysis), but different from those proposed by Ford and Fulkerson§ (Ford L.R.,Jr. and Fulkerson D. R. 1962). 'However, flow graphs introduced here are different from those proposed by Ford and Fulkerson for optimal flow analysis, because they model rather, e.g., flow distribution in a plumbing network, than the optimal flow. The flow graphs considered in this paper are basically meant not to physical media (e.g water) flow analysis, but to information flow examination in decision algorithms. To this end branches of a flow graph are interpreted as decision rules'.

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* Industrial strength techniques for system and software verification (e.g. combinatorial testing, the classification tree method, static analysis, etc.).

† See Subsection 5.7: Thomas Bayes (1701 – 1761)

‡ (Łukasiewicz J. 1913): Jan Łukasiewicz (1878 – 1956) , see also (Adams E.W. 1975)

§ Lester Raudolph Ford Jr. (1927 – 2017), Delbert Ray Fulkerson (1924 – 1976)

Conflict analysis using rough set approach was illustrated in the last part (Pawlak Z., *Rough sets*: Chapter 5: p. 45). ‘Conflict analysis and resolution play an important role in business, governmental, political and lawsuits disputes, labor-management negotiations, military operations and others. To this end many mathematical formal models of conflict situations have been proposed and studied … Various mathematical tools, e.g., graph theory, topology, differential equations and others, have been used to that purpose. Needless to say that game theory can be also considered as a mathematical model of conflict situations. In fact there is no, as yet, “universal” theory of conflicts and mathematical models of conflict situations are strongly domain dependent. We are going to present in this paper still another approach to conflict analysis, based on some ideas of rough set theory’.There was presented a simple tutorial example of voting analysis in conflict situations.

Some other applications related to rough sets are cited as follows: ‘*Discovery of concurrent data models from experimental tables: A rough set approach*’(Suraj Z. 1996), ‘*A parallel algorithm for real-time decision making: a rough set approach*’(Skowron A. and Suraj Z. 1996), ‘*Time and clock information systems: concepts and roughly fuzzy Petri net models*’ (Peters J.F. 1997), ‘*Guarded Transitions in Rough Petri Nets*’ (Peters, J.F. et al. 1999),[*] ‘*Conflict logic with degrees, rough fuzzy hybridisation – a new trend in decison-making*’ (Nakamura A. 1999), ‘*The rough set theory and applications*’ (Wu C. et al. 2004): ‘This paper presents a comprehensive review of the available literature on applications of the rough set theory. Concepts of the rough set theory are discussed for approximation, dependence and reduction of attributes, decision tables and decision rules. The applications of rough sets are discussed in pattern recognition, information processing, business and finance, industry, environment engineering, medical diagnosis and medical data analysis, system fault diagnosis and monitoring and intelligent control systems. Development trends and future efforts are outlined. An extensive list of references is also provided to encourage interested readers to pursue further investigations’.

9. Fuzzy rough sets and other nonstandard approaches

Some nonstandard approaches such as fuzzy rough sets and other non-standard ones (e.g. intuitionistic fuzzy sets, complex fuzzy sets and complex intuitionistic fuzzy classes, fuzzy rough sets and intuitionistic fuzzy rough sets, interval type-2 fuzzy sets, near sets, nested sets, forcing sets, non-wellfounded and paraconsistent sets) are briefly presented below.

9.1. Intuitionistic fuzzy sets, complex fuzzy sets and complex intuitionistic fuzzy classes

The notion of an *intuitionistic fuzzy set* was introduced by Atanasov K.T. in 1983[†], e.g. see: (Atanasov K.T. 1986). Intuitionistic fuzzy sets and *interval - valued fuzzy sets*[‡] (abbreviated as IFS and IVFS, respectively) are two intuitively straightforward extensions of Zadeh’s fuzzy sets that were conceived independently to alleviate some of the drawbacks of the latter (Cornelis C. et al. 2004): ‘IFS theory basically defies the claim that from the fact that an element x belongs to a given degree (say μ) to a fuzzy set A, naturally follows that x should not belong to A to the extent $1 - \mu$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element of the universe both a degree of membership μ and one of non – membership ν such that $\mu + \nu \leq 1$, thus relaxing the enforced duality $\nu = 1 - \mu$ from fuzzy set theory. Obviously, when $\mu + \nu = 1$ for all elements of the universe, the traditional fuzzy set concept is recovered. IFSs owe their name to the fact that

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[*] ‘This paper considers the construction of Petri nets to simulate conditional computation in various forms of systems. Coloured Petri nets underly the definition of a family of Petri nets based on rough set theory. Two families of guards on transitions are introduced: Lukasiewicz guards and rough guards. Lukasiewicz guards provide a basis for transitions with a form of continuous enabling. The notion of level-of-enabling of transitions is introduced. Rough guards on transitions are derived from approximations of our knowledge of input values.’

[†] Atanasov K.T., *Intuitionistic fuzzy sets*. In: VII ITKR’s Session, Sofia deposed in Central Sci.-Technical Library of Bulg. Acad. of Sci., 1697/84 (1983): in Bulgarian. See also: (Atanasov K.T. and Stoeva S. 1983)

‡ [‡] Dubois D. and Prade H., *Interval-valued fuzzy sets, possibility theory and imprecise probability*.6pp. file:///C:/Users/user/Downloads/Dubois%20and%20Prade.pdf

this latter identity is weakened into an inequality, in the other words: a denial of the *Aristotelian law of excluded middle* occurs, one of the main ideas of intuitionism'[*].

'IVFS theory emerged from the observation that in a lot of cases, no objective procedure is available to select the crisp membership degrees of elements in a fuzzy set. It was suggested to alleviate that problem by allowing to specify only an interval $[\mu_1,\mu_2]$ to which the actual membership degree is assumed to belong … Both approaches, IFS and IVFS theory, have the virtue of complementing fuzzy sets, that are able to model vagueness, with an ability to model uncertainty as well' (Cornelis C. et al. 2004): A related approach, *second-order fuzzy set theory*, introduced in (Zadeh L.A. 1975), was also cited.

According to the last work: 'Both approaches, IFS and IVFS theory, have the virtue of complementing fuzzy sets, that are able to model vagueness, with an ability to model uncertainty as well'[†].

Another approach was proposed in (Takeuti G. and Titani S. 1984)[‡]. In accordance with fuzzy set theory (Zadeh L.A. 1965), 'the characteristic of fuzzy sets is that the range of truth value of the membership relation is the closed interval [0,1] of real numbers' (Takeuti G. and Titani S. 1984). The logical connectives (called here operators) $\Rightarrow, \sim$ used in Zadeh's fuzzy sets 'seem to be *Łukasiewicz's logic*, where $p \Rightarrow q = \min\{1, 1-p+q\}$, $\sim p = 1 - p$.' The last logic was extended in (Hay L. S. 1963) to a *predicate logic* and proved its *weak completeness theorem* … from a logical standpoint, each logic has its corresponding set theory in which each logical operation for set theory; namely, the relation $\subseteq$ and $=$ on sets are translation of the logical operations $\Rightarrow$ and $\Leftrightarrow$. For Łukasiewicz's logic, $p \wedge (p \Rightarrow q) \Rightarrow q$ is not valid[§]. Translating it to the set version, it follows that the axiom of exstensionality does not hold. Thus this very basic principle of set theory is not valid in the corresponding set theory'. It was shown that the *sheaf model*[**] associated with each *complete Heyting algebra* (a bounded lattice), defined in the closed interval [0,1] is a model of intuitionistic set theory ZF_I[††] (Grayson R.J. 1977). The last algebra, denoted in short by cHa [0,1], was used for referring to the I-valued logic as intuitionistic fuzzy one (Takeuti G. and Titani S. 1984). Here, the logical value of implication '$p \Rightarrow q$' $=_{df}$ if $p \leq q$ then 1 else q. Similarly, the logical value of negation '$\sim p$' = '$p \Rightarrow 0$' = if $p = 0$ then 1 else 0. The obtained I - valued logic was referred as 'intuitionistic fuzzy logic IF in a language with proposisional variables'. In accordance with the last work, it was shown the *consistency* and *strong completeness*: if a *sequent*[‡‡] $\Delta \vdash \Gamma$ is valid in every IF then $\Delta \vdash \Gamma$ is provable (Δ is a finite or infinite sequence of formulae and Γ is a sequence of at most one formula).

The *global intuitionistic analysis*[§§] and globalisation *of intuitionistic set theory* were presented in the next two works: (Takeuti G. and Titani S. 1986) and (Takeuti G. and Titani S. 1987), respectively. In particular, in the first work it was briefly presented the (*stronger*) i*ntuitionistic set theory* given by Grayson R.J. (1977), the *intuitionistic linear algebra* (Ruitenburg W.B.G. 1982) and the *global intuitionistic set theory* (Takeuti G. and Titani S. 1987).

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[*] 'The term i*ntuitionistic* is to be read in a broad sence, alluding loosely to the denial of the law of the excluded middle on element level (since $\mu + \nu < 1$ is possible)' (Cornelis C. et al. 2004): see also (Dubois D. et al. 2005).

[†] There were juxtaposed the *vagueness* and *uncertainty* as two important aspects of *imprecision*. 'Some authors … prefer to speak of *non-specificity* and reserve the term *uncertainty* for the global notion of *imprecision*'.

[‡] A narrow graded extension of intuitionistic logic proper has also been proposed by Takeuti G. and Titani S. (1984) … 'it bears no relationship to Atanassovs notion of IFS theory' (Cornelis C. et al. 2004).

[§] The rule of omitting an implication, i.e. modus ponens: see Subsection 1.2.

[**] '*Sheaf models* and *toposes* have by now become an important means for studying intuitionistic systems. They provide a unifying generalization of earlier semantic notions, such as Kripke models, topological (Beth) models, and realizability interpretations. Moreover, higher order languages with arbitrary functionand power-types can be interpreted naturally in these models.' (Van Der Hoeven G. and Moerdijk I. 1984). See also: (Fourman M.P. 1980). 'In mathematics, a *sheaf* is a tool for systematically tracking locally defined data attached to the open sets of a topological space. The data can be restricted to smaller open sets, and the data assigned to an open set is equivalent to all collections of compatible data assigned to collections of smaller open sets covering the original one.'(T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.).

[††] Denotes: intuitionistic Zermelo - Fraenkel set theory.

[‡‡] See: Subsection 1.8.

[§§] A global intuitionistic set theory with some constraints.

Grayson's *stronger intuitionistic set theory* ZF_I is a first-order intuitionistic theory with relation symbols '$\in$' and '=', and with nine *axioms of*: *equality*, *extensionality*, *pairing*, *union*, *power set*, *$\in$-iduction*, *separation*, *collection* and *infinity*. As an example, the axiom of extensionality corresponds to Zermelo's axiom A1 (see: Subsection 5.1) or equivalently to the first axiom in ZF's system*. The sets of natural numbers and rational numbers, i.e. $\mathbb{N}$ and $\mathbb{Q}$, are introduced as usual. Real numbers are introduced in ZF_I as pairs of subsets of $\mathbb{Q}$. In the next considerations, some basic notions concerning Ruitenburg's *intuitionistic linear algebra* are given (e.g. such as: apartness†, ring, apartness ring, apartness F-module, apartness vector space, etc.) and several properties are shown: left to the reader. Some considerations related to *complex fuzzy sets* are given below.

Complex fuzzy sets were introduced (Ramot D. et al. 2002). The objective of this work was 'an investigation of the innovative concept of complex fuzzy sets. The novelty of the complex fuzzy set lies in the range of values its membership function may attain. In contrast to a traditional fuzzy membership function, this range is not limited to [0, 1], but extended to the unit circle in the complex plane. Thus, the complex fuzzy set provides a mathematical framework for describing membership in a set in terms of a complex number. The inherent difficulty in acquiring intuition for the concept of complex-valued membership presents a significant obstacle to the realization of its full potential. Consequently, a major part of this work is dedicated to a discussion of the intuitive interpretation of complex-valued grades of membership. Examples of possible applications, which demonstrate the new concept, include a complex fuzzy representation of solar activity (via measurements of the *sunspot number*‡), and a signal processing application. A comprehensive study of the mathematical properties of the complex fuzzy set is presented. Basic set theoretic operations on complex fuzzy sets, such as complex fuzzy complement, union, and intersection, are discussed at length. Two novel operations, namely set rotation and set reflection, are introduced. Complex fuzzy relations are also considered'.

'To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: the theory of probability, the theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But mathematical tools may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties' (Jun Y.B. 2008). As it was observed in the last work, ' all of these theories have their own difficulties pointed out in (Molodtsov D.A. 1999)§'. The soft set concept was used in (Jun Y.B. 2008) as 'a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches'. And hence, the use of soft sets to the theory of BCK/BCI - algebras**. The notions of *soft BCK/BCI-algebras* and *soft subalgebras* were first introduced. Some basic properties related to the last soft algebraic systems were also derived.†† As an illustration, the above notions of BCK/BCI - algebras were defined as follows (provided there is no ambiguity and for convenience, instead of '$x, y, z \in X$' (Jun Y.B. 2008), equivalently there are used below: '$a,b,c \in A$').

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* $\forall_x \forall_y (\forall_z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$. In general, sets are commonly denoted with a capital letter. Without loss of generality, in the ZF axiomatic system sets are also denoted by smail letters and they are always on the right hand side of the relation '$\in$' ('*is an element of*'): eg. '$z \in x$' corresponds to '$z \in X$'. Instead of Z(X) and Z(Y), there are used two additional quantifiers (see A1 of Subsection 5.1).

† The *apartness relation* (usually denoted by $\neq$ or #) is a binary relation (defined on a given set X) satisfying the following three conditions: (1) # is antireflexive, (2) # is symmetric and the following implication (3) $x \# y \Rightarrow x \# z \vee y \# z$ (for any $x,y,z \in X$). The complement of this relation is equivalence. In particular, if this equivalence is equality, then # is said to be *tight*.

‡ *Daily total sunspot number*: DTSN $=_{df}$ (the number of spots) + 10 × (the number of groups counted over the entire solar disk).

§ Dbmitrii Anatolevich Molodtsov, born: 1949: 'A pair (S,A) will be called a *soft set* over X if S is a mapping from set A to the set of all subsets of X, i.e. $S : A \rightarrow 2^X$.' See also: (Molodtsov D.A. 2018): The universe U is here denoted by X.

** Introduced in: (Iséki K. and Tanaka S. 1978).

†† An *algebraic system* is defined as an ordered set $\mathscr{A} =_{df} (A ; \alpha_1, \dots, \alpha_n ; o_1, \dots, o_m)$, where A is an arbitrary set, $\alpha_i \in A$ ($i = \overline{1, n}$) are the *distinguished elements* of A (*constants* of the algebraic system) and $o_j : A^{n_j} \rightarrow A$ ($j = \overline{1, m}$) are the *system operations*. The system $\mathscr{A}$ is *finite* if A is finite. The following sequence $(0, 0, \dots, 0, n_1, n_2, \dots, n_m)$ is called *type* of $\mathscr{A}$ (0's are related to the number of constants and n_j correspond to the number of arguments of o_j, for any j). Two algebraic systems $\mathscr{A}$ and $\mathscr{B}$ are s*imilar* if they are of the same type (Kerntopf P. 1967). Algebraic systems will be presented in Part II of this book.

Definition 9.1 (*BCK/BCI – algebras*)

The algebraic system $\mathscr{A}$ $=_{df}$ (A ; 0 ; o) of type (0,2) is said to be a *BCI - algebra* if the following four conditions are satisfied (for any a,b,c ∈ A):

(1) (a o b) o (a o c) o (c o b) = 0,

(2) (a o (a o b)) o b = 0,

(3) a o a = 0 and

(4) (a o b = 0) ∧ (b o a = 0) ⇒ a = b.

The above BCI - algebra is a *BCK algebra* if in addition the following condition holds:

(5) 0 o x = 0.

A more formal treatment is omitted: left to the reader. Some considerations related to the 'complex intuitionistic fuzzy classes' are given below.

'A complex fuzzy class is characterized by a pure complex fuzzy grade of membership. Pure complex fuzzy classes are paramount in providing rich semantics for cases where the fuzzy data is periodic with a fuzzy period. Often, however, the available data is contaminated by noise, opposing expert opinions, ambiguity, and false information. This opens the door for using intuitionistic fuzzy sets theory: representing the false information via a degree of non-membership. Several researchers have identified the benefits of integrating the two concepts of complex fuzzy sets and intuitionistic fuzzy sets. Nevertheless, complex fuzzy sets allow for only one component of the degree of membership to be fuzzy. In this paper, we introduce the *concept of complex intuitionistic fuzzy classes*, which are characterized by *pure complex intuitionistic fuzzy grade of membership*. We define the basic terms and operations on complex intuitionistic fuzzy classes and provide a motivating example of relevant application' (Ali M. et al. 2016)[*]. Here, A *complex fuzzy class* Γ, defined on $\mathscr{U}$ (the *universum*), is characterised by a *pure complex fuzzy grade of membership* $\mu_\Gamma(V,z)$ $=_{df}$ $\mu_r(V) + j\cdot\mu_i(z)$[†] , where $\mu_r(V)$ and $\mu_i(z)$ are the *real* and *imaginery components* of the last grade of membership (obviously: $\mu_r(V), \mu_i(z) \in [0,1]$). And hence, a *pure complex fuzzy class* $\Gamma =_{df} \{(V, z, \mu_\Gamma(V,z)) / V \in 2^{\mathscr{U}}, z \in \mathscr{U}\}$. In accordance with the last work, some applications were also given.

'Intuitionistic fuzzy sets are useful for modeling uncertain data of realistic problems' (Ngan R.T. et al.). Here, there was expanded the utility of *complex intuitionistic fuzzy sets* using the space of *quaternion numbers*[‡]. 'The proposed representation can capture composite features and convey multi-dimensional fuzzy information via the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership'. There were analysed 'the order relations and logic operations of the complex intuitionistic fuzzy set theory' and also introduced 'new operations based on *quaternion numbers*'. There were also presented 'two *quaternion distance measures* in *algebraic* and *polar forms*[§]'. An analysis of the obtained properties was given. The obtained quaternion representations and measures were used in decision-making models. In particular, this

[*] There was used Atanassov's intuitionistic fuzzy set theory (Atanassov K.T. 1986). Moreover, instead of *sets*, there are used *classes* ('to be avoided some set theoretical paradoxes').

[†] In general, the algebraic form of a *complex number* $z \in \mathbb{C}$ is defined as: $z =_{df} a + i\cdot b$, where $a =_{df} Re(z)$ and $b =_{df} Im(z)$ ($a, b \in \mathbb{R}$, $i^2 = -1$). For convenience, instead of 'i' , here the *imaginary unit* is denoted by 'j'.Any complex number can be represented by *Argand's diagram* (Jean-Robert Argand 1768 – 1822): first described by Caspar Wessel (1745 – 1818) and also called *Gauss' plane* (Carl Friedrich Gauss (1777 – 1855). See also: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

[‡] The last numbers (an extension of complex numbers ℂ) were first introduced in (Hamilton W.R. 1844: William Rowan Hamilton 1805 – 1865). Traditionally, this set is denoted by H.

[§] The same terminology is used as in the case of complex numbers where the *rectangular* (or: *algebraic*) and the *polar forms* of $z \in \mathbb{C}$ are given as follows: $z = a + i\cdot b$ and $z = r\cdot(\cos\varphi + i\cdot\sin\varphi)$, respectively.

model was 'experimentally validated in medical diagnosis, which is an emerging application for tackling patient's symptoms and attributes of diseases'.

There are various forms of *quaternion representation*, e.g. such as: *algebraic sum representation*, *matrix representations* or e.g. *pairs of complex numbers*.

Definition 9.2 (*quaternion as an algebraic sum*)

A *quaternion*, denoted by 'q', is defined by the following algebraic sum: $q =_{df} a \cdot \mathbf{e} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$, where $a, b, c, d \in \mathbb{R}$ and $\mathbf{e}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are the *basic quaternion units*.

In accordance with the last definition, we have: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{e}$. The *quaternion sum and product* are realised in a similar way as for polynomials (having the above basic quaternion units as variables). Moreover, the quaternion product is not a *commutative binary operation*. The last product is defined in Figure 9.1 given below.

×	e	i	j	k
e	**e**	**i**	**j**	**k**
i	**i**	**–e**	**k**	**–j**
j	**j**	**–k**	**–e**	**i**
k	**k**	**j**	**–i**	**–e**

Figure 9.1 The quaternion product

According to the last figure, '**e**' is a *neutral element* (or *identity*) wrt to the *binary operation* '×' , i.e. $\mathbf{e} \times \mathbf{a} = \mathbf{a} \times \mathbf{e} = \mathbf{a}$ ($\mathbf{a} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$): the neutral element is often not included in quaternion's definition. The following expression is then obtained: $q =_{df} a + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$. And hence, in the next example it is assumed that $\mathbf{e} = 1$. Then: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ (as in $\mathbb{C}$ wrt 'i').

Example 9.1 (*quaternion addition and multiplication*)

Let $q_1 =_{df} 2 + 4\mathbf{i} + 3\mathbf{k}$ and $q_2 =_{df} 3 + 2\mathbf{j} + 4\mathbf{k}$. The following sum and product can be obtained.

$$\begin{aligned} q_1 + q_2 &= (2 + 4\mathbf{i} + 3\mathbf{k}) + (3 + 2\mathbf{j} + 4\mathbf{k}) \\ &= 5 + 4\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}. \end{aligned}$$

$$\begin{aligned} q_1 \times q_2 &= (2 + 4\mathbf{i} + 3\mathbf{k}) \times (3 + 2\mathbf{j} + 4\mathbf{k}) \\ &= 2 \times (3 + 2\mathbf{j} + 4\mathbf{k}) + 4\mathbf{i} \times (3 + 2\mathbf{j} + 4\mathbf{k}) + 3\mathbf{k} \times (3 + 2\mathbf{j} + 4\mathbf{k}) \\ &= 6 + 4\mathbf{j} + 8\mathbf{k} + 12\mathbf{i} + 8\mathbf{ij} + 16\mathbf{ik} + 9\mathbf{k} + 6\mathbf{kj} + 12\mathbf{kk} \\ &= 6 + 4\mathbf{j} + 8\mathbf{k} + 12\mathbf{i} + 8\mathbf{k} - 16\mathbf{j} + 9\mathbf{k} - 6\mathbf{i} - 12 \\ &= -6 + 6\mathbf{i} - 12\mathbf{j} + 25\mathbf{k}. \quad \square \quad \{\text{Df. '×'}\} \end{aligned}$$

As an example, an interesting application, related to the *quaternion space geometry*, was given in (Yefremov A. et al. 2007)*: left to the reader.

9.2. Fuzzy rough sets and intuitionistic fuzzy rough sets

* *Yang-Mills field from quaternion space geometry, and its Klein-Gordon representation* (Chen Ning Yang: born 1922, Robert Mills: 1927 – 1999, Oskar Benjamin Klein: 1894 – 1977, Walter Gordon: 1893 – 1939).

'Traditional rough set theory uses equivalence relations to compute lower and upper approximations of sets. The corresponding equivalence classes either coincide or are disjoint[*]. This behaviour is lost when moving on to a fuzzy T-equivalence relation[†]. However, none of the existing studies on fuzzy rough set theory tries to exploit the fact that an element can belong to some degree to several "soft similarity classes" at the same time. In this paper we show that taking this truly fuzzy characteristic into account may lead to new and interesting definitions of lower and upper approximations. We explore two of them in detail and we investigate under which conditions they differ from the commonly used definitions. Finally we show the possible practical relevance of the newly introduced approximations for query refinement' (De Cock M. et al. 2007).

According to the last work, it was observed that the introduced definitions of the above two approximations require use of some fuzzy t-equivalence relation and hence a selection of a corresponding t-norm. In applications the often used as a t-norm is the classical Łukasiewicz's such one because the notion of fuzzy t-equivalence relation is dual to that of a pseudo-metric. And so, as an appropriate was proposed the *Łukaszewicz's t-norm*. In fact, the Łukaszewicz's t-norm is considered as one of the tree most important in fuzzy logic systems, in common with *Gödel's* and p*roduct logic* systems (Hájek P. 1998)[‡].

'Triangular norms are a generalisation of the classical two-valued conjunction. They were originally introduced for definition of the probabilistic (statistical) metric spaces as a generalisation of the classical triangle inequality for ordinary metric spaces. The next investigations were related with axiomatic of these norms. In this paper, a new t-norm is proposed which is generalisation of the Łukaszewicz's t-norm[§]. Some selected properties of this generalised t-norm are first presented. Next , it is shown a possibility of generalisation of the notions of lower and upper approximations used in fuzzy rough sets and also of obtaining better such approximations' (Tabakow I.G. 2014). In accordance with this work, some selected properties of this generalised t-norm were first presented. Next, it was shown a possibility of generalisation of the notions of lower and upper approximations used in fuzzy rough sets and also of obtaining better such approximations.

In general, the *lower* and *upper approximations* of a given subset $X \subseteq U$ are computed using ρ and defined as follows: $\underline{A}(X) =_{df} \{x \in U / [x]_\rho \subseteq X\}$ and $\overline{A}(X) =_{df} \{x \in U / [x]_\rho \cap X \neq \varnothing\}$, respectively. An equivalent version of the above two approximations was originally proposed in (De Cock M. et al. 2007). And so, we have: $y \in \underline{A}(X) \Leftrightarrow_{df} \forall_{x \in U}(x \rho y \Rightarrow x \in X)$ and $y \in \overline{A}(X) \Leftrightarrow_{df} \exists_{x \in U}(x \rho y \wedge x \in X)$.

By definition, it follows that $y \in \underline{A}(X) \Leftrightarrow [y]_\rho \subseteq X$ and $y \in \overline{A}(X) \Leftrightarrow [y]_\rho \cap X \neq \varnothing$. And hence, the following two properties should be satisfied: $[y]_\rho \subseteq X \Leftrightarrow \forall_{x \in U}(x \rho y \Rightarrow x \in X)$ and $[y]_\rho \cap X \neq \varnothing \Leftrightarrow \exists_{x \in U}(x \rho y \wedge x \in X)$. A more formal treatment is given below.

Proposition 9.1

$[y]_\rho \subseteq X \Leftrightarrow \forall_{x \in U}(x \rho y \Rightarrow x \in X)$

Proof (*if-condition*)*:*

(1)	$[y]_\rho \subseteq X$	{a}
(2)	$\sim \forall_{x \in U}(x \rho y \Rightarrow x \in X)$	{aip}
(3)	$\exists_{x \in U}(x \rho y \wedge x \notin X)$	{ N∀*, NC, SR : 2}

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[*] See Subsection 8.1.

[†] The questions 'whether approximate equality should satisfy transitivity, and whether fuzzy T-equivalence relations resolve the Poincar'e paradox' were discussed in (De Cock M. and Kerre E. 2003): The 'approximate equality, or indistinguishability, can only be modelled by means of a T-equivalence relation when the following condition does not occur: 'two objects can also be approximately equal even if they are not exactly equal'.

[‡] See Subsection 2.2 / p.76.

[§] $x \hat{\otimes} y =_{df} \max\{0, x^\alpha + y^\alpha - 1\}^{1/\alpha}$ and $x \hat{\oplus} y =_{df} \min\{1, x^\alpha + y^\alpha\}^{1/\alpha}$ (See: Subsection 2.2 / p. 90 – 91.

(4)	$a \in U$	
(5)	$a\,\rho\,y$	$\{-\exists^*, -K : 3\}$
(6)	$a \notin X$	
(7)	$a \in [y]_\rho$	{df.' $[y]_\rho$': 5}
(8)	$a \in X$	{df.' $\subseteq$': 1,7}
	contr. □	{6,8}

Proof (only *if-condition*):

(1)	$\forall_{x \in U}(x\,\rho\,y \Rightarrow x \in X)$	{a}
(2)	$[y]_\rho \nsubseteq X$	{aip}
(3)	$\sim([y]_\rho \subseteq X)$	{df.' $\nsubseteq$': 2}
(4)	$\sim \forall_{x \in U}(x \in [y]_\rho \Rightarrow x \in X)$	{df.' $\subseteq$', SR : 3}
(5)	$\exists_{x \in U}(x \in [y]_\rho \wedge x \notin X)$	{ $N\forall^*$, NC, SR : 4}
(6)	$a \in U$	
(7)	$a \in [y]_\rho$	$\{-\exists^*, -K : 5\}$
(8)	$a \notin X$	
(9)	$a \in U \Rightarrow (a\,\rho\,y \Rightarrow a \in X)$	$\{-\forall^* : 1\}$
(10)	$a\,\rho\,y \Rightarrow a \in X$	$\{-C : 6,9\}$
(11)	$a\,\rho\,y$	{df.' $[y]_\rho$': 7}
(12)	$a \in X$	$\{-C : 10,11\}$
	contr. □	{8,12}

Proposition 9.2

$$[y]_\rho \cap X \neq \varnothing \Leftrightarrow \exists_{x \in U}(x\,\rho\,y \wedge x \in X)$$

Proof (*if-condition*):

(1)	$[y]_\rho \cap X \neq \varnothing$	{a}
(2)	$\sim \exists_{x \in U}(x\,\rho\,y \wedge x \in X)$	{aip}
(3)	$\forall_{x \in U}(x\,\rho' y \vee x \notin X)$	{ $N\exists^*$, NK, SR : 2}
(4)	$\exists_{x \in U}(x \in [y]_\rho \cap X)$	{1}
(5)	$a \in U$	
(6)	$a \in [y]_\rho$	$\{-\exists^*$, df.'$\cap$', $-K : 4\}$
(7)	$a \in X$	
(8)	$a\,\rho\,y$	{df.' $[y]_\rho$': 6}
(9)	$a \in U \Rightarrow a\,\rho' y \vee a \notin X$	$\{-\forall^* : 3\}$
(10)	$a\,\rho' y \vee a \notin X$	$\{-C : 5,9\}$
(11)	$a \notin X$	$\{-A : 8,10\}$
	contr. □	{7,11}

Proof (only *if-condition*):

(1)	$\exists_{x \in U}(x\,\rho\,y \wedge x \in X)$	{a}

(2)	$[y]_\rho \cap X = \varnothing$	{aip}
(3)	$\forall_{x \in U} (x \notin [y]_\rho \cap X)$	{2}
(4)	$a \in U$	
(5)	$a \rho y$	$\{-\exists^*, -K : 1\}$
(6)	$a \in X$	
(7)	$a \notin [y]_\rho \cap X$	$\{-\forall^* : 3\}$
(8)	$a \notin [y]_\rho \vee a \notin X$	{ df.'∩', NK, SR : 7}
(9)	$a \notin [y]_\rho$	{– A : 6,8}
(10)	$a \rho' y$	{df.' $[y]_\rho$': 9}
	contr. □	{5,10}

The considered in (De Cock M. et al. 2007) relation $\rho(x,y) =_{df} \max\{0, 1 - |x - y|\}$ remains a fuzzy t-equivalence for any $\alpha \geq 1$. In fact, the following proposition holds.

Proposition 9.3

Let $\alpha \geq 1$. Then $\rho(x,y) =_{df} \max\{0, 1 - |x - y|\}$ is a fuzzy t-equivalence with respect to the generalised Łukasiewicz's t-norm.

Proof:

Assume that $\alpha \geq 1$. It is sufficient to show that ρ is t-transitive, i.e. $\rho(x,z) \geq \rho(x,y) \hat{\otimes} \rho(y,z)$ (for any x, y, z $\in \mathbb{R}$), where $\hat{\otimes}$ is the generalised Łukasiewicz's t-norm. Equivalently, the following inequality should be shown: $\max\{0, 1 - |x - z|\} \geq \max\{0, \max\{0, 1 - |x - y|\}^\alpha + \max\{0, 1 - |y - z|\}^\alpha - 1\}^{1/\alpha}$. And hence: $\max\{0, 1 - |x - z|\}^\alpha \geq \max\{0, \max\{0, 1 - |x - y|\}^\alpha + \max\{0, 1 - |y - z|\}^\alpha - 1\}$. Since any absolute value $|x - z|, |x - y|$ and $|y - z|$ may be greater than, equal to, or less than 1, in general, $3^3 = 27$ cases should be considered (eventually reduced to $2^3 = 8$). However, the most important is the case when $|x - z| = 1$, $|x - y| < 1$ and $|y - z| < 1$. Hence, the following inequality should be shown: $(1 - |x - y|)^\alpha + (1 - |y - z|)^\alpha \leq 1$. This case is considered below.

Since $1 = |x - z| \leq |x - y| + |y - z|$, $|x - y|, |y - z| < 1$ and $1 - |x - z| \geq 1 - (|x - y| + |y - z|)$, the above inequality is always satisfied. In fact, for any $\alpha \geq 1$ we have: $(1 - |x - y|)^\alpha + (1 - |y - z|)^\alpha \leq (1 - |x - y| + 1 - |y - z|)^\alpha = (1 - (|x - y| + |y - z|) + 1)^\alpha \leq (1 - |x - z| + 1)^\alpha = (1 - 1 + 1)^\alpha = 1^\alpha = 1$ (since $\lfloor\alpha\rfloor \leq \alpha \leq \lceil\alpha\rceil$, where $\lfloor\alpha\rfloor$ and $\lceil\alpha\rceil$ are the corresponding floor and ceiling functions and $a^n + b^n \leq (a + b)^n$). □

Some other considerations are left to the reader. The intuitionistic fuzzy rough sets are briefly presented below.

'The properties of the *intuitionistic fuzzy rough sets* are very complicated and inadequate in the sense of the extension of intuitionistic properties. In order to overcome this unnaturalness, we introduce a new definition of intuitionistic fuzzy rough sets and investigate important properties about the image and inverse image of an intuitionistic rough sets under a mapping. All the results obtained from this new definition are different from the results in other papers, and will be proven useful in expanding the related theory' (Yun S.M. and Lee S.J. 2020). In accordance with the last work, some introductory considerations are cited below.

The idea of *fuzzy rough sets* was proposed in (Nanda S. and Majumdar S. 1992)[*] and (Coker D. 1998). The next research was oriented mainly in combining 'the concepts of fuzzy rough sets and intuitionistic fuzzy sets' (Samanta S.K. and Mondal T.K. 2001). 'Many attempts at combining fuzziness and roughness have been made. … However, the properties of the intuitionistic fuzzy rough sets are very complicated and inadequate in the sense of the extension of intuitionistic properties. This is because of the unnaturalness of the definition of fuzzy rough sets. For example, the double complement of a fuzzy rough set is different from itself. The property that the double complement of a set becomes the set itself is one of the essential properties of Boolean algebra. Hence this flaw is critical in expanding the related theory. In order to overcome this unnaturalness, we need a new approach to intuitionistic fuzzy rough sets'.

In accordance with (Yun S.M. and Lee S.J. 2020), it was introduced 'a new definition of intuitionistic fuzzy rough sets and investigated important properties about the image and inverse image of an intuitionistic rough sets under a mapping. This new approach enables us to manipulate fuzzy rough sets more simply and easily. All the results obtained from this new definition are different from the results in other papers, and will be proven useful in expanding the related theory'. Some results obtained in this work are given below.

The notion of a fuzzy rough set on X, introduced in (Nanda S. and Majumdar S. 1992), was considered as 'an object of the form $A = (A_L, A_U)$, where A_L and A_U are characterised by a pair of maps: $A_L : X_L \rightarrow L$ and $A_U : X_U \rightarrow L$ with $A_L(x) \leq A_U(x)$, for all $x \in X_L$, where $(L, \leq)$ is a *fuzzy lattice*[†]' (it was assumed $X_L \subseteq X_U$).

As it was observed in (Yun S.M. and Lee S.J. 2020), 'the double complement of a fuzzy rough set *A* is different from *A*, because X_L and X_U are different, i.e., $X_L \leq X_U$, The property that double complement of a set becomes the set itself is one of the essential properties of Boolean algebra. Hence this flaw is critical in expanding the related theory. Thus we are going to introduce the new definition of a fuzzy rough set by weakening the condition of the old definition. Then the properties we obtain in this paper are different from the results in the above paper'. Moreover, 'the properties of the *intuitionistic fuzzy rough sets* are very complicated and inadequate in the sense of the extension of intuitionistic properties. This is because of the unnaturalness of the definition of *fuzzy rough sets*. Hence this flaw is critical in expanding the related theory. In order to overcome this unnaturalness, we introduce a new definition of *intuitionistic fuzzy rough sets* and investigate important properties about the image and inverse image of an intuitionistic rough sets under a mapping. This new approach enables us to manipulate fuzzy rough sets more simply and easily. All the results obtained from this new definition are different from the results in other papers, and will be proven useful in expanding the related theory'. The notions of fuzzy rough set and intuitionistic fuzzy rough are illustrated below (Yun S.M. and Lee S.J. 2020).

Definition 9.3 (*fuzzy rough set*)

Let X be an underlying set and $(L, \leq)$ a fuzzy lattice. A *fuzzy rough set* in X is an object of the form $A =_{df} (A_L, A_U)$, where A_L and A_U are defined by a pair of maps $A_L : X \rightarrow L$ and $A_U : X \rightarrow L$ with $A_L(x) \leq A_U(x)$, for all $x \in X$. The *null* and the *whole fuzzy rough sets* in X are defined by $\underline{0} =_{df} (\underline{0}_L, \underline{0}_U)$ and $\underline{1} =_{df} (\underline{1}_L, \underline{1}_U)$, respectively.

Definition 9.4 (*fuzzy rough set complement*)

The *complement* $\bar{A} = ((\bar{A})_L, (\bar{A})_U)$ of a fuzzy rough set $A = (A_L, A_U)$ in X is defined by $(\bar{A})_L(x) =_{df} (A_U(x))'$ and $(\bar{A})_U(x) =_{df} (A_L(x))'$ for all $x \in X$.

Corollary 9.1 (*closeness of complement*)

$(\bar{A})_L(x) = (A_U(x))' \leq (A_L(x))' = (\bar{A})_U(x)$. □ {Df.3, Df.4}

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[*] The proposed here model corresponds to the *L – fuzzy set* model developed by Atanassov K.T. (1986): see: (Coker D. 1998).

[†] A *fuzzy lattice* is a pair (L,μ), where L is a crisp lattice and $(L \times L, \mu)$ is a fuzzy set with membership function $\mu : L \times L \rightarrow [0,1]$ such that $\mu(x,y) = 1$ iff $x \leq y$ (Kaburlasos V.G. et al. 2007). Algebraic structures will be considered in the next part of this book.

In accordance with the last corollary $\overline{A}$ is a fuzzy rough set. It was also shown the following double complement property: $\overline{\overline{A}} = A$. The notion of an *ituitionistic fuzzy rough set* (in short: *IF rough set*) was introduced as follows (Yun S.M. and Lee S.J. 2020).

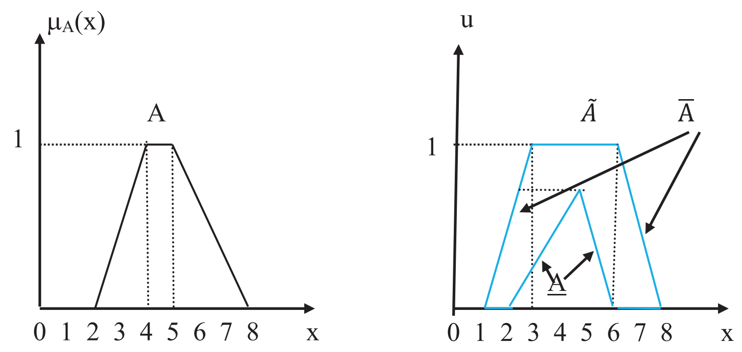
Definition 9.5 (*IF rough set*)

If $A = (A_L, A_U)$ and $B = (B_L, B_U)$ are two fuzzy rough sets in X with $B \subseteq \overline{A}$ then the ordered pair (A,B) is called an *intuitionistic fuzzy rough set* (briefly *IF rough set*) in X. The condition $B \subseteq \overline{A}$ is called the *intuitionistic condition* (briefly IC).

It is easily to show that $B \subseteq \overline{A} \Leftrightarrow A \subseteq \overline{B}$ (in accordance with *Remark 3.2* given in the last work). A more formal treatment is left to the reader*.

9.3. Interval type-2 fuzzy sets

Fuzzy sets were introduced introduced independently in (Zadeh L.A. 1965) and (Klaua D. 1965}, 'as extension of the classical notion of set'. The last theory, known also as '*type-1* (in short: T1) *fuzzy set theory*' was successfully applied in many areas (e.g. see Subsection 7.5). An illustration of the basic ideas of *interval type-2* (in short: IT2) *fuzzy sets* was given in (Wu D. 2010)†. According to the last tutorial, an 'obstacle in learning IT2 fuzzy logic is its complex notions': some considerations introduced in this work are cited below. In particular, examples of a T1 FS and IT2 FS are illustrated in Figure 9.2 below‡ (Wu D. 2010).



(a) T1 FS (b) IT2 FS

Figure 9.2

Let first consider the T1 FS A, shown in Figure 9.2(a). This set is represented as follows: {0/2, 0.5/3, 1/4, 1/5, 0.67/6, 0.33/7, 0/8}. Here, the used abbreviation "a/b" denotes that $\mu_A(b) =_{df} a$, e.g. 0.33/7 denotes that $\mu_A(7) =_{df} 0.33$. In a similar way: $\mu_A(2) = 0$, $\mu_A(4) = 1$, etc. On the other hand, for a crisp set, the membership degree of any element in it $\in \{0,1\}$.

The membership function $\mu_A(x)$ of a T1 FS 'can either be chosen based on the users opinion ((hence, the membership functions from two individuals could be quite different depending upon their experiences, perspectives, cultures, etc.) or it can be designed using optimisation procedure'. On the other hand, there were observed 'limitations

* This interesting work includes: 2 Corollaries, 10 Definitions, 4 Examples, 6 Remarks and 13 Theorems.

† See also: (Wu D. and Tan W.W. 2006) and (Mendel J.M. 2007).

‡ According to the above Figure 9.2, instead of 'X' , the considered fuzzy set is denoted by 'A'. 'X' is used as a *space point set* (see Definition 5.45 of Subsection 7.1.).

concerning the ability of T1 FS, to model and minimise the effect of uncertainties (certain in the sense that its membership grades are crisp values)'. *Type-2 FSs* introduced by Zadeh L.A. (1975) were characterised by fuzzy membership functions. *IT2 FSs* (Mendel J.M. 2001) as 'a special case of Type-2 FSs, are currently the most widely used for their reduced computational cost'.

Consider now the example IT2 FS $\tilde{A}$ of the above Figure 9.2(b). Instead of *numbers* (as in the previous T1 FS), the membership of an IT2 FS is an *interval*, e.g. instead of 0.5/3 , 1/5, 0.67/6 and 0.33/7 we can obtain: [0.25, 1]/3, [0.75, 1]/5, [0,1]/6 and [0, 0.5]/7, respectively. It can be observed that $\tilde{A}$ is bounded from above and below by the following two T1 FSs: $\bar{A}$ and $\underline{A}$, called *upper* and *lower membership functions* (in short: UMF and LMF, respectively). The area between $\bar{A}$ and $\underline{A}$ is said to be the *'footprint of uncertainty'* (FOU: here is marked in blue).

IT2 FSs 'are practically useful, when it is difficult to determine the exact MF, or in modelling the diverse options from different individuals. The MFs can be constructed from surveys or using optimisation algorithms'. A block diagram of an *IT2 fuzzy logic system* (IT2 FLS) is shown in Figure 9.3 below. This diagram 'is similar to its T1 counterpart, the major difference being that at least one of the FSs in the rule base is an IT2 FS. Hence, the outputs of the inference engine are IT2 FSs, and a type-reducer is needed to convert them into a T1 FS before defuzzification can be carried out' (Wu D. 2010).

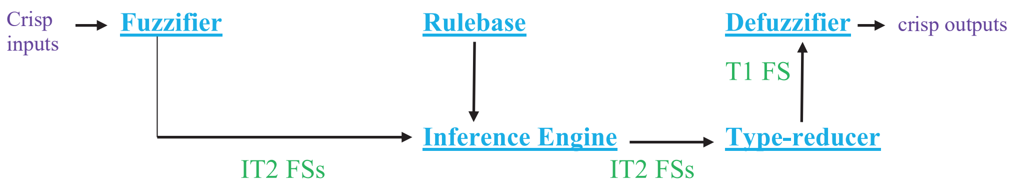


Figure 9.3 IT2 fuzzy logic system

As it was shown in the last work, in practice the computations in an IT2 FLS can be significantly simplified. The presented here rule-base for IT2 FLS is consisting of the following N *rules*:

$$R^n \ : \ \text{if} \ \ a_1 \ \text{is} \ \tilde{A}_1^n \ \text{and} \ \cdots \ \text{and} \ \ a_I \ \text{is} \ \tilde{A}_I^n \ \text{then} \ b \ \text{is} \ B^n \quad (n = 1, 2, \ldots, N),$$

where $\tilde{A}_i^n$ $(i = 1, 2, \ldots , I)$ are IT2 FSs and $B^n =_{df} [\underline{b}^n , \bar{b}^n]$ 'is an interval, which can be understood as the centroid of a consequent IT2 FS, or the simplest TSK model*, for its simplicity'. In accordance with the last work it is used: $\underline{b}^n = \bar{b}^n$ (each rule consequent is a crisp number).

In accordance with (Wu D. 2010) , typical computations in an IT2 FLS involve four (generalised) steps using as a sub procedure two times the *Karnik - Mendel* (*KM*) *algorithms* (Mendel J.M. 2001): computing y_l (switch point from the upper firing level to the lower firing level) and y_r (switch point from the lover firing level to the upper firing level). And finaly, the obtained defuzzified output y is defined as an arithmetic mean, i.e. $y =_{df} (y_l + y_r) / 2$. A more formal treatment is omitted: left to the reader.

9.4. Near sets and nested sets

The approximation of sets of objects that are 'qualitatively but not necessarily *spatially near*† each other' was considered in (Peters J.F. 2007): 'The term *qualitatively near* is used here to mean closeness of descriptions or distinctive characteristics of objects. The solution to this problem is inspired by the work of Zdzisaw Pawlak‡

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* See: (Takagi T. and Sugeno M. 1985) and (Sugeno M. and Kang G. 1988).

† 'By contrast, the proposed approach to assessing the nearness {closeness) of objects is not defined directly using a *distance metric*' (Peters J.F. 2007).

‡ Zdzislaw Pawlak (1926 – 2006)

during the early 1980s on the classification of objects by means of their attributes. This article introduces a special theory of the nearness of objects that are either static (do not change) or dynamic (change over time). The basic approach is to consider a link relation, which is defined relative to measurements associated with features shared by objects independent of their spatial relations. One of the outcomes of this work is the introduction of new forms of approximations of objects and sets of objects. The nearness of objects can be approximated using rough set methods. The proposed approach to approximation of objects[*] is a straightforward extension of the rough set approach[†] to approximating objects, where approximation can be considered in the context of information granules (neighborhoods). In addition, the usual rough set approach to concept approximation has been enriched by an increase in the number of granules (neighborhoods) associated with the classification of a concept as near to its approximation. A byproduct of the proposed approximation method is what we call a near set. It should also be observed that what is presented in this paper is considered a special (not a general) theory about nearness of objects. The contribution of this article is an approach to nearness as a vague concept which can be approximated from the state of objects and domain knowledge'. In accordance with the last work, 'a view of approximation spaces' is introduced 'in a slightly modified manner in comparison with the original definition' given in (Pawlak Z. 1982, 1991). Some basic notions related to near set theory, introduced by Peters J.F.(2007) are illustrated below.

Let $\rho \subseteq X \times X$ be a binary relation and $Y \subseteq X$. By $[Y]_\rho$ and $_\rho[Y]$ we shall denote the sets $\{x \in X \,/\, \exists_{y \in Y} (y\,\rho\,x)\}$ and $\{x \in X \,/\, \exists_{y \in Y} (x\,\rho\,y)\}$, respectively. The notion of an approximation space was introduced as follows: $AS =_{df} (U, \mathfrak{F}, \nu)$, where $\mathfrak{F}$ is 'a *covering* of a finite universe of objects, i.e:

$$\bigcup \mathfrak{F} = U \quad \text{and} \quad \nu : \mathbb{P}(U) \times \mathbb{P}(U) \to [0,1].^{‡}$$

It is assumed in the sequel that ν is a *standard rough inclusion function*[§]. Let $X \subseteq U$. The $\mathfrak{F}$ - *lower* and $\mathfrak{F}$ – *upper approximations* of X are defined as follows.

$$\mathfrak{F}_* =_{df} \bigcup \{ Y \in \mathfrak{F} \,/\, \nu(X,Y) = 1 \} \quad \text{and}$$
$$\mathfrak{F}^* =_{df} \bigcup \{ Y \in \mathfrak{F} \,/\, \nu(X,Y) > 0 \}.$$

The following binary link relation $link_{\mathfrak{F}} \subseteq U \times U$ was also introduced.

$$x\ link_{\mathfrak{F}}\ y \Leftrightarrow_{df} \exists_X \ ((X \in \mathfrak{F}) \wedge (x,y \in X)).$$

The *neighbourhood* of $x \in U$ was defined as: $[x]link_{\mathfrak{F}}y =_{df} \{y \in U \,/\, x\ link_{\mathfrak{F}}\ y\}$. According to (Peters J.F. 2007), each approximation space $AS =_{df} (U, \mathfrak{F}, \nu)$ defines an approximation space $AS_{link_{\mathfrak{F}}} =_{df} (U, I, \nu)$, where the *neighborhood function* $I : U \to \mathbb{P}(U)$ is defined by $I(x) =_{df} [x]_{link_{\mathfrak{F}}}$ for $x \in U$. The above $link_{\mathfrak{F}}$ relationon on subsets of U was also extended to a *nearness relation* as follows.

$$X\,\delta_{\mathfrak{F}}Y \Leftrightarrow_{df} \exists_{x \in X} \exists_{y \in Y} (x\ link_{\mathfrak{F}}\ y).$$

In accordance with (Peters J.F. 2007), there exists a possibility of introducing 'many other possible nearness relations defined by $link_{\mathfrak{F}}$, e.g. $X\,\delta_{\mathfrak{F}}^*\,Y$ iff $\{ (x,y) \,/\, x\ link_{\mathfrak{F}}\ y\}$ is *sufficiently large*. Some differences concerning

[*] It was suggested by Pawlak Z. (1982, 1991) that objects can be classified by their attributes. In accordance with (Peters J.F. 2007), 'the term *object* denotes something that has parts. A *feature* of an object is a *form* (configuration, design) or distinguishing characteristic in the make-up of the object … The focus in this article is on what it means for one object to be close (qualitatively near) to another object'. In this approach, the assertion 'x and y are very *close in F - relevant respects* means that the objects x, y are *near* (*close*) wrt the relevant features for the approximation of a concept represented by the vague predicate F' (i.e. the predicate F that can be applied to this pair of objects).

[†] See Subsection 8.1.

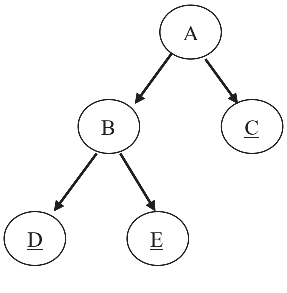
[‡] $\mathbb{P}(U)$ denotes the *powerset* of U (see Definition 5.21 of Subsection 5.4).

[§] *Rough inclusion functions* are mappings considered in rough set theory with which one can measure the degree of inclusion of a set (information granule) in a set (information granule) in line with rough mereology (Gomolińska A. and Wolski M. 2014).

the notions *nearness* and *roughness* were also presented ('distinction between near sets and rough sets'), considering the notion of an information system.

Let S $=_{df}$ (U, A) be an information system* and $N_r(A) =_{df} \bigcup_{B \in P_r(A)} U / IND(B)$ (for: $1 \leq r \leq |A|$), where $P_r(A) =_{df} \{B \subseteq A \ / \ |B| = r\}$. Here, by IND (B) it is denoted the *indiscernibility relation* defined. As it was shown in (Peters J.F. 2007), 'a set X relative to the family of neighborhoods $N_r(B)$ is a *B - near set* iff $|BND_{N_r(B)}(X)| \geq 0$. Then X is *B - near* wrt its *lower approximation* $(N_r(B))_*(X)$ or *B - near* wrt its *upper approximation* $(N_r(B))^*(X)$, if $BND_{N_r(B)}(X) = \varnothing$.' In accordance with the last work, 'nearness diminishes as the size of $BND_{N_r(B)}(X)$ increases' and X is a rough one iff $|BND_{N_r(B)}(X)| > 0$. The roughness of X increases as $|BND_{N_r(B)}(X)|$ increases. A more formal treatment is omitted: left to the reader.

Some considerations concerning *nested sets* (known also as *trees* or *hierarchies*) are given below. Informally, any such set can be considered as a chain of subsets. 'In the *nested sets model* each data item is stored as a row in the database table in the normal manner. However, additional columns are present in the table to express the hierarchical relationship between the data items. Each data item is referred to as a *node* and the collection of nodes can be thought of as forming a *tree*. Nodes can have zero, one or many *child nodes*. A node that has no children is referred to as a *leaf node*. A child node can itself have children and this nesting can carry on to arbitrary depth. The table is assumed to hold a single tree with a single node being allocated as the *root node*'.†



(a)

Tail	Head
A	B
B	D
B	E
A	C

(b)

Figure 9.4

Example 9.2 (*nested set*)

The *nested structure* associated with the *binary tree* of the above Figure 9.4(a) having as a *root node* A and as *leaf nodes*: C, D and E. The *mean length* of this tree (2 + 2 + 1) / 3 ≈ 1.66(6). The corresponding *adjacency relation* is shown in Figure 9.4(b), where: A $=_{df}$ {a, b, c, d, e}, B $=_{df}$ {b,d,e), C $=_{df}$ {c}, D $=_{df}$ {d} and E $=_{df}$ {e}, It can be observed that, e.g. $D \subseteq B$ and $B \subseteq A$ (i.e. $D \subseteq B \subseteq A$: like in *Russian dolls*‡). Another way of representing nested sets structure can be obtained by using 'set elements as *boxes* and sets as the *ovals* including them': left to the reader. □

In general, in applications, nested sets are used as a model for storing hierarchical information in relational databases. On the other hand, the combinatorics of nested sets and set partitions play an important role in discrete mathematical structures.

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* See Definition 8.1 of Subsection 8.3: The information system is here denoted by S.

† [Using nested sets – Joomla! Documentation]. See also: Nested sets and relational division, https://vadimtropashko.wordpress.com/why-relational-division-is-so-uncommon/nested-sets-and-relational-division/.

‡ Known also e.g. as: *babushka* or *matryoshka dolls*.

‘Let $D_{n+1,k-1}$ be the number of dissections of a convex polygon with $n + 1$ labelled edges by $k - 1$ diagonals, such that no two of the diagonals intersect in their interior. The formula $D_{n+1,k-1}$ dates back to Kirkman T.P.[*] (1857) and Cayley A.[†] (1890): who gave the first complete proof’ (Gaiffi G. 2015). The above formula is presented as follows.

$$D_{n+1,k-1} = \frac{1}{k}\binom{n-2}{k-1}\binom{n+k-1}{k-1}$$

In fact, there exist various proofs of the last formula. In accordance with the last work, the obtained proof ‘consists in the description of two explicit bijections and is based on the combinatorics of *nested sets* and *set partitions*.

According to (Gaiffi G. 2015), the notion of nested set ‘appeared in geometry in connection with models of configuration spaces, first in (Fulton W. and MacPherson R. 1994), with more generality in De Concini and Procesi’s papers’, e.g. (De Concini C. and Procesi C. 1995 /459 – 535)[‡].

Let now $\mathbb{P}_2(\{1, 2, \dots , n\}) \subseteq \mathbb{P}(\{1, 2, \dots , n\})$ be a subset with elements of cardinality greater than or equal to 2.[§] ‘The following definition of nested set of $\mathbb{P}_2(\{1, 2, \dots , n\})$ is a special case of the more general definition given in (De Concini C. and Procesi C. 1995 /459 – 535)’.

Definition 9.6 (*nested set*)

Let $n \geq 2$. A subset S of $\mathbb{P}_2(\{1, 2, \dots , n\})$ is a *nested set* iff it contains $\{1, 2, \dots , n\}$ and for any $I, J \in S$ we have that either $I \subset J$ or $J \subset I$ or $I \cap J = \varnothing$. By $\mathfrak{S}_2(n,k)$ it is denoted the set of the nested sets S of $\mathbb{P}_2(\{1, 2, \dots , n\})$ such that $|S| = k$.

It was also observed a bijective correspondence between $\mathfrak{S}_2(n,k)$ and the set $\mathcal{T}_2(n + k - 1, k)$ of partitions of the set $\{1, 2, \dots , n + k - 1\}$ into k blocks of cardinality greater than or equal to 2: ‘can be obtained as a particular case of the bijection between *rooted trees* on n leaves and partitions, proven in (Erdős P.L. and Székely L. 1989)’. Moreover, in accordance with (Gaiffi G. 2015) and for fixed n, ‘the cardinalities $|\mathfrak{S}_2(n,k)| = |\mathcal{T}_2(n + k - 1, k)|$ are the *Ward*[**] *numbers* (Ward M. 1934): see the sequence A134991 of OEIS[††]) and can be read along the diagonals in the table of the *2-associated Stirling numbers of the second kind* (Comtet L. 1974). They can as well be interpreted as the face numbers in the tropical Grassmannian[‡‡] $G(2, n + 1)$, i.e. the space of phylogenetic trees T_{n+1} (Billera L.J. et al. 2001; Speyer D. and Sturmfels B. 2004; Feichtner E. 2006).

9.5. Paraconsistent sets

‘It is known from *Russell’s paradox*[§§] that the first-order axiomatization of the *naive set theory* (defined in natural language, without using any axiomatics) of Cantor and Frege is inconsistent in classical logic[*] … Naïve set

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[*] Thomas Penyngton Kirkman (1806 – 1895).

[†] Arthur Cayley (1821 – 1895).

[‡] ‘Some generalisations of De Concini and Procesi’s definition successively appeared in various combinatorial contexts’ (Gaiffi G. 2015).

[§] See Definition 5.21(the notion of a *power set*) of Subsection 5.2.

[**] Henry Morgan Ward (1901 – 1963).

[††] On – line Encyclopedia of Integer Sequences (Neil James Alexander Sloane, born 1939).

[‡‡] Hermann Günther Grassmann (1809 – 1877).

[§§] Consider the set $V =_{df} \{ X \ / \ X \notin X\}$. Is V an element of its own? Let $V \in V$. We have a contradiction with the definition of the set V. Assume now that $V \notin V$. According to the same definition we have: $V \in V$. And hence: $V \notin V \Leftrightarrow V \in V$. This paradox (previously observed by Zermelo in 1899: Ernst Friedrich Ferdinand Zermelo 1871 – 1953) was next independently studied in (Russell B. 1903).

theory is thus one of the simplest examples of an intuitively correct theory leading so readily and desperately to such a contradiction … Although it fortunately appeared that these contradictory sets are by no means essential ocfor the foundations of mathematics, certain logicians have expressed a desire that such inconsistent objects be handled and studied within suitable theories, namely para(in)consistent ones … To end this introduction, let us indicate how such a membership ambiguity can easily be concocted within a classical context' (Libert T. 2005). Some considerations given in the last work are cited below.

Let $\mathscr{U}$ be an *universum* consisting of a 'collection U of objects together with a topology on it which might materialise some notion of indiscernibility on U.' The following two definitions were introduced (for any $x \in U$ and $S \subseteq U$): $x \in_U S \Leftrightarrow_{df} x \in \bar{S}$ and $x \notin_U S \Leftrightarrow_{df} x \in \overline{U - S}$, where $\overline{(\cdot)}$ is the *closure operator* on U. Obviously: $x \in S \Rightarrow x \in_U S$ and $x \in_U U - S \Leftrightarrow x \notin_U S$ (for some $x \in U$ and $S \subseteq U$. In consequence, the following conjunction becomes possible: $x \in_U S$ and $x \notin_U S$ (for some $x \in U$ and $S \subseteq U$). 'Note that the non-contradictory subsets of U nothing but the *clopen subsets*[†] in $\mathscr{U}$.' Moreover, ''$\in_U$, $\notin_U$ are actually not independent … it is hopeless to define a model for paraconsistent set theory assuming that $\in_U , \notin_U \subseteq U \times \mathbb{P}(U)$, not of $U \times U$. … Nevertheless, we have a situation here in which a *paraconsistent set* can already be thought of as a covering pair of closed subsets of the universe'. Some structures for a paraconsistent set theory are given below (Libert T. 2005).

Let M be a set and $\mathbb{P}_p(M)$ be the set of ordered pairs of subsets covering M, defined as follows.

$$\mathbb{P}_p(M) =_{df} \{ (X,Y) / X \cup Y = M \}$$

By $\mathscr{M}$ it is denoted the *structure of a paraconsistent set theory*, formally defined below.

$$\mathscr{M} =_{df} (M , [\cdot]_{\mathscr{M}}) ,$$

where: $[\cdot]_{\mathscr{M}} : M \rightarrow \mathbb{P}_p(M)$ is an extension function, which applies to any $a \in M$ the ordered pair of its *positive* $[a]_{\mathscr{M}+}$ and its *negative* $[a]_{\mathscr{M}-}$ extension, i.e. $[a]_{\mathscr{M}} =_{df} ([a]_{\mathscr{M}+} , [a]_{\mathscr{M}-})$.

In accordance with (Libert T. 2005), 'this conception of a paraconsistent set leads naturally to the related following notion of (*strong*) *extensionality*'.

$$\forall_{a,b \in M} \; (([a]_{\mathscr{M}+} = [b]_{\mathscr{M}+}) \wedge ([a]_{\mathscr{M}-} = [b]_{\mathscr{M}-}) \Rightarrow a = b)$$

According to the last expression, the above structure $\mathscr{M}$ is strongly extensional if the extension function $[\cdot]_{\mathscr{M}}$ is injective and hence M 'can be identified with a subset of $\mathbb{P}_p(M)$, namely the range of $[\cdot]_{\mathscr{M}}$.

The following two definitions were introduced (for any $a,b \in M$): $a \in_{\mathscr{M}} b \Leftrightarrow_{df} a \in [b]_{\mathscr{M}+}$ and $a \notin_{\mathscr{M}} b \Leftrightarrow_{df} a \in [b]_{\mathscr{M}-}$. Hence, the obtained structure $\mathscr{M}$ was equivalently defined by the following two binary relations on M: $\mathscr{M} =_{df} (M; \in_{\mathscr{M}} , \notin_{\mathscr{M}})$, where $\in_{\mathscr{M}} \cup \notin_{\mathscr{M}} = M \times M$. A more formal treatment is omitted: left to the reader.

9.6. Forcing sets, non well-founded sets and stationary sets

It is first introduced below the notion of a *forcing set* (Sujana J. G. and Rajalaxmi T.M. 2020). Here is considered a *graph model* of an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has at least one neighbor in S. The minimum cardinality of a dominating set of G is its *domination number*, denoted by $\gamma(G)$ (Ferrero D. et al. 2011).

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[*] Bertrand Russell (1872 – 1970), Georg Ferdinand Ludwig Philipp Cantor (1845 – 1918), Friedrich Ludwig Gottlob Frege (1848 – 1925)

[†] A *clopen set* in a *topological space* is a set which is both *closed* and *open* (in short: *clopen*): See: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*.

‘In a graph G, suppose S is a subset of vertices which are all colored and the rest of the vertices are not colored. The dynamic coloring of the vertices is defined as, at each discrete time interval, a colored vertex forces exactly one vertex which is not colored to be colored. This process continues to make all the vertices colored. The subset S is called a *forcing set* of G. The *forcing number* ζ(G) of a graph G is the minimum cardinality of a set S with colored vertices which forces the set V(G)* to be colored after some time. If the subset S has an additional property that it induces a subgraph of G whose components are all edges, then S is called a *ζP_2-forcing set* of G. The minimum cardinality of a P_2-forcing set of G with a is called the *P_2-forcing number* of G and is denoted by $\zeta P_2(G)$. Analogous to the *P_2-forcing set*, we define set S as a *P_3-forcing set* if all components of S are vertex disjoint paths on 3 vertices. The minimum cardinality of a P_3-forcing set is called the *P_3-forcing number* of G and is denoted by $\zeta P_3(G)$. We compute the P_2-forcing number and P_3-forcing number of the *triangular grid network*’ (Sujana J. G. and Rajalaxmi T.M. 2020).

The following definition was introduced in (Ferrero D. et al. 2011).

Definition 9.7 (*closure of a subset of vertices*)

For a graph G and a set $T \subseteq V(G)$, the *closure* of T in G denoted by $C_G(T)$ is recursively defined as follows: Start with $C_G(T) = T$. As long as exactly one of the neighbors of some element of $C_G(T)$ is not in $C_G(T)$, add that neighbor to $C_G(T)$. If $C_G(T) = V(G)$ at some stage, then T is a *zero forcing set* of G. A forcing set of minimum cardinality is called the *forcing number* of G and is denoted by ζ(G).

The notion of a P_2 - forcing set of G is introduced as follows (Sujana J. G. and Rajalaxmi T.M. 2020).

Definition 9.8 (*P_2 - forcing set of G*)

Let G be a graph. For a set K of independent P_2 - paths in G†, define T to be the set of all end vertices of edges in K. Proceeding as in Definition 9.7, if at some stage $C_G(T) = V(G)$ then K is called a *P_2 - forcing set* of G. The minimum cardinality of a P_2 – forcing set of G is the *P_2 - forcing number* of G , denoted by $\zeta P_2(G)$.

In accordance with (Sujana J. G. and Rajalaxmi T.M. 2020), an additional condition on a forcing set that it is composed of paths of length 2 or 3 ensures more reliability. The next considerations of this work are related to the study of P_2 - and P_3 - forcing problems in *triangular grid networks*: left to the reader.

‘*Non-well-founded set theories* are variants of *axiomatic set theory* that allow sets to be elements of themselves and otherwise violate the rule of *well-foundedness*. In non-well-founded set theories, the *foundation axiom* of ZFC is replaced by axioms implying its negation. The study of non-well-founded sets was initiated by Dmitry Mirimanoff‡ in a series of papers between 1917 and 1920, in which he formulated the distinction between well-founded and non-well-founded sets§; he did not regard well-foundedness as an axiom. Although a number of axiomatic systems of non-well-founded sets were proposed afterwards, they did not find much in the way of applications until Peter Aczel’s** *hyperset theory*’†† (Aczel P. 1988). Some considerations related to the last work are given below‡‡.

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* V(G) denotes the set of all vertices of G.

† The *$P_k(G)$-path graph* corresponding to a graph G has for vertices the set of all paths of length k in G. Two vertices are joined by an edge if and only if the intersection of the corresponding paths forms a path of length $k - 1$ in G, and their union forms either a cycle or a path of length $k + 1$. *Path graphs*, introduced in (Broersma H.J. and Hoede C. 1989) are a generalisation of *line graphs* (the lst can be obtained for $k = 1$). Graphs will be presented in the next Part II of this book.

‡ Dmitry Semionovitch Mirimanoff (1861 – 1945)

§ Eg. see: (Mirimanoff D. 1917): There was ‘first stated the fundamental difference between *well-founded* and *non-well-founded* sets. He called sets with no infinite descending membership sequence *well-founded* and others *non-well-founded*’.

** Peter Henry George Aczel, born 1941.

†† See: *The Free Encyclopaedia, The Wikimedia Foundation, Inc.*

‡‡ See also: Grausi R., *Non well-founded set theory*. The Panthéon-Sorbonne University, 7pp: file:///C:/Users/user/Downloads/non%20well-founded%20sets.pdf.

In accordance with the last work, ‘a *non-well-founded set* is an extraordinary set in the sense of Mirimanoff. Such a set has an infinite descending membership sequence; i.e. an infinite sequence of sets, consisting of an element of the set, an element of that element, an element of that element of that element and so on ad infinitum. What is extraordinary about such a set is that it would seem that it could never get formed; for in order to form the set we would first have to form its elements, and to form those elements we would have to have previously formed their elements and so on leading to an infinite regress … If non-well-founded sets are not needed for the development of mathematics then it may well seem natural to leave them out of consideration when formulating an axiomatic basis for mathematics. Sometimes a stronger view is expressed. According to that view there is only one sensible coherent notion of set. That is the iterative conception in which sets are arranged in levels, with the elements of a set placed at lower levels than the set itself. For the iterative conception only well-founded sets exist and FA and the other axioms of ZFC are true when interpreted in the iterative universe of pure sets’ (Aczel P. 1988).

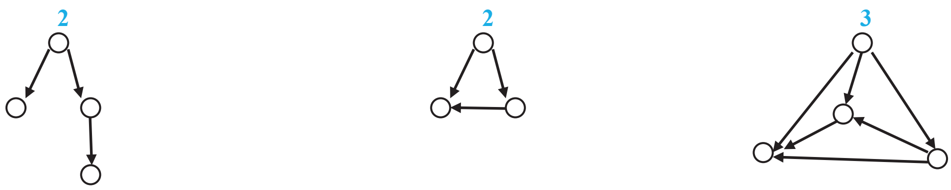
According to the above work, ‘there has been yet one more reason … why FA has been routinely included among the axioms of axiomatic set theory. This is the fact that the cumulative hierarchy of the iterative universe has an enticingly elegant mathematical structure. This structure was already revealed by Mirimanoff and over the years it has been powerfully exploited by set theorists in a great variety of model constructions. There is a natural reluctance to forgo the pleasure of working within this structure’.

The last study was stimulated by some earlier works (concerning non-well-founded sets, mathematical theory of concurrent processes and situation theory), such as: (Boffa M. e.g.1969), (Milner R. 1980) and (Barwise K.J. 1986), respectively[*]. In particular, there was proposed (in Part I of the last work) a *modified ZFC axiom system*, denoted by ‘ZFC^{-}+ AFA’, where ZFC^{-} is the system ZFC without FA (the *foundation axiom*): replaced by AFA (*Anti-Foundation Axiom*). This modified ZFC system is presented below.

As an introduction to AFA, it was considered a pictural representation of sets (different from Venn diagrams[†]), e.g. using ‘ the standard set theoretical representation of the natural numbers[‡], where the natural number n is represented as the set of natural numbers less than n’

Example 9.3 (*pictural representation of sets*)

As an example, the natural number 2 can be represented by the set {0,1}, 3 by the set {0,1,2}, etc. Any such representation is related to some oriented tree, e.g. the corresponding tree associated with 2 is shown in Figure 9.5(a) below. A more general, called ‘*pointed graph representation*’ is shown in Figure 9.5(b). In a similar way, the pointed graph representation associated with 3 is shown in Figure 9.5(c): see (Aczel P. 1988). □



(a) (b) (c)

Figure 9.5

Let $X \neq \varnothing$ be a set and $\rho \subseteq X \times X$. Here, in accordance with Figure 9.5, any *pointed graph*[§] $G = (X, \rho)$ is defined as follows: $(x,y) \in \rho$ iff x is a *parent* of y (or equivalently: y is a *child* of x).

[*] Robin Milner (1934 – 2010), Maurice Boffa (1939 - 2001), Kenneth Jon Barwise (1942 – 2000).

[†] See: e.g. Figure 5.1 of Subsection 5.2.

[‡] In fact: $\mathbb{N} \cup \{0\}$.

[§] Some basic elements of *graph theory* will be presented in the next part of this book.

Provided there is no ambiguity and for convenience, in the next considerations ordered pairs (x,y) will be denoted by (n_i, n_{i+1}): the last *edge* is also written by $n_i \rightarrow n_{i+1}$, where n_{i+1} is the *child* of n_i ($i \in \mathbb{N} \cup \{0\}$). Any path (of finite or infinite length) in G is considered as a sequence $n_0 \rightarrow n_1 \rightarrow n_2 \rightarrow \ldots$ of *nodes* $n_0, n_1, n_2, \ldots$ linked by *edges* $(n_0, n_1), (n_1, n_2), \ldots$ (Aczel P. 1988). According to this work, the following basic notions were also introduced:

• *pointed graph* (a graph together with a distinguished node called its *point*: in diagrams always located at the top),

• *accessible pointed graph* (*apg*: if for any node n there is a path from the point n_0 to the node n),

• *rooted tree* (an accessible pointed graph having always unique path from the point n_0 to the node n: the point n_0 is the *root* of the tree),

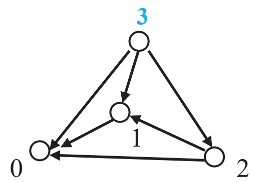
• *decoration of a graph* (an assignment of a set to each node of the graph in such a way that the elements of the set assigned to a node are the sets assigned to the children of that node) and

• *picture of a set* (an apg which has a decoration in which the set is assigned to the point}.

In accordance with the above approach, there is only one way to decorate the apgs.

Example 9.4 (*decoration of a graph*)

Consider the oriented graph of the above Figure 9.5(c). The obtained decoration is shown in Figure 9.6 below. □



node label	children (sets)
0	$\varnothing$
1	{0}
2	{0,1}
3	{0,1,2}

Figure 9.6

The following definition was introduced in (Aczel P. 1988).

Definition 9.9 (*well-founded graph*)

A graph G is said to be *well-founded* one if it has no infinite path.

As an example, the graph G of the above Figure 9.6 is a well-founded one. The following property is satisfied.

Lemma 9.1 (*Mostowski's*[*] *collapsing lemma*)

Every well-founded graph has an unique decoration. □

This lemma was proved 'by a simple application of definition by recursion on a *well-founded relation*[†] to obtain the unique function d defined so that $dn =_{df} \{dn' \,/\, n \twoheadrightarrow n'\}$ for each node n of G. The decoration d assigns the set dn to the node n'. As a consequence of the above Lemma 9.1: ''Every well-founded apg is a

[*] Andrzej Stanisław Mostowski (1913 – 1975).

[†] A binary relation $\prec$ is said to be *well-founded* if it has no infinite descending chains. An *infinite descending chain* is an infinite sequence of elements $a_0, a_1, a_2, \ldots$ such that $a_{i+1} \prec a_i$ for all $i \geq 0$. Note that a well-founded relation cannot be reflexive. As an example, the *less-than* '<' and the *greater-than* '>' *binary relations*, defined in $\mathbb{N} \cup \{0\}$, are well-founded and not well-founded ones, respectively. See: Myers A., Well-founded induction. CS 6110 Lecture 7 (2013) 6 pp: https://www.cs.cornell.edu/courses/cs6110/2013sp/lectures/lec07-sp13.pdf.

picture of an unique set' and 'Every set has a picture' (Aczel P. 1988). In accordance with the last work, the following *anti-foundation axiom* was introduced.

The Anti-Foundation Axiom

Every graph has an unique decoration.

Corollary 9.2

(1) Every apg is a picture of an unique set.
(2) Non- well-founded sets exist. □

Let G be a *labelled graph*, i.e. 'a graph together with an assignment of a set $a\downarrow$ of labels to each node a'. The *labelled decoration* of G is introduced as an assignment d of a set da to each node* a such that: $da =_{df} \{db / a \rightarrow b\} \cup a\downarrow$.

The Labelled Anti-Foundation Axiom

Every labelled graph has a unique labelled decoration.

The following notions are also introduced (Aczel P. 1988): a system ('a *class* M of *nodes* together with a class of *edges*, consisting of ordered pairs of nodes … write $a \rightarrow b$ if (a,b) is an *edge* of M'). Provided there is no ambiguity and for convenience, 'M' was also used to refer the system. The system M was also required to satisfy the following condition: the class $a_M =_{df} \{b \in M / a \rightarrow b\}$ of children of a is a set†. The following binary relation was also introduced: $a \equiv b$ iff 'there is an apg that is a picture of both a and b'.

Definition 9.10 (*bisimulation*)

A binary relation ρ on the system M is a *bisimulation* on M if $\rho \subseteq \rho^+$, where for $a, b \in M$: $a \rho^+ b \Leftrightarrow_{df} (\forall_{x \in a_M} \exists_{y \in b_M} x \rho y) \wedge (\forall_{y \in b_M} \exists_{x \in a_M} x \rho y)$.

Any system map $G \rightarrow M$ is said to be *M - decoration* of G and M is a *complete* system if every graph has an unique decoration. Some notions used in construction of a complete system are cited below (Aczel P. 1988). Every apg is of the form Ga where G is a graph and a is a node of G. 'The class of apgs form a system V_0 with edges (Ga, Gb) whenever G is a graph and a is a node of G'. The notion of a *strongly extensional quotient* of V_0 was introduced as follows: $\pi_C : V_0 \rightarrow V_C$ (Acz.el P. 1988): it was shown that for each system M there is an unique system map $M \rightarrow V_C$. According to Theorem 3.4 of the last work, the following are equivalent for a system M: For any system M′ there is an unique system map $M' \rightarrow M$, M is complete and $M \cong V_C$ (the symbol '≅' is used here to denote the *isomorphism relation* between apgs). In accordance with the last considerations, 'the system V_0 of apgs has an edge (Ga, Gb) whenewer $a \rightarrow b$ in the graph G'.

Definition 9.11 (*regular bisimulation*)

A bisimulation relation ~ on V_0 is a *regular bisimulation relation* if:

(1) ~ is an equivalence relation on V_0,

(2) $Ga \cong G'a' \Rightarrow Ga \sim G'a'$ and

(3) $a_G = a'_G \Rightarrow Ga \sim Ga'$ (for $a, a' \in G$)‡.

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* *Graph vertices* are traditionally denoted by: x, y, etc. or equivalently by: a, b, etc.

† For convenience, e.g. the sets: a_M, b_M, etc. are sometimes denoted also by a, b, etc.

‡ Provided there is no ambiguity and also for convenience wrt the above presented work, the *logical negation symbol* '~' was used to represent an *equivalence relation* on V_0.

In general, the ordinary anti-foundation axiom (AFA) may be considered as a special case ‘by treating ordinary graphs as labelled graphs with an empty set of labels attached to each node’. As an extension, the notion of system was also considered ‘as a class of nodes together with a class of edges consisting of ordered pairs of nodes’ (Aczel P. 1988). Some variants of AFA were presented in the next Part II of this work (in particular, variants using a regular bisimulation). Some applications of AFA (e.g. such as: ’*fixed points of set continuous operators*, the special final coalgebra theorem and an application to communicating systems) were illustrated in the last Part III of this work: left to the reader.

Stationary sets are important tools in proofs of properties in sets of uncountable cardinality. Some basic notions are cited below*.

Topology: is concerned with the properties of a geometric object that are preserved under continuous deformations.

Order topology: a certain topology that can be defined on any totally ordered set.

Limit ordinal: an ordinal number that is neither zero nor a successor ordinal, i.e. an ordinal $\lambda > 0$ is a limit ordinal iff it has no immediate predecessor.

Club† *set*: a subset of a limited ordinal that is closed under the order topology and is unbounded relative to the limit ordinal.

Stationary set: A set that is not too small in the sense that it intersects all club sets.

Let A be a partially ordered set (in short: *poset*)‡ and $B \subseteq A$. The subset B is said to be *cofinal* in A if the following condition is satisfied: $\underset{a \in A}{\forall} \ \underset{b \in B}{\exists} \ (a \leq b)$.

The *cofinality* cf(A) is the least of the cardinalities of the cofinal subsets of A.

A *regular cardinal* is a cardinal number that is equal to its own cofinality.

Let κ be a cardinal of uncountable cofinality, $S \subset \kappa$ and S intersects every club set in κ, then S is called a stationary set. The last set is a *bistationary* if both S and $\kappa - S$ are stationary.

Lemma 9.1 (*Fodor’s lemma*)§

If κ is regular, uncountable cardinal, $S \subset \kappa$ is stationary and $f : S \to \kappa$ is regressive (i.e. $f(\alpha) < \alpha$ for any $\alpha \in S - \{0\}$) then there is some γ and some stationary $S_0 \subseteq S$ such that $f(\alpha) = \gamma$ for any $\alpha \in S_0$. □

Theorem 9.1 (*Ulam S.M. and Solovay R.M.*)**

Every stationary subset of κ is the union of κ disjoint stationary subsets. □

10. Bunch Theory

‘Perhaps because set theory is so useful in mathematics, computer science has adopted it for some purposes. Sets appear explicitly in some programming languages (e.g. SETL, Pascal), and very commonly in reasoning about programs … A *programming language* is a set of programs, but we do not specify a language in set

* See: T*he Free Encyclopaedia, The Wikimedia Foundation, Inc*., see also: (Vermeulen J. 2020).

† Is an abbreviation from: ‘closed’ and ‘unbounded’.

‡ See: Definition 5.25 of Subsection 5.4.

§ Géza Fodor (1927 – 1977}

** Stanisław Marcin Ulam (1909 – 1984), Robert Martin Solovay (born 1938).

notation. Instead, we use a grammar in the form of rewriting rules … Set theory in its entirety is more powerful than necessary for most of computer science … We may, of course, use only those parts of set theory that are useful to us, and ignore the rest …But the notation of set theory, designed for power we don't want, is sufficiently cumbersome that we have tended instead to invent many special notations where a weaker set theory would have served well. These dissatisfactions lead to propose a set theory that has the right power for computer science, and that is notationaly more convenient for our purposes than mathematical set theory. The new theory called *bunch theory*, does not exclude set theory, when appropriate, we can form sets of bunches, or bunches of sets' (Hehner E.C.R. 1981)[*]. Some basic notions related to this theory are illustrated belw.

10.1. Basic notions and definitions

Let $\{1, 3, 6\}$ be a set containing three integers. The obtained *bunch* is written as follows: 1, 3, 6 (i.e. the set without curly braces). Obviously, in accordance with the last notation: 'an element and a bunch containing only that element are indistinguishable. More generally, a bunch built from other bunches does not retain the structure, as does a set whose elements are sets. This corresponds well to Pascal[†], in which sets of sets are illegal ... '. The following three bunch axioms were introduced (Hehner E.C.R. 1981).

(A1) There exist bunches that are considered '*elementary*'. Each such bunch is called an *element.*

(A2) A predicate p on the elements elements specifies the bunch consisting of those elements for which this predicate is true. This bunch is denoted as $x \$ p(x)$.

(A3) If a and b are bunches then a,b is the bunch consisting the elements of both a and b.

The above three axioms A1, A2 and A3 are called *existence, specification* and *union*, respectively. There are no other bunches. In accordance with the last work, the *null* (or *empty*) *bunch* and the *universal bunch* are introduced as follows: *null* $=_{df}$ $x \$ \text{false}$ and *universe* $=_{df}$ $x \$ \text{true}$, respectively. As it was observed, in set theory, 'postulating the universal set is inconsistent, leading to paradoxes. But the universal bunch causes no problems, and for this reason our specification axiom is simpler … By choosing a countable universe of elements, no bunches of uncountable cardinality can be constructed'.

10.2. Bunch operations

Similarly as in set theory, the following three relations were introduced (Hehner E.C.R. 1981): $\in$ (element of), $\subseteq$ (sub - bunch of) and $=$ (equal to). The binary relation $x \in a$ is defined separately, i.e. in accordance with the next three possible ways (in which a was constructed):

$$x \in a \quad \text{means} \quad \begin{cases} x = a, & \text{if } a \text{ is an element} \\ p(x), & \text{if } a \text{ is } y \$ p(y) \\ x \in b \vee x \in c, & \text{if } a \text{ is } b,c. \end{cases}$$

For any two bunches a and b: $a \subseteq b$ iff $\forall_x (x \in a \Rightarrow x \in b)$ and $a = b$ iff $(a \subseteq b) \wedge (b \subseteq a)$. In the case 'when the left operand is an element, the last two relations $\in$ and $\subseteq$ coincide. For equality, the order and multiplicity of elements is irrelevant'. The following example was given in (Hehner E.C.R. 1981).

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[*] See also: (Hehner E.C.R. 1984, 1993): Hehner Eric C.R., born 1947., the file:///C:/Users/user/Downloads/Eric%20Hehner%20Bunch%20Theory%20Oryginal.pdf

[†] Is a programming language.

Example 10.1 (*bunch name*, *bunch operations*)

The *name* of a bunch is first written, followed by a *colon*, followed by the *bunch*: e,g. a : 1, 3, 6 means that whenever a occurs, it stands for 1, 3, 6. In accordace with the last definition, writing e.g. b : 6 , a, 4 is equivalent to writing b : 6, 1, 3, 6, 4 which in turn is equivalent to writing b : 1, 3, 4, 6.

Let '○' be a *binary operation* on the elements of our *universe* and X, Y be two bunches. We have: $X \circ Y =_{df} z \,|\, S \exists_x \exists_y (x \in X \wedge y \in Y \wedge z = x \circ y)$. As an example: (1,2) + (10,20) = 11, 12, 21, 22. (1,2) + 10 = 11, 12, 1 + 1 = 2, etc. In accordance with the first two examples, it is assumed that ' + ' has higher precedence than ' , '. □

In accordance with the last work, the following two applications were also presented: within a *programming language* and as *language description*. In the first application, it was suggested 'a few of the possible uses of bunch within an ALGOL – like programming language. As an example, 'in addition to the notations of the specification and union axioms, it seems useful to introduce the notation i to j, for suitable expressions i and j to mean the bunch $k \,|\, S\ i \le k \le j$, that is, i, i + 1, … , j −1, j '. In the second application, the considered language was described as a bunch program of strings. *CFG*[*] *formalism* was also discussed (left to the reader).

[*] Context – Free Grammar, see also: the *notation technique for context-free grammars*: John Backus (1924 – 2007), Peter Naur (1928 – 2016).

Conclusions

Starting with the classical-oriented theories (concerning propositions, predicates and sets), there were also presented and some non-classical systems. And so, the considered here basic theories may be used as a part of lectures, predestined at first for computer science students, however it can be useful in other areas, e.g. such as system techniques and control, technical cybernetics, telecommunication, managing etc. The considered here systems may be also useful for any researcher who is interested in the above given area. The expected effects can be summarised as twofold. First, thorough knowledge of the sense of using natural deduction methods in computer science and second, a possibility of obtaining a knowledge for the purpose of efficient bibliographic search in this field of application and also with respect to future scientific investigations and/or practical applications.

References

Ackermann W.F., *Begründung einer strengen* i*mplication*. Journal of Symbolic Logic, vol. 21 (1956) 113 – 128.

Ackermann W.F., *Zur axiomatik der mengenlehre*. Mathematische Annalen, Springer, vol.131 (1956) 336-345.

Aczel P., *Non-well-founded sets*. Foreword by Jon Barwise, CSLI (Center for the Study of Language and Information), Stanford, Lecture Notes no. 14 (1988) 159pp.

Adams E.W., *The logic of conditionals, an application of probability to deductive logic*. D.Reidel Publishing Company, Dordrecht, Boston (1975) 155pp.

Aerts D., *Description of compound physical systems and logical interaction of physical systems*. Current Issues on Quantum Logic, Beltrametti E.G. and Fraassen B.C. van eds., Physical Sciences, Dordrecht, Kluwer Academic, vol. 8 (1981) 381 – 405.

Alchourrón C., Gärdenfors P. and Makinson D., *On the logic of theory change: partial meet contraction and revision functions*. Journal of Symbolic Logic, vol. 50, issue 2 (1985) 510–530.

Ali M., Tamir D.E., Rishe N.D. and Kandel A., *Complex Intuitionistic Fuzzy Classes*. IEEE International Conference on Fuzzy Systems (2016) 2027 – 2034.

Aliev I.M. and Kara Z., *Fuzzy system reliability analysis using time dependent fuzzy sets.* Control and Cybernetics, vol. 33, no. 4 (2004) 653 – 662.

Alodat M., *Intuitionistic fuzzy logic applied to real-time traffic: 373odelling and optimization* (abstract). PhD, University of Craiova (2015) 15pp.

Alur R. and Madhusudan P., *Adding nesting structure to words*. Journal of the ACM, vol. 56, issue 3 (2009) 1 – 45.

Anderson A.R., *A reduction of deontic logic to alethic modal logic*. Mind vol. 67 no. 265, Oxford Univ. Press (1958) 100 – 103.

Anderson A.R. and Belnap N.D., *Entailment: the logic of relevance and necessity*. Princeton Univ. Press, Princeton, N.J., vol. 1 (1975).

Anderson A.R., Belnap N.D. and Dunn J.M., *Entailment: the logic of relevance and necessity*. Princeton Univ. Press, Princeton, N.J., vol. 2 (1992).

André C., *The Behaviour of a Petri Net on a subset of transitions*. R.A.I.R.O. Systems Analysis and Control 17,France (1983) 5 – 21.

André D., *Sur les permutations alternées*. J. Math. Pures Appl., no. 7 (1881) 167 – 184.

Anitha K., *Gene selection based on rough set applications in computational biology*. International Journal of Computing Algorithm, vol. 1, issue 2 (2012) 38 – 41.

Arieli O. and Avron A., *Reasoning with logical bilattices*, Journal of Logic, Language and Information, vol. 5 (1996) 25 – 63.

Arruda A.I., *N.A. Vasiliev: a forerunner of paraconsistent logic*. Philosophia Naturalis, vol. 21 (1984) 472 – 491.

Asenjo F.G., *A calculus of antinomies*. Notre Dame Journal of Formal Logic, vol.7, no.1(1966) 103 – 105.

Asenjo F.G., *Antinomicity and the axiom of choice*. To the memory of Stanisław *Jaśkowski*, Logic and Logical Philosophy (1996) 53 – 95.

Asenjo F.G., *The general concept of antinomicity*. Foundations of Science, Dordrecht, Kluwer, vol.3 (1999) 429 – 465.

Atanassov K.T. and Stoeva S., *Intuitionistic fuzzy sets*. Proc. of Polish Symp. on Interval and Fuzzy Mathematics, Poznan (1983) 23 – 26.

Atanassov K.T., *Intuitionistic fuzzy sets*. Fuzzy Sets and Systems, vol. 20, no. 1 (1986) 87 – 96.

Atanassov K.T., *Two variants of intuitionistic fuzzy propositional calculus*. Preprint IMMFAIS-5-88, Sofia (1988).

Atanassov K.T. and Gargov G., *Elements of intuitionistic fuzzy logic: part I.*, Fuzzy Sets and Systems, vol. 95, no.1 (1998) 39 – 52.

Atmanspacher H., *Determinism is ontic, determinability is epistemic*, Univ. of Pittsburgh (2001) 49 – 54.

Aumann G., *Kontakt-Relationen*. Sitzungsberichte der Bayerischen Akademie der Wissenschaften, MathematischNaturwissenschaftliche Klasse (1970) 67 – 77.

Avron A., *A constructive analysis of RM*. Journal of Symbolic Logic, vol. 52, no.4 (1987) 939 – 951.

Avron A., *Hypersequents, logical consequence and intermediate logics for concurrency*. Annals of Mathematics and Artificial Intelligence, vol.4 (1991) 225 – 248.

Avron A., *The method of hypersequents in the proof theory of propositional non-classical logics*. Logic: Foundations to Applications, Hodges W., Hyland M., Steinhorn C. and Truss J. eds., Oxford Science Publications (1996) 36pp: file:///C:/Users/user/Documents/AVRON%20SURVEY%20PAPER%20IN%20OXFORD.pdf.

Baaz M. and Zach R., *Hypersequents and the proof theory of intuitionistic fuzzy logic*. In Computer Science Logic (CSL'2000), LNCS, vol.1862, (2000) 187 – 201.

Baaz M., Ciabattoni A. and Fermüller C.G., *Hypersequent c Journal for Gödel logics - a survey*. Journal of Logic Computation, vol. 13, no. 6 (2003) 835 – 861.

Babuska R. and Mamdani E.H., *Fuzzy control*. Scholarpedia, vol. 3, no.2 (2008) 14pp: http://www.scholarpedia.org/article/Fuzzy_control

Back R.J.R. and Wright J. von., *Refinement calculus: a systematic introduction*. In: Graduate Texts in Computer Science, Springer-Verlag, New York (1998) 519pp.

Badia A., *Quantifiers in action: generalised quantification in query*. Logical and Natural Languages, Springer - Verlag (2009) 74 – 76.

Ball T. and Rajamani S.K., *Boolean programs: a model and process for software analysis*, Tech. Rep. MSR-TR-2000-14, Microsoft Research, 2000, 29 pp.

Baltag A. and Smets S., LQP: *The dynamic logic of quantum information*. Mathematical Structures in Computer Science. Special issue on quantum programming languages, vol. 16, issue 3 (2006) 491 – 525.

Baltag A. and Smets S., *Complete axiomatizations of quantum actions*. International Journal of Theoretical Physics, vol.44, issue 12 (2005) 2267–2282.

Barlow R.E. and Proschan F., *Statistical theory of reliability and life testing: probability models*. New York: Holt, Rinehart and Winston, New York (1975) 290 pp.

Barnum H., Mueller M., Ududec C., Graydon M. and Wilce A., *Quantum and Jordan-algebraic theories. Single-system characterizations, and Jordan-algebraic composites* (2014) 39pp: http://www.physics.usyd.edu.au/~ericc/SQF2014/slides/Howard%20Barnum.pdf.

Baruah H. K., *Fuzzy membership with respect to a reference function*. Journal of the Assam Science Society, vol. 40, no. 3 (1999) 65-73.

Barwise K.J., *Admissible sets and structures. An approach to definability theory*. Springer - Verlag, Berlin, Heidelberg, New York (1975) 394pp.

Barwise K.J., *Situations, sets and the axiom of foundation*. Logic Colloquium'84, Studies in Logic, North Holland (1986) 21 – 36.

Barwise K.J. and Cooper R., *Generalized Quantifiers and Natural Language,* Linguistics and Philosophy 4, (1981) 159-219.

Barwise K.J. and Perry J., *Situations and attitudes*. The David Hume series. CSLI Publications,.Stanford,Calif.Originally publ. Cambridge,Mass. MIT Press (1983) 352pp.

Başkent C., *A quick introduction to dynamic logic*. The City Univ. of New York / Istanbul Kultur Univ (2010) 21pp.

Bazhanov V.A., *The imaginary geometry of N.I. Lobachevski and the imaginary logic of N.A. Vasiliev*. Modern Logic, vol. 4 (1994) 148 – 156.

Becker O., *Zur logic der modalitäten*. Jahrbuch für Philosophie und Phenomenologische Forschung, XI (1930) 497 – 518.

Bednář J., *Fuzzy distances*. Kybernetika, vol. 41, no.3 (2005) 375 – 388.

Bell J.S., On the Einstein-Podolsky-Rosen Paradox, Physics, vol. 1 (1964) 195 – 200.

Belnap N.D., *A useful four-valued logic*. Modern uses of multiple-valued logic, Dunn J.M. and Epstein G. eds., Reidel, Dordrecht (1977) 8 – 37.

Bělohlávek R., *Binary Fuzzy Relations*. In: Fuzzy Relational Systems. International Federation for Systems Research International Series on Systems Science and Engineering, Springer, Boston, MA, vol 20 (2002)

181 – 213.

Benthem J. van., *Questions about quantifiers*. Journal of Symbolic Logic, vol. 49 (1986) 443 – 466.

Berge C., *Graphes et hypergraphes*.Dunod,Paris (1973) 516pp.

Berghammer R. and Guttmann W., *A relation - algebraic approach to multirelations*. In: Mathematics of Program Construction. 12th International Conference MPC 2015, Königswinter, Germany, June 29 – July 1, Proceedings, Springer International Publishing, Switzerland, Hinze R. and Voigtländer J. eds. (2015) 50 – 70.

Bernardinello L., Pomello L. and Rombolà S., *Orthomodular lattices in occurrence nets*. Applications and theory of Petri nets - 30th International Conference, Paris, France, Proceedings, Franceschinis G. and Wolf K. eds., Springer (2009) 162 – 181:
https://books.google.pl/books?id=axosKVugYq4C&pg=PA165&lpg=PA165&dq=%22relations+li+and+co%22&source=bl&ots=FKxhRjcf8y&sig=ACfU3U1wP24-Fp6L3Qe2Xm-pyqO87e40nw&hl=pl&sa=X&ved.pdf.

Berto F., *How to Rule Out Things with Words: Strong Paraconsistency and the Algebra of Exclusion*. In: New Waves in Philosophical Logic. New Waves in Philosophy, Restall G. and Russell G. eds., Palgrave Macmillan, London (2012).

Best, E. and Devillers R., *Sequential and concurrent behaviour in petri net theory*. Theoretical Computer Science, vol. 55, issue 1 (1987) 87 – 136.

Bezdek J.C., *Fuzzy partitions and relations and axiomatic basis for clustering*. Fuzzy Sets and Systems, vol.1, no.2 (1978) 111–127.

Bezdek J.C., *Pattern recognition with fuzzy objective function algorithm*. Plenum Press (1981) 256pp.

Bezhanishvili N. and Jongh D. de., *Intuitionistic Logic*. Institute for Logic, Language and Computation Universiteit van Amsterdam (2006) 57pp: https://eprints.illc.uva.nl/200/1/PP-2006-25.text.pdf.

Béziau J-Y., *Logica universalis: towards a general theory of logic*. Springer (2007) 246pp.

Bhattacharyya A., Li B. and Randell B., *Time in structured occurrence nets*. International Workshop on Petri Nets and Software Engineering: PNSE'16 (2016) 55pp:
file:///C:/Users/user/Documents/OCCURRENCE%20NETS%20DALEJ%20PDF.pdf.

Biamonte J. and Perkowski M., *Automated Test Pattern Generation for Quantum Circuits*, McNair Research Journal, vol. 1, issue 1 (2005) 10pp.

Bienvenu M., Herzig A. and G. Qi., Prime implicate-based belief revision operators. A note (2006) 2pp:
http://www.lirmm.fr/~meghyn/papers/BieHerQi_ECAI08.pdf.

Billera L.J., Holmes S.P. and Vogtmann K.,, *Geometry of the space of phylogenetic trees*, Advances in Applied Mathematics, Elsevier, vol. 27, no. 4 (2001) 733 – 767.

Birkhoff G. and Neumann J. von., *The Logic of Quantum Mechanics*, Annals of Mathematics vol. 37 (1936) 823 – 843.

Bishop E., *Foundations of Constructive Analysis*. McGrow-Hill (1967) 370pp.

Bjorvand A.T., Fractal logic. A new approach to intelligent systems theory. Project report, The University of Trondheim (1995) 47pp.

Black M., The elusiveness of sets. Review of metaphysics, vol. 24, no. 4 (1971) 614 – 636.

Blizard W.D., *Multiset Theory*. Notre Dame Journal of Formal Logic, vol. 30, no. 1 (1989) 36 – 66.

Boffa M., *Sur la théorie des ensembles sans axiome de fondement*. Bulletin de la Société Mathématique de Belgique. Société mathématique de Belgique, Bruxelles, Belgique., vol. 31 (1969) 16 – 56.

Bonatti P.A. and Olivetti N., *Sequent calculi for propositional nonmonotonic logics*. ACM Transactions on Computational Logic, vol. 3, issue 2 (2002) 226 – 278.

Bonferroni C, *Sulle medie multiple di potenze*. Bolletino Matematica Italiana, vol. 5 (1950) 267 – 270.

Borkowski L. and Słupecki J., *A logical system based on rules and its application in teaching mathematical logic*. Studia Logica 7 (1958).

Bradfield J. and Stirling C., Modal logics and mu-calculi: an introduction. Univ. of Edinburgh,41pp: http://www.dis.uniroma1.it/~degiacom/didattica/semingsoft/SIS05-06/materiale/3-servizi/altro%20materiale/bradfield-stirling-HPA-mu-intro.pdf. See also: Bergstra J.A. , Ponse A. and Smolka S.A. eds., Handbook of process algebra, Elsevier (2001) 1329pp.

Bradistilov G., *High level mathematics*.Part III, Technika, Sofia (1961) 422pp.

Brady R.T., *Universal logic*. Stanford, Centre for the Study of Language and Information: CSLI Publications (2006) 346pp.

Brams G.W., *Reseaux de Petri: Theorie et Pratique* (G.W. Brams denote name of collective).Tome I: Theorie et analyse. Masson, Paris (1983)121 – 147.

Braüner T., *Introduction to Linear Logic*. Basic Research In Science, BRICS Lecture Series LS966, Centre of the Danish National Research Foundation, Technical Report (1996) iiiv + 55pp.

Breuer M.A. and Friedman A.D., *Diagnosis and Reliable Design of Digital Systems*. Pitman,London (1977)36 – 97.

Brodie M.L. and Mylopoulos J. eds., *On knowledge base management systems: integrating artificial intelligence and database technologies*. New York, Springer-Verlag (1986) 660pp.

Broersma H.J. and Hoede C., *Path graphs*. Journal of Graph Theory, John Wiley & Sons, vol. 13 (1989) 427 – 444.

Bronstein I.N., Semendjajew K.A., Musiol G. and Mühlig H., Taschenbuch der Mathematik. Verlag Harri Deutsch (2001) 1149pp.

Brouwer, L. E. J., *On the Foundations of Mathematics*. Thesis, Amsterdam (1907). English translation in Heyting A. ed. (1975) 11 – 101.

Brown C. and Gurr D., *A categorical linear framework for Petri nets*. In: Proceedings of the 5th LICS Symposium, IEEE Press (1990) 208 – 218.

Bull R. and Segerberg K., *Basic modal logic*. Gabbay D. and Guenthner F. eds., Handbook of Philosophical Logic, vol.2: Extensions of Classical Logic, Reidel, Dordrecht (1984) 1 – 88.

Burgess J.P., Philosophical logic. Princeton University Press (2009) 153pp.

Cantor G., *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. Journal für die reine und angewandte Mathematik, vol. 77 (1874) 258 –

262.

Cantor G., Ein Beitrag zur Mannigfaltigkeitslehre. Journal für Reine und Angewandte Mathematik, vol. 84 (1878) 242 – 258.

Cardone F. and Hindley J.R., History of Lambda-calculus and Combinatory Logic. Swansea Univ., Mathematics Depart., Research Report no. MRRS-05-06 (2006) 93pp.

Carnap R., *Einführung in die symbolische Logik, mit besonderer Berücksichtigung ihrer Anwendungen*. Springer-Verlag, Wien (1954) 209pp.

Carson J., *Modal logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (1995) 16pp.

Cattaneo G., Chiara M.L.D., Giuntini R. and Paoli F., *Quantum logic and nonclassical logics*. Semantic Scholar, Università di Milano – Bicocca, 98pp: https://pdfs.semanticscholar.org/9e4a/2d6900cec7671dd68adc5de6c44aa33858a9.pdf.

Cayley A.,, *On the partitions of a polygon*, Proc. Lond. Math. Soc. 1 (1890) 237 – 264.

Chang C.-L. and Lee R.C.-T., *Symbolic logic and mechanical theorem proving*. Academic Press, New York,San Francisco,London (1973) 358pp.

Chaudhan A., *Fuzzy clustering*. Technology, Health & Medicine (2013) 18pp. https://www.slideshare.net/AkshayChaudhari2/fuzzy-dm

Chaum D.L., *Untraceable electronic mail, return addresses and digital pseudonyms*. Programming Techniques and Data Structures. Technical Note, Rivest R. ed., Communications of the ACM, vol.24, no.2 (1981) 84 – 88.

Chen Z-H., He H-C.and Mao M-Y., *Exploration of fractal and chaos logics based on universal logic*. National Natural Science Foundation of China, Grant No. 60273087 (2010) 11pp: https://www.sinoss.net/qikan/uploadfile/2010/1130/1375.pdf.

Chiriţă C., *A completeness theorem for three-valued temporal predicate logic*. Annals of the University of Craiova, Math. Comp. Sci. Ser., vol.35, (2008) 41 – 53.

Choquet G., *Theory of capacities*. Annales de l'Institut Fourier, vol. 5 (1953) 131 – 295.

Church A., *An unsolvable problem of number theory*. Amer. Journal Math., no. 58 (1936) 345 – 363.

Church A., *The Calculi of Lambda Conversion*. Princeton Univ.Press, Princeton, New Jersey, U.S.A. (1941) 19pp.

Church A., The weak theory of implication, Kontrolliertes Denken, Aber, München (1951) 22 – 37.

Ciabattoni A., *A Proof-theoretical Investigation of global intuitionistic (fuzzy) logic*. TU Wien, Springer-Verlag (2004) 23pp.

Ciabattoni A. and Metcalfe G., *Bounded Łukasiewicz logics*. Mayer M.C. and Pirri F. eds., TABLEAUX 2003, Lecture Notes in Computer Science, vol. 2796, Springer (2003) 32 – 47.

Cignoli, R., Esteva, F., Godo, L., and Torrens, A., *Basic logic is the logic of continuous t-norms and their residua.* Soft Computing, 4 (2000) 106–112.

Cintula P., Navara M., *Compactness of fuzzy logics.* Fuzzy Sets and Systems, vol. 143 (2004) 59 – 73.

Cintula P., Fermüller C.G., and Noguera C., *Fuzzy logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2017) 17pp.

Clark H.H. and Brennan S.E., *Grounding in communication*: from Perspectives on socially shared cognition. Resnick L.B., Levine J.M., and Teasley S.D. eds., Washington Psychological Association (1991) 127 – 149.

Cocchiarella N.B., *Notes on deontic logic* (based on course on modal logic). State Univ. of California, San Francisco (in the late 1960s) 19pp.

Cockett J.R.B. and Seely R.A.G., Weakly distributive categories. Journal of Pure and Applied Algebra, Elsevier, vol.114, issue 2 (1997) 133 – 173.

Coecke B., Structure, logicality and sense in quantum theory (a quick draft version of a course, Univ. of Oxford, 23pp: http://www.cs.ox.ac.uk/people/bob.coecke/SLS.pdf.

Coker D., *Fuzzy rough sets are intuitionistic L-fuzzy sets*. Fuzzy Sets and Systems, vol. 96, no. 3 (1998) 381 – 383.

Colmerauer A. and Roussel P., *The birth of Prolog*. History of programming languages II, ACM, New York (1996) 331 – 367.

Comtet L., *Advanced combinatorics: the art of finite and infinite expansions*. Dordrecht, Boston, D. Reidel Pub. Co. (1974) 343pp. / Springer Science & Business Media (2012) 343pp.

Cornelis, C. and Deschrijver G., *The compositional rule of inference in an intuitionistic fuzzy logic setting*. Proceedings of the 13th European Summer School in Logic, Language and Information, Striegnitz K. ed., ESSLLI'2001 (2001) 83 – 94.

Cornelis C., Deschrijver G. and Kerre E.E., *Implication in intuitionistic fuzzy and interval - valued fuzzy set theory: construction, classification, application*. International Journal of Approximate Reasoning, Elsevier, vol. 35 (2004) 55 – 95.

Craus M., Bîrlescu V.S. and Agop M., *Fractal aspects in classical parallel computing*. Computers, Multidisciplinary Digital Publishing Institute, vol. 5, issue 19 (2016) 1 – 17.

D'Alfonso D., *Generalised quantifiers: logic and language*. Logic and Philosophy Science, vol. 9, no. 1 (2011) 85 – 94.

Da Costa N.C.A., *On the theory of inconsistent formal systems*. Notre Dame Journal of Formal Logic, vol. 15, no. 4 (1974) 497 – 510.

Da Costa N.C.A. and Wolf R.G., *Studies in paraconsistent logic I: the dialectical principle of the unity of opposites*. Philosophia, Springer, vol. 9 (1980) 189 – 217.

Da Costa N.C.A. and Wolf R.G., *Studies in paraconsistent logi II: Quantifiers and the unity of opposites*. Revista Colombiana de Matemáticas, vol. 19 (1985) 59 – 67.

Dan S., Kar M.B., Majumder S., Roy B., Kar S. and Pamucar D., *Intuitionistic type 2 – fuzzy set and its properties*. Symmetry, MDPI (2019) 18pp.

Das R., Mishra D. and Mishra S., *Rough set applications for the classification of software industries using rule based approach*. International Journal of Computer Applications, vol. 175 (2017) 26 – 28.

Davenport H. and Hasse H., *Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen. (On the zeros of the congruence zeta-functions in* *some cyclic cases)*, Journal für Reine und Angewandte Mathematik 172 (1935) 151–182, in German.

Davis M. and Putnam H., *A computing procedure for quantification theory*. Journal Assoc. Comput. Mach., vol. 7, no. 3 (1960) 201 – 215.

De Cock M. and Kerre E., *Why fuzzy T- equivalence relations do not resolve the Poincare paradox and related issues*. Fuzzy Sets and Systems, vol. 133 (2003) 181 – 192.

De Cock M., Cornelis C. and Kerre E.E., *Fuzzy rough sets: the forgotten step*. IEEE Trans. on Fuzzy Systems, vol. 15, no.1 (2007) 121 – 130.

De Concini C. and Procesi C., *Wonderful models of subspace arrangements*, Selecta Mathematica, Springer, vol. 1 (1995) 459–494.

De Concini C. and Procesi C., *Hyperplane arrangements and holonomy equations*. Selecta Mathematica, Springer, vol. 1 (1995) 495 – 535.

De Luca A. and S Termini S., *A definition of non probabilistic entropy in the settings of fuzzy set theory*. Information and Control, vol. 20 (1972) 301-312.

De Paiva V., *Categorical multirelations, linear logic and Petri nets* (draft). Technical Report no. 225, Univ. of Cambridge (1991) 31pp.

Delgado M., Sánchez D. and Vila M. A., *Fuzzy cardinality based evaluation of quantified sentences*. International Journal of Approximate Reasoning, vol. 23, no.1 (2000) 23 – 66.

Detyniecki M., Lesot M-J. and Moncuquet P., *Intuitionistic fuzzy tautology definitions for the validity of intuitionistic fuzzy implications: an experimental sstudy* (2014) 12pp: http://www-poleia.lip6.fr/~marcin/web/wp-content/papercite-data/pdf/Detyniecki_IS_2014.pdf.

Deutsch D., *Quantum theory, the Church -Turing principle and the universal quantum computer*. Proc. Royal Society, London, A400, vol.97 (1985).

Dhar M., *Cardinality of fuzzy sets: An Overview*. International Journal of Energy, Information and Communications, vol. 4, issue 1 (2013) 15 – 22.

Di Cosmo R. and Miller D., Linear logic. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2016) 9pp.

Diamond J., Pedrycz W. and McLeod D., *Fuzzy J-K flip-flops as computational structures: design and implementation*. IEEE Trans. on Circuit and Systems - II: Analog and Digital Signal Processing, vol. 41, no. 3 (1994) 215 – 226.

Dijkstra E.W., *A Discipline of Programming*. Englewood Cliffs, Prentice Hall Inc. (1976) 217pp.

Dimitrescu D., Lazzerini B. and Jain L.C., *Fuzzy sets and their application to clustering and training*. The CRC Press, International Series on Computational Intelligence. Boca Raton, London, New York, Washington D.C. (2000) 622pp.

Dubois D., Gottwald S., Hajek P., Kacprzyk J. and Prade H., *Terminological difficulties in fuzzy set theory - the case of "intuitionistic fuzzy sets*." Fuzzy Sets and Systems, vol. 156 (2005) 485 – 491.

Dunn, J. C., *A fuzzy relative of the ISODATA Process and its use in detecting compact well-separated clusters*. Journal of Cybernetics, vol.3, no. 3 (1973) 32–57.

Dunn J.M., *Intuitive semantics for first-degree entailments and 'coupled trees'*.

Philosophical Studies 29 (1976) 149 – 168.

Dunn J.M. and Restall G., *Relevance logic*. http://consequently.org/papers/rle.pdf, 136pp.

Dvořák A. and Holčapek M., Fuzzy quantifiers: from Zadeh's fuzzy quantification to theory of fuzzy quantifiers. FSTA: The Fourteenth Int. Conference on Fuzzy Set Theory and Applications (2018) 57pp.

Eichelberger E.B., *Hazard detection in combinational and sequential switching circuits*. IBM J.Res.Develop.,vol.9 (1965) 90 – 99.

El-Hakeim K. and Zeyada F., *Comments on some versions of intuitionistic fuzzy propositional calculus due to K. Atanassov and G. Gargov*. Fuzzy Sets and Systems, vol.110, no. 3 (2000) 451 – 452.

El-Naschie M.S., *Computational fractal logic for quantum physics and cosmology*. American Journal of Astronomy and Astrophysics, vol.4, issue 4, (2016) 42 – 53.

Encyclopedia of Mathematics, Kluwer Academic Publishers (2002): https://www.encyclopediaofmath.org/index.php/Main_Page.

Erdős P.L. and Székely L., *Applications of antilexicographic order.* I. *An enumerative theory of trees*, Advances in Applied Mathematics, Elsevier, vol. 10, no. 4 (1989) 488 – 496.

Ervin J.C. and Altekin S.E., *Techniques for developing large scale fuzzy logic systems*. 12th Int. Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU'08 proceedings, Magdalena L., Ojeda-Aciego M. and Verdegay J.L. (eds), Málaga, Spain (2008) 806 – 811.

Erwig M., Schneider M. and Güting R.H., *Temporal objects for spatio - temporal data models and a comparison of their representations*. International Workshop on Advances in Database Technologies. LNCS, vol. 1552, Springer (1998) 454 – 465.

Eslami E., *An algebraic structure for intuitionistic fuzzy logic*. Iranian Journal of Fuzzy Systems, vol. 9, no. 6 (2012) 31 – 41.

Etchemendy J., *The concept of logical consequence*.Cambridge,Mass. Harward Univ. Press (1990) 174pp. Also in: CSLI publications (1999).

Feichtner E., *Complexes of trees and nested set complexes*. Pacific Journal of Mathematics, vol. 227, no. 2 (2006) 271–286.

Ferrero D., Varghese A. and Vijayakumar A., Power domination in honeycomb networks. Journal of Discrete Mathematical Sciences and Cryptography, vol. 14, no. 6 (2011) 521 – 529.

Finch P.D., *Quantum logic as an implication algebra*. Bull. Austral. Math. Soc., vol. 2 (1970) 101 – 106.

Fisz M., *Probability theory and mathematical statistics*. PWN, Warsaw (1969) 694pp; English translation: Huntington, N.Y., Krieger Pub. Co. (1980) 677pp.

Fischer M.J. and Ladner R.E., *Propositional dynamic logic of regular programs*. Journal of computer and system sciences, vol.18, issue 2 (1979) 194 – 211.

Fitting M., *A Kripke – Kleene semantics for general logic programs*. Logic Programming 2 (1985) 295 – 312.

Fitting M., *Kleene's tree valued logics and their children*. Proc. of the Bulgarian Kleene 90 Conference, 1990,21pp.An enlarged version: NSF Grant CCR-9104015.

Floyd R.W., *Assigning meanings to programs*. Proceedings of the American Mathematical Society Symposia on Applied Mathematics, vol. 19 (1967) 19–31.

Ford L.R., Jr. and D. R. Fulkerson D.R., *Flows in Networks*. Princeton University Press, Princeton, New Jersey, (1962) 194pp.

Formal logic. Encyclopedical outline with applications to informatics and linguistics, Marciszewski W. ed., PWN, Warsaw (1987) 472pp,in Polish.

Fourman M.P., *Sheaf models for set theory*. Journal of Pure and Applied Algebra, North-Holland, vol.19 (1980) 91 – 101.

Frege G., *Begriffsschrift, eine der arithmetischen nachbildete Formelsprache des reinen Denkens*. Halle (1879) 105pp.

Fulton W. and MacPherson R., *A compactification of configuration spaces*. Annals of Mathematics, Princeton University, vol. 139, no. 1 (1994) 183 – 225.

Furusawa H., Tsumagari N. and Nishizawa K., *A non-probabilistic relational model of probabilistic Kleene algebras*. Springer-Verlag, Berlin, Heidelberg, New York, Berghammer R., Möller B. and Struth G. eds. (2008) 110 – 122.

Gabbay D.M., Rodrigues O.T. and Russo A., *Revision, acceptability and context: theoretical and algorithmic aspects*. Springer Science & Business Media (2010) 386pp.

Gaiffi G., *Nested sets, set partitions and Kirkman – Caley dissection numbers*. European Journal of Combinatorics, vol. 43 (2015) 279 – 288.

Galindo J., Carrasco R.A. and Almagro A.M^{a}., *Fuzzy quantifiers with and without arguments for databases: definition, implementation and application to fuzzy dependencies*. Proceedings of IPMU'08 (Information Processing and Management of Uncertainty in Knowledge-Based Systems), Torremolinos, Spain (2008) 227 – 234.

Geldenhuys A.E., van Rooyen H.O. and Stetter F., *Introduction to the theory of relation nets*. In: Knowledge Representation and Relational Nets, The Kluwer International Series in Engineering and Computer Science, vol. 506, Springer, Boston, MA (1999) 193 – 208.

Gentzen G.K.E., *Untersuchungen über das logische Schliessen*. Math. Zeitschrift 39 (1934) 176 – 210 / (1935) 405 – 431.

Ghilardi S., *Unification in intuitionistic logic*. Journal of Symbolic Logic, vol. 64, issue 2 (1999) 859 – 880.

Ginis A., *Non-monotonic logic* (2008) 52pp:
http://www.cs.tau.ac.il/~annaz/teaching/TAU_winter08/Seminar/ArielLogic.pdf

Ginsberg M.I., *Multivalued logics: a uniform approach to inherence artificial inteligence*. Computational Inteligence vol.4, no.3(1988) 265 – 316.

Glivenko V., *Sur qulques points de la logique de M. Brouwer*. Academie Royale de Belgique, Bulletins de la classe des sciences, vol. 5, issue 15 (1929) 183–188.

Glottopedia, the free encyclopedia of linguistics. Naumann S. and Wohlgemuth J. eds.

Glöckner I., *A framework for evaluating approaches to fuzzy quantification*. Technical Report TR99-03, Universität Bielefeld (1999).

Glöckner I. and Knoll A., *A formal theory of fuzzy natural language quantification and its role in granular computing*. W. Pedrycz ed., Granular computing: An emerging paradigm. Studies in Fuzziness and Soft Computing, vol. 70, Physica-Verlag (2001) 215 – 256.

Glöckner I., *Fuzzy Quantifiers in Natural Language: Semantics and Computational Models*. PhD thesis, Universität Bielefeld (2004) 465pp.

Glöckner I., *Fuzzy quantifiers. A computational theory*. Springer-Verlag, Berlin, Heidelberg (2006) 460pp.

Glöckner I., *Generalised fuzzy quantifiers and the modelling of fuzzy branching quantification*. International Journal of Intelligent Systems, vol. 24, no. 6 (2009) 624 – 648.

Glushkov V.M., *Introduction to Cybernetics* Ukranian Academy of Sciences, Kiev (1964),in Russian.

Gniewek L. and Kluska J., *Family of fuzzy J-K flip-flops based on bounded product, bounded sum and complementation*. IEEE Trans. on Systems, Man, and Cybernetics, Part B: Cybernetics, vol. 28, no. 6 (1998) 861 – 868.

Gniewek L. and Kluska J., *Hardware implementation of fuzzy Petri net as a controller*. IEEE Trans. on Systems, Man, and Cybernetics, Part B: Cybernetics, vol. 34, no. 3 (2004) 1315 – 1324.

Gniewek L., *Modeling and synthesis of control systems using fuzzy interpreted Petri nets*. Reszów (2012) 157pp., in Polish.

Goebel G., *An introduction to fuzzy control systems*. *Public domain*, vol.0.4 (2003).

Gomolińska A. and Wolski M., *Rough inclusion functions and similarity indices*. Fundamenta Informaticae. IOS Press, vol. 133, no. 2/3 (2014) 149 – 163.

Gosaina A. and Dahiyab S., *Performance Analysis of Various Fuzzy Clustering Algorithms: A Review*. In: 7th International Conference on Communication, Computing and Virtualization, Thakur College of Engineering and Technology, Mumbai, Maharashtra, India, Procedia Computer Science, Elsevier B.V.,vol. 79 (2016) 100 – 111.

Gottwald S., *Many-valued logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2000) 14pp.

Gödel K., *Die vollständigkeit der axiome des logischen funktionenkalküls*. Monatshefte für Math. und Physik XXXVII (1930) 349 – 360.

Gödel K., *Zum intuitionistischen aussagenkalkul*. Ergebnisse eines mathematischen kolloquiums, vol. 4 (1933) 34 – 38.

Grabisch M., *Fuzzy integral in multicriteria decision making*. Fuzzy Sets and Systems, vol.69 (1995) 279–298.

Grabisch M., Nguyen H.T. and Walker E.A., *Fundamentals of uncertainty calculi, with applications to fuzzy inference*. Kluwer Academic,Dordrecht, Boston (1995) 346pp.

Grayson, R.J., *Heyting-valued models for intuitionistic set theory*. In: Applications of Sheaves (Lecture Notes in Mathematics, vol. 753), M.P. Fourman, C.J. Mulvey, and D.S. Scott (eds.), Berlin: Springer (1979) 402 – 414.

Grinshpan A., *The number of partitions of a set*. Drexel University, Philadelphia, 1pp: http://www.math.drexel.edu/~tolya/partitions.pdf.

Grzegorczyk A., *The mathematical logic outline*.PWN, Warsaw (1969) 505pp.,in Polish.

Gurov D. and Huisman M., *Reducing behavioural to structural properties of programs with procedures*. Theoretical Computer Sciience, Nord-Holland, vol. 480 (2013) 69 – 103.

Haar S., On Occurrence Net Semantics of Petri Nets. RR-3718, INRIA, inria-00072949 (1999) 36pp: file:///C:/Users/user/Documents/OCCURRENCE%20NET.pdf.

Hamilton W.R., *On quaternions, or on a new system of imaginaries in algebra*. Philosophical Magazine, vol. 25, no. 3 (1844) 489 – 495.

Han Q., Li W., Song Y., Zhang T. and Wang R., *A new method for MAGDM based on improved TOPSIS and a novel Pythagorean fuzzy soft entropy*. Symmetry, MDPI (2019) 14pp.

Handbook of philosophical logic. Gabbay D.M. and Guenthner F. eds., Springer Science and Business Media, Dodrecht, Boston, Kluwer Academic Publishers, 2nd. edition,vol.4 (2002) 431pp.

Handbook of quantum logic and quantum structures. *Quantum logic*. Engesser K., Gabbay M. and Lehmann D. eds., Elsevier (2009) 703pp.

Harel D. and Sherman R., Propositional dynamic logic of flowcharts. Information and Control, vol. 64 (1985) 119 – 135.

Harel D., Kozen D.C. and Tiuryn J., *Dynamic logic*. MIT Press (2000) 476pp.

Hay L.S., *Axiomatization of the infinite-valued predicate calculus*. Journal of Symbolic Logic, vol. 28 (1963) 77 – 86.

Hájek P. and Godo L., *Deductive systems of fuzzy logic (a tutorial*).Tatra-mountains mathematical publications, vol. 13 (1997) 35 – 66.

Hájek P., *Metamathematics of fuzzy logic*. Kluwer Acad.Publ., Dordrecht (1998) 297pp.

Hájek P., *Fuzzy Logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2002) 7pp.

Hájek P., *Arithmetical Complexity of Fuzzy Predicate Logics - A Survey*. Soft Computing, vol. 9 (2005) 935-941, see also: *Complexity of models of fuzzy predicate logics with witnessed semantics*. CAS, 23pp: file:///C:/Users/user/Documents/Fuzzy_Modal_Temporal_Intit%20and%20Paracons%20quantifiers/Peter%20Hajek%20COMPLEXITY%20of%20fuzzy%20pred%20logic.pdf.

Hehner E.C.R., Bunch Theory: *A Simple Set Theory for Computer Science*. Information Processing Letters, Elsevier, vol.12 no.1 (1981) 26 – 30.

Hehner E.C.R., *The logic of programming*. Englewood Cliffs, N.J., Prentice Hall International (1984) 361pp.

Hehner E.C.R., *A Practical Theory of Programming*. Springer-Verlag, New York (1993) 243pp.

Hella L., *Definability hierarchies of generalised quantifiers*. Annals of Pure and Applied Logic, vol.43, no.3 (1989) 235 – 271.

Hendrics V. and Symons J., *Epistemic logic*, The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., Center for the Study of Language and Information, Stanford University, Stanford, CA (2006) 6pp.

Henkin L., *Some remarks on infinitely long formulas*. *Infinitistic Methods. Proceedings of the Symposium on Foundations of Mathematics, Warsaw (1959)*: PWN and Pergamon Press, Warsaw (1961) 167–183.

Hennessy M., Milner R., *Algebraic laws for nondeterminism and concurrency*. Journal of the ACM , vol. 32 , issue 1 (1985)137 – 161.

Herbrand J., *Recherches sur la théorie de la demonstration*. PhD thesis, University of Paris (1930). Translated as: *Investigations in proof theory*, Goldfarb W ed., Jacques Herbrand – *Logical writings*. Harvard University Press (1971), see also: Handbook of automated reasoning, Robinson A. and Voronkov A. eds., vol. I, Elsevier (2001) 2115pp.

Heyting, A., *Die formalen Regeln der intuitionistischen Logik*. In three parts. Sitzungsberichte der preussischen Akademie der Wissenschaften (1930) 42–71, 158–169. English translation of Part I in Mancosu (1998) 311–327.

Hewitt C., *ORG's for scalable, robust, privacy-friendly client cloud computing*. IEEE Internet computing,vol.12 no.5 (2008) 96.

High-level Petri Nets – Concepts, Definitions and Graphical Notion. Final Draft International Standard ISO / IEC 15909 version 4.7.1 October 28 (2000) 43pp. Software and Systems Engineering – High-level Petri Nets, Part 2: Transfer Format. International Standard ISO / IEC 15909 – 2, WD version 0.9.0 June 23 (2005) 50pp.

Hilbert D. and Bernays P., Die Grundlagen der Mathematik, in two volumes, Springer, Berlin (1934, 1939).

Hintikka J., *Knowledge and belief. An introduction to the logic of the two notions*. Cornell Univ. Press, Ithaca, New York (1962) 179pp.

Hirota K. and Ozawa K., *The concept of fuzzy flip-flop*. IEEE Trans. on Systems, Man, and Cybernetics, vol. 19, no. 5 (1989) 980 – 987.

Hitchcock P. and Park D.M.R., *Introduction rules and termination proofs*. Proc. 1st Int. Colloq. on Automata, Languages and Programming. North-Holland, Amsterdam (1973) 225 – 251.

Hoare C. A. R., *An axiomatic basis for computer programming*. Communications of the ACM, vol. 12, issue 10 (1969) 576–580.

Holický M., Fuzzy probabilistic models in structural reliability. Maintenance and Reliability, vol.18, no. 2 (2006) 11 – 13.

Holiday W.H., *Epistemic logic and epistemology*. Handbook of Formal Philosophy, Hansson S.O. and Hendrics V.F. eds., Springer (2016) 18pp.

Holmblad L.P. and Ostergaard J.J. *Control of a cement kiln by fuzzy logic*. In: Fuzzy Information and Decision Processes, Gupta M.M. and Sanchez E. eds., North-Holland (1982) 389 – 399.

Holmes M.R., *Alternative axiomatic set theories*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., Stanford, CA (2017) 42pp.

Horty J.F., *Moral dilemmas and nonmonotonic logic*. Journal of Philosophical Logic, vol. 23 (1994) 35 – 65.

Horty J.F., *Nonmonotonic logic* (2001) 33pp:
http://www.umiacs.umd.edu/~horty/articles/2001-nonmon.pdf.

Howard W.A., *The formulae-as-type notion of construction*. To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, Seldin J.P. and Hindley J.R. eds., London, New York, Academic Press (1980) 606pp.

Huacan He. ed., *Universal logics principle*. China Science Publ. House, Beijing (2001).

Huisman M., Aktug I. and Gurov D., *Program models for compositional verification*, LNCS, vol. 5256, Springer (2008) 147–166.

Huzar Z., *Elements of logics for computer scientists*. Of.Wyd. TU Wroclaw, Wroclaw University of Technology ISBN 83-7085-625-X (2002),266pp..

Immanuel B. and Rangarajan K., *System diagnosis and k-distinguishability in Petri nets,* Private communication, India (2001) 14pp.

Iséki K. and Tanaka S., *An introduction to the theory of BCK-algebras*. Mathematica Japonica, Japanese Association of Mathematical Sciences, vol. 23 (1978) 1 – 26.

International Organization for Standardization, ISO Central Secretariat, Geneva, Switzerland, *High-level Petri Nets - Concepts, Definitions and Graphical Notation*. Final Draft International Standard ISO/IEC 15909, Version 4.7.1, October 28 (2000) 43pp. and Software and Systems Engineering - *High-level Petri Nets, Part 2: Transfer Format*, International Standard ISO/IEC 15909-2, WD Version 0.9.0, June 23 (2005) 50pp.

Jani ˇs V. and Montes S., *Distance between fuzzy sets as a fuzzy quantity*. Acta Universitatis Matthiae Belii, ser. Mathematics, vol. 14 (2007) 41 – 49.

Jankov V. A., *On the extension of the intuitionistic propositional calculus to the classical calculus, and the minimal calculus to the intuitionistic calculus*. Journal of Symbolic Logic, vol. 38, issue 2 (1973) 331 – 332.

Japaridze G., *Introduction to computability logic.* Annals of Pure and Applied Logic, Elsevier, vol. 123, issues 1-3 (2003) 1 – 99.

Japaridze G., *Intuitionistic computability logic*. Acta Cybernetica, vol. 18 (2007) 77 – 113.

Jaśkowski S., *Propositional calculus for contradictory deduction systems*. Studia Societatis Scientiarum Torunniensis, sec. A I (1948) 57 – 77., in Polish.

Jensen K., *Coloured Petri nets and the invariant method*. Theoretical Computer Science, Nord-Holland, 14 (1981) 317 – 336.

Jensen K., *Coloured Petri nets*. LNCS, vol. 254, Springer (1987) 248 – 299.

Jensen K., *Coloured Petri nets - basic concepts, analysis methods and practical use*. EATCS Monographs on Theoretical Computer Science, Berlin, Springer-Verlag, vol. 1: *Basic concepts* (1992) 234 pp., vol. 2: *Analysis methods* (1994) 173pp. and vol. 3: *Practical use* (1997) 265pp.

Jensen K. and Kristensen L.M., *Coloured Petri nets: modelling and validation of concurrent systems*. Dordrecht, New York, Springer (2009) 384pp.

Jun Y.B., *Soft BCK/BCI-algebras*. Computers and Mathematics with Applications, Elsevier, vol. 56 (2008) 1408 – 1413.

Kaburlasos V.G., Athanasiadis I.N. and Mitkas P.A., *Fuzzy lattice reasoning (FLR) classifier and its application for ambient ozone estimation.* International Journal of Approximate Reasoning, Elsevier, vol. 45 (2007) 152 – 188.

Kalmbach G., *Orthomodular lattices*. Academic Press, London, New York (1983) 390pp.

Kamide N. and Wansing H., *Symmetric and dual paraconsistent logics*. Logic and Logical Philosophy, vol 19 (2010) 7 – 30.

Kamp H., *Tense logic and the theory of linear order*. University of California dissertation, Los Angeles, USA (1968).

Kamp H. and Reyle U., *From discourse to logic: introduction to modeltheoretic semantics of natural language, formal logic and discourse representation theory*. Kluwer Academic, Dordrecht, Boston (1993) 713pp.

Kanger S., *Provability in logic*. Stockholm Studies in Philosophy. vol. 1, Almqvist and Wiksell, Stockholm (1957).

Katarzyniak R.P., *Grounding modal language in communications of agent systems*. EXIT, Warsaw (2007) 320pp.,in Polish.

Katsuno H. and Mendelzon A.,, *Propositional knowledge base revision and minimal change*, Artificial Intelligence, vol.52, issue 3 (1991) 263–294.

Kavitha B., Karthikeyan S. and Sheeba Maybell P., *Emerging Intuitionistic Fuzzy Classifiers for Intrusion Detection System*. Journal of advances in information technology, vol.2, no. 2 (2011) 99 – 108.

Kawaguchi M.F., Watari O. and Miyakoshi M., Fuzzy Logics and Substructural Logics without Exchange. Proceedings of the Joint 4th Conference of the European Society for Fuzzy Logic and Technology and the 11th Rencontres Francophones sur la Logique Floue et ses Applications, EUSFLAT-LFA, Barcelona, Spain, (2005) 973 – 978.

Keenan, E.L. and Stavi J., 1986. *A semantic characterization of natural language determiners,* Linguistics and Philosophy (1986) 253-326.

Keisler H.J., *Logic with the qwuantifier "there exists uncountably many".* Annals of Mathematical Logic, vol. 1 (1970) 1 – 93.

Kerntopf P., *Fundamental mathematical notions in automata theory: relations, algebraic systems, lattices.* PWN , Warsaw (1967) 65pp.,in Polish.

Kirkman T.P.,, *On the k-partitions of the r-gon and r-ace*, Philos. Trans. Roy. Soc. London 147 (1857) 217 – 272.

Klaua D., *Über einen ansatz zur mehrwertigen mengenlehre*. Monatsberichte Deutsche Akademie Wissensschaften zu Berlin, 7 (1965) 859 – 867.

Kleene S.C., *Introduction to Metamathematics*. Groningen,P.Noordhoff N.V., North-Holland Pub (1952) 550pp.

Kleene S.C., *Representation of Events in Nerve Nets and Finite Automata*. Automata Studies, Shannon C.E. and McCarthy J. eds., Princeton University Press (1956) 3 – 42.

Klement E.P., Navara M., *A survey of different triangular norm-based fuzzy logics.* Fuzzy Sets and Systems, vol. 101 (1999) 241 – 251.

Knuth D.E., *The art of computer programming*. Reading, Mass.,Addison-Wesley, vol.1 (1997) 650pp.

Kohlenbach U., *Applied proof theory: proof interpretations and their use in mathematics*. Springer monographs in mathematics, Berlin, New York, Springer-Verlag (2008) 532pp.

Kolmogorov A.N., *O principie tertium non datur*. Mat. Sbornik, XXXII (1925) 646 – 667.

Kolmogorov A.N., *Foundations of the theory of probability*. Morrison N. ed. , New York. Chelsea Pub.Co. (1956) 84pp.

Kowalczyk W., *Rough data modelling: A new technique for analyzing data*. In: Rough Sets and Knowledge Discovery 1: Methodology and Applications, Heidelberg, Physica-Verlag, Ch.20 (1998) 400–421.

Kozen D.C., *Results on the propositional μ-calculus*. Theoretical Computer Science 27 (1983) 333 – 354.

Kozen D.C., *A completeness theorem for Kleene algebras and the algebra of regular events*. Information and Computation, vol.110 (1994) 366 – 390.

Kripke S., *Semantical Analysis of Modal Logic (abstract).* The Journal of Symbolic Logic, 24 no.4 (1959) 323 – 324.

Kripke S., *Semantical analysis of modal logic I: normal modal propositional calculi*. Zeitschrift für Math. Logic und Grundlagen der Math, no. 9 (1963) 67 – 96.

Kripke S., *Transfinite recursion on admissible ordinals*. The Journal of Symbolic Logic, 29 (1964) 161 – 162.

Kripke S., *Semantical analysis of modal logic II: non normal propositional calculi.* The theory of models.North-Holland Publishing, Amsterdam (1965) 206 – 220.

Kripke S., *Semantical analysis of intuitionistic logic*. In: Formal systems and recursive function theory, Crossley J.N. and Dummett M.A.F. eds., North - Holland Publishing, Amsterdam (1965) 320pp.

Kroger F., *LAR: a logic of algorithmic reasoning*. Acta Informatica, vol. 8 (1977) 243 – 246.

Kryszkiewicz M. and Rybinski H., *Finding reducts in composed information systems*. In: Rough Sets, Fuzzy Sets and Knowledge Discovery, Ziarko W.P. ed., British Computer Society (1994) 261 – 273.

Kummer O. and Stehr M - O., *Petri's axioms of concurrency. A selection of recent results*. 18th International Conference, TCATPN'97: Applications and Theory of Petri Nets 1997, Azéma P. and Balbo G. eds.,Toulouse, France, Proceedings, Springer (1997) 195 – 214.

Kuratowski K., *Sur la notation de l'ordre dans la théorie des ensembles*. Fundamenta Mathematicae, vol. 2, no.1 (1921) 161 – 171.

Kuratowski K., Introduction to set theory and topology. Warszawa, PWN (1966) 275pp: Oxford, New York, Pergamon Press (1972) 352pp.

Kuratowski K. and Mostowski A., Set theory. Mathematical Monographs, vol. 27, PWN Warszawa (1966) 375pp.

Lafont Y., *Introduction to linear logic*. Lecture notes from TEMPUS Summer School on Algebraic and Categorical Methods in Computer Science, Brno, Czech Republic (1993).

Lafont Y., *Linear logic pages*. Université d'Aix-Marseille, (1999,last correction: 2017),17pp: http://iml.univ-mrs.fr/~lafont/pub/llpages.pdf.

Lambert K.A., Nance D.W. and Naps T.L., *Introduction to Computer Science with C++*. West Publishing Company, Minneapolis/St. Paul, New York ,Los Angeles, San Francisco (1996) 962pp.

Larsen H.L., *Fundamentals of fuzzy sets and fuzzy logic*.Course in fuzzy logic Part I – II, Aalborg University, Esbjerg, Denmark (1998).

Lemmon E.J. and Scott D., An introduction to modal logic. American Philosophical Quarterly, Monograph Series, Oxford (1977) 94pp.

Lewis C.I., *A survey of symbolic logic*. Univ. of California Press, Berkeley (1918) 406pp. New edition (1960).

Lewis C.I. and Langford C.H., Symbolic logic. Dover, New York (1932). New edition (1959).

Libert T., *Models for a paraconsistent set theory*. Journal of Applied Logic, Elsevier, vol. 3 (2005) 15 – 41.

Lichtenstein O., Pnueli A. and Zuck L., *The glory of the past*. Conference on Logics of Programs, LNCS, vol. 193, Springer (1985) 196 – 218.

Lindley D., *The quantum word*, New Scientist, Reed Business Information, (1998) 64pp.

Lindström P., *First order predicate logic with generalised quantifiers*. Theoria 32 (1966) 186 – 195.

Liouville J., *Sur des classes très étendues de quantités don't la valeur n´est ni algebraique, ni même reducible à des irrationnelles algébriques*. J. Math. Pures Appl, vol. 16 (1851) 133 – 142.

Løken T. and Komorowski J., *Rough modelling - a bottom - up approach to model construction*. International Journal of Applied Mathematics and Computer Science, vol. 11, no. 3 (2001) 675 – 690.

Losada D.E., Diaz-Hermida F. and Bugarin A., *Semi-fuzzy quantifiers for information retrieval*. Soft Computing in Web Information Retrieval. Models and Applications, Herrera-Viedma E., Pasi G. and Crestani F. eds., Springer (12006) 195 – 220.

Lu H. and Lee S.C., *Fault detection in m-logic circuits using the M-diff*erence. Proceedings of the IEEE International Symposium on Multiple-Valued Logic (1984)62 – 70.

Lu H. and Lee S.C., *M-algebra*. Proceedings of the IEEE International Symposium on Multiple-Valued Logic (1985)272 – 284.

Łukasiewicz J., *Die logishen Grundlagen der Wbilityahrscheinilchkeitsrechnung*. Kraków (1913), in: 'Jan Łukasiewicz - selected works', North Holland Publishing Company, Amsterdam, London, Polish Scientific Publishers, Warsaw, Borkowski L. ed. (1970) 405pp.

Łukasiewicz J., Die Logic und das Grundlagenproblem. Les entretiens de Zurich sur les fondement et la méthode des sciences mathématiques, XII (1938) 6 – 9, Zurich (1941) 82 – 100.

Łuszczewska - Rohmanowa S., *Analysis and generalisations of Venn diagrams method for testing of logical formulae*. Studia Logica, T. L (1953), in Polish.

MacLane S. and Birkhoff G., *Algebra*. New York, Macmillan (1967).

Mallion R.B., *The six (or seven) bridges of Kaliningrad: a personal Eulerian walk*. MATCH Communications in Mathematical and in Computer Chemistry, no. 58 (2007) 529 – 556.

Mally E., *Grundgesetze des Sollens: Elemente der Logik des Willens*, Leuschner und Lubensky, Universitäts-Buchhandlung, Graz (1926). Reprinted in Mally E., *Logische Schriften: Großes Logikfragment, Grundgesetze des Sollens*, Karl Wolf and Paul Weingartner eds., Dordrecht (1971) 227 – 324.

Mamdani E.H. and Assilian S., *An experiment in linguistic synthesis with a fuzzy logic controller*. International Journal of Man-Machine Studies,vol. 7 (1975) 1 – 13.

Mandelbrot B.B., *The fractal geometry of nature*. W.H. Freeman and Company

(1983) 468pp.

Manna Z. and Pnueli A., *The Temporal Logic of Reactive and Concurrent Systems. Specification*. Springer-Verlag, New York (1992) 427pp.

Manna Z. and Pnueli A., *Temporal verification of reactive systems : safety*. Springer-Verlag, New York (1995) 512pp.

Mares E.D., *Relevance Logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (1998) 6pp.

Martin C.E., Curtis S.A. and Rewitzky I., *Modeling angelic and demonic nondeterminism with multirelations*. Science of Computer Programming, Elsevier, vol. 65 (2007) 140 – 158.

Mayeda W., *Graph Theory*. John Wiley & Sons,Inc.,New York (1972)523 – 557.

Mazurkiewicz A., *Introduction to Trace Theory*.in *The Book of Traces*, V. Diekert, G. Rozenberg eds., World Scientific, Singapore (1995) 3 – 41.

McArthur R.P., Tense Logic. Dordrecht, Holland, Boston, D. Reidel Pub. Co. (1976) 84pp.

McCarthy J., *Circumscription – a form of non-monotonic reasoning*. Artificial Intelligence, vol. 13 (1980) 27 – 29.

McColl H., *Symbolic logic and its applications*. London (1906) 141pp.

McDermott D. and Doyle J., Non-monotonic logic I. Intelligence, vol. 13 (1980) 41 – 72.

McGill V.J. and Parry W.T., *The unity of opposites: a dialectical principle*. Science and Society, vol. 12 (1948) 418 – 444.

McMillan K.L., *Using unfoldings to avoid the state explosion problem in the verification of asynchronous circuits*. In: International Conference on Computer Aided Verification, CAV 1992, New York, LNCS, vol. 663, Springer, Heidelberg, Probst D.K. and von Bochmann G. eds. (1993) 164 – 177.

Megill N.D., *The existential uniqueness quantifier*. (2005) 11pp: http://us.metamath.org/downloads/megillaward2005eu.pdf.

Megill N.D. and Pavičić M., *Quantum implication algebras.* Int. Journal of Theoretical Physics, vol. 42, issue 12 (2003) 21pp.

Mendel J.M., *Uncertain rule-based fuzzy logic systems: introduction and new directions.* Upper Saddle River, NJ: Prentice Hall (2001) 555pp.

Mendel J.M., *Advances in type-2 fuzzy sets and systems*. Information Sciences, vol. 177, issue 1 (2007) 84 – 110.

Menger K., *Statistical metrics*. Proc. Nat. Acad. Sci. USA, 28 (1942) 535 – 537.

Metcalfe G., Olivetti N. and Gabbay D., Proof theory for fuzzy logics. Springer Science – Business Media B.V. (2009) 276pp.

Meyden R. van der., *Proof, computation and agency*. Synthese Library, vol.352, Springer Science + Business Media B.V. (2011) 133 – 144.

Milner R., *A calculus of communicating systems*. LNCS, Springer - Verlag, Berlin, Heidelberg, vol. 92 (1980) 165pp.

Mirek R., *The systems of relevance logic*. Argument J., vol.1 , issue 1 (2011) 87 – 102.

Mirkowska G. and Salwicki A., *Algorithmic Logic*. Warszawa & Boston: PWN & D. Reidel Publ. (1987) 372pp.

Mittelstaedt P., *Quantenlogische interpretation orthokomplementärer quasimodularer verbände*. Z. Naturforsch, vol. 25 (1970) 1773 – 1778.

Moh Shaw-Kwei., The deduction theorem and two logical systems. Methodos, vol.2 (1950) 56 – 75.

Molodtsov D.A., *Soft set theory – first results*. Computers and Mathematics with Applications, Elsevier, vol. 37 (1999) 19 – 31.

Molodtsov D.A., *Equivalence and correct operations for soft sets*. International Robotics & Automation Journal, vol. 4, issue 1 (2018) 18 – 21.

Monaghan P.X., *Property possession as identity. An essay in Metaphysics*. Ontos Verlag, Frankfurt, Paris, Lancaster, New Brunswick (2011) 19

Montague R., *Formal philosophy: selected papers of Richard Montague, edited and with an introduction by Richmond H. Thomason,* Yale University Press, New Haven (1974).

5
Montague R., *Universal grammar*. Theoria, vol. 36 (1970) 373 – 398.

Moore E.G., External and internal implications. Proceedings of Aristotelian Society, n.s. XX (1920) 40 – 62.

Moore R.C., *Possible-world semantics for autoepistemic logic*. Proc. of the Workshop on non-monotonic reasoning, Association of the Advancement of Artificial Intelligence (1984) 344 – 354.

Morgan C.C., Programing from specifications. Prentice Hall (1998) 331pp.

Moschovakis J.R., *Intuitionistic Logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2015) 5pp.

Mostowski A.S., *On a generalisation of quantifiers*. Fundamenta Mathematicae, vol. 44 (1957) 12 – 36.

Mostowski A.W. and Pawlak Z., *Logics for engineers*. PWN, Warsaw (1970) 315pp.,in Polish.

Moszkowski B., *A temporal logic for multi-level reasoning about hardware*. Report Nos: STAN-CS-82-952, Stanford, CA. (1982) 30pp.

Moszkowski B., *Executing temporal logic programs*. Cambridge University Press (1983), ACM SIGART Bulletin, ACM, New York, issue 98, Oct. (1986) p.15.

Möller B. and Beer M., *Fuzzy Randomness*. Springer, Berlin, Heidelberg (2004) 307pp.

Munarini E., *Generalised q-Fibonacci numbers*. Fibonacci Quarterly 43, no.3 (2005) 234 – 242.

Murata T., *Petri nets and their application*. Journal Soc. Instrum. Control Eng. vol. 22 (1983), 65 – 72.

Murata T., *Petri nets: properties, analysis and applications*. Proceedings of the IEEE, vol.1, no.4 (1989) 541 – 580.

Muravitsky A.Y., *New mathematical foundations for knowledge maintenance systems: research program.* National Science Foundatin Oct.(1994) 13pp.

Muzio J.C. and Wesselkamper T.C., Multiple-valued switching theory. Adam Hilger Ltd, Bristol and Boston (1986) 195pp.

Nakamura A., *Conflict logic with degrees, rough fuzzy hybridisation – a new trend in decison-making*, (Pal S. K. and Skowron A. eds.), Springer (1999) 136 – 150.

Nanda S. and Majumdar S., *Fuzzy rough sets*. Fuzzy Sets and Systems, vol. 45, no. 2 (1992) 157 – 160.

Navara M., *Satisfiability in fuzzy logics.* Neural Network World vol.10 (2000) 845 – 858.

Nelson D., *Constructible falsity*. Journal of Symbolic Logic, vol. 14 (1949) 16 – 26.

Neumann J. von., *Grundlagen der Quantenmechanik*, Springer Verlag, Berlin (1932): *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, New Jersey, Beyer R.T. ed.,translated from the German (1955) 445pp.

Ngan R.T., Son L.H, Ali M., Tamir D.E., Rishe N.D. and Kandel A., *Representing complex intuitionistic fuzzy set by quaternion numbers*. Applied Soft Computing Journal, Elsevier, vol. 87 (2020) 1 – 15.

Olivetti N., *Introduction to non-monotonic reasoning*. Master Recherche SIS Université Paul Cezanne, CNRS LSIS (2010/11) 36pp: http://www.lsis.org/olivetti/TEACHING/MASTER/INTRONMR1.pdf.

Parikh R.J., *On context-free languages*. Journal of the ACM, vol. 13, no. 4 (1966) 570 – 581.

Parikh R.J., *Propositional logics of programs: new directions*. International Symposium on Fundamentals of Computation Theory, LNCS, Springer, vol. 158 (1983) 347 – 359.

Parikh R.J., *Belief revision and splitting languages* . Logic, Language, and Computation, CSLI Publications, vol. 2 (1999) 19pp.

Parikh R.J. and Pauly M., *Game logic – an overview*. Studia Logica, vol. 72, issue 2 (2003) 165 – 182.

Pacuit E., *Dynamic epistemic logic II: Logics of information change*. Philosophy Compass (2013) 1 – 19.

Paraconsistency: the logical way to the inconsistent. The second world congress on Paraconsistency, Juquehy-Sao Sebasstiao, Sao Paulo, Brazil, Camielli W.A., Coniglio M. and D'ottaviano I.M.L. eds., CRC Press (2002) 376pp.

Parry W.T., Implication, Thesis, Harvard Univ. (1932).

Parry W.T., *Ein axiomensystem für eine neue art von 392mplication (analytische implication).* Ergebnisse eines mathematischen Kolloquiums, vol. 6 (1933) 4 – 5.

Pawlak Z., *Automated theorem proving*. PZWS Warsaw (1965),in Polish.

Pawlak Z., *Mathematical machines.* PZWS Warsaw (1971),in Polish.

Pawlak Z., *Rough sets*. International Journal of Computer and Information Sciences, vol. 11 (1982) 341 -356.

Pawlak Z., *Rough sets-theoretical aspect of reasoning about data.* Kluwer Academic Publishers, Dodrecht, Boston (1991) 229pp.

Perrussel L., Marchi J. and Zhang D., *Characterizing relevant belief revision operators*. AI 2010, In: LNAI, Springer-Verlag, Berlin / Heidelberg, LNAI vol. 6464 (2010) 42 – 51.

Peters J.F., *Time and clock information systems: concepts and roughly fuzzy Petri net models*. In: Rough Sets in Knowledge Discovery, Heidelberg:Physica-Verlag, vol. 2, pp. 385-418, 1997.

Peters J.F., *New frontiers in scientific discovery - Commemorating the life and work of Zdzislaw Pawlak*. Fundamenta Informaticae, vol. 75, no.1 - 4 (2007) 407 – 433.

Peters J.F., Skowron A., Suraj, Z. and Ramanna, S., *Guarded Transitions in Rough Petri Nets*. In: Proc. of the 7th European Congress on Intelligent Techniques & Soft Computing, Aachen, Germany, September 13-16 (1999) 1 – 6.

Peters S. and Westerståhl D., *Quantifiers*. Draft (2002) 77pp: https://web.stanford.edu/group/nasslli/courses/peters-wes/PWbookdraft2-3.pdf.

Peterson G.L., *Myths about the mutual exclusion problem*. Information Processing Letters, vol. 12, no. 3 (1981) 115 – 116.

Peterson J.L., *Petri net theory and the modeling of system*s. Prentice Hall, Englewood Cliffs, N.J. (1981) 280pp.

Petri C. A., *Kommunikation mit Automaten*, Institut für Instrumentelle Mathematik, Schriften des IIM Nr. 3, Bonnn (1962) 132pp.

Petry F.E., *Fuzzy databases : principles and applications ; with contribution by Patrick Bosc*. Kluwer Academic Publishers, Boston, Dordrecht, London (1996) 226pp.

Pickover C.A. ed., *Chaos and Fractals*. A Computer Graphical Journey, Elsevier (1998) 451pp.

Pigozzi D., *Data types over multiple-valued logics*. Theoretical Computer Science 77 (1990) 161 – 194.

Pinna G.M., *How much is worth to remember? A taxonomy based on Petri nets unfoldings*. 32nd Int. Conference Petri Nets, Newcastle, UK, Proceedings, Kristensen L.M. and Petrucci L eds., Springer, Heidelberg, Dordrecht, London, New York (2011) 109 – 128.

Platek R.A., *Foundation of recursion theory*. PhD thesis, Department of Mathematics, Stanford University (1966) 219pp.

Plaza J.A., *Logic of public communication*. 4th International Symposium on Methodologies for Intelligent Systems. Emrich M.L., Pfeifer M.S., Hadzikadic M. and Ras Z.W. eds. (1989) 201 – 216.

Pnueli A., *The temporal logic of programs*. The 18th Annual Symposium on Foundations of Computer Science, IEEE (1977) 46 – 57.

Pogonowski J. and Smigerska J., *Generalised quantifiers*. Adam Mickiewicz University, Poland (2008) 88pp., in Polish: https://www.yumpu.com/xx/document/view/37113011/o-kwantyfikatorach-uogalnionych-zakaad-logiki-stosowanej-uam/23.

Poliard S., *Philosophical introduction to set theory*. University of Notre Dame Press (1990) 180pp., Courier Dover Publications (2015) 192pp.

Portoraro F.D., *Automated Reasoning*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed.,University of Toronto,Canada (2001) 23pp.

Pottinger G., *Uniform cut-free formulations of T, S4 and S5* (abstract). Journal of Symbolic Logic, vol. 48, no.3 (1983) 900.

Pratt V.R., *Six Lectures on Dynamic Logic*, Foundations of Computer Science III, Part 2, 53-82, Mathematical Centre Tracts 109, Amsterdam (1980) 53 – 82.

Pratt V.R., A decidable μ-calculus (Preliminary Rept.). Proc. 22nd IEEE Symp. on Foundations of Computer Science (1981) 421 – 427.

Priest G., Tanaka K. and Weber Z., *Paraconsistent logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2013) 20pp.

Prior A.N., Time and Modality, Clarendon, Oxford (1957), Westport Conn., Greenwood Press (1979) 148pp.

Prior A. N., Past, Present and Future, Clarendon, Oxford (1967) 217pp.

Radhika C., R. Parvathi R. and Karthikeyani Visalakshi N., *Noise removal from images using intuitionistic fuzzy logic controller*. Annals of Fuzzy Mathematics and Informatics (2016) 15pp.

Rajarajeswari P. and Uma N., *Advanced fuzzy intuitionistic logic techniques in image processing*. Indian Journal of Computational and Applied Mathematics, vol.1, no.1 (2013) 9 – 20.

Ramot D., Milo R., Friedman M. and Kandel A., *Complex fuzzy sets*. IEEE Transaction on Fuzzy Systems, vol.10, issue 2 (2002) 171 – 186.

Rasiowa H. and Sikorski R., *The mathematics of metamathematics*.Polish Scientific Publishers, Warsaw (1963) 519pp.

Read S., *Hugh MacColl and the algebra of strict implication*. Nordic Journal of Philosophical Logic 3,UK (1998) 59 – 83.

Reisig W., *Petri Nets. An Introduction*; Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1985) 161pp.

Reisig W., *A primer in Petri net design*. Springer - Verlag, Berlin, Heidelberg (1992) 120pp.

Reiter R., *A logic for default reasoning*. Artificial Intelligence, vol. 13 (1980) 81 – 132.

Rewitzky I., *Binary multirelations*. In: Theory and Applications of Relational Structures as Knowledge Instruments. Springer-Verlag, Berlin, Heidelberg, New York, De Swart H., Orlowska E., Schmidt G. and Roubens M. eds. (2003) 259 – 274.

Roberts F. S., *Discrete mathematical models with applications to social, biological and environmental problems*. Englewood Cliffs, Prentice – Hall, N.J. (1976) 559pp.

Robinson J.A., *Theorem proving on the computer*. Journal. Assoc. Comput. Mach., vol. 10, no. 2 (1963) 163 – 174.

Robinson J.A., *Automatic deduction with hyper-resolution*. Internat. Journal. Comput. Math., no.1 (1965) 227 – 234.

Roger Jang J.S. and Gulley N., *MATLAB. Fuzzy logic toolbox. Computation. Visualization. Programming.* User's Guide, Version 1, The MathWorks,Inc. (1997) 208pp.

Ross A.N.C., *Imperatives and Logic*. Theoria vol. 7 (1941) 53-71.

Ross K.A. and Wright C.R.B., *Discrete mathematics*. Prentice Hall Inc. (1992) 753pp.Polish translation, PWN, Warszawa (1999) 899pp.

Ruitenburg W.B.G., *Intuitionistic algebra: theory and sheaf models*. Rijksuniversiteit te Utrecht (1982) 143pp.

Rushdi A.M., Zarouan M., Alshehri T.M. and Rushdi A., *A Modern Syllogistic Method in Intuitionistic Fuzzy Logic with Realistic Tautology*. The Scientific World Journal, Hindawi Publishing Corporation (2015) 12pp: http://dx.doi.org/10.1155/2015/327390

Ruspini E.H.: *A new approach to clustering*. Information and Control 15 (1969) 22–32.

Russell B. *Principles of mathematics*. London (1903), New York (1938) 534pp.

Rybakov V.V., *Rules of inference with parameters for intuitionistic logic*. Journal of Symbolic Logic, vol. 57, issue 3 (1992) 912 – 923.

Saeed B. I. and Mehrdadi B., *Design of an iterative auto-tunning algorithm for fuzzy PID controller*. 25th Int. Congress on Condition Monitoring and Diagnostic Engineering, IOP Publishing, Journal of Physics: Conference Series 364 (2012) 14pp.

Salii V.N., *Binary L-relations*. Izvestiya Vyshych Uchebnych Zavedenii, Matematika 44 / 1 (1965) 133 –145.

Sallak M., Simon C. and Aubry J – F., *A Fuzzy Probabilistic Approach for Determining Safety Integrity Level*. IEEE Transactions on fuzzy systems, vol. 16, no. 1 (2008) 239 – 248.

Samanta S.K. and Mondal T.K., *Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets.* Journal of Fuzzy Mathematics, vol. 9, no. 3 (2001) 561 – 582.

Sart F., *A purely combinatorial approach to deontic logic*. Logique & Analyse, 206 (2009) 131 – 138.

Sasaki U., *Orthocomplemented lattices satisfying the exchange axiom*. J. Sci. Hiroshima Univ, A 17 (1964) 293 – 302.

Schneider M. and Kandel A., *Properties of the fuzzy expected value and the fuzzy expected interval in fuzzy environment*. In: Readings in Fuzzy Sets for Intelligent Systems, Science Direct, Elsevier, Dubois D., Prade H. and Yager R.R. eds. (1993) 258 – 264.

Schütte K., *Intuitionistic predicate calculus*. In: Proof Theory. Grundlehren der mathematischen Wissenschaften (A series of Comprehensive Studies in Mathematics),Springer Verlag, Berlin, Heidelberg (1977) 36 – 55.

Schweizer B. and Sklar A., *Statistical metric spaces*. Pacific J. Math. 10 (1960) 313 – 334.

Schweizer B. and Sklar A., *Associative functions and abstract semi-groups*. Publ. Math. Debrecen 10 (1963) 69 – 81.

Schweizer B. and Sklar A., *Probabilistic metric spaces*. North Holland (1983), see also: Mineola N.Y.,Dover Publications (2005) 313pp.

Scott D., *Advice in modal logic*. In. Philosophical Problems in Logic, Lambert K. (1970) 143 – 173.

Segerberg, K., *A completeness theorem in the modal logic of programs*. Notices of the American Mathematical Society, vol.24 (1977) 522.

Shapiro S., *Classical Logic*. The Stanford Encyclopaedia of Philosophy, Zalta E.N. ed., The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (1995) 26pp.

Sher G., *Logical quantifiers*. EBSCO Publishing (2015) 571 – 595: http://philosophyfaculty.ucsd.edu/faculty/gsher/logical_quantifiers.pdf.

Shor P.W., *Algorithms for quantum computation: discrete logarithms and factoring*. Proc. 35th Symp. on Foundations of Computer Science (1994) 124 – 134.

Skowron A. and Rauszer C., *The discernibility matrices and functions in information systems*, In: Intelligent Decision Support Systems, Handbook of Applications and Advances in Rough Set Theory, Słowiński R. ed., Dordrecht, Kluwer Academic Publishers (1991) 331–362.

Skowron A. and Suraj Z., *A parallel algorithm for real-time decision making: a rough set approach*. Journal. of Intelligent Information Systems, vol. 7 (1996) 5 – 28.

Słupecki J. and Borkowski L., *Elements of mathematical logic and set theory*. International series of monographs in pure and applied mathematics, Oxford, New York, Pergamon Press, vol. 96 (1967) 349pp.

Smarandache F., *A unifying field in logics: neutrosophic logic*. Multiple-Valued Logic. An International Journa*l*, vol. 8, no.3 (2002) 385 – 438.

Soleimanifard S., Gurov D. and Huisman M., *ProMoVer: Modular verification of temporal safety properties*. SEFM 2011, LNCS , Barthe G., Prado A. and Schneider G. eds., Springer-Verlag, Berlin, Heidelberg (2011) 366 – 381.

Speyer D. and Sturmfels B., *The tropical Grassmannian*. Advances in Geometry, Walter de Gruyter, vol. 4, no. 3 (2004) 389–411.

Spivey J.M., *The Z notation: a reference manual*. Programming Research Group, University of Oxford, England (1998) 158 pp.

Stefanoiu D. and Ionescu F., *Fuzzy – statistical reasoning in fault diagnosis*. In: Computational Intelligence in Fault Diagnosis, Springer-Verlag, London Limited, Palade V., Bocaniala C.D. and Jain L. eds. (2006) 125 – 178.

Ştefanoiu D., Culiţă J. and Ionescu F., *Fast methods for identification of vibration defects.* University Politechnica of Bucharest, Sci. Bull., Series C, vol. 71, no 1 (2009) 33 – 44.

Štepnička M., *Fuzzy transform and its applications to problems in engineering practice.* Ph.D. thesis, University of Ostrava, Faculty of science, Depattment of mathematics (2007) 194pp.

Stillwell J., *Roads to infinity. The mathematics of truth and proof*. CRC Press, (2010) 250pp.

Stirling C.P., *Modal and temporal logics of processes*. Springer, Heidelberg (2001) 190pp.

Suber P., *A bibliography of non-standard logics*. Electronic hand-out for the course, Logical Systems.Earlham College (1997 - 2000) 15pp.

Suber P., *Paradoxes of material implication*. Electronic hand-out for the course, Logical Systems.Earlham College (1997) 4pp.

Sugeno M., *Theory of fuzzy integrals and its applications*. Tokyo Institute of Technology, Ph.D. thesis (1974) 124pp.

Sugeno M. and Kang G., *Structure identification of fuzzy model*. Fuzzy Sets and Systems, vol. 26 (1988) 15–33.

Sujana J. G. and Rajalaxmi T.M., *Forcing sets in triangular grid networks*. Procedia Computer Science, Elsevier, vol. 172 (2020) 159 – 164.

Sulkunen P. and Törrönen J., The production of values. The concept of modality in textual discourse analysis. Semiotica 113 - 1/2 (1997) 43 – 69.

Suraj Z., *Discovery of concurrent data models from experimental tables: A rough set approach*. Fundamenta lnformaticae, vol. 28 (1996) 3 – 4.

Surikova S., *Model of developing pupils'social competence and its algorithm of introducing in the primary school teaching/learning process*. Proc. of the ATEE Spring University,Klaipeda University (2007) 253 – 263.

Svozil K., *Quantum logic*. Springer-Verlag (1998) 221pp.

Swart H.C.M. de., *An intuitionistically plausible interpretation of intuitionistic logic*. Journal of Symbolic Logic, vol. 42, issue 4 (1977) 564 – 578.

Symons F.J.W., *Modelling and analysis of communication protocols using numerical Petri nets*. PhD thesis, Dep. Elec. Eng. Sci., Univ. Essex, Telecommun. Syst. Group Rep. 152, May 1978. See also: *Introduction to numerical Petri nets, a general graphical model of concurrent processing systems*. Australian Telecommunications Research, vol.14, no.1 (1980) 28 – 32.

Szmidt E. and Kacprzyk J., *Distances between intuitionistic fuzzy sets*, Fuzzy Sets Syst., vol. 114, no. 3 (2000) 505 – 518.

Tabakow I.G., *Boolean Interpreted Petri Nets: The Behaviour Equivalence, Reduction and PN-Fault Detectability Considerations*. Petri Net Newsletter 32,Germany (1989) 8 -17.

Tabakow I.G., *Using D-algebra to generate tests for m-logic combinational circuits*. International Journal of Electronics 75, UK (1993) 897 – 906.

Tabakow I.G., *Using Petri net invariants in system diagnosis*; Petri Net Newsletter 58 (2000) 21 – 31.

Tabakow I.G., *Digital circuits and systems: test generation and fault distinguishability. A mathematical approach*. Oficyna Wyd. TU Wroclaw, Wroclaw University of Technology, ISBN 83-7085-551-2 (2001) 193pp.

Tabakow I.G., *Using place invariants to isolate faults in concurrent systems*; Petri Net Newsletter 68, Germany (2005) 10 – 20.

Tabakow I.G., *Fault Diagnosis of Discrete Event Systems Using Place Invariants*; Ninth International Conference on Knowledge-Based & Intelligent Information & Engineering Systems KES'2005, Invited Session on Communicative Intelligence, Melbourne, Australia, September 14 - 16 (2005) in: LNCS, Springer-Verlag Vol. 3682 (2005) 541 - 547.

Tabakow I.G., *An introduction to fuzzy propositional calculus using proofs from assumptions*. +10th Int. Conference on Knowledge Based Intelligent Information & Engineering Systems, Bournemouth, UK Main Conference date: 9, 10 & 11 October 2006 Invited Session on Communicative Intelligence 2006. In: LNAI, Springer-Verlag, Berlin / Heidelberg, LNAI vol. 4252, Proceedings, Part II, Gabrys B., Howlett R.J., and Jain L.C. eds. (2006) 187 – 194.

Tabakow I.G., *Fault Distinguishability of Discrete Event Systems*. Int.Conf.on Computational Science, Workshop on Knowledge and Information Management in Computer Communication Systems KIMCCS'06, University of Reading, UK, May 28-31 (2006). In: LNCS, Springer-Verlag, Berlin / Heidelberg Vol. 3993 , Proceedings, Part III, Alexandrov V.N., G.D. van Albada, P.M.A. Sloot and Dongarra J. Eds. (2006)168 – 175.

Tabakow I.G., *Using Place Invariants and Test Point Placement to Isolate Faults in Discrete Event Systems*. Journal of Universal Computer Science, vol. 13, no.2 (2007) 224 – 243.

Tabakow I.G., *Using place and transition fault net models for sequential diagnosis time assessment in discrete event systems*. 21st International Conference on Industrial, Engineering and Other Applications of Applied Intelligent Systems, IEA / AIE, Wroclaw, Poland, LNCS 5027 (2008) 677 – 686.

Tabakow I.G., *A generalised Łukasiewicz's t-norm system: definition, properties and applications*. The Seventeenth International Conference on System Science, Wrocław (2010). In Advances in Systems Science, Academic Publishing House EXIT, Warsaw (2010) 373 – 383.

Tabakow I.G., *Using generalised Łukasiewicz's t-norm to represent and improve fuzzy rough approximations*. Advances in Computer Science: an International Journal vol.3, issue 6, no 12 (2014) 1 – 9.

Takagi T. and Sugeno M., *Fuzzy identification of systems and its application to modeling and control.* IEEE Transactions on Systems, Man, and Cybernetics, vol. 15 (1985) 116–132.

Takeuti G. and Titani S., *Intuitionistic Fuzzy Logic and Intuitionistic Fuzzy Set Theory*. The Journal of Symbolic Logic, vol. 49, issue 3 (1984) 851 – 866.

Takeuti G. and Titani S., *Global intuitionistic analysis*. Annals of pure and applied logic, vol. 31 (1986) 307 – 339.

Takeuti G. and Titani S., *Globalization of intuitionistic set theory*. Annals of Pure and Applied. Logic, vol. 33 (1987) 195 – 211.

The Free Encyclopaedia. The Wikimedia Foundation, Inc., San Francisco, California, USA

The internet encyclopaedia of philosophy. Fieser J. and Dowden B. eds. (2001).

The little encyclopaedia of logic. Publishing House Ossolińskich , Warsaw (1988) 256pp.,in Polish.

Tripathy B.K., Satapathy M.K. and Choudhury P.K., *Intuitionistic Fuzzy Lattices and Intuitionistic Fuzzy Boolean Algebras*. International Journal of Engineering and Technology, vol.5, no. 3 (2013) 2352 – 2361.

Turing A.M., *On computable numbers, with an application to the entscheidungsproblem*. Proc. London Math. Soc.,no. 42 (1936) 230 – 265.

Turner W., *Logic*. *The catholic encyclopaedia*, Knight K. ed.. Robert Appleton Comp..Online (1999). 10pp.

Turunen E., *On Brouwerian valued fuzzy logic*. Università di Pisa. Dipartimento di Mathematica, vol.2, issue 14 (1989) p.461. See also: *On fuzzy intuitionistic logic*. Annales Univ. Sci., Sect. Comp. 12, Budapest (1991) 241 – 247.

Udrescu-Milosav M., *Quantum circuits engineering: efficient simulation and reconfigurable quantum hardware*. Ph.D. Thesis, Politehnica Univ. of Timişoara (2005) 156pp.

Umezawa T.,. *On logics intermediate between intuitionistic and classical predicate logic*. Journal of Symbolic Logic, vol. 24, issue 2 (1959) 141 – 153.

Umezawa T., *On some properties of intermediate logics*. Journal of Symbolic Logic, vol. 25, issue 3 (1960) 297 – 298.

Van Der Hoeven G. and Moerdijk I., *Sheaf models for choice sequences*. Annals of Pure and Applied Logic, North-Holland, vol. 27 (1984) 63 – 107.

Väänänen J., *Generalised quantifiers*. Bulletin of the European Association for Theoretical Computer Science, vol. 62 (1997) 115 – 136: reprinted in *Generalised Quantifiers and Computation*, Lecture Notes in Computer Science, vol. 1754, Springer (1999).

Venema Y., *Lectures on the modal μ-calculus*. Inst. For Logic, Language and Computation, Univ. of Amsterdam (2008) 156pp.

Venkata S.R.P., Fuzzy temporal predicate logic for incomplete information. International Conference on Fuzzy Theory and Its Applications (IEEE),Yilan, Taiwan (2015) 86 – 90.

Vermeulen J., *Stationary sets and on the existence of homeomorphisms between them*. BSc report Applied mathematics, Delft University of Technology, Delft (2020) 34pp.

Vetter B., *Potentiality. From dispositions to modality*. Oxford Univ. Press (2015) 330pp.

Vilenkin N. Ya. and Shreider Yu. A., Majority spaces and majority quantifier. Semiotics and Informatics, vol. 8, Moscow (1977), in Russian.

Vollmer H., *A generalised quantifier* concept *in computational complexity theory*. ESSLLI'97, Lecture Notes in Computer Science, vol. 1754 (1999) 99 – 123.

Wald A., *On a statistical generalization of metric spaces*. *Proc. Nat. Acad. Sci. USA* , 29 (1943), 196 – 197.

Walukiewicz I., *Completeness of Kozen's axiomatisation of the propositional μ-calculus*. Logic in Computer Science, LICS'95, IEEE (1995) 14 – 24.

Wang D., *A CUSUM control chart for fuzzy quality data*. Advances in Soft Computing 6, Springer-Verlag, Berlin, Heidelberg (2006) 357 – 364.

Wang Weijie, Ren G. and Wang Wei., *The applications of rough set theory in civil engineering*. Artificial Intelligence and Computational Intelligence: International Conference, AICI 2010, Sanya, China, , Proceedings, vol.1 (2010) 27 – 31.

Wang Z.-X., *Fuzzy measures and measures of fuzziness*. Journal of Mathematical Analysis and Applications, Elsevier, vol. 104 (1984) 589 – 601.

Wansing H., *Constructive negation, implication, and co-implication*. Journal of Applied Non-Classical Logics, vol. 18 (2008) 341 – 364.

Westerståhl D., *Quantifiers*. Göteborg University (2001) 26 pp: file:///F:/HP%20info%20programs%20and%20papers%202018%20state/2001%20BlackwellQuant%20Quantifiers%20WESTERSH.pdf.

Westerståhl D., *Generalised quantifiers*. The Stanford Encyclopaedia of Philosophy, The Metaphysics Research Lab at the Center for the Study of Language and Information, Stanford University, Stanford, CA (2011) 25pp. See also: *Generalised quantifiers: linguistics meets model theory*. Stockholm University,,41pp: https://www.philosophy.su.se/polopoly_fs/1.165898.1391710202!/menu/standard/file/2013%20CambridgeHandbookGQ.pdf.

Whitehead A.N. and Russell B., *Principia mathematica*. Cambridge Univ. Press, Cambridge, A three-volume work (1913).

Wittgenstein L., *Tractatus Logico-Philosophicus*. Routledge and Kegan Paul, London (1922) 75pp.

Wright G.H. von., *Deontic logic*. Mind vol. 60 no. 237, Oxford Univ. Press (1951) 1 – 15.

Wright G.H. von., *An essay in modal logic*. North-Holland Publishing, Amsterdam (1951) 90pp.

Wu C., Yue Y., Li M. and Adjei O., *The rough set theory and applications*. Engineering Computations, Emerald Group Publishing, vol. 21, no. 5 (2004) 488 – 511.

Wu D. and Tan W.W., *A simplified type-2 fuzzy controller for real-time control*. ISA Transactions, Elsevier, vol. 15, issue 4 (2006) 503 – 516.

Wu D., *A brief tutorial on interval type-2 fuzzy sets and systems.* University of Southern California, Scholar articles (2010) 1 – 13: in Fuzzy Sets and S, ystems (2010).

Xu Z.S. and Yager R.R., *Some geometric aggregation operators based on intuitionistic fuzzy sets*. Int. J. Gen. Syst., vol. 35, no. 4 (2006) 417 – 433.

Xu Z.S. and Yager R.R., *Intuitionistic Fuzzy Bonferroni Means*. IEEE Trans. on Systems, Man, and Cybernetics, Part B: Cybernetics, vol.41, no.2 (2011) 568 – 578.

Yager R.R., *Connectives and quantifiers in fuzzy sets*. Fuzzy Sets and Systems, vol. 40 (1991) 39 – 75 .

Yager R.R., *A general approach to rule aggregation in fuzzy logic control*. Applied Intelligence, vol. 2 (1992) 333 – 351.

Yager R.R., *Aggregation operators and fuzzy systems modeling*. Fuzzy Sets and Systems 67 (1994) 129 – 145.

Yager R.R., *Pythagorean fuzzy subsets.* Proceedings of the 9th Joint World Congress on Fuzzy Systems and NAFIPS Annual Meeting , IFSA/NAFIPS (2013) 57 – 61.

Yager R.R. and Abbasov A.M., *Pythagorean membership grades, complex numbers and decision making*. International Journal of Intelligent Systems, vol. 28, no.5 (2013) 436 – 452.

Yang M. - S. and Hussain Z., *Fuzzy entropy for Pythagorean fuzzy sets with application to multicriterion decision making*. Wiley / Hindawi (2018) 14pp: https://doi.org/10.1155/2018/2832839

Yefremov A., Smarandache F. and Christianto V., *Yang-Mills field from quaternion space geometry, and its Klein-Gordon representation*. Progress in Physics, IOP publishing, vol. 3 (2007) 42 – 50.

You L., Zhang J., Li Q. and Ye N., *Structural reliability analysis based on fuzzy random uncertainty*. Maintenance and Reliability, vol.21, no. 4 (2019) 599 – 609.

Yun S.M. and Lee S.J., *New approach to intuitionistic fuzzy rough sets*. International Journal of Fuzzy Logic and Intelligent Systems. Korean Institute of Intelligent Systems, vol. 20, no. 2 (2020) 129 – 137. http://www.ijfis.org/journal/view.html?uid=915&&vmd=Full

Zadeh L.A., *Fuzzy sets*. Information and Control 8 (1965) 338 – 353.

Zadeh L.A., *Fuzzy logic and its application to approximate reasoning*. Information Processing 74 (1974) 591 – 594.

Zadeh L.A., *The concept of a linguistic variable and its application to approximate reasoning-1*. Information Sciences, vol. 8 (1975) 199 – 249.

Zadeh L.A., *A computational approach to fuzzy quantifiers in natural languages.* Computer Mathematics with Applications, vol. 9 (1983) 149 – 183.

Zermelo E., *Untersuchungen über die grundlagen (Axiome) der Mengenlehre I*. Mathematische Annalen, vol. 65 (1908) 261 – 281.

Zhang X.L. and Xu Z.S., *Extension of TOPSIS to multiple criteria decision making with pythagorean fuzzy sets*. International Journal of Inteligent Systems, vol. 29, no. 12 (2014) 1061 – 1078.

Zhou M.C. and DiCesare F., *Petri net synthesis for discrete event control of manufacturing systems*; Kliwer Academic, Dordrecht, Boston, London (1993) 233pp.

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