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THE CAPITAL ASSET PRICING MODEL OVERVIEWED[1]

Abstract

One of the dogmas of contemporary financial risk management comprises the assumption that the returns of an asset can be modelled as linearly dependent on the risk-free rate and the market returns; and the resulting Capital Asset Pricing Model (CAPM) has gained great popularity. The contribution focuses upon the CAPM and discusses its appropriateness, with attention being paid to validity of the model.

1. The exposition

Every textbook on financial risk management (although the exception makes the rule) covers as one of its topics the Capital Asset Pricing Model (CAPM) which has become throughout the last four decades one of the basic theoretical instruments to explain the workings of capital markets and the behaviour of investors. Its derivation and theoretical justification lie in the hands of the founders of financial economics Jack Treynor, William Sharpe, John Lintner and Jan Mossin who independently of one another formalized, under a set of assumptions, the existence of this model. The CAPM implies that the expected return of a risky asset or of a portfolio is linearly linked to the expected return of the market portfolio in terms of excess returns above risk-free rate. Much is said of the rationale behind the CAPM and of its implications for the decision making of investors; yet little attention is paid to the issue of its validity and stability. The contribution discusses the question of the validity of the model. To this end, a series of statistical tests fit for testing whether the CAPM empirically holds is introduced and the said tests are performed on a set of U.S. data.

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2. The formula and the assumptions

It is not the aim of the contribution to ponder on the legitimacy of the model insomuch as its full derivation or the interpretation is readily available in literature. It is hence assumed that the reader is familiar with the interpretation issues concerning the CAPM. However, if this be not the case, the reader may acquaint himself with the theoretical derivation, e.g. in [4] or [5]. It must be said that there are many versions of the model and numerous modifications stemming from attempts to best reflect the reality of financial markets. In principle, the CAPM may appear either in the Sharpe[2] and Lintner[3] variant, or in the Black[4] version. The distinction consists in their attitude towards the existence of lending and borrowing facilities at a riskless rate of interest. Whilst Sharpe and Lintner admit that an investor is able to borrow or lend at a riskless rate of interest, the Black version operates in the milieu with the absence of riskless assets.

– The Sharpe and Lintner formula of the CAPM therefore reads for the expected return of the *i*-th asset at a given time in this way:

$$\mathrm{E}R_i = R_f + \beta_i(\mathrm{E}R_m - R_f)\,, \tag{1a}$$

$$\beta_i = \frac{\operatorname{cov}(R_i, R_m)}{\mathrm{D}R_m}\,, \tag{1b}$$

where R_i stands for the return of risky asset i, R_m denotes the return on the market portfolio and R_f is the return on the risk-free asset (i.e. the riskless rate of interest). Here it needs be accentuated that in the traditional *ansatz* the riskless rate R_f is deemed as non-stochastic and constant over time. Contrariwise, it is observed that the riskless rate R_f is stochastic which motivates, particularly in practical implementations, the treatment of (1a) and (1b) under excess returns. If $\gamma_i := R_i - R_f$ represents the return on the i-th asset in excess of the risk-free rate and, similarly $\gamma_m := R_m - R_f$ signifies the excess return on the market portfolio (often referred to as market premium or risk premium), equations (1a) and (1b) go into this form:

$$\mathrm{E}\gamma_i = \beta_i \mathrm{E}\gamma_m\,, \tag{2a}$$

$$\beta_i = \frac{\operatorname{cov}(\gamma_i, \gamma_m)}{\mathrm{D}\gamma_m}\,, \tag{2b}$$

[2] Sharpe's derivation is to be found in the article W.F. Sharpe, "Capital asset prices: A theory of market equilibrium under conditions of risk", *Journal of Finance* 1964, 19 (3), pp. 425-442.

[3] Lintner presented his derivation in the article "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics 1965*, 47 (1), pp. 13-37.

[4] Black's approach was published in: F. Black, "Capital market equilibrium with restricted borrowing", *Journal of Business*, 1972, 45, pp. 444-454.

Providing that riskless rate R_f were non-stochastic and constant, equations (1a), (1b) and (2a), (2b) would be equivalent. To reflect the true conditions of capital markets, it is recommendable to employ the variant presented under equations (2a) and (2b), which removes the stochastic influence of the riskless rate on the estimation of β_i.

– The formula of Black is more general as it operates with the zero-beta[5] portfolio rather than with the riskless rate of interest. In the environment where there is no risk-free asset, the expected return on the i-th asset is supposed to be linearly related to its beta by virtue of the excess of the return on the zero-beta portfolio. The Black version assumes that the expected return of the i-th asset at a certain time is governed by the formula

$$\mathrm{E}R_i = \mathrm{E}R_{om} + \beta_i^* (\mathrm{E}R_m - \mathrm{E}R_{om}), \tag{3a}$$

$$\beta^* = \frac{\operatorname{cov}(R_i, R_m)}{\mathrm{D}R_m}. \tag{3b}$$

The econometric analysis of this model is comparatively complicated as the zero-beta portfolio is not observable and its return is an unobserved quantity.

The CAPM model was derived to describe economic equilibrium between rational agents who make decisions founded on the expectations of their future wealth under the belief that the standard deviation best assesses and measures their risks. A set of assumptions was therewith made which can be found in [4]. One of them is the assumption of mean-variance efficiency of capital markets, which underlies the validity of the CAPM.

The contribution focuses upon the Sharpe and Lintner version of the CAPM. The following explication presents, in line with the above-declared aim, an econometric model of this version and develops suitable statistical tests for its empirical verification. For this purpose the model of linear regression

$$\gamma_i = \alpha_i + \beta_i \gamma_m + \varepsilon_i \tag{4}$$

is taken under advisement and it is tested whether the intercept α_i can be thought of as zero. Should it be possible to find the intercept to be zero, it would be suggestive of the empirical trueness of the CAPM. In contrast, statistical evidence against the null hypothesis of its insignificance (nullity) would be at odds with the theoretical development and empirical validity of the model. This must be done not individually for each asset available but for all assets in the market, though.

[5] The zero-beta portfolio is defined to be the portfolio of the minimum variance amongst all portfolios uncorrelated with the market portfolio.

3. The econometric model of the Sharpe and Lintner CAPM

Equations (2a) and (2b) pertain to one asset at a given time and the time subscript was thus omitted. For the CAPM to be valid, it needs be true at an arbitrary time and for all capital assets and sub-sets of them available in the market (as well as for their portfolios). Accordingly, in the contribution N capital assets over T time periods are considered and the model – in the sense of Sharpe and Lintner's equations – reads at any time t

$$\gamma_t = \boldsymbol{\alpha} + \boldsymbol{\beta}\,\gamma_{mt} + \boldsymbol{\varepsilon}_t, \tag{5a}$$

whereas it is required

$$\mathrm{E}\boldsymbol{\varepsilon}_t = \mathbf{0}, \;\&\; \mathrm{cov}\boldsymbol{\varepsilon}_t = \textstyle\sum, \;\&\; \mathrm{E}\gamma_{mt} = \mu_m, \;\&\; \mathrm{D}\gamma_{mt} = \sigma_m^{\,2}, \;\&\; \mathrm{cov}(\boldsymbol{\varepsilon}_t, \gamma_{mt}) = \mathbf{0}. \tag{5b}$$

In (5a) and (5b) $\boldsymbol{\gamma}_t$ is an $(N \times 1)$ vector of excess returns for N assets (or portfolios of assets) at time t, $\boldsymbol{\alpha}$ denotes the $(N\times 1)$ vector of the intercepts and $\boldsymbol{\beta}$ the $(N\times 1)$ vector of the beta coefficients of the regression, γ_{mt} is a scalar to represent the aforesaid market premium at time t, and finally $\boldsymbol{\varepsilon}_t$ stands for the $(N \times 1)$ vector of disturbances.

The estimates for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are obtainable via the ordinary least squares (OLS) method, i.e. by minimizing the expression

$$\left(\boldsymbol{\Gamma}-\overline{\boldsymbol{\Gamma}}\right)^T\left(\boldsymbol{\Gamma}-\overline{\boldsymbol{\Gamma}}\right)=\left(\begin{pmatrix}\boldsymbol{\gamma}_1\\ \vdots \\ \boldsymbol{\gamma}_T\end{pmatrix}-\begin{pmatrix}\boldsymbol{\alpha}+\boldsymbol{\beta}\gamma_{m1}\\ \vdots \\ \boldsymbol{\alpha}+\boldsymbol{\beta}\gamma_{mT}\end{pmatrix}\right)^T\left(\begin{pmatrix}\boldsymbol{\gamma}_1\\ \vdots \\ \boldsymbol{\gamma}_T\end{pmatrix}-\begin{pmatrix}\boldsymbol{\alpha}+\boldsymbol{\beta}\gamma_{m1}\\ \vdots \\ \boldsymbol{\alpha}+\boldsymbol{\beta}\gamma_{mT}\end{pmatrix}\right)=\sum_{t=1}^{t=T}\left(\boldsymbol{\gamma}_t-\boldsymbol{\alpha}-\boldsymbol{\beta}\gamma_{mt}\right)^T\left(\boldsymbol{\gamma}_t-\boldsymbol{\alpha}-\boldsymbol{\beta}\gamma_{mt}\right)$$

with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The vector $\boldsymbol{\Gamma}=(\gamma_1^{T,\dots,}\gamma_T^{T})^T$ denotes the $(NT \times 1)$ vector of T periods for all N assets and $\overline{\boldsymbol{\Gamma}}$ with the components $\boldsymbol{\alpha}+\boldsymbol{\beta}\gamma_{mt}$ represents its OLS counterpart with the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be chosen yet. The OLS estimates $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively are the solution to the equations

$$\left.\frac{\partial\left(\boldsymbol{\Gamma}-\overline{\boldsymbol{\Gamma}}\right)^T\left(\boldsymbol{\Gamma}-\overline{\boldsymbol{\Gamma}}\right)}{\partial\boldsymbol{\alpha}}\right|_{\substack{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}\\ \boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}} = -2\sum_{t=1}^{t=T}\left(\boldsymbol{\gamma}_t-\hat{\boldsymbol{\alpha}}-\hat{\boldsymbol{\beta}}\gamma_{mt}\right) \equiv \mathbf{0}, \tag{6a}$$

$$\left.\frac{\partial\left(\boldsymbol{\Gamma}-\overline{\boldsymbol{\Gamma}}\right)^T\left(\boldsymbol{\Gamma}-\overline{\boldsymbol{\Gamma}}\right)}{\partial\boldsymbol{\beta}}\right|_{\substack{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}\\ \boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}} = -2\sum_{t=1}^{t=T}\left(\boldsymbol{\gamma}_t-\hat{\boldsymbol{\alpha}}-\hat{\boldsymbol{\beta}}\gamma_{mt}\right)\gamma_{mt} \equiv \mathbf{0}. \tag{6b}$$

Having introduced the notation $\overline{\boldsymbol{\gamma}}=\frac{1}{T}\sum_{t=1}^{t=T}\boldsymbol{\gamma}_t$, $\bar{\gamma}_m=\frac{1}{T}\sum_{t=1}^{t=T}\gamma_{mt}$ and $s_m^2=\frac{1}{T}\sum_{t=1}^{t=T}(\gamma_{mt}-\overline{\gamma}_m)^2$, it follows nstantly from (6a) that the OLS estimator $\hat{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ is

$$\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{\gamma}} - \hat{\boldsymbol{\beta}}\overline{\gamma}_m , \tag{7a}$$

and combining it with (6b) after small re-arrangements it is arrived at the OLS estimator $\hat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = \frac{\frac{1}{T}\sum_{t=1}^{t=T}(\boldsymbol{\gamma}_t - \overline{\boldsymbol{\gamma}})(\gamma_{mt} - \overline{\gamma}_m)}{s_m^2} . \tag{7b}$$

It is beyond pointing out that these OLS estimates coincide with the maximum likelihood (ML) estimates if

(A) multivariate (N-variate) normality of the excess returns $\gamma_1, \ldots, \gamma_T$ conditional on market premiums $\gamma_{m1}, \ldots, \gamma_{mT}$, and

(B) independence and identity of the distribution of the excess returns $\gamma_1, \ldots, \gamma_T$ are jointly assumed. Assuming (A) and (B) it is possible to obtain the ML estimate $\hat{\sum}$ for $\sum$ (see [3], pp. 190-191)

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{t=T}\left(\boldsymbol{\gamma}_t - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}}\gamma_{mt}\right)^T\left(\boldsymbol{\gamma}_t - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}}\gamma_{mt}\right) . \tag{7c}$$

Thus, regardless of the assumptions (A) and (B), one has the OLS estimators for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined by equations (7a) and (7b); however, if the assumptions are employed, one has at hand the ML estimators for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$ set by equations (7a), (7b), and (7c). In the latter case it is possible to utilize the theoretical results on the maximum likelihood method.

4. The tests of the validity of the Sharpe and Lintner CAPM

Since all γ_t's are assumed N-dimensional normal, also the $\hat{\boldsymbol{\beta}}$ is – as a linear combination of $\gamma_1, \ldots, \gamma_T$ – N-dimensional normal. Furthermore, for the same reason, the distribution of the $\hat{\boldsymbol{\alpha}}$ is N-dimensional normal, the $\hat{\boldsymbol{\alpha}}$ being a linear combination of the γ_t's and $\hat{\boldsymbol{\beta}}$. By definition (see e.g. [7], p. 534), $T\,\hat{\boldsymbol{\Sigma}}$ is governed by the Wishart distribution. It may be demonstrated that, using the Fisher information matrix (according to Theorem 7.100 stated by [1], pp. 159-160),

$$\hat{\boldsymbol{\alpha}} \sim N_N\left(\boldsymbol{\alpha} ; \frac{1}{T}\left[1 + \frac{\overline{\gamma}_m^2}{s_m^2}\right]\boldsymbol{\Sigma} \right), \tag{8a}$$

$$\hat{\boldsymbol{\beta}} \sim N_N\left(\boldsymbol{\beta} ; \frac{1}{T}\left[\frac{1}{s_m^2}\right]\boldsymbol{\Sigma} \right), \tag{8b}$$

and that

$$T\hat{\boldsymbol{\Sigma}} \sim W_N\left(T-2 ; \boldsymbol{\Sigma}\right) . \tag{8c}$$

To test the null hypothesis H_0: $\boldsymbol{\alpha} = \mathbf{0}$ against the alternative hypothesis H_A: $\boldsymbol{\alpha} \neq \mathbf{0}$ three tools are considered in the contribution: (a) Wald test, (b) likelihood ratio test, and (c) MacKinlay test. The pillar for the first two tests can be found in [1], pp. 178-179, under Remark 8.17, and for the last of the tests in [7], p. 542, under (8b.2.14). The rigorous derivation of the tests is here, for the sake of convenience, omitted.

The Wald test statistics is

$$W = \hat{\boldsymbol{\alpha}}^T \left[\operatorname{cov} \hat{\boldsymbol{\alpha}}\right]^{-1} \hat{\boldsymbol{\alpha}} = \frac{T s_m^2}{s_m^2 + \bar{\gamma}_m^2} \hat{\boldsymbol{\alpha}}^T \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\alpha}}, \tag{9a}$$

where, however, the unknown $\boldsymbol{\Sigma}$ must be replaced by its consistent ML estimator $\hat{\boldsymbol{\Sigma}}$. Under the null hypothesis the Wald statistics W follows asymptotically a chi-square distribution with N degrees of freedom.

The likelihood ratio statistic is constructed by comparing the values of the logarithmic likelihood functions of the unconstrained model (5a) and of the model constrained under the null hypothesis. It may be derived that criterion

$$LR = T \log \left[1 + \frac{s_m^2}{s_m^2 + \bar{\gamma}_m^2} \hat{\boldsymbol{\alpha}}^T \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}} \right] \tag{9b}$$

is distributed asymptotically under a chi-square distribution with N degrees of freedom.

And, finally, MacKinlay, Gibbons, Ross, and Shanken[6] derived that the test statistics

$$MK = \frac{T - N - 1}{N} \frac{s_m^2}{s_m^2 + \bar{\gamma}_m^2} \hat{\boldsymbol{\alpha}}^T \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}} \tag{9c}$$

is, if the null hypothesis is true, distributed central F with N degrees of freedom in the numerator and $T - N - 1$ degrees of freedom in the denominator.

5. An empirical demonstration

For an illustration of the aforesaid statistical issues, the estimation procedure and the tests were conducted on a set of U.S. data. Available were monthly stock prices of 10 U.S. companies represented in S&P 500 Index. Out of the monthly stock prices of Exxon Mobil Corp., General Electric Co., Microsoft Corp., Chevron Corp., AT&T Inc., Procter & Gamble Co., Johnson & Johnson, International Business Machines Corp., Apple Inc., and Conocophillips logarithmic returns were constructed and so

[6] See A.C. MacKinlay, „On multivariate tests of the CAPM", *Journal of Financial Economics 1987*, 18, pp. 341-372, and M. Gibbons, S. Ross, J. Shanken,. „A test of the efficiency of a given portfolio", Econometrica 1989, 57, pp. 1121-1152.

was done with the monthly values of S&P 500 Index. However, it is needful to note that logarithmic returns of S&P 500 Index (or of any stock index whatsoever) are chosen merely as a proxy to market returns, which are unobservable. The series of stock monthly returns and of S&P 500 Index monthly returns were accompanied by the corresponding series of BBA Libor USD 6 Month rate (stated at % p.a.) to represent the riskless rate. To ensure comparability, the stock returns and index returns were annualized. The series of returns encompassed 240 monthly observations over 20 calendar years from 29 January 1988 up to 31 December 2007.[7] All computations were performed in the environment of Microsoft ® Excel 2003.

The methodological procedure ran in this fashion: The series of 20 years of observations was divided into four non-overlapping 5-year sub-periods, each counting 60 monthly observations, to which two non-overlapping 10-year sub-periods corresponded, each counting 120 monthly observations. The estimation of α's and β's and the application of the three described tests were first effected on the 5-year sub-periods (M1/1988 – M12/1992, M1/1993 – M12/1997, M1/1998 – M12/2002, M1/2003 – M12/2007), further on the 10-year sub-periods (M1/1988 – M12/1997, M1/1998 – M12/2007), and eventually on the entire 20-year period (M1/1988 – M12/2007).

The results of the tests are reported in Table 1. It is self-evident that the results diverge across the periods. It is straightforward that especially during the period of M1/1998 – M12/2002 the tests all together are not rejective of the validity of the Sharpe–Lintner version of the CAPM (and, as this sub-period has its influence upon the period of M1/1998 – M12/2007, being part of it, the results there are alike). The validity of the Sharpe–Lintner version of the CAPM in the other periods may be seen as rather inconclusive for the interpretation depends heavily upon the choice of the subjective level of significance. At the level of 5% at least one of the tests rejects the null hypothesis and suggests that the Sharpe–Lintner version of the CAPM does not preserve its validity. The consequence of this finding is strong and tied up with the rationale behind the CAPM – if the CAPM does not hold, the market portfolio is not efficient. This is, of course, with relation to the selection of the ten assets.

[7] The authors owe thanks to František Štulajter for support and assistance in obtaining data for the empirical illustration.

Table 1. The results of the testing on the sub-periods and the entire period

The period	No. of observations	Wald statistics	Significance	Likelihood ratio statistics	Significance	MacKinlay statistics	Significance
5-year sub -periods							
M1/1988- M12/1992	60	24.20	0.0071	20.33	0.0263	1.98	0.0567
M1/1993- M12/1997	60	20.35	0.0261	17.52	0.0636	1.66	0.1173
M1/1998- M12/2002	60	4.55	0.9193	4.38	0.9284	0.37	0.9532
M1/2003- M12/2007	60	24.16	0.0072	20.30	0.0265	1.97	0.0572
10-year sub -periods							
M1/1988-M12/1997	120	28.66	0.0014	25.68	0.0042	2.60	0.0072
M1/1998-M12/2007	120	13.81	0.1818	13.07	0.2196	1.255	0.2653
20-year period							
M1/1988-M12/2007	240	22.36	0.0133	21.38	0.0186	2.13	0.0229

The implementation was accomplished without inspecting the fulfilment of the assumptions. For instance, upon effecting Mardina's 1970 tests of multivariate normality (to be found e.g. in [6]), it was found that although γ_1, ..., γ_T may be symmetric, their kurtosis is far from being that of a multivariate normal distribution. It is needless to say that this manifestation is typical of market returns and is consistent with other empirical studies.

6. Conclusion

Over the last three or four decades enough empirical evidence has been obtained against validity of the Capital Asset Pricing Model. Nonetheless, the model still enjoys great popularity for its intuitive and understandable construction resulting from a set of assumptions peculiar to the environment of developed capital markets. Much literature is devoted to the promotion of the model; yet scarcely is its validity questioned. The idea that its advocating cannot go without investigating its validity incites this contribution. Thus motivated, the contribution presents the framework for the estimation of the parameters of the CAPM after Sharpe and Lintner, and acquaints with three statistical tests for judging its validity. The practical part of the contribution contains the results of empirical verification of the Sharpe and Lintner version of the CAPM. The results, as it happens, are diverse and indicative that the CAPM during most of the 20-year period from 1988 to 2007 does not hold with respect to the chosen portfolio of 10 assets. The implication is notable – during 15 of the set 20 years the market portfolio (though with respect to the chosen 10 assets) is not efficient in the sense of Harry Markowitz.

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DYSKUSJA O MODELU WYCENY AKTYWÓW I PASYWÓW

Streszczenie

Jednym z dogmatów zarządzania rynkami kapitałowymi jest założenie, że stopa zwrotu może być modelowana jako funkcja liniowa stopy zwrotu bez ryzyka i rynkowej stopy zwrotu. Cieszący się dużą popularnością model CAPM poddano analizie ze względu na jego odpowiedniość i trafność.