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TWO-LEVEL AGGREGATION OF LEONTIEF MODEL "INPUT-OUTPUT"

1. Introduction

Production and distribution of products in economy are connected by balance correlations which make a basis of all economic-mathematical models of process of reproduction. Any macromodel of economy as complete developing system can be considered as some set of macrocorrelations relative to macrovariables, that is a result of aggregation of the certain microcorrelations. Therefore it is urgent to develop the theory of aggregation of the economic descriptions, as the methods of aggregating are the tool of a substantiation of mathematical models of economy. So, the opening by W.W. Leontief of relative stability of proportions of interindustry inputs promoted distribution of models of economic growth.

The Leontief model "input-output" for a long time is recognized as one of the basic tools of the empirical analysis of structure of economy and mathematical analysis of economic problems. W.W. Leontief has formulated the linear balance model as a result of empirical researches of the balance tables of interindustry connections widely used for the analysis of structure of economy [Leontief 1966; 1990].

In the balance table production is submitted by a set of the conditionally allocated pure branches – productive units producing a homogeneous product. Allocation of pure branches, as well as calculation of outputs of conditionally homogeneous products, till now remain rather subject of art of the economic analysis than strictly proved mathematical procedure [Petrov et al. 1996].

The balance table contains values of total output x_i (t) of pure branches, deliveries of production of branch i into branch j $x_{ij}(t)$, and also the quantities of products $y_i(t)$, which go on final consumption. Let us denote by n the number of the allocated pure branches, $i, j = 1, 2, ..., n$. Formally the table expresses balances of production and distribution of produced output

$$x_i(t)=\sum_{j=1}^{n}x_{ij}(t)+y_i(t),\ \ i,j=1,2,...,n\,. \tag{1}$$

Studying the balance tables during many consecutive intervals of time $t = 1, 2, ..., T$, W.W. Leontief has noticed that it is possible to allocate pure branches so that the values x_{ij}/x_j will depend feebly on time, though x_j and x_{ij} change in the long run. W.W. Leontief has understood that under certain conditions it is possible to consider that with sufficient accuracy

$$x_{ij}(t)=a_{ij}x_j(t)\,, \tag{2}$$

where a_{ij} are constant norms of input of production of branch i on output of a unit of production of branch j.

The matrix A, composed of elements a_{ij}, is called Leontief's matrix of direct inputs. Subject to made assumption about constancy of direct inputs balances of production and distribution of outputs are

$$x(t)=Ax(t)+y(t)\,, \tag{3}$$

where $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ are vectors-columns. Hence, the vectors of total inputs $x(t)$ and final inputs $y(t)$ are connected by a linear correlation (3), which is called Leontief model "input-output".

W.W. Leontief emphasized that the calculation on the matrix of direct inputs is insignificantly small part of general work. The basic part of this work is collecting data and constructing a matrix. The matrix is constructed during the long laborious analysis of economic connections in the field of production, distribution and consumption of output [Petrov et al. 1996].

The matrix of direct inputs can be used for prediction calculations so long as there will be no structural improvements in economy. Therefore the problem of substantiation of Leontief model "input-output" is urgent.

The Leontief's balance model is the result of aggregating of some initial microdescription of production. To be internally consistent, Leontief's model should be kept at repeated aggregating to the one-dimensional equation. Thus natural aggregating operator in Leontief's model "input-output" should become a vector of the prices.

2. General formulation of the problem

In modern economic science consisting of two various parts – microeconomy and macroeconomy, the search for connections between the microeconomic descriptions operating detailed parameters and macroeconomic descriptions operating global parameters is actively carried out. The system approach to construction of economic-mathematical models demands to open features and opportunities of the theory of aggregating. The linear balance Leontief's model "input-output" existing in three kinds with a different degree of aggregation – interproduct balance, interindustry balance and macrobalance, is simple enough and at the same time widely used object illustrating opportunities of such system approach.

Thus the consideration of two-level Leontief's of static model "input-output" is natural. Aggregation according to the plan "products → pure branches → national economy" should be considered as two-level aggregation with the specific methods.

3. Results

As to the first level of aggregation of linear static Leontief's model "input-output", the so-called model of interproduct balance, which can be conditionally written down as system of balance correlations(ratio)

$$x_i(t) = \sum_{j=1}^{N} a_{ij} x_j(t) + y_i(t), \qquad i = 1, 2, ..., N, \tag{4}$$

where $N > n$ – number of various products (or various technologies of production), the answer here was given by W.W. Leontief, when he spoke about "collecting" of model of interindustry balance (3) and formation of $n < N$ pure branches as a result of empirical researches of the balance tables of interindustry connections (2).

Therefore here in detail we shall stop only at the second level of aggregation: transition from interindustry model with n pure branches (3) to aggregated in a uniform economic product – GDP and FP.

Let us consider interindustry balance Leontief's model "input-output" (3), written down as scalar correlations

$$x_i = \sum_{j=1}^{N} a_{ij} x_j + y_i, \qquad i = 1, 2, ..., n. \tag{5}$$

Thus we shall consider that a technological matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{6}$$

is non-negative and indecomposable [Ponomarenko et al. 1995]. Non-negativity of elements of the matrix (6) is the evidence that in all branches of considered economy the by-products are not produced. Indecomposability of the matrix A is the evidence that each of branches uses (directly or indirectly) production of all branches of the given economic system. In other words, indecomposability means that there is no subset of branches, functioning of which is isolated without connections with other branches of economic system.

For non-negative indecomposable matrices the theorem of spectral properties by Perron-Frobenius is proved.

Theorem 1 [Ponomarenko et al. 1995]. *Let matrix A of dimension $n \times n$ is non-negative and indecomposable, and $\Lambda(A)$ is a set of its eigen-values: $\Lambda(A) = \{\lambda_1, \lambda_2, ..., \lambda_m\}$, $m \le n$. Then in the set $\Lambda(A)$ there is a positive number $\lambda_A > 0$ such that*

$$|\lambda_k| \le \lambda_A, \quad k = 1, 2, ..., m.$$

Besides to eigen-value λ_A corresponds unique (accurate to scalar multiplier) right x_A and left p_A eigen-vectors such that $(x_A)_i > 0$, $(p_A)_i > 0$ for all $i = 1, 2, ..., n$.

In this case $\lambda_A > 0$ is called Frobenius root, and $x_A > 0$ and $p_A > 0$ – right and left Frobenius vectors accordingly.

The results of the above mentioned theorem can be written down as such correlations:

$$Ax_A = \lambda_A x_A, \quad \lambda_A > 0, \quad x_A > 0;$$
$$p_A A = \lambda_A p_A, \quad \lambda_A > 0, \quad p_A > 0;$$

where: $x = (x_1, x_2, ..., x_n)^T$ – vector-column,
$p = (p_1, p_2, ..., p_n)$ – vector-line.

Definition [Ponomarenko et al. 1995]. *If for any non-negative vector of demand the system (3) is compatible, that is has the non-negative solution $x \ge 0$, the appropriate Leontief's model (and also technological matrix) is called productive.*

It is found out that the productivity of Leontief's model is completely determined by the value of Frobenius root λ_A of matrix A of factors of direct inputs.

Theorem 2 [Ponomarenko et al. 1995] (criterion of efficiency of Leontief's model "input-output"). *For efficiency of Leontief's model (3) it is necessary and enough that the Frobenius root λ_A of matrix A satisfied to the inequality $\lambda_A < 1$.*

In this case matrix $(I - A)^{-1}$ exists and thus $(I - A)^{-1} \ge 0$.

Further we shall consider that the technological matrix A is not only non-negative and indecomposable, but also is productive. Thus, for it the inequalities

$$0 < \lambda_A < 1, \;\; x_A > 0, \;\; p_A > 0 \tag{7}$$

are right.

Let us consider the problem of aggregation of balance interindustry model (5) or in the vector form

$$x = Ax + y, \quad y \ge 0 \tag{8}$$

in one product. Let $p = (p_1, p_2, ..., p_n)$ be aggregating vector-line of prices of products. Multiply both parts of the equation (8) at the left on a vector p, as a result we shall receive the aggregated model

$$X = pAx + Y, \tag{9}$$

where

$$X = px = \sum_{i=1}^{n} p_i x_i, \quad Y = py = \sum_{i=1}^{n} p_i y_i.$$

At the same time, aggregated economic balance we shall present as

$$X = aX + Y, \tag{10}$$

where: $a > 0$ – numerical coefficient of economic direct inputs of production,
X – GDP,
Y – FP.

It is possible also to copy equation (10) as

$$X = apx + Y. \tag{11}$$

Comparing correlations (9) and (11), we obtain a correlation

$$pA = ap, \tag{12}$$

that is a condition of exact aggregation and represents modification of the condition, known as Hatanaka condition [Hatanaka 1952]. Thus such economic requirements should be carried out:

a) $p > 0$, since p is a vector of prices;

b) $A \neq aI$, since A is non-negative indecomposable technological matrix.

Both these requirements are carried out if to choose $p = p_A$, $\lambda_A = a$. Summarizing all mentioned above, we come to the following theorem.

Theorem 3. *For productive indecomposable Leontief's model "input-output" $x = Ax + y$, $y \geq 0$ unique exact aggregation in the prices to one-dimensional model $X = aX + Y$, $Y > 0$ always exists. In it $a = \lambda_A < 1$ is a coefficient of direct economic industrial expenses – Frobenius root, and aggregating vector-line of the prices on production $p_A > 0$ – left Frobenius vector of matrix A.*

Let us investigate the correlation (12) in a different way, namely we shall write it down (12) as correlation of proportions

$$\sum_{i=1}^{n} p_i a_{i1} : \sum_{i=1}^{n} p_i a_{i2} : \ldots : \sum_{i=1}^{n} p_i a_{in} = p_1 : p_2 : \ldots : p_n .$$

Hence, the following theorem is true.

Theorem 4. *Necessary and sufficient condition of exact aggregation in the prices of Leontief's model "input-output" to one-dimensional model is a condition, direct industrial expenses per unit were proportional to the prices of production p. Then the vector of the prices p will be aggregating operator.*

The correlation connecting GDP and FP is obvious:

$$X = \frac{1}{1-\lambda_A} Y .$$

4. Conclusions

Thus, in this work two levels of aggregation of linear static balance Leontief's model "input-output" are described. The first level is "collecting"" of interindustry balance model, where the main moment is concept of "pure branch". The forming of interindustry model is carried out by grouping the nomenclature of separate products into pure branches, for which the interindustry coefficients of direct industrial inputs are calculated. The second level is aggregation of interindustry balance model to one-dimensional model. The existence of unique aggregation of interindustry model into one-dimensional model is shown, when the operator of aggregation is an equilibrium vector of prices, which represents the left Frobenius vector of non-negative indecomposable productive technological matrix of inter-industry model. Thus aggregated coefficient of direct industrial inputs is numerically equal to Frobenius root of interindustry technological matrix.

Thus, to a certain extent it was possible to overcome the break between micro-economic and macroeconomic theory and to learn to deduce balance macromodels from the initial balance microdescriptions on the basis of clear initial hypotheses.

As it is marked in [Petrov et al. 1996], the processes of public reproduction develop elementary processes in numerous cooperating industrial cells. The cumulative results of elementary processes are expressed in the aggregated parameters (macroparameters) of economic development. They are measured by economic statistics. It is necessary to aspire to deduce macroparameters and correlation between them (macromodels) from the initial microdescriptions of elementary processes, developing methods of aggregation. At that, first, the sense of macroparameters will be clear; secondly, it is possible to outline areas of applicability of macrodescriptions; thirdly, the opportunity will appear to correlate variable of macromodels with real economic parameters and to design such parameters. Besides, it is intuitively clear that at the microdescriptions it is easier to build independent hypotheses than at macrodescriptions. Therefore macro-descriptions, obtained from the microdescriptions, as though already pass check on consistency. The opportunity of aggregated description of economic system depends, in the end, on mechanisms of self-regulation of processes, proceeding in it. Apparently, if the system supposes the aggregated description, it has the existing structure that stabilizes activity of the economic agents.

Literature

Hatanaka M., “Note on consolidation within a Leontief system”, *Econometrica* 1952, Vol. 20, No. 2.

Leontief W.W., *Input-Output Economics*, Oxford University Press, New York 1966.

Leontief W.W., *Ekonomicheskiye esse*. *Teoria*, *issledovaniya*, *fakty i politika*, Moscow 1990.

Petrov A.A., Pospelov I.G., Shananin A.A., *Opyt matematicheskogo modelirovaniya ekonomiki*, Moscow 1996.

Ponomarenko O.I., Perestiuk M.O., Burym V.M., *Osnovy matematichnoy ekonomiki*, Kyiv 1995.