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Witold Chmielarz

Management Faculty of University of Warsaw, Poland
vitec@post.pl

VES FUNCTION (VARIABLE ELASTICITY OF SUBSTITUTION) IN THE EVALUATION OF FACTORS SUBSTITUTION SHAPING A WEBSITE'S USABILITY

Abstract: The main aim of this article is presenting a new form of a production function of VES class as a potential tool for evaluating the substitutability of factors influencing the websites' performance. After remarks placing this paper in the series of studies into websites, a proposed form of VES function is characterized. Then the author presents the outline of a proof that this function fulfils the assumptions of the production function. The article is concluded with an attempt to present an economic interpretation of the function coefficients and conclusions concerning further research in this direction.

1. Introduction

The basic aim of the conducted research is the analysis of websites' usability for the final client. The study was started in 2003 with a series of articles concerning comparative analyses performed in the most popular internet sectors [Chmielarz 2006, pp. 251-363]. In an extensive from – only with the application of a scoring method - the studies were conducted in all most important branches of electronic business till mid-2007 [Chmielarz (Ed.) 2007], which allowed for the monitoring of electronic business development in Poland in the last five years. Subsequently, the author began an intensive phase which consisted in searching the best method to evaluate websites. Owing to a limited number of team members and allocated resources we had to concentrate on selected branches of electronic industry. These were Internet computer shops and electronic banking. The findings presented in 2008 open a third – in relation to the one performed last year – series of analyses concerning the evaluation of electronic banking, where the author tried to eliminate methodological inconvenience which occurred previously and the problems connected with obtaining rational experts' evaluation [Chmielarz 2008, pp. 361-368]. Essentially, it amounts to the experts' team analysis of a set of identified criteria in selected internet banks and calculating, on the basis of their individual judgements, a series of combinations according to various methods of websites' evaluations. Next, the author has performed a comparative analysis of the results received in such a way, and conclusions have been drawn. In the research presented so far three methods of websites' evaluation have been applied: a scoring method, AHP method (*Analytic Hierarchy Process* – T.L. Saaty [Saaty 1990, pp. 9-26]) and the author's own conversion method which is based on measuring average distances obtained from the scoring method. The conversion method has been established as a compromise between AHP method and a scoring method, and it takes into consideration all answers to reported claims for the improvement of the scoring method (above all, it limits the subjectivity in experts' judgements).

Even though the evaluations of the same websites according to various methods were performed by the same experts' team, rankings obtained in previous studies – despite keeping within the main trend (leading in one ranking usually meant taking first positions in another), differed slightly.

Table 1. Collective evaluation of the possibilities of methodologies applications in the evaluation of websites

Characteristic feature	Scoring method	AHP method	Conversion method
Ease of application	high	low	high
Ease of acquisition	high	low	n/a
Ease of performing calculations	high	high, with appropriate software	high, with appropriate software
Objectivity	low	high	medium
Findings interpretation	high	medium	medium

And with a large number of applied criteria, it is sometimes difficult to identify the actual reasons for the differences.

Generally:

– scoring method, though regarded as subjective, despite applying a large number of criteria and a traditional linear scoring scale, was evaluated by experts in a positive way, as a rational evaluation method which is easy to acquire. After taking into consideration the preference scale, experts claimed that – in their view – the impression of subjectivity and equivalence of radically different criteria is not as significant as the evaluations of academics suggest;
– in the experts' view, AHP method turned out to be more troublesome in the case of the necessity to compare many websites, by means of a larger number of criteria. A declarative objectivity of this method was losing out with an expert's fatigue; therefore, frequently the websites which were examined first were

evaluated higher than subsequent websites (the change of the order of websites' evaluation produced entirely different results). The presented score was often regarded by experts as ambiguous, owing to its relativity and extended Likert's scale. Work consumption of this method was increasing, in comparison to the scoring method, exponentially in relation to the number of used evaluation criteria and the number of branch websites which were examined. We should consider the question whether the benefits of the limited evaluation subjectivity are worthwhile, taking into account the increase in expert's work consumption;

– conversion method combining advantages of a scoring method (unequivocal, easy criterion evaluation) and Saaty's method (specifying the relation of one criterion to other criteria), consisting in defining the relation of one criterion with reference to other criteria based on averaged distances from a potential maximum value on the basis of the earlier scoring method, is regarded as a reasonable compromise between these methods.

Moreover, the experts observed the following regularity. A website's performance (calculated for example by means of a number of the Internet users doing shopping using the website) depends not only, and not mainly, on its technical perfection (visualisation, navigation, functionality), but equally on its economic factors (relative attractiveness of a price range – competitive prices of products, reasonable transport costs, etc.). In experts' view, there occurs a substitution between "capital" factors (economic factors) and the labour which has to be put into designing a website that would be perfect with regard to its technical and functional properties. This phenomenon is clearly visible in the case of branches of shops selling clothes and shoes as well as cosmetics shops. However, it turned out that also electronic banking is not free from it. The necessity to examine this matter suggested analogies with the substitution shaping the results of the production function.

In the mid-80's the author carried out research concerning sector production substitutability in particular regions of Poland, using his own "geometric" form of VES function to forecast the phenomenon. Variable elasticity – in the conditions of multi-dimensional, structural and dynamic economic changes – allowed to use the function to make much more accurate estimations than in the case of CES type or Cobb-Douglas functions[1]. Because the phenomenon of substitutability between websites' evaluation factors belongs to a similar class of problems, it appears that this function, after certain modifications, can be used to examine the described phenomenon. In the first stage we have to show that this modified function meets the conditions of the production function, and, additionally, differs from functions of VES type which have been applied in such analyses so far. The latter has become the aim of this article. Appropriate estimations and conclusions resulting from them will be the subsequent phase of the study.

[1] Contrary to the studies by [Kemme 1984, p. 59-66], where the author obtained the best specifications for Cobb–Douglas's function. However, he used a different kind of function class VES than presented in this article.

2. Characteristics of "geometric" VES function

The name of the function was derived from applying initial assumptions concerning its form, which, from a technical point of view, is similar to a circle section. After modifying and adjusting the function to the examined situation, its final formal form is as follows:

$$F(K,L)_i=\alpha_i\left[\delta_i K_i+(1-\delta_i)L_i-\beta_i\sqrt{\omega_i K_i^2+(1-\omega_i)L_i^2}\right], \qquad (1)$$

where:

F(*K*, *L*) – production, here: denoted as the value of the shopping done by means of a website, or an estimated number of clients visiting the website,

K – involvement of elements related to economic factors, e.g. the costs of using the service, and the price range for a client,

L – involvement of elements connected with designing and using the website to sell goods and services (design+maintenance/operating costs), estimated evaluations of technological, visual and functional factors,

i – index of an analysed branch,

α – parameter of effectiveness of the production function,

β – parameter of substitution of the production function,

δ – parameter of the division between the elements of the linear part of the production function,

ω – parameter of the division between the elements of the nonlinear part of the production function.

Creating the new function form the author aimed at its maximal simplification. In contrast to generally applied approaches [Lovell, Knox 1973, pp. 678-692; Revanhar 1971; Sato, Hoffmann 1970; Nakatani, 1973, pp. 394-396] the author did not initially assume the form of the dependence of the substitution elasticity on technological innovations. In this case this relation was a secondary matter. The primary task was constructing a new form of the production function and checking if it fulfils all theoretical assumptions imposed on the production function. Only in subsequent steps – from this relation – the author derived the formula for the elasticity of substitution and attempted at an economic interpretation of its parameters.

3. Analysis of the fulfilment of assumptions for the production function

The basic steps of the procedure are presented below in a very simplified manner. In order to simplify calculations, an initially transformed, technical form of the function has been applied:

$$F(K,L) = x_1K + x_2L - \sqrt{x_3K^2 + x_4L^2}, \tag{2}$$

where:

K – involvement of economic factors in website application,

L – involvement of technological factors in website application,

x_1, x_2, x_3, x_4 – technical coefficients.

The assumptions held for the production function were adopted on the basis of work by [Barkałow 1981]: positive values of the function for function factors >0, concavity of the production function by co-ordinates, homogeneity of degree $\gamma > 0$, asymptotic properties.

3.1. The condition of the positive values of the function for factors greater than zero

Firstly, function $Y = F(K, L)$ should take positive values for $K, L > 0$

$$F(K, L) > 0, \forall K, L > 0 \tag{3}$$

and it should have positive first partial derivatives:

$$r = \frac{\partial Y}{\partial K} > 0, \tag{4}$$

$$w = \frac{\partial Y}{\partial L} > 0. \tag{5}$$

Parameters x_3 and x_4 must have non-negative values. It is the condition of definiteness of the expression $\sqrt{x_3K^2 + x_4L^2}$ for all $K, L > 0$. The necessary condition for $F(K, L) > 0$ is

$$x_1, x_2 > 0.$$

Further, if:

$$x_1K + x_2L - \sqrt{x_3K^2 + x_4L^2} > 0, \tag{6}$$

then

$$x_1K + x_2L > \sqrt{x_3K^2 + x_4L^2}\ , \tag{7}$$

hence,

$$x_1^2K^2 + x_2^2L^2 + 2x_1x_2K \quad > x_3K^2 + x_4L^2\ . \tag{8}$$

Thus, conditions:

$$x_1^2 \geq x_3 \tag{9}$$

$$x_2^2 \geq x_4 \tag{10}$$

together with the conditions of parameters' non-negativity are sufficient to ensure positive values of the function (2). We can also justify that they are necessary. We assume their fulfilment, and in the subsequent part it will be proved that the remaining conditions usually held for the production function occur.

$$\frac{\partial Y}{\partial K} = x_1 - \frac{x_3 K}{\sqrt{x_3 K^2 + x_4 L^2}}, \tag{11}$$

$$x_1 - \frac{x_3 K}{\sqrt{x_3 K^2 + x_4 L^2}} > 0, \tag{12}$$

$$x_1 > \frac{x_3}{\sqrt{1 + \frac{x_4}{x_3}\left(\frac{L}{K}\right)^2}}. \tag{13}$$

The above implication substantiates that the condition (4) is met.

$$\text{(denominator) } M > 1 \rightarrow \left(\frac{\sqrt{x_3}}{M} < \sqrt{x_3} < x_1 \right), \tag{14}$$

$$\frac{\partial Y}{\partial L} = x_2 - \frac{x_4 L}{\sqrt{x_3 K^2 + x_4 L^2}}, \tag{15}$$

$$x_2 > \frac{x_4}{\sqrt{1 + \frac{x_3}{x_4}\left(\frac{K}{L}\right)^2}}, \tag{16}$$

$$\text{(denominator) } M > 1 \rightarrow \left(\frac{\sqrt{x_4}}{M} < \sqrt{x_4} < x_2 \right). \tag{17}$$

The above supports the condition fulfilment (5) in an analogical way.

3.2. The conditions of the concavity of the production function by co-ordinates

$$r_2 = \frac{\partial^2 Y}{\partial^2 K^2} < 0, \tag{18}$$

$$w_2 = \frac{\partial^2 Y}{\partial^2 L^2} < 0, \quad \forall K, L > 0, \tag{19}$$

$$\frac{\partial^2 Y}{\partial^2 K^2}=\frac{-x_3\sqrt{x_3K^2+x_4L^2}+x_3K\left(\frac{x_3K}{x_3K^2+x_4L^2}\right)}{\sqrt{x_3K^2+x_4L^2}}=-\frac{x_3x_4L^2}{\left(x_3K^2+x_4L^2\right)^{\frac{3}{2}}}, \tag{20}$$

$$\frac{\partial^2 Y}{\partial^2 L^2}=-\frac{x_3x_4K^2}{\left(x_3K^2+x_4L^2\right)^{\frac{3}{2}}}. \tag{21}$$

As $K,L>0$, and from the properties of the function: $x_3,x_4>0$, then the conditions (18) and (19) are fulfilled.

3.3. The condition of homogeneity of degree $\gamma > 0$

The condition of homogeneity of degree γ comes down to the equation:

$$F(\lambda K,\lambda L)=\lambda^{\gamma}F(K,L) \text{ for any } \lambda > 0 \tag{22}$$

$$x_1\lambda K + x_2\lambda L - \sqrt{x_3\lambda^2K^2 + x_4\lambda^2L^2} = \lambda\left[x_1K + x_2L - \sqrt{x_3K^2 + x_4L^2}\right] \tag{23}$$

for $\gamma = 1$ it means that the function (2) is of degree 1.

It is a limit, being the result of aiming at the simplicity of the function.

3.4. The conditions of asymptotic properties of the production function

The last conditions which should be satisfied by the production function are the so-called K. Inada asymptotic characteristics [Otani 1970]. The extreme values of the function with *K* approaching zero, *L* approaching zero, and *K* approaching infinity, and *L* approaching infinity are presented in Table 2.

The combined findings are in accordance with general results obtained by K. Inada for the production function, because during the estimation there appears no claim to fulfil conditions (9) and (10) equally. However, in all cases where we arrive at proper results of the estimation of the function parameters (2) for the previously analysed cases, these conditions were fulfilled almost equally. The values in the first row in Table 2 then mean that together with an unlimited increase of one production factor and with the established value of another, *F*(*K*, *L*) it should stabilize at a certain specified level.

3.5. Calculating the substitution elasticity σ

For the production function denoted by the expression (2) the substitution elasticity σ is calculated as a function resulting from work innovativeness (influences (in

Table 2. Asymptotic values of VES production function for various cases of the values of its parameters

		lim $F(K, L)$ for $K\to 0^+$	lim $F(K, L)$ for $L\to 0^+$	lim $F(K, L)$ for $K\to\infty$	lim $F(K, L)$ for $L\to\infty$
$x_1=\sqrt{x_3}$	$x_2=\sqrt{x_4}$	0	0	$\frac{2x_1x_2L}{x_1+\sqrt{x_3}}$	$\frac{2x_1x_2L}{x_2+\sqrt{x_4}}$
$x_1=\sqrt{x_3}$	$x_2>\sqrt{x_4}$	$(x_2-\sqrt{x_4})L$	0	$\frac{2x_1x_2L}{x_1+\sqrt{x_3}}$	+
$x_1>\sqrt{x_3}$	$x_2=\sqrt{x_4}$	0	$(x_1-\sqrt{x_3})K$	+	$\frac{2x_1x_2L}{x_2+\sqrt{x_4}}$
$x_1>\sqrt{x_3}$	$x_2>\sqrt{x_4}$	$(x_2-\sqrt{x_4})L$	$(x_1-\sqrt{x_3})K$	+	+

the sense of calculated unit price * turnover) determining the value of the function), here treated in a traditional way as a technical (capital) production input $z=\frac{K}{L}$.

In the beginning the extreme substitution rate R is established.

$$0=x_1\frac{dK}{dL}+x_2-\frac{x_2K\frac{dK}{dL}+x_4L}{\sqrt{x_3K^2+x_4L^2}}\ , \tag{24}$$

$$0=x_1\frac{dK}{dL}\sqrt{x_3K^2+x_4L^2}+x_2\sqrt{x_3K^2+x_4L^2}-x_3\frac{dK}{dL}K-x_4L\ . \tag{25}$$

After transformations, from the formula (8), using simultaneously the definition of the extreme substitution rate, we arrive at:

$$R\Rightarrow\frac{dK}{dL}=\frac{x_2\sqrt{x_3z^2+x_4}-x_4}{x_1\sqrt{x_3z^2+x_4}-x_3z}\ . \tag{26}$$

In order to derive the formula for the substitution elasticity [Allen 1974] we need a derivative form of the extreme substitution rate.

$$\frac{dR}{dz}=\frac{x_3x_4\left(x_1z+x_2-\sqrt{x_3z^2+x_4}\right)}{\sqrt{x_3z^2+x_4}\left(x_1\sqrt{x_3z^2+x_4}-x_3z\right)^2}\ . \tag{27}$$

From the definition, the substitution elasticity σ is expressed by a formula:

$$\sigma=\frac{dz}{dR}\times\frac{R}{z}\ . \tag{28}$$

Thus:

$$\sigma(z)=\frac{\sqrt{x_3z^2+x_4}\left(x_1\sqrt{x_3z^2+x_4}-x_3z\right)\left(x_2\sqrt{x_3z^2+x_4}-x_4\right)}{zx_3x_4\left(x_1z+x_2-\sqrt{x_3z^2+x_4}\right)}\ . \tag{29}$$

In the earlier study [Chmielarz, Stachurski 1986, pp. 367-381] concerning the estimation of the function parameters it has been proved that the substitution elasticity takes only positive values, as long as parameters fulfil assumed conditions (9) and (10).

Production function written as a formula (2) can be also presented as follows:

$$F(K,L)=\alpha\left[\delta K+(1-\delta)L-\beta\sqrt{\omega K^2+(1-\omega)L^2}\right]. \tag{30}$$

After employing substitutions:

$$x_1=\alpha\,\delta, \tag{31}$$

$$x_1+x_2=\alpha, \tag{32}$$

$$\frac{x_1}{x_1+x_2}=\delta, \tag{33}$$

$$x_3=\alpha^2\beta^2\omega, \tag{34}$$

$$x_3+x_4=\alpha^2\beta^2, \tag{35}$$

$$\frac{x_3}{x_3+x_4}=\omega\cdot \tag{36}$$

In this case coefficient α can be interpreted as a measure of management effectiveness. However, δ and ω are parameters of the share of production function factors in constructing this function, appearing respectively in the linear and non-linear part of this function. And in this case parameter β can be referred to as "substitution parameter". Substitution elasticity and parameter β are interdependent – the increase in the value of this parameter results in the fall in the elasticity of substitution under the assumption of constancy of parameters α, δ and ω[2].

4. Conclusions

The presented considerations concerning a new form of VES production function, together with the proof that it fulfils the assumptions of the production function, and general interpretation of its coefficients should be treated as a preparatory stage for further research into evaluation methods of websites' usability. The substitution between factors influencing the websites' economic results in particular branches

[2] What has been proved in: [Chmielarz, Stachurski 1983, pp. 251-265].

– discovered during experts' individual evaluations and confirmed by their subsequent judgements – appears to be so significant that it requires a special, separate examination. The best approach to conduct the applied experiments can be ensured by the "geometric" VES production function form, examined in the author's earlier studies and modified with regard to possibilities of its application. At present – after methodology specification – the research is at the stage of data collection (findings were gathered from May till June 2008 and now combined tables need to be constructed), and experts' groups are performing evaluations of electronic services of e-banking websites in Poland (40 such groups have been established, and they, on average, consist of two people). Because in the analysed case it is difficult to ensure a sufficient time span for the research, we concentrated on the estimates involving groups and sectors, a procedure which has been already applied in estimating the coefficients of the production function[3]. The findings obtained from the estimates of VES production function and conclusions resulting from them will be the subject of the author's further studies; the author also considers the possibility of comparing the results with estimates according to the assumed form of CES function and Cobb-Douglas function.

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[3] E.g. in the cited studies [Kemme 1984; Zestos 1996, pp. 43-51].

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