Politechnika Wrocławska
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Modele samopodobne w teorii ryzyka

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Rozprawa doktorska przygotowana pod kierunkiem
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Wrocław, 25 maja 1998

Self-similar models in risk theory

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Chapter 1

Introduction

Self-similar processes, introduced by Lamperti [23], are the ones that are invariant under suitable translations of time and scale. In the past few years there has been an explosive growth in the study of self-similar processes, cf. e.g. Taqqu [40], Maejima [24], Samorodnitsky and Taqqu [38], Willinger et al. [42], Michna [27], and Rogers [33].

This caused that various examples of such processes have been found and relationships with distinct types of processes have been studied. In Chapter 2 we establish the uniqueness of the Lamperti transformation leading from self-similar to stationary processes, and conversely. We discuss α-stable processes, which allow to understand better the difference between the Gaussian and non-Gaussian cases. As a by-product we get a natural construction of two distinct α-stable Ornstein–Uhlenbeck processes via the Lamperti transformation for $0 < \alpha < 2$. Also a new class of mixed linear fractional α-stable motions is introduced which is further exploited in the next chapter.

It seems natural to try to find all H-ss processes and to characterize them. In Chapter 3 we establish a spectral representation of any symmetric stable self-similar process in terms of multiplicative flows and cocycles. A structure of this class of self-similar processes is studied. Applying the Lamperti transformation we obtain a unique decomposition of a symmetric stable self-similar process into three independent parts: mixed fractional motion, harmonizable and evanescent. This decomposition is illustrated by graphical presentation of corresponding kernels of their spectral representations.

Self-similar processes are closely connected with limit theorems for identical and in general strongly dependent variables. Moreover, since they allow heavy-tailed distributions and provide an additional "adjusting" parameter H they appear to be interesting in the area of risk models. Chapters 4 and 5 are devoted to some applications of self-similar processes in insurance and finance mathematics. In Chapter 4 we prove that only self-similar processes with stationary increments appear naturally as weak limits of a risk reserve process and conversely every finite mean H-self-similar process with stationary increments, for $\frac{1}{2} < H \leq 1$ can result as the weak approximation. A lower bound for general self-similar processes with drift is also provided. In Chapter 5 we illustrate a test of self-similarity (namely variance–time plots) on a DJIA index data in order to justify the use of self-similar processes in financial modelling. Last but not least we propose an alternative model for stock price movements incorporating a martingale which generates the same filtration as the fractional Brownian motion. This leads to an option pricing formula different from the Black–Scholes one. Chapters 2 and 3 are based on two author's papers : Burnecki et al. [6] and Burnecki et al. [7].

1.1 Foundations

Stochastic processes $X = (X(t))_{t\in T}$ (another equivalent notation used is $X = \{X_t\}_{t\in T}$) in this thesis are always assumed to be defined for $t \in T$, where $T = [0, \infty)$ or $\mathbf{R}$. By $(X(t)) \stackrel{d}{=} (Y(t))$ we mean the equality of all finite dimensional distributions. Sometimes we simply write $X(t) \stackrel{d}{=} Y(t)$. $X(t) \stackrel{d}{\sim} Y(t)$ means the equality of one-dimensional distributions for fixed t. We also mean, by $X_n(t) \stackrel{d}{\Rightarrow} Y(t)$, the convergence of all finite-dimensional distributions as $n \to \infty$, and by $\zeta_n \stackrel{d}{\to} \zeta$ the convergence in distribution of real-valued random variables (ζ_n) to ζ.

Definition 1.1.1 *X is said to be* degenerate *if $X(t) = X(0)$ a.s. for any $t \in T$, and* non-degenerate *otherwise.*

Definition 1.1.2 (Lamperti [23]) *A process $X = (X(t))_{t\geq 0}$ is* self-similar *(ss) if for some $H > 0$,*

$$X(ct) \stackrel{d}{=} c^H X(t) \quad for \ \ every \ \ c > 0. \tag{1.1}$$

We call this X an H-ss process. The parameter H is called the index or the exponent of the self-similarity. X is said to be trivial if $X(t) = t^H X(1)$ a.e., $t \geq 0$.

Remarks.

1. Notice that (1.1) indeed means "scale-invariance" of the finite-dimensional distributions of X. It does not imply this property for the sample paths. Therefore, pictures trying to explain self-similarity by zooming in and out on one sample path, are by definition misleading. A convenient tool to observe self-similarity are so-called quantile lines (see Section 2.1).

2. If we interpret t as "time" and $(X(t))$ as "space" then (1.1) tells us that every change of time scale $c > 0$ corresponds to a change of space scale c^H. The bigger H, the more dramatic the change of the space coordinate.

3. Self-similarity is convenient for simulations: a sample path of $(X(t))$ on $[0, 1]$ multiplied by c^H and re-scaling of the time axis by c immediately provide a sample path on $[0, c]$ for any $c > 0$.

Definition 1.1.3 *$X = (X(t))_{t\geq 0}$ is said to have* stationary increments *if for any $b > 0$,*

$$(X(t+b) - X(b)) \stackrel{d}{=} (X(t) - X(0)).$$

We call X simply a si process.

Definition 1.1.4 *$(X(t))_{t\geq 0}$ is said to have* independent increments *if for any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, $X(t_2) - X(t_1)$, $X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ are independent. We call X simply an ii process.*

There exist the following relations between the moment condition and the parameter H of ss si processes.

Proposition 1.1.1 (Maejima [24]) *Let X be H-ss, si, $H > 0$ and non-degenerate.*

(i) If $E|X(t)|^\gamma < \infty$ for some $\gamma < 1$, then $H < 1/\gamma$.

(ii) If $E|X(t)| < \infty$, then $H \leq 1$.

(iii) If $E|X(t)| < \infty$ and $0 < H < 1$, then $E[X(t)] = 0$.

Proposition 1.1.2 (Vervaat [41]) *If X is 1-ss, si and $E|X(t)| < \infty$, then $X(t) = tX(1)$ a.s.*

The study of ss processes is mainly focused on processes with strongly dependent increments (cf. Section 5.1).

Fix now $0 < H < 1$. Since the function $\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H},\ t_1, t_2 \in \mathbf{R}\}$ is non-negative definite, there exist a Gaussian process $(X(t))_{t\geq 0}$ with mean zero and autocovariance function

$$R(t_1, t_2) = Cov(X(t_1), X(t_2)) = \frac{1}{2}\left\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\right\} VarX(1). \tag{1.2}$$

It is easy to check that this process is H-ss, si. It is called a fractional Brownian motion (FBM) and is often denoted by $B_H(t)$ (or equivalently B_t^H). Are there any other Gaussian H-ss si processes for $0 < H < 1$? The following proposition yields the negative answer.

Proposition 1.1.3 (Maejima [24]) *Let X be an H-ss si Gaussian process with $0 < H < 1$. Then X is essentially equivalent (see Definition 2.2.1) to the fractional Brownian motion $B_H(t)$.*

Remark. An integral representation of a FBM is given by (2.1) for $\alpha = 2$.

Definition 1.1.5 *A process $Y = (Y(t))_{t\in\mathbf{R}}$ is* stationary *if*

$$Y(t + \sigma) \stackrel{d}{=} Y(t) \quad for \ every \quad \sigma \in \mathbf{R}.$$

Y is said to be trivial if $Y(t) = Y(0)$ a.e., $t \in \mathbf{R}$ (it is degenerate).

The following theorem makes clear that self-similarity is very closely related to stationarity: a logarithmic time transform translates shift invariance of the stationary process into scale invariance of the self-similar process.

Proposition 1.1.4 (Lamperti [23]) *If $Y = (Y(t))_{t\in\mathbf{R}}$ is a stationary process and if for some $H > 0$*

$$X(t) = t^H Y(\log t), \quad for \ t > 0, \ X(0) = 0,$$

then $X = (X(t))$ is H-ss. Conversely, every non-trivial ss-process with $X(0) = 0$ is obtained in this way from some stationary process Y.

This tool is extensively used in Chapters 2 and 3. For further applications see Maejima and Sato [25], where the transformation is proved to be a link between semi-selfsimilar and periodically stationary processes.

Self-similar processes are of interest in probability theory because they are closely connected with limit theorems. Namely, every limiting process with scaling is self-similar, and all self-similar processes are characterized in such way, as was observed by Lamperti [23].

Proposition 1.1.5 (Lamperti [23]) *Suppose $X = (X(t))_{t\geq 0}$ is continuous in probability at $t = 0$ and the distribution of $X(t)$ is non-degenerate for each $t > 0$.*

(i) If there exist a stochastic process $Y = (Y(t))_{t\geq 0}$ and reals $A(\lambda))_{\lambda\geq 0}$ with $A(\lambda) > 0$, $\lim_{\lambda\to\infty} A(\lambda) = \infty$ such that as $\lambda \to \infty$,

$$\frac{1}{A(\lambda)} Y(\lambda t) \stackrel{d}{\Rightarrow} X(t), \tag{1.3}$$

then for some $H > 0$, X is H-ss. Furthermore, $A(\lambda)$ is of the form $A(\lambda) = \lambda^H L(\lambda)$, $L(\lambda)$ being a slowly varying function.

(ii) If X is H-ss, then there exist Y and $(A(\lambda))_{\lambda>0}$ satisfying (1.3).

Remark. Notice that part (ii) is trivial by taking $Y = X$ and $A(\lambda) = \lambda^H$.

Proposition 1.1.5 can be specialized to the following result.

Proposition 1.1.6 (Lamperti [23]) *Let $(\zeta)_{k=1}^{\infty}$ be a stationary sequence of $\mathbf{R}$-valued random variables with the partial sum process $Y(t) = \sum_{k=1}^{[t]} \zeta_k$ for $t \geq 0$. If*

$$\frac{1}{a_n} Y(nt) \stackrel{d}{\Rightarrow} X(t) \quad \text{as } n \to \infty \text{ through the reals,}$$

where $a_n > 0$, $a_n \to \infty$ and $X(1) \neq 0$ with positive probability, then there is an $H > 0$ such that

$$a_n = n^H L(n),$$

for L being a slowly varying function and X is H-ss, si. Conversely, all H-ss si X with $H > 0$ can be obtained in this way.

Remark. For the last statement take $a_n = n^H$ and $\zeta_k = X(k) - X(k-1)$ for $k \in \mathbf{N}$.

If we strengthen the assumption of stationary increments to stationary independent increments, we enter a classical domain of probability. Note that Proposition 1.1.6 remains true with the same substitution and (ζ_k) a sequence of independent identically distributed random variables. Thus, almost by definition, self-similar processes X with stationary and independent increments are strictly stable motions (see Definition 1.1.9 below). It turns out that $H \geq \frac{1}{2}$, with $H = \frac{1}{2}$ corresponding to the Brownian motion.

In order to state other relations between self-similar and α-stable processes we start with some necessary definitions.

Definition 1.1.6 *A random variable X is said to have a* stable *distribution if for any positive numbers A and B, there is a positive number C and a real number D such that*

$$AX_1 + BX_2 \stackrel{d}{\sim} CX + D, \tag{1.4}$$

where X_1 and X_2 are independent copies of X.

A stable random variable is called *strictly* stable if (1.4) holds with $D = 0$. A stable random variable is called *symmetric* stable if its distribution is symmetric, that is, if X and $-X$ have the same distribution. A symmetric stable random variable is obviously strictly stable. Moreover the constant C in (1.4) can be taken as $C = (A^\alpha + B^\alpha)^{1/\alpha}$ for some $\alpha \in (0, 2]$. Hence α is one of the characteristics of X, and in this case X is said to be α-stable. When $\alpha = 2$, 2-stable is Gaussian. In the following, α always satisfies $0 < \alpha \leq 2$.

Explicit forms of stable density functions only exist in the cases $\alpha = \frac{1}{2}$ (Lévy distribution), $\alpha = 1$ (Cauchy distribution) and $\alpha = 2$ (Normal distribution). The tails of non-Gaussian stable distributions decrease like a power function. The rate of decay mainly depends on the parameter α. The smaller the α, the slower the decay and the heavier the tails. For a stable random variable X with index $\alpha < 2$ one has $E|X|^\delta = \infty$ for any $\delta \geq \alpha$ and $E|X|^\delta < \infty$ for $0 < \delta < \alpha$.

The second definition states that stable distributions are the only distributions that can be obtained as limits of normalized sums of i.i.d. variables (compare Proposition 1.1.6 for self-similar processes).

Definition 1.1.7 (Equivalent to Definition 1.1.6) *A random variable X is said to have a stable distribution if it has a* domain of attraction, *i.e. if there is a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers (d_n) and real numbers (a_n), such that*

$$\frac{Y_1 + Y_2 + \cdots Y_n}{d_n} + a_n \xrightarrow{d} X.$$

The definition of stability in $\mathbf{R}^d$ is analogous to that in $\mathbf{R}^1$.

Definition 1.1.8 *A random vector $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ is said to be a* stable random vector in $\mathbf{R}^d$ *if for any positive numbers A and B there is a positive number C and a vector $\mathbf{D} \in \mathbf{R}^d$ such that*

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} \stackrel{d}{=} C\mathbf{X} + \mathbf{D}, \tag{1.5}$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of $\mathbf{X}$.

The vector $\mathbf{X}$ is called *strictly stable* if (1.5) holds with $\mathbf{D} = 0$ for any $A > 0$ and $B > 0$. The vector is called *symmetric stable* if it is stable and satisfies in addition the relation $P\{\mathbf{X} \in \mathbf{A}\} = \mathbf{P}\{-\mathbf{X} \in \mathbf{A}\}$ for any Borel set A of $\mathbf{R}^d$. As in $\mathbf{R}^1$, a symmetric stable vector is strictly stable.

Definition 1.1.9 *A stochastic process $(X(t))_{t \in T}$ is* stable *if all its finite-dimensional distributions are stable. It is* strictly stable *or* symmetric stable *of all its finite distributions are, respectively, strictly stable or symmetric stable.*

If the finite-dimensional distributions are stable, then, by consistency they must all have the same index of stability α. We use the term α-stable when we wish to specify the index of stability. We will often refer to the symmetric case. Thus, we recall the following. For $\alpha \in (0, 2]$, a process $(X(t))_{t \in T}$ is symmetric α–stable (which will be referred to as $S\alpha S$) if for arbitrary $n \in N$, $a_1, \ldots, a_n \in \mathbf{R}$, $t_1, \ldots, t_n \in T$ a random variable $\sum_{i=1}^{n} a_i X(t_i)$ has a $S\alpha S$ distribution. A $S\alpha S$ process $(X(t))_{t \in T}$ is called a $S\alpha S$ Lévy motion if it has stationary and independent increments, is continuous in probability and $X(0) = 0$ a.e. We denote it by $Z_\alpha = (Z_\alpha(t))_{t \in T}$. For a comprehensive survey of properties of α-stable random variables and processes we refer to Janicki and Weron [20] and Samorodnitsky and Taqqu [38].

If an α-stable process is self-similar, then the self-similarity parameter H can never exceed $\max(1, 1/\alpha)$. It is easy to see that strictly α-stable Lévy motions are $\frac{1}{\alpha}$-ss si α-stable processes. Are there any others? In the Gaussian case $\alpha = 2$, the answer is easily to be negative (when $\alpha = 2$ only Brownian motion has such property). The answer is positive when $1 < \alpha < 2$ (see Maejima [24]). The answer is positive when $\alpha = 1$ as well, because if $X(1)$ has a 1-stable law then the linear function with random slope $X(t) = tX(1)$, $t \geq 0$, is 1-ss, si. The problem for the case $0 < \alpha < 1$ is settled through the following result.

Proposition 1.1.7 (Samorodnitsky and Taqqu [37]) *The only non-degenerate α-stable $\frac{1}{\alpha}$-ss si processes with $0 < \alpha < 1$ are the strictly α-stable Lévy motions.*

Chapter 2

The Lamperti transformation

Lamperti defined a transformation which changes stationary processes to the corresponding self-similar ones (see Proposition 1.1.4). In this context a question arises whether the transformations proposed by Lamperti are unique. In this chapter we search for functions ϕ, ψ, ζ and η such that

$$X(t) = \phi(t)Y(\psi(t)) \text{ is } H\text{-ss for a non-trivial stationary process } Y$$

and

$$Y(t) = \zeta(t)X(\eta(t)) \text{ is stationary for a non-trivial } H\text{-ss process } X.$$

There are two theorems presented in Section 2.2 which lead to the conclusion that essentially $\phi(t) = t^H$, $\psi(t) = a \log t$, $\zeta(t) = e^{-bHt}$ and $\eta(t) = e^{bt}$ for some $a, b \in \mathbf{R}$ according to our convention (see Definition 2.2.1). In Section 2.1, a computer visualization of the Lamperti transformation is provided. Section 2.3 is devoted to the study of the influence of various a's and b's on distributions of corresponding processes. This is illustrated by four processes chosen to express a difference between the Gaussian and non-Gaussian case. As a result of this investigation, we construct, in a natural way, a pair of distinct α-stable Ornstein-Uhlenbeck processes for $\alpha < 2$, already known in the literature (Adler et al. [1]). This supports the conjecture that there are only two such processes. In the last section (Section 2.4), we discuss a new class of self-similar stable processes whose corresponding stationary processes Y through the Lamperti transformation are stable mixed moving averages. The class is called mixed fractional motions and is precisely defined and exploited in Section 3.3.1.

2.1 Computer visualization of the Lamperti transformation

Before we present main results of this chapter we find it interesting to illustrate the Lamperti transformation by demonstrating graphically self-similar processes and corresponding stationary ones. We generate the fractional stable motion with parameters H and α, applying its integral representation, that is,

$$X(t) = \int_{-\infty}^{0} (|t-u|^{H-\frac{1}{\alpha}} - |u|^{H-\frac{1}{\alpha}}) Z_\alpha(du) + \int_0^t |t-u|^{H-\frac{1}{\alpha}} Z_\alpha(du), \tag{2.1}$$

which is well defined for $0 < H < 1$ and $0 < \alpha \leq 2$.

In order to approximate the integral, we use the method introduced by Mandelbrot and Wallis [26] replacing a sequence of Gaussian with α-stable random variables. In Fig. 2.1 we can see four trajectories of the process (thin lines) for $\alpha = 1.8$ and $H = 0.7$. To give the insight view on the nature of the process, we follow Janicki and Weron [20]. We evaluate a large number of realizations of the process and compute quantiles in the points of discretization for some fixed p $(0 < p < 0.5)$, i.e. we compute $F^{-1}(p)$ and $F^{-1}(1-p)$, where F is the distribution function. Fig. 2.1 and Fig. 2.2 have the same graphical form of output. The number of considered realizations is 4000. The thin lines represent four sample trajectories of the process. The thick lines stand for quantile lines, the bottom one for $p = 0.1$ and the top one for $1 - p = 0.9$. The lines determine the subdomain of $\mathbf{R}^2$ to which the trajectories of the approximated process should belong with probabilities 0.8 at any fixed moment of time. In Fig. 2.2 we can see the corresponding process transformed by the Lamperti transformation for the parameter $H = 0.7$. We can see that the quantile lines are "parallel". This means they are time invariant, demonstrating the stationarity of the process.

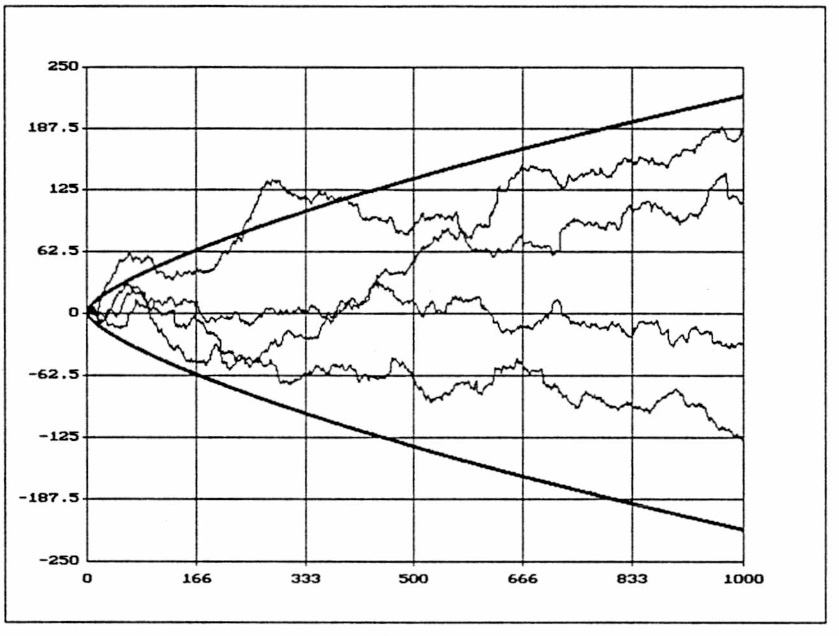

Figure 2.1: Visualization of the fractional stable motion for $H = 0.7$ and $\alpha = 1.8$.

2.2 Uniqueness of the Lamperti transformation

Definition 2.2.1 *When for two stochastic processes $X = (X(t))$ and $Y = (Y(t))$, $X(t) \stackrel{d}{=} aY(t)$ for some $a \in \mathbf{R}\backslash\{0\}$, we say that X and Y are* essentially equivalent.

Henceforth we will not distinguish between such processes. Furthermore, we will assume that all considered processes throughout this chapter are stochastically continuous.

In this section we establish the uniqueness of the Lamperti transformations leading from stationary to self-similar processes, and conversely. The following lemma on stationary processes makes a technical argument used in the proof of Theorem 2.2.1 (ii).

Lemma 2.2.1 *Let $(Y(t))_{t\in\mathbf{R}}$ be a non-trivial stochastically continuous stationary process and let $f : \mathbf{R} \to \mathbf{R}$ be a continuous monotone increasing function. If*

$$Y(f(t)) \stackrel{d}{=} Y(t), \tag{2.2}$$

then $f(t) = t + h$ for some $h \in \mathbf{R}$.

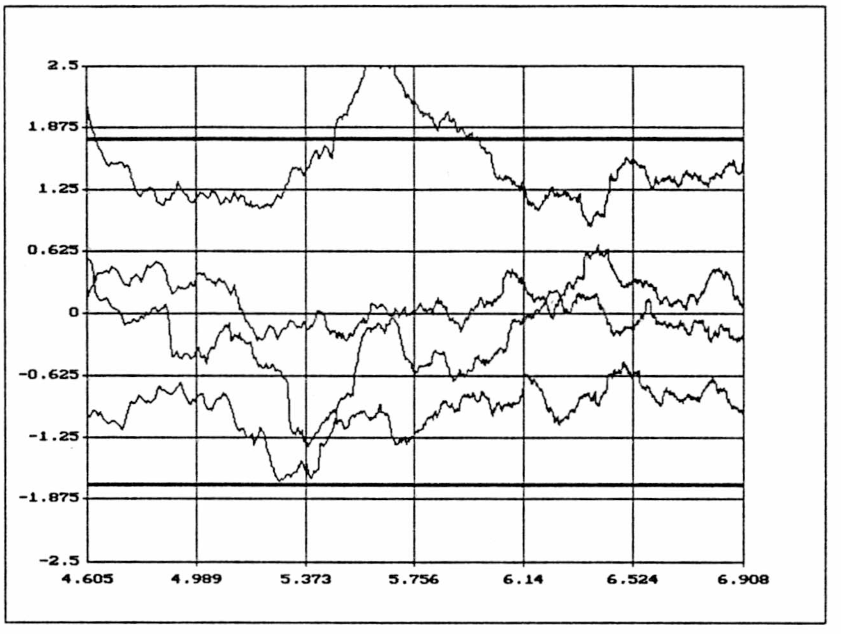

Figure 2.2: Visualization of the stationary process obtained from the fractional stable motion by the Lamperti transformation.

Proof. Suppose that the conclusion is not true. Then (i) there exist an interval $[a, b]$ and $\theta \in (0, 1)$ such that for every $t \in [a, b]$, $0 \leq f(t) - f(a) < \theta(t - a)$, or (ii) there exist an interval $[a, b]$ and $\theta > 1$ such that for every $t \in [a, b]$, $f(t) - f(a) > \theta(t - a)$. Note that since f is continuous and monotone increasing, it follows from (2.2) that $Y(f^{-1}(t)) \stackrel{d}{=} Y(t)$. Thus without loss of generality, we suppose (i).

For any $t_0 \in (0, b - a]$, define $t_1 = f(a + t_0) - f(a)$. Then $0 \leq t_1 < \theta t_0$. From the assumption and the stationarity of Y, we have

$$(Y(0), Y(t_0)) \stackrel{d}{=} (Y(a), Y(a + t_0)) \stackrel{d}{=} (Y(f(a)), Y(f(a + t_0))) \stackrel{d}{=} (Y(0), Y(t_1)).$$

For every $n = 2, 3, \ldots$, define $t_n = f(a + t_{n-1}) - f(a)$. Then $0 \leq t_n < \theta t_{n-1}$ and by the same argument as above, we have

$$(Y(0), Y(t_0)) \stackrel{d}{=} (Y(0), Y(t_n)).$$

Since $t_n \to 0$ as $n \to \infty$, it follows from the stochastic continuity of Y that

$$(Y(0), Y(t_0)) \stackrel{d}{=} (Y(0), Y(0)).$$

Namely

$$Y(t_0) = Y(0) \ \ a.s.$$

Since $t_0 \in (0, b - a]$ was taken arbitrary, this together with the stationarity of Y gives us that

$$Y(t) = Y(0) \ \ a.s \ for \ every \ t \in \mathbf{R},$$

which is an contradiction to that Y is non-trivial. Therefore it must be that for some $h \in \mathbf{R}$

$$f(t) = t + h \ \ for \ any \ t \in \mathbf{R}. \ \Box$$

Theorem 2.2.1 *Let* $0 < H < \infty$.

(i) If $(Y(t))_{t\in\mathbf{R}}$ *is a stationary process and* $a \in \mathbf{R}$, *then*

$$X(t) = \begin{cases} t^H Y(a\log t), & for\ t > 0 \\ 0, & for\ t = 0 \end{cases}$$

is H*-ss.*

(ii) Conversely, if for some continuous functions ϕ, ψ *on* $(0,\infty)$ *and for a non-trivial stationary process* $Y = (Y(t))_{t\in\mathbf{R}}$,

$$X(t) = \begin{cases} \phi(t)Y(\psi(t)), & for\ t > 0 \\ 0, & for\ t = 0 \end{cases} \tag{2.3}$$

is H*-ss, then* $\phi(t) = t^H$ *and* $\psi(t) = a\log t$ *for some* $a \in \mathbf{R}$.

Proof. (i) Note that

$$X(ct) = c^H t^H Y(a\log t + a\log c) \stackrel{d}{=} c^H X(t),$$

hence we conclude that $(X(t))_{t\geq 0}$ is H-ss.

(ii) Since $(X(t))_{t\geq 0}$ in (2.3) is H-ss, we have

$$\phi(ct)Y(\psi(ct)) \stackrel{d}{=} c^H\phi(t)Y(\psi(t)) \quad for\ every\ c > 0, \tag{2.4}$$

which leads to

$$\phi(ct) = c^H\phi(t) \quad for\ every\ t > 0\ and\ c > 0,$$

since (2.4) must agree with respect to marginal distributions as well. Consequently, $\phi(t) = t^H\phi(1), t > 0$. The constant $\phi(1)$ is of no importance by Definition 2.2.1, thus we consider $\phi(t)$ only of the form $\phi(t) = t^H, t > 0$. Now (2.4) can be phrased as

$$c^H t^H Y(\psi(ct)) \stackrel{d}{=} c^H t^H Y(\psi(t)),$$

namely

$$Y(\psi(ct)) \stackrel{d}{=} Y(\psi(t)) \quad for\ every\ c > 0. \tag{2.5}$$

This yields that ψ is monotone on $(0,\infty)$. In order to see it, suppose a'contrario that $\psi(t_1) = \psi(t_2)$ for some $t_1 \neq t_2$. Since

$$(Y(\psi(ct_1)), Y(\psi(ct_2))) \stackrel{d}{=} (Y(\psi(t_1)), Y(\psi(t_2))) \stackrel{d}{=} (Y(0), Y(0))$$

for every $c > 0$, $\psi(ct_1) - \psi(ct_2)$ is continuous with respect to variable c and Y is stationary, we infer that $Y(t) = Y(0)$ a.s. for every $t \in \mathbf{R}$. Thus Y is trivial. Therefore ψ must be monotone on $(0,\infty)$. Furthermore, ψ takes every value in $\mathbf{R}$. One can see this from (2.5) letting $c \to 0$ and $c \to \infty$.

Taking

$$f_c(t) = \psi(c\psi^{-1}(t)), \tag{2.6}$$

we obtain by Lemma 2.2.1 that for some $h \in \mathbf{R}$

$$\psi(c\psi^{-1}(t)) = t + h, \quad for\ every\ t \in \mathbf{R}. \tag{2.7}$$

Notice that ψ can be either decreasing or increasing. Nevertheless f_c defined by (2.6) is always increasing. Clearly, (2.7) can be rewritten as

$$\psi(ct) = \psi(t) + h(c), \quad \textit{for any } t > 0 \textit{ and } c > 0,$$

where $h(c)$ is a function depending only on c. From this and Definition 2.2.1, one can easily see that for some $a \in \mathbf{R}$

$$\psi(t) = a \log t, \quad t > 0. \quad \square$$

Theorem 2.2.2 *Let* $0 < H < \infty$.

(i) If $(X(t))_{t \geq 0}$ *is an* H*-ss process and* $b \in \mathbf{R}$*, then*

$$Y(t) = e^{-bHt} X(e^{bt}), \quad t \in \mathbf{R},$$

is stationary.

(ii) Conversely, if for some continuous functions ζ, η, *where* η *is invertible, and for a non-trivial* H*-ss process* $(X(t))$,

$$Y(t) = \zeta(t) X(\eta(t)), \quad t \in \mathbf{R},$$

is stationary, then

$$\zeta(t) = e^{-bHt} \textit{ and } \eta(t) = e^{bt} \textit{ for some } b \in \mathbf{R}.$$

Proof. (i) We have

$$Y(t+\sigma) = e^{-bH(t+\sigma)} X(e^{b(t+\sigma)}) \stackrel{d}{=} e^{-bH(t+\sigma)} e^{bH\sigma} X(e^{bt}) = Y(t).$$

Thus we conclude that Y is stationary.

(ii) Since $Y(t) = \zeta(t) X(\eta(t))$ is stationary and η is invertible, one can easily claim that the process

$$\frac{1}{\zeta(\eta^{-1}(t))} Y(\eta^{-1}(t)) = X(t)$$

is H-ss. Thus, by Theorem 2.2.1 we obtain

$$\eta^{-1}(t) = a \log t \quad \textit{for some } a \in \mathbf{R} \backslash \{0\}.$$

This is equivalent to

$$\eta(t) = e^{bt}, \quad \textit{for some } b \in \mathbf{R}.$$

Using the same arguments for ζ, we have $\zeta(a \log t) = t^{-H}$. This yields $\zeta(t) = e^{-bHt}$. $\square$

Remarks.

1. Marginal distributions do not depend on the choice of a and b, that is,

   $$X(t) = t^H Y(a \log t) \stackrel{d}{\sim} t^H Y(1)$$

   since Y is stationary, and

   $$Y(t) = e^{-bHt} X(e^{bt}) \stackrel{d}{\sim} X(1)$$

   since X is H-ss.

2. The parameters a and b are meaningful when considering finite-dimensional distributions. The influence of a and b will be discussed in the sequel.

2.3 Finite-dimensional distributions in the α-stable case

We want to establish the influence of a's and b's on distributions of the corresponding processes. To this end we need the following lemma.

Lemma 2.3.1 *If $Y = (Y(t))_{t\in\mathbf{R}}$ is a non-trivial stationary stochastic process and if*

$$Y(ct) \stackrel{d}{=} Y(t), \quad \textit{for some } c \in \mathbf{R}\backslash\{0\}, \tag{2.8}$$

then either $c = -1$ or $c = 1$.

Proof. It is enough to prove that if Y satisfies (2.8) for some c with $0 < |c| < 1$, then Y is trivial. Since

$$(Y(t_1), \ldots, Y(t_m)) \stackrel{d}{=} (Y(c^n t_1), \ldots, Y(c^n t_m))$$

for $0 \le t_1 < \ldots < t_m$, and $n \ge 1$, it follows from the stochastic continuity that

$$(Y(t_1), \ldots, Y(t_m)) \stackrel{d}{=} (Y(0), \ldots, Y(0)) \quad \square$$

The following theorem is a direct consequence of Lemma 2.3.1.

Theorem 2.3.1 *Let $0 < H < \infty$.*

(i) If $Y = (Y(t))_{t\in\mathbf{R}}$ is a non-trivial stationary process and if for some $a,\ a' \in \mathbf{R}\backslash\{0\}$

$$t^H Y(a \log t) \stackrel{d}{=} t^H Y(a' \log t),$$

then either $a = a'$ or $a = -a'$.

(ii) If $X = (X(t))_{t\ge 0}$ is a non-trivial H-ss process and if for some $b,\ b' \in \mathbf{R}\backslash\{0\}$

$$e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't}),$$

then either $b = b'$ or $b = -b'$.

Proof. Part (i) follows directly from Lemma 2.3.1. In order to prove (ii) it is enough to apply Lemma 2.3.1 to $Y(t) = e^{-Ht} X(e^t)$. $\square$

Up to now we have considered processes merely assuming that they are stochastically continuous. In order to gain insight into the influence of different a's and b's on finite-dimensional distributions of corresponding processes we are to concentrate on α-stable processes. We will study Gaussian and non-Gaussian examples to take a different view of the foregoing results.

Note that for Gaussian stationary processes $Y(t) \stackrel{d}{=} Y(-t)$. Hence if Y is Gaussian, then the statement (i) in Theorem 2.3.1 can be replaced by that $t^H Y(a \log t) \stackrel{d}{=} t^H Y(a' \log t)$ if and only if $a = \pm a'$, and if X is Gaussian, then (ii) can be replaced by that $e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't})$ if and only if $b = \pm b'$. Therefore we have the following.

Example 2.3.1 *Let $0 < H < \infty$ and $(Y_\lambda(t))_{t\in\mathbf{R}}$ be a Gaussian Ornstein–Uhlenbeck process, namely*

$$Y_\lambda(t) = \int_{-\infty}^{t} e^{-\lambda(t-x)} B(dx), \ t \in \mathbf{R},$$

where $(B(t))$ is a standard Brownian motion. Then

$$t^H Y_\lambda(a \log t) \stackrel{d}{=} t^H Y_\lambda(a' \log t), \quad t > 0$$

if and only if $a = \pm a'$.

Example 2.3.2 *Let $(X(t))_{t\geq 0}$ be a Gaussian H-ss process and $0 < H < 1$. (If, in addition, it has stationary increments, it is the fractional Brownian motion defined by (1.2) and the stochastic integral with $\alpha = 2$ in (2.1)). Then*

$$e^{-bHt}X(e^{bt}) \stackrel{d}{=} e^{-b'Ht}X(e^{b't}), \quad t \in \mathbf{R},$$

if and only if $b = \pm b'$.

Remarks.

1. Let us recall that the Gaussian Ornstein–Uhlenbeck process can be obtained by transforming the Brownian motion by the Lamperti transformation and there exists only one such process (this was observed by Doob [12] and Itô [19]). How does this fact match the above theorems and examples? Comparing the covariance functions of the transformed Brownian motion and the Gaussian Ornstein–Uhlenbeck process (characterized by parameter λ) leads to the conclusion

Brownian motion		G.O.U. process
$B(t)$	$\stackrel{gen.Lamp.tr.\ with\ a}{\Longrightarrow}$	$Y_\lambda(at)$
		(where $\lambda = \frac{1}{2}$).

   $Y_\lambda(at)$ and $Y_\lambda(a't)$ are different processes when $a \neq \pm a'$ (with respect to finite-dimensional distributions) but nevertheless they are still in the same class of processes because $Y_\lambda(at) \stackrel{d}{=} \sqrt{a}Y_{a\lambda}(t)$, (see Example 2.3.1).

2. Due to the above generalization of the Lamperti theorem we are able to obtain the complete class of Ornstein–Uhlenbeck processes from the standard Brownian motion.

3. Using the generalized Lamperti transformation with different a's, one can generate the entire class of H-ss Gaussian Markov processes starting from the standard Ornstein–Uhlenbeck process with $\lambda = 1$, (see Example 2.3.1). They are given by the covariance function in the following way:

   $$E[X(t)X(s)] = t^H s^H E[Y_1(a\log t)Y_1(a\log s)] = t^H s^H e^{-a(\log t - \log s)} = t^{H-a}s^{H+a},$$

   where $a > 0$ and $s < t$.

We proceed to non-Gaussian stable cases.

Example 2.3.3 *Let $0 < H < \infty$ and $(Y_\lambda(t))_{t\in\mathbf{R}}$ be a $S\alpha S$ Ornstein–Uhlenbeck process, namely*

$$Y_\lambda(t) = \int_{-\infty}^{t} e^{-\lambda(t-x)}Z_\alpha(dx),\ t \in \mathbf{R}$$

where $0 < \alpha < 2$. Then

$$t^H Y_\lambda(a\log t) \stackrel{d}{=} t^H Y_\lambda(a'\log t), \quad t > 0, \tag{2.9}$$

if and only if $a = a'$.

Proof. We compute the characteristic function of vector $(Y_\lambda(as), Y_\lambda(at))$. Fixing $s < t$ and $a > 0$, we have the following equations :

$$
\begin{aligned}
&E\exp\{i(\theta_1 Y_\lambda(as) + \theta_2 Y_\lambda(at))\}\\
&\quad = E\exp\{i([\theta_1 + \theta_2 e^{-\lambda a(t-s)}]Y_\lambda(as) + \theta_2[Y_\lambda(at) - e^{-\lambda a(t-s)}Y_\lambda(as)])\}\\
&\quad = E\exp\{i(\theta_1 + \theta_2 e^{-\lambda a(t-s)})\int_{-\infty}^{as} e^{-\lambda(as-x)} Z_\alpha(dx)\} \cdot E\exp\{i\theta_2 \int_{as}^{at} e^{-\lambda(at-x)} Z_\alpha(dx)\}\\
&\quad = \exp\{-(|\theta_1 + \theta_2 e^{-\lambda a(t-s)}|^\alpha \int_{-\infty}^{as} e^{-\alpha\lambda(as-x)}dx + |\theta_2|^\alpha \int_{as}^{at} e^{-\alpha\lambda(at-x)}dx)\}\\
&\quad = \exp\{-\frac{1}{\alpha\lambda}[(1 - e^{-\alpha\lambda a(t-s)})|\theta_2|^\alpha\\
&\qquad + |1 + e^{-2\lambda a(t-s)}|^{\alpha/2} \cdot \left|\frac{\theta_1}{|1 + e^{-2\lambda a(t-s)}|^{1/2}} + \frac{\theta_2 e^{-\lambda a(t-s)}}{|1 + e^{-2\lambda a(t-s)}|^{1/2}}\right|^\alpha]\}.
\end{aligned}
$$

Thus the spectral measure of vector $(Y_\lambda(as), Y_\lambda(at))$ is given by the formula

$$
\Gamma = \frac{1}{2\alpha\lambda}[(1 - e^{-\alpha\lambda a(t-s)})(\delta(0,1) + \delta(0,-1)) + (1 + e^{-2\lambda a(t-s)})^{\alpha/2}(\delta(c,d) + \delta(-c,-d))],
$$

where

$$
c = \frac{1}{(1 + e^{-2\lambda a(t-s)})^{1/2}}, \quad d = \frac{e^{-\lambda a(t-s)}}{(1 + e^{-2\lambda a(t-s)})^{1/2}}
$$

and $\delta(p,q)$ is the delta measure at $(p,q) \in \mathbf{R}^2$. Similarly, when $a < 0$ the spectral measure of vector $(Y_\lambda(as), Y_\lambda(at))$ is given by

$$
\Gamma = \frac{1}{2\alpha\lambda}[(1 - e^{-\alpha\lambda a(s-t)})(\delta(1,0) + \delta(-1,0)) + (1 + e^{-2\lambda a(s-t)})^{\alpha/2}(\delta(d,c) + \delta(-d,-c))].
$$

Because of the uniqueness of the spectral measure Γ, formula (2.9) (as concerns bivariate distributions) holds only if $a = a'$. This completes the proof. □

Example 2.3.4 *Let* $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ *and* $(Z_\alpha(t))_{t \geq 0}$ *be a* $S\alpha S$ *Lévy motion. Then*

$$
e^{-bHt} Z_\alpha(e^{bt}) \stackrel{d}{=} e^{-b'Ht} Z_\alpha(e^{b't}), \quad t \in \mathbf{R}
$$

if and only if $b = b'$.

Proof. By Theorem 2.3.1 it is enough to show that

$$
e^{-Ht} Z_\alpha(e^t) \stackrel{d}{\neq} e^{Ht} Z_\alpha(e^{-t}),
$$

which is equivalent to

$$
Z_\alpha(t) \stackrel{d}{\neq} t^{2H} Z_\alpha(t^{-1}).
$$

For that, we show that the process on the right hand side does not have independent increments. To this end, it suffices to represent the process by a stable integral $t^{2H} \int_0^{t^{-1}} dZ_\alpha(u)$ and to check its increments. Use the fact that two non-Gaussian stable random variables $\int f dZ_\alpha$ and $\int g dZ_\alpha$ are independent if and only if $f \cdot g = 0$ *a.e.* □

Remarks.

1. As in the Gaussian case there is a correspondence between the $S\alpha S$ Lévy motion (characterized by the parameter α) and the $S\alpha S$ Ornstein–Uhlenbeck process (determined by α and λ) through the Lamperti transformation:

$S\alpha S$ Lévy motion $\quad\quad\quad\quad$ $S\alpha S$ O.U. process

$$Z_\alpha(t) \overset{gen.Lamp.tr.\ with\ a}{\Longrightarrow} Y_\lambda(at)$$

(where $\lambda = \frac{1}{\alpha}$).

(See Adler et al. [1], Theorem 5.1 for $1 < \alpha < 2$ and for general $0 < \alpha < 2$ compute and compare the characteristic functions of processes $\{e^{-at/\alpha}Z_\alpha(e^{at})\}$ and $\{Y_{1/\alpha}(at)\}$, which can be calculated in a way similar to the above proof of Example 2.3.3.)

2. Contrary to the Gaussian case, $Y_\lambda(at)$ defines distinct processes for a and for $-a$ (see Example 2.3.3). For example, $a = 1$ and $a' = -1$ produce the $S\alpha S$ Ornstein–Uhlenbeck and the reverse $S\alpha S$ Ornstein–Uhlenbeck process, respectively (which are different when $0 < \alpha < 2$), (see Adler et al. [1]). Since $Y_\lambda(at) \overset{d}{=} a^{1/\alpha}Y_{a\lambda}(t)$, so we can construct only two different Ornstein–Uhlenbeck processes.

2.4 Mixed linear fractional α-stable motions

In the paper, Surgailis et al. [39], a new class of stationary non-Gaussian $S\alpha S$ processes, namely stable mixed moving averages, is introduced. This includes the well-studied class of moving averages. In this section, we discuss the self-similar stable processes whose corresponding stationary processes $(Y(t))$ through the Lamperti transformation are stable mixed moving averages.

Although more general class is introduced in Surgailis et al. [39], we focus here only on the following type of stable mixed moving averages (which are sums of independent usual moving averages):

$$Y(t) = \sum_{k=1}^{K} \int_{-\infty}^{\infty} f_k(t-v)Z_\alpha^{(k)}(dv), \quad t \in \mathbf{R}, \tag{2.10}$$

where the $Z_\alpha^{(k)}$'s are independent $S\alpha S$ Lévy motions, $f_k \in L^\alpha(-\infty,\infty)$ and where the f_k's are not "equivalent" in the sense that for $k \neq \ell$, there do not exist c and τ such that $f_k(\cdot) = cf_\ell(\cdot - \tau)$. We call the process (2.10) the K-sum stable moving average. It is observed in Surgailis et al. [39] that K-sum stable moving average with $K \geq 2$ is different in law from the ordinary moving average.

We remark here that (2.10) is a special case of stable mixed moving averages introduced in Surgailis et al. [39], but finite sums of independent $S\alpha S$ moving averages as in (2.10) are dense in the class of stable mixed moving averages.

In the following, we give examples of self-similar processes with stationary increments, whose corresponding stationary processes are K-sum stable moving averages.

Definition 2.4.1 *Let* $0 < H < 1, 0 < \alpha < 2, H \neq \frac{1}{\alpha}$, *and*

$$X(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \left\{p_n[(t-u)_+^{H-\frac{1}{\alpha}} - (-u)_+^{H-\frac{1}{\alpha}}] + q_n[(t-u)_-^{H-\frac{1}{\alpha}} - (-u)_-^{H-\frac{1}{\alpha}}]\right\} Z_\alpha^{(n)}(du), \tag{2.11}$$

where a_+ *and* a_- *stand for* $\max\{a,0\}$ *and* $\max\{-a,0\}$, *respectively. The process* $(X(t))$ *is called mixed linear fractional stable motion (cf. Section 3.3.1).*

It is easy to check that $(X(t))$ is H-self-similar and has stationary increments. When $N = 1$ and $p_n = 1, q_n = 1$, it is a linear fractional stable motion in (2.1). The distribution of $(X(t))$ is distinct for different collection of $\{p_n, q_n, n = 1, \cdots, N\}$ unless $p_n = p, q_n = q$ for all n.

In the following, we restrict ourselves to the stationary process $Y_+(t) = e^{-Ht}X(e^t)$. However, as we pointed out in Section 2.3, $(Y_+(t))$ is distinct from $(Y_-(t))$, where $Y_-(t) = e^{Ht}X(e^{-t})$, since we are dealing with non-Gaussian stable case. As to $(Y_-(t))$, we have a similar argument. We shall write below $Y(t)$ for $Y_+(t)$ and $\beta = H - \frac{1}{\alpha}$ for the notational simplicity.

Theorem 2.4.1 *The mixed linear fractional stable process $X(t)$ given by (2.11) corresponds via the Lamperti transformation to a K-sum stable moving average for some $K \leq 2N$.*

Proof. From (2.11), we have

$$\begin{aligned}
Y(t) &= e^{-Ht}X(e^t) \\
&= \sum_{n=1}^{N} e^{-Ht}\int_{-\infty}^{\infty}\Big\{p_n[(e^t-u)_+^\beta-(-u)_+^\beta] \\
&\qquad +q_n[(e^t-u)_-^\beta-(-u)_-^\beta]\Big\}Z_\alpha^{(n)}(du) \\
&= \sum_{n=1}^{N} e^{-Ht}\Big\{p_n\int_{-\infty}^{0}[(e^t-u)^\beta-(-u)^\beta]Z_\alpha^{(n)}(du) \\
&\qquad +\int_0^{e^t}[p_n(e^t-u)^\beta-q_nu^\beta]Z_\alpha^{(n)}(du) \\
&\qquad +q_n\int_{e^t}^{\infty}[(u-e^t)^\beta-u^\beta]Z_\alpha^{(n)}(du)\Big\} \\
&= \sum_{n=1}^{N} e^{-Ht}\Big\{\int_{-\infty}^{0}p_n[(e^t-u)^\beta-(-u)^\beta]Z_\alpha^{(n)}(du) \\
&\qquad +\int_0^{\infty}\Big(I[0<u<e^t][p_n(e^t-u)^\beta-q_nu^\beta] \\
&\qquad +I[e^t<u]q_n[(u-e^t)^\beta-u^\beta]\Big)Z_\alpha^{(n)}(du)\Big\}.
\end{aligned}$$

Thus, for $c_j \in \mathbf{R}$,

$$\begin{aligned}
&-\log E[\exp\{i\sum_j c_jY(t_j)\}] \\
&= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{0}\Big|\sum_j c_je^{-Ht_j}p_n[(e^{t_j}-u)^\beta-(-u)^\beta]\Big|^\alpha du \\
&\qquad +\int_0^{\infty}\Big|\sum_j c_je^{-Ht_j}\{I[0<u<e^{t_j}][p_n(e^{t_j}-u)^\beta-q_nu^\beta] \\
&\qquad +I[e^{t_j}<u]q_n[(u-e^{t_j})^\beta-u^\beta]\}\Big|^\alpha du\Big\}
\end{aligned}$$

by the change of variables $|u| = e^v$,

$$\begin{aligned}
&= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{\infty}\Big|\sum_j c_je^{-Ht_j}p_n[(e^{t_j}+e^v)^\beta-e^{\beta v}]\Big|^\alpha e^v dv \\
&\qquad +\int_{-\infty}^{\infty}\Big|\sum_j c_je^{-Ht_j}\{I[v<t_j][p_n(e^{t_j}-e^v)^\beta-q_ne^{\beta v}] \\
&\qquad +I[t_j<v]q_n[(e^v-e^{t_j})^\beta-e^{\beta v}]\Big|^\alpha e^v dv\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \left\{ \int_{-\infty}^{\infty} \left| \sum_j c_j e^{-Ht_j+\beta v} p_n[(e^{t_j-v}+1)^\beta - 1] \right|^\alpha e^v dv \right. \\
&\quad + \int_{-\infty}^{\infty} \left| \sum_j c_j e^{-Ht_j+\beta v} \{ I[t_j - v > 0][p_n(e^{t_j-v} - 1)^\beta - q_n] \right. \\
&\quad\quad \left. \left. + I[t_j - v < 0] q_n[(1 - e^{t_j-v})^\beta - 1] \} \right|^\alpha e^v dv \right\} \\
&= \sum_{n=1}^{N} \left\{ \int_{-\infty}^{\infty} \left| \sum_j c_j e^{-H(t_j-v)} p_n[(e^{t_j-v}+1)^\beta - 1] \right|^\alpha dv \right. \\
&\quad + \int_{-\infty}^{\infty} \left| \sum_j c_j e^{-H(t_j-v)} \{ I[t_j - v < 0] q_n[(1 - e^{t_j-v})^\beta - 1] \right. \\
&\quad\quad \left. \left. + I[t_j - v > 0][p_n(e^{t_j-v} - 1)^\beta - q_n] \} \right|^\alpha dv \right\} \\
&= \sum_{n=1}^{N} \left\{ \int_{-\infty}^{\infty} \left| \sum_j c_j f_n(t_j - v) \right|^\alpha dv + \int_{-\infty}^{\infty} \left| \sum_j c_j g_n(t_j - v) \right|^\alpha dv \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
f_n(t) &= e^{-Ht} p_n[(e^t + 1)^\beta - 1] \\
g_n(t) &= e^{-Ht} \{ I[t < 0] q_n[(1 - e^t)^\beta - 1] + I[t > 0][p_n(e^t - 1)^\beta - q_n] \}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
Y(t) \stackrel{d}{=} & \sum_{n=1}^{N} \int_{-\infty}^{\infty} f_n(t - v) Z_\alpha^{(n)}(dv) \\
& + \sum_{n=1}^{N} \int_{-\infty}^{\infty} g_n(t - v) Z_\alpha^{(N+n)}(dv),
\end{aligned}
$$

where $Z_\alpha^{(n)}, n = 1, 2, \cdots, 2N$ are independent stable motions. □

Example 2.4.1 *If $N = 1, p_1 = 0, q_1 \neq 0$, then*

$$
Y(t) \stackrel{d}{=} \int_{-\infty}^{\infty} g_1(t - v) Z_\alpha(dv),
$$

and hence $K = 1$. The linear fractional stable motion corresponds to a stable moving average.

Example 2.4.2 *If $N = 1, p_1 \neq 0$ (whatever q_1 is), then $f_1(\cdot) = \pm c g_1(\cdot + \tau)$ is not true. Hence*

$$
Y(t) \stackrel{d}{=} \int_{-\infty}^{\infty} f_1(t - v) Z_\alpha^{(1)}(dv) + \int_{-\infty}^{\infty} g_1(t - v) Z_\alpha^{(2)}(dv),
$$

which is 2-sum stable moving average. Thus, the linear fractional stable motion can also correspond to a stable mixed moving average.

Example 2.4.3 *Let $K \geq 3$ and choose N such that $2N \geq K$. Then by choosing p_n and q_n, zero or non-zero suitably, we can construct K-sum stable moving average from the mixed linear fractional stable motion.*

Next we consider the case of $H = \frac{1}{\alpha}$.

Example 2.4.4 *Let* $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ *and* $X(t) = Z_\alpha(t)$. *Then*

$$\begin{aligned} Y(t) &= e^{-\frac{1}{\alpha}t} X(e^t) = e^{-\frac{1}{\alpha}t} Z_\alpha(e^t) \\ &= e^{-\frac{1}{\alpha}t} \int_0^{e^t} Z_\alpha(du) = e^{-\frac{1}{\alpha}t} \int_{-\infty}^{s} Z_\alpha(e^v dv) \\ &\stackrel{d}{=} e^{-\frac{1}{\alpha}t} \int_{-\infty}^{s} e^{\frac{1}{\alpha}v} Z_\alpha(dv) = \int_{-\infty}^{s} e^{-\frac{1}{\alpha}(t-v)} Z_\alpha(dv) \\ &= \int_{-\infty}^{\infty} f(t-v) Z_\alpha(du), \end{aligned}$$

where

$$f(t) = e^{-\frac{1}{\alpha}t} I[t > 0].$$

Example 2.4.5 *Let* $1 < \alpha < 2, H = \frac{1}{\alpha}$ *and*

$$X(t) = \int_{-\infty}^{\infty} \log \left| \frac{t-u}{u} \right| Z_\alpha(du).$$

This $(X(t))$ *is called a log-fractional stable motion. (See Kasahara et al. [21].) Then*

$$\begin{aligned} Y(t) &= e^{-\frac{1}{\alpha}t} X(e^t) \\ &= e^{-\frac{1}{\alpha}t} \int_{-\infty}^{0} \log \left| \frac{e^t - u}{u} \right| Z_\alpha(du) + e^{-\frac{1}{\alpha}t} \int_{0}^{\infty} \log \left| \frac{e^t - u}{u} \right| Z_\alpha(du) \\ &\stackrel{d}{=} e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| \frac{e^t + e^v}{-e^v} \right| Z_\alpha^{(1)}(-e^v dv) + e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| \frac{e^t - e^v}{e^v} \right| Z_\alpha^{(2)}(e^v dv) \\ &\stackrel{d}{=} e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| e^{t-v} + 1 \right| e^{\frac{1}{\alpha}v} Z_\alpha^{(1)}(dv) + e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| e^{t-v} - 1 \right| e^{\frac{1}{\alpha}v} Z_\alpha^{(2)}(dv) \\ &= \int_{-\infty}^{\infty} f_1(t-v) Z_\alpha^{(1)}(dv) + \int_{-\infty}^{\infty} f_2(t-v) Z_\alpha^{(2)}(dv), \end{aligned}$$

where

$$f_1(t) = e^{-\frac{1}{\alpha}t} \log |e^t + 1|$$

and

$$f_2(t) = e^{-\frac{1}{\alpha}t} \log |e^t - 1|$$

Thus, the log-fractional stable motion also corresponds to a 2-sum moving average as in the case of the linear fractional stable motion in Example 2.4.2.

Chapter 3

Integral representation of stable H-ss processes

In this section we establish and exploit the connection between theory of self-similar stable processes and ergodic theory of nonsingular flows. Using this connection and the Lamperti transformation, a special decomposition of self-similar processes is obtained. In Section 3.2 we show that a minimal spectral representation of an H-self-similar $S\alpha S$ process $\{X_t\}_{t\in\mathbf{R}_+}$ is of the form

$$X_t = \int_S t^H [a_t f \circ \phi_t] m_t^{1/\alpha} dM, \quad t \in \mathbf{R}_+.$$

Here $\{\phi_t\}_{t\in\mathbf{R}_+}$ is a nonsingular multiplicative flow on (S,μ), $\{a_t\}_{t\in\mathbf{R}_+}$ is a cocycle for this flow taking values in $\{-1,1\}$, $m_t = d(\mu\circ\phi_t)/d\mu$, $f \in L^\alpha(S,\mu)$ and M is a $S\alpha S$ random measure.

As a consequence we prove in Section 3.3 that every stable self-similar process admits a unique decomposition into three independent parts

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \quad t \in \mathbf{R}_+,$$

such that $\{X_t^{(1)}\}_{t\in\mathbf{R}_+}$ corresponds to a superposition of moving averages in the theory of stationary processes, the second class $\{X_t^{(2)}\}_{t\in\mathbf{R}_+}$ is harmonizable and $\{X_t^{(3)}\}_{t\in\mathbf{R}_+}$ is called evanescent. This result shows how rich the class of stable self-similar processes actually is.

3.1 Preliminaries and definitions

Definition 3.1.1 *A map $t \to f_t$, where $\{f_t\}_{t\in T} \subset L^\alpha(S,\mathcal{B},\mu)$, $(S,\mathcal{B},\mu)$ is a standard Lebesgue space, is said to be a spectral representation of a $S\alpha S$ process $\{X_t\}_{t\in T}$ if*

$$\{X_t\}_{t\in T} \stackrel{d}{=} \left\{\int_S f_t(s) M(ds)\right\}_{t\in T}, \tag{3.1}$$

where M is an independently scattered random measure on $\mathcal{B}$ such that

$$E\exp\{iuM(A)\} = \exp\{-|u|^\alpha \mu(A)\}, \quad u \in \mathbf{R},$$

for every $A \in \mathcal{B}$ with $\mu(A) < \infty$. The family of functions $\{f_t\}_{t\in T}$ is called the kernel of a spectral representation.

It is well-known that every separable in probability $S\alpha S$ process admits a spectral representation such that S is a unit interval or a countable discrete set or the union of the latter two and μ is the direct sum of Lebesgue measure acting on the unit interval and a counting measure acting on the discrete part of S (cf. Hardin [16]). On the other hand, many interesting classes of stable processes are defined by explicitly given families of functions f_t and control measure μ given on various spaces. Because a spectral representation is a natural characterization of a stable process, the question of its uniqueness is important. Spectral representation is not unique, even when S and μ are fixed. We study the problem of uniqueness in the context of so-called *minimal representations*. Minimal representations are unique and other representations of the same process are transformations of the minimal ones (see, e.g. Rosiński [35]).

Definition 3.1.2 *A spectral representation $t \to f_t$ is said to be minimal if $\sigma\{f_t/f_u : t, u \in T\} = \mathcal{B}$ modulo μ.*

Every separable in probability $S\alpha S$ process has a minimal representation (see Hardin [16], and Janicki and Weron [20]).

We will also consider complex stable processes. However, in the complex case we restrict our attention to those processes $\{X\}_{t\in T}$ for which all linear combinations $\sum a_i X_{t_i}$, $a_i \in \mathbf{C}$, $t_i \in T$ have *rotationally invariant* stable distributions. In that case, a family of complex α-integrable functions $\{f_t\}_{t\in T}$ defined on a standard Lebesgue space $(S, \mathcal{B}, \mu)$ is called the kernel of a spectral representation of the process $\{X\}_{t\in T}$ if (3.1) holds with a complex independently scattered random measure M such that

$$E\exp\{i\Re(u\overline{M(a)})\} = \exp\{-|u|^\alpha \mu(A)\}, \quad u \in \mathbf{C}.$$

3.2 Minimal spectral representation of stable self-similar processes

From now on we will consider processes indexed by $T = \mathbf{R}_+ = (0, \infty)$. In this section we will characterize the kernel of a spectral representation of an H-self-similar $S\alpha S$ stochastic process. Without loss of generality we may and do assume that underlying measure space $(S, \mathcal{B}, \mu)$ for the kernel is Borel. A collection $\{\phi_t\}_{t>0}$ of measurable maps from S onto S such that

$$\phi_{t_1 t_2}(s) = \phi_{t_1}(\phi_{t_2}(s)) \tag{3.2}$$

and $\phi_1(s) = s$ for all $s \in S$ and $t_1, t_2 > 0$ is called a *multiplicative flow.* Such flow is said to be *measurable* if the map $\mathbf{R}_+ \times S \ni (t, s) \mapsto \phi_t(s) \in S$ is measurable. Given a σ-finite measure μ on $(S, \mathcal{B})$, $\{\phi_t\}_{t>0}$ is said to be *nonsingular* if $\mu(\phi_t^{-1}(A)) = 0$ if and only if $\mu(A) = 0$ for every $t > 0$ and $A \in \mathcal{B}$.

Let A be a locally compact second countable group. A measurable map $\mathbf{R}_+ \times S \ni (t, s) \to a_t(s) \in A$ is said to be a *cocycle* for a measurable flow $\{\phi_t\}_{t>0}$ if for every $t_1, t_2 > 0$

$$a_{t_1 t_2}(s) = a_{t_2}(s) a_{t_1}(\phi_{t_2}(s)) \quad \textit{for all } s \in S. \tag{3.3}$$

Theorem 3.2.1 *Let $\{f_t\}_{t>0} \subset L^\alpha(S, \mu)$ be the kernel of a measurable minimal spectral representation of a measurable H-ss $S\alpha S$ process $\{X_t\}_{t>0}$. Then there exist a unique modulo μ nonsingular flow $\{\phi_t\}_{t>0}$ on (S, μ) and a cocycle $\{a_t\}_{t>0}$ taking values in $\{-1, 1\}$ ($\{|z| = 1\}$ in the complex case) such that for each $t > 0$*

$$f_t = t^H a_t \left\{\frac{d\mu \circ \phi_t}{d\mu}\right\}^{1/\alpha} (f_1 \circ \phi_t) \quad \mu - a.e. \tag{3.4}$$

Proof. Since $t \to f_t$ is minimal, then, for each $c > 0$ $\{1/c^H f_{ct}\}_{t>0}$ and $\{f_t\}_{t>0}$ are kernels of minimal representations of the the same H-ss $S\alpha S$ process. Applying Theorem 2.2 in Rosiński [34] there exist a one-to-one and onto function $\Phi_c : S \to S$ and a function $h_c : S \to \mathbf{R} - \{0\}$ such that, for each $t > 0$,

$$f_{ct} = (c^H)(h_c)(f_t \circ \Phi_c) \quad \mu - a.e., \tag{3.5}$$

and

$$\frac{d(\mu \circ \Phi_c)}{d\mu} = |h_c|^\alpha, \quad \mu - a.e. \tag{3.6}$$

Since, for every $t, c_1, c_2 > 0$, it is true that, $\mu - a.e$

$$f_{c_1 c_2 t} = (c_2^H)(h_{c_2})(f_{c_1 t} \circ \Phi_{c_2}) = (c_2^H c_1^H)(h_{c_2})(h_{c_1} \circ \Phi_{c_2})(f_t \circ \Phi_{c_1} \circ \Phi_{c_2}) \tag{3.7}$$

and

$$f_{c_1 c_2 t} = (c_1^H c_2^H)(h_{c_1 c_2})(f_t \circ \Phi_{c_1 c_2}),$$

we infer from Theorem 2.2 in Rosiński [34] that, for every $c_1, c_2 > 0$,

$$h_{c_1 c_2} = (h_{c_2})(h_{c_1} \circ \Phi_{c_2}), \quad \mu - a.e., \tag{3.8}$$

and

$$\Phi_{c_1 c_2} = \Phi_{c_1} \circ \Phi_{c_2}, \quad \mu - a.e. \tag{3.9}$$

In order to conclude the proof it is enough to rewrite the arguments of the proof of Theorem 3.1 in Rosiński [34] replacing the additive group $\mathbf{R}$ with the multiplicative $\mathbf{R}_+$. Therefore, $\phi_t = \Phi_t$ is the map and putting $a_t = h_t/|h_t|$ ends the proof. □

Remark. It is possible to present an alternative proof of the theorem using the Lamperti transformation. That is, first we need to see that the Lamperti transformation leading from self-similar to stationary processes preserves the minimality of a spectral representation. To this end it is enough to verify condition (iii) of Theorem 3.8 in Rosiński [35] with $F = \{e^{-tH} f_{e^t}\}_{t \in \mathbf{R}}$. It is trivially satisfied as the condition is fulfilled for $F = \{f_t\}_{t \in \mathbf{R}_+}$. Now, taking $Y_t = e^{-tH} X_{e^t}$ we obtain a stationary process which minimal representation is defined by Theorem 3.1 in Rosiński [34] in terms of a unique flow and a corresponding cocycle on the additive group $\mathbf{R}$. In order to conclude the proof we apply the reciprocal transformation $X_t = t^H Y_{\log t}$ which leads to the minimal spectral representation of the process X as stated in Theorem 3.2.1. □

Corollary 3.2.1 *Since there is a correspondence between self-similar and stationary processes through Lamperti transformation every minimal representation $t \to f_t$ (3.4) given in terms of a flow ϕ_t and a cocycle a_t defines the kernel of a minimal spectral representation $\{f_t^1\}_{t \in \mathbf{R}}$ of the corresponding stationary process as follows*

$$f_t^1 = a_t^1 \left\{ \frac{d\mu \circ \phi_t^1}{d\mu} \right\}^{1/\alpha} (f_0 \circ \phi_t^1), \quad \mu - a.e. \tag{3.10}$$

such that

$$\phi_t^1(s) = \phi_{e^t}(s), \; a_t^1(s) = a_{e^t}(s), \; f_0^1(s) = f_1(s) \quad \textit{for all } s \in S \textit{ and } t \in \mathbf{R}.$$

Conversely if (3.10) is the kernel of a minimal spectral representation of a stationary process then (3.4) defines the kernel of a minimal representation of an H-ss process in terms of a pair $\{a_t, \phi_t\}_{t>0}$ such that

$$\phi_t(s) = \phi_{\log t}^1(s), \; a_t(s) = a_{\log t}^1(s), \; f_1(s) = f_0^1(s) \quad \textit{for all } s \in S \textit{ and } t > 0.$$

Remark. Combining results of Theorem 3.1 in Rosiński [34] and Theorem 3.2.1 we may try to prove Theorems 2.2.1 and 2.2.2 describing classes of transformations leading from self-similar to stationary processes and conversely. Let us concentrate on the (ii) part of Theorem 2.2.1. We will support the thesis that $\theta = t^H$ and $\psi = a \log t$ using Theorem 3.1 in Rosiński [34] and Theorem 3.2.1 which concern minimal spectral representations of stationary and self-similar processes, respectively. First we notice that any transformation of the form $X_t = \theta(t) Y_{\psi(t)}$ for a non-trivial stationary process Y and functions θ, $\psi : \ (0, \infty) \to \mathbf{R}$ such that ψ is onto preserves minimality of the spectral representation. It is obvious since $F = \{\theta(t) f^1_{\psi(t)}\}_{t>0}$ satisfies condition (iii) of Theorem 3.8 in Rosiński [35] as $\{f^1_t\}_{t \in \mathbf{R}}$ (the spectral representation of process Y) is rigid in $L^\alpha(S, \mu)$. Thus X is H-ss with the spectral representation as follows

$$f_t = \theta(t) a^1_{\psi(t)} \left\{ \frac{d\mu \circ \phi^1_{\psi(t)}}{d\mu} \right\}^{1/\alpha} (f_0 \circ \phi^1_{\psi(t)}) \quad \mu - a.e.$$

Now we use the fact that the process X has a spectral representation defined by (3.4) and compare them. We immediately obtain that $\theta(t) = t^H$. Furthermore, it is easy to see that the spectral representations are equivalent if

$$\phi^1_{\psi(t_1 t_2)} = \phi^1_{\psi(t_1) + \psi(t_2)} \quad and \quad \psi(1) = 0.$$

This yields either

$$\psi(t_1 t_2) = \psi(t_1) + \psi(t_2) \quad for\ all\ t_1, t_2 > 0 \tag{3.11}$$

or

$$\psi(t_1 t_2) = \psi(t_1) + \psi(t_2) + c \quad for\ some\ t_1, t_2 > 0\ and\ c \neq 0.$$

Since ψ is continuous the latter implies that Y is trivial. The equivalence (3.11) leads to the statement $\psi(t) = a \log t$ for some real constant a.

3.3 Decomposition of stable self-similar processes

Every measurable stable self-similar process is generated by a nonsingular flow. First we will show that certain standard decompositions of flows in ergodic theory induce natural decompositions of stable self-similar processes. To this end let us recall basic definitions and facts concerning nonsingular maps and flows.

A nonsingular map $V : S \to S$ is said to be *conservative* if there is no *wandering* set of positive μ measure (a set is called wandering if the sets $V^{-k}B$ are disjoint). Given a nonsingular map V, there exist a decomposition of S into two disjoint measurable sets C and D – the conservative and the dissipative parts – such that

(i) C and D are V-invariant,

(ii) the restriction of V to C is conservative and

(iii) $D = \cup_{k=-\infty}^{\infty} V^k B$ for some wandering set B.

The decomposition of S into C and D is unique (modulo μ) and is called the *Hopf decomposition.* Given a nonsingular flow $\{\phi_t\}_{t \in T}$, for each $t \in T - \{0\}$ one has the Hopf decomposition of S, $S = C_t \cup D_t$, generated by the map ϕ_t. Since all C_t (D_t, resp.) are equal to each other modulo μ (see Krengel [22]), one can choose a set C that is invariant under $\{\phi_t\}_{t \in T}$ and such that $C = C_t$, and $D = S - C = D_t$ modulo μ for every $t \in T - \{0\}$. This is the Hopf decomposition of S

corresponding to the flow $\{\phi_t\}_{t\in T}$. A flow is called *dissipative* if $S = D$ and *conservative* if $S = C$ modulo μ.

Similarly as in the case of stationary $S\alpha S$ processes, Theorem 3.2.1 allows one to use ergodic theory ideas in the study of $S\alpha S$ self-similar processes. In particular, the Hopf decomposition of the underlying space S of the spectral representation (3.4) into invariant parts C and D, such that the flow ϕ_t is conservative on C and dissipative on D, generates a decomposition of $\{X_t\}_{t>0}$ into two independent $S\alpha S$ H-ss processes $\{X_t^C\}_{t>0}$ and $\{X_t^D\}_{t>0}$. We will characterize the latter process.

3.3.1 Mixed fractional motions

The simplest H-ss $S\alpha S$ process is obtained from a kernel of the form

$$f_t(s) = t^{H-\frac{1}{\alpha}} f\left(\frac{s}{t}\right), \qquad t, s > 0, \tag{3.12}$$

considered with Lebesgue control measure on $(0,\infty)$, $f \in L^\alpha((0,\infty), Leb)$. A $S\alpha S$ process with such representation will be called a *fractional motion* (FM). A superposition of independent FM processes of type (3.12) is called a *mixed fractional motion* (MFM).

Definition 3.3.1 *An H-ss $S\alpha S$ process $\{X_t\}_{t>0}$ is said to be a MFM if it admits a spectral representation with a kernel $\{g_t\}_{t>0}$ defined on $(W \times (0,\infty), \mathcal{B}_W \otimes \mathcal{B}_{(0,\infty)}, \nu \otimes Leb)$, for some Borel measure space $(W, \mathcal{B}_W, \nu)$, such that*

$$g_t(w,u) = t^{H-\frac{1}{\alpha}} g\left(w, \frac{u}{t}\right), \tag{3.13}$$

$(w,u) \in W \times (0,\infty), \quad t > 0.$

Theorem 3.3.1 *$\{X_t^D\}_{t>0}$ is a MFM and one can choose a minimal representation of $\{X_t^D\}_{t>0}$ of the form (3.13). Furthermore, $\{X_t^D\}_{t>0}$ is a FM if and only if $\{\phi_t\}_{t>0}$ restricted to D is ergodic.*

Proof. Using Corollary 3.2.1 we infer that the process $\{X_t^D\}_{t>0}$ corresponds, by Lamperti transformation, to a stationary $S\alpha S$ process $\{Y_t\}_{t\in\mathbf{R}}$ generated by a dissipative flow. From Theorem 4.4 in Rosiński [34] we get that $\{Y_t\}_{t\in\mathbf{R}}$ is a mixed moving average, implying that $\{X_t^D\}_{t>0}$ is a MFM.

We will now prove the second part of the theorem. Since a moving average representation kernel is minimal (see, e.g, Rosiński [35]), (3.12) is minimal as well. Since f_t in (3.4) is minimal, then also f_t restricted to D is minimal. By Theorem 3.6 in Rosiński [34] we infer that the (multiplicative) flow ϕ_t is equivalent to the flow $\psi_t(s) = t^{-1}s, \ t, s > 0$. Since $\{\psi_t\}$ is ergodic, so is $\{\phi_t\}$. Now suppose that $\{\phi_t\}$ is ergodic. By the first part of this theorem, $\{X_t\}$ admits a minimal representation of the form (3.13) whose flow is given by $\psi_t(w,u) = (w, t^{-1}u)$. Since the latter flow is equivalent to $\{\phi_t\}$ by the foregoing theorem, it must be ergodic which is only possible when ν is a point-mass measure. Thus (3.13) reduces to (3.12). □

We will give a few examples of FM and MFM processes. We begin with the simplest one.

Example 3.3.1 *Let $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ and $\{X\}_{t>0}$ be a Lévy motion. Then*

$$X_t = \int_0^t M(ds) = \int_0^\infty f(s/t) M(ds),$$

where

$$f(s) = I[0 < s < 1]$$

and M is $S\alpha S$ on $(0,\infty)$ with Lebesgue control measure (see Figure 3.1).

Example 3.3.2 *Let* $f \in L^\alpha(\mathbf{R}^d, Leb)$. *Let*

$$f_t(s) = t^{H-\frac{d}{\alpha}} f\left(\frac{s}{t}\right), \quad s \in \mathbf{R}^d,\ t > 0,$$

and let M *be a* $S\alpha S$ *random measure on* $\mathbf{R}^d$ *with Lebesgue control measure. It is easy to check that a* $S\alpha S$ *process* $\{X_t\}_{t>0}$ *with such spectral representation is* H*-ss. We will show that* $\{X_t\}_{t>0}$ *is a MFM. Indeed, let* $W = S_d$ *be the unit sphere in* $\mathbf{R}^d$ *equipped with the uniform probability measure* ν *and let*

$$g(w,u) = (c_d u^{d-1})^{1/\alpha} f(uw), \quad (w,u) \in S_d \times (0,\infty),$$

where $c_d = 2\pi^{d/2}/\Gamma(d/2)$ *is the surface area of* S_d. *Using polar coordinates, we get for every* $a_1, \ldots, a_n \in \mathbf{R}$, $t_1, \ldots, t_n > 0$,

$$\int_{\mathbf{R}^d} |\sum a_j f_{t_j}(s)|^\alpha \, ds$$

$$= c_d \int_{S_d} \int_0^\infty |\sum a_j t_j^{H-\frac{d}{\alpha}} f\left(\frac{uw}{t_j}\right)|^\alpha u^{d-1} \, du \nu(dw)$$

$$= \int_{S_d} \int_0^\infty |\sum a_j t_j^{H-\frac{1}{\alpha}} g\left(w, \frac{u}{t_j}\right)|^\alpha \, du \nu(dw),$$

which proves the claim.

Comparing the kernel from the above example with the general form (3.4) we get that $S = \mathbf{R}^d \setminus \{0\}$, $\phi_t(s) = t^{-1}s$, $f_1(s) = f(s)$, and $\frac{d\mu\circ\phi_t}{d\mu} = t^{-d}$. The following well-known H-ss processes are special cases of Example 3.3.2.

Example 3.3.3 *Let* $1 < \alpha < 2$ *and* $H = \frac{1}{\alpha}$. *Then a log-fractional motion (cf. Kasahara et al. [21])* $\{X_t\}_{t>0}$ *is defined by*

$$X_t = \int_{-\infty}^{\infty} \log\left|\frac{t-s}{s}\right| M(ds) = \int_{-\infty}^{\infty} f(s/t) M(ds),$$

where

$$f(s) = \log|1/s - 1|$$

and M *is* $S\alpha S$ *on* $\mathbf{R}$ *with Lebesgue control measure (see Figure 3.2).*

Example 3.3.4 *Let* $0 < H < 1$, $0 < \alpha < 2$, $H \neq \frac{1}{\alpha}$. *Put* $\beta = H - \frac{1}{\alpha}$. *Then a linear fractional stable motion (cf. Cambanis et al. [8])* $\{X_t\}_{t>0}$ *is defined by*

$$\begin{aligned} X_t = &\int_{-\infty}^0 p[(t-s)^\beta - (-s)^\beta]\, M(ds) + \\ &\int_0^\infty \left(I[0<s<t][p(t-s)^\beta - qs^\beta] + I[t<s]q[(s-t)^\beta - s^\beta]\right) M(ds) \\ = &\int_{-\infty}^\infty t^\beta f(s/t) M(ds), \end{aligned}$$

where

$$\begin{aligned} f(s) = &I[s<0]p[(1-s)^\beta - (-s)^\beta] + \\ &I[0<s<1][p(1-s)^\beta - qs^\beta] + I[s>1]q[(s-1)^\beta - s^\beta], \end{aligned}$$

and M *is* $S\alpha S$ *on* $\mathbf{R}$ *with Lebesgue control measure (see Figure 3.3).*

Next Theorem shows that the kernel of a spectral representation of any MFM can be defined on $\mathbf{R}^2$ in a canonical way.

Theorem 3.3.2 (Canonical representation of a MFM) . *Let σ be a σ-finite measure on the unit circle S_2 of $\mathbf{R}^2$ and let μ be a measure on $\mathbf{R}^2 \setminus \{0\}$ whose representation in polar coordinates is*

$$\mu(dr, d\theta) = r^{\alpha H - 1} dr\, \sigma(d\theta), \qquad r > 0,\ \theta \in S_2. \tag{3.14}$$

Let $f : \mathbf{R}^2 \setminus \{0\} \mapsto \mathbf{R}$ (or $\mathbf{C}$) be such that

$$\int_{\mathbf{R}^2 \setminus \{0\}} |f(z)|^\alpha \mu(dz) < \infty.$$

Then the family of functions $\{f_t\}_{t>0} \subset L^\alpha(\mathbf{R}^2 \setminus \{0\}, \mu)$ given by

$$f_t(z) = f(t^{-1} z) \tag{3.15}$$

is the kernel of a spectral representation of a $S\alpha S$ process, which is H-ss and MFM. Conversely, every MFM admits a (canonical) representation (3.14)-(3.15).

Proof. We are to show only the converse part. Consider a MFM with a representation (3.13). Since S is a Borel space, S is measurably isomorphic to a Borel subset S_2. Let $\Phi : S \mapsto S_2$ denote this isomorphism and let $\sigma = \nu \circ \Phi^{-1}$. Define a function f on $\mathbf{R}^2 \setminus \{0\}$ as follows

$$f(z) = \begin{cases} g\left(\Phi^{-1}\left(\frac{z}{|z|}\right), |z|\right) |z|^{1/\alpha - H}, & \text{if } \frac{z}{|z|} \in \Phi(S) \\ 0, & \text{otherwise.} \end{cases}$$

Let μ be a measure on $\mathbf{R}^2 \setminus \{0\}$ given by (3.14). Then

$$\int_{\mathbf{R}^2 \setminus \{0\}} |\sum a_j f_{t_j}(z)|^\alpha \mu(dz) = \int_{\mathbf{R}^2 \setminus \{0\}} |\sum a_j f(t_j^{-1} z)|^\alpha \mu(dz)$$

$$= \int_{S_2} \int_0^\infty |\sum a_j f(t_j^{-1} r\theta)|^\alpha r^{\alpha H - 1}\, dr \sigma(d\theta)$$

$$= \int_S \int_0^\infty |\sum a_j f(t_j^{-1} r \Phi(s))|^\alpha r^{\alpha H - 1}\, dr \nu(ds)$$

$$= \int_S \int_0^\infty |\sum a_j t_j^{H - 1/\alpha} g(s, t_j^{-1} r)|^\alpha\, dr \nu(ds),$$

for every $t_1, \ldots, t_n > 0$ and $a_1, \ldots, a_n \in \mathbf{R}\,(\mathbf{C})$. This ends the proof. □

Remark. The Lamperti transformation maps FMs onto moving average processes and MFMs onto mixed moving averages (see Surgailis et al. [39]). Considering above examples it seems that MFMs appear more naturally than FMs. This is quite opposite to the relation between mixed and the usual moving averages.

It is clear that a stable process may have many spectral representations with different kernels defined on various measure spaces. However, we can identify one property, common to all such representations, which characterizes MFMs.

Theorem 3.3.3 *Let $\{X_t\}_{t>0}$ be a $S\alpha S$ H-ss process with an arbitrary representation (3.1). Then X is MFM if and only if*

$$\int_0^\infty t^{-\alpha H-1}|f_t(s)|^\alpha dt < \infty \quad \mu - a.e. \tag{3.16}$$

Proof. The condition (3.16) is equivalent to

$$\int_{-\infty}^\infty e^{-\alpha Ht}|f_{e^t}(s)|^\alpha dt < \infty \quad \mu - a.e.$$

By Theorem 2.1 in Rosiński [36] and the Lamperti transformation this concludes the proof. □

3.3.2 Harmonizable processes

The class generated by conservative flows consists of harmonizable processes and processes of a third kind (evanescent).

Definition 3.3.2 *An H-ss $S\alpha S$ process $\{X_t\}_{t>0}$ is said to be* harmonizable *if it admits the representation*

$$\{X_t\}_{t>0} \stackrel{d}{=} \left\{\int_{\mathbf{R}} t^{H+is} N(ds)\right\}_{t>0}, \tag{3.17}$$

where N is a complex-valued rotationally invariant $S\alpha S$ measure with the finite control measure ν on S.

Notice that the representation (3.17) is minimal and it is generated by an identity flow acting on S with $a_t(s) = t^{is}$ as the corresponding multiplicative cocycle. It is easy to prove the converse:

Proposition 3.3.1 *Let $\{X_t\}_{t>0}$ be a measurable complex-valued H-ss $S\alpha S$ process generated by an identity flow. Then $\{X_t\}_{t>0}$ is harmonizable.*

Proof. Let

$$S_0 = \{s : a_{t_1 t_2}(s) = a_{t_1}(s)a_{t_2}(s) \;\; for \;\; Leb \otimes Leb \;\; a.a. \; (t_1, t_2)\}.$$

Now it is enough to show that for each $s \in S_0$ there exist a unique $k(s) \in \mathbf{R}$ such that

$$a_t(s) = t^{ik(s)}.$$

To this end we follow the proof of Proposition 5.1 in Rosiński [34] and next define a finite measure $\mu_0(ds) = |f(s)|^\alpha \mu(ds)$ on S. Therefore, (3.17) holds with $\nu = \mu_0 \circ k^{-1}$. □

Theorem 3.3.4 *Let $\{f_t\}_{t>0}$ be the kernel of a minimal spectral representation of the form (3.4) for a complex-valued $S\alpha S$ harmonizable process $\{X_t\}_{t>0}$. Then $\{\phi_t\}_{t>0}$ is the identity flow and (3.4) reduces to*

$$f_t(s) = t^{H+is} f(s) \tag{3.18}$$

Proof. Since (3.18) follows from the proof of the previous proposition, we only need to show that $\{\phi_t\}_{t>0}$ is the identity flow. However, the representation (3.17) is minimal and is induced by the identity flow $\psi_t(s) = s$, for all t, s, so that by Theorem 3.6 in Rosiński [34], ϕ_t being equivalent to the identity flow must be identity. □

Example 3.3.5 *Let*

$$\{X_t\}_{t>0} \stackrel{d}{=} \left\{ \int_{-\infty}^{\infty} t^{H+is} \frac{e^{is}-1}{is} |s|^{-(H-1+1/\alpha)} M(ds) \right\}_{t>0},$$

where M is a complex-valued rotationally invariant $S\alpha S$ measure. The process X corresponds via the Lamperti transformation to the complex harmonizable fractional stable noise (cf. Samorodnitsky and Taqqu [38]).

Remark. There can not be any non-zero real-valued stationary harmonizable process. Using Lamperti transformation, the same statement is valid about real-valued harmonizable self-similar processes. However, the class of real-valued self-similar processes whose spectral representation is generated by the identity flow is slightly larger. Any process of this class must be of the form $X_t = t^H X_1$ (cf. Proposition 5.2 in Rosiński [34]).

As it was in the case of a MFM (see Theorem 3.3.3) we can verify whether an H-ss $S\alpha S$ process is harmonizable given its arbitrary spectral representation.

Theorem 3.3.5 *Let $\{X_t\}_{t>0}$ be a $S\alpha S$ H-ss process with an arbitrary representation (3.1). Then X is harmonizable if and only if*

$$f_{t_1 t_2}(s) f_1(s) = f_{t_1}(s) f_{t_2}(s) \text{ for } (Leb \otimes Leb \otimes \mu) \text{ a.a. } (t_1, t_2, s) \in \mathbf{R}_+ \times \mathbf{R}_+ \times S.$$

Proof. It is a direct consequence of Theorem 2.4 in Rosiński [36] and the Lamperti transformation. □

3.3.3 Evanescent processes

Definition 3.3.3 *A stochastic process whose minimal representation (3.4) contains a conservative flow without fixed points will be called* evanescent.

This class is not well understood at present. The next theorem is useful to verify whether or not a process is evanescent.

Theorem 3.3.6 *Let $\{X_t\}_{t>0}$ be a $S\alpha S$ H-ss process with an arbitrary representation (3.1). Then $\{X_t\}_{t>0}$ is evanescent if and only if*

$$\mu\{s \in S : \int_0^{\infty} t^{-\alpha H-1} |f_t(s)|^{\alpha}\, dt < \infty\} = 0$$

and

$$\mu\{s \in S : f_{t_1 t_2}(s) f_1(s) = f_{t_1}(s) f_{t_2}(s) \text{ for a.a. } t_1, t_2 > 0\} = 0$$

Proof. Theorems 3.3.3 and 3.3.5 combined with the results of Section 6 in Rosiński [34] yield the thesis. □

We will give two examples of evanescent processes.

Example 3.3.6 *Let*

$$\{X_t\}_{t>0} \stackrel{d}{=} \left\{ \int_0^1 t^H \cos \pi[\log t + s] M(ds) \right\}_{t>0},$$

where $[x]$ denotes the largest integer not exceeding x (see Figure 3.4). Then X does not have a corresponding harmonizable nor mixed moving average component, so provides an example of an evanescent component.

Example 3.3.7 *Let $\{X_t\}_{t>0}$ be the real part of a harmonizable process, i.e.*

$$\{X_t\}_{t>0} \stackrel{d}{=} \left\{ \int_{[0,2\pi)\times\mathbf{R}} t^H \cos(s + w\log t)\, Z(ds, dw) \right\}_{t>0},$$

where Z is a real-valued $S\alpha S$ random measure with control measure $Leb \otimes \nu$ and ν is a finite measure on $\mathbf{R}$ (see Rosiński [35], Example 4.9). Here $\phi_t(s,w) = (s +_{2\pi} w\log t, w)$, where " $+_{2\pi}$ " denotes addition modulo 2π.

Now we conclude this section with the following theorem.

Theorem 3.3.7 *Every $S\alpha S$ self-similar process $\{X_t\}_{t>0}$ admits a unique decomposition into three independent parts*

$$\{X_t\}_{t>0} \stackrel{d}{=} \{X_t^{(1)}\}_{t>0} + \{X_t^{(2)}\}_{t>0} + \{X_t^{(3)}\}_{t>0},$$

where the first process on the right-hand side is a MFM, the second is harmonizable, and the third one is an H-ss evanescent process.

Proof. Since the set D of Hopf decomposition and the set of fixed points for a flow are invariant, we obtain a decomposition of self-similar processes analogous to the decomposition of stationary processes (see Theorem 6.1 in Rosiński [34]). □

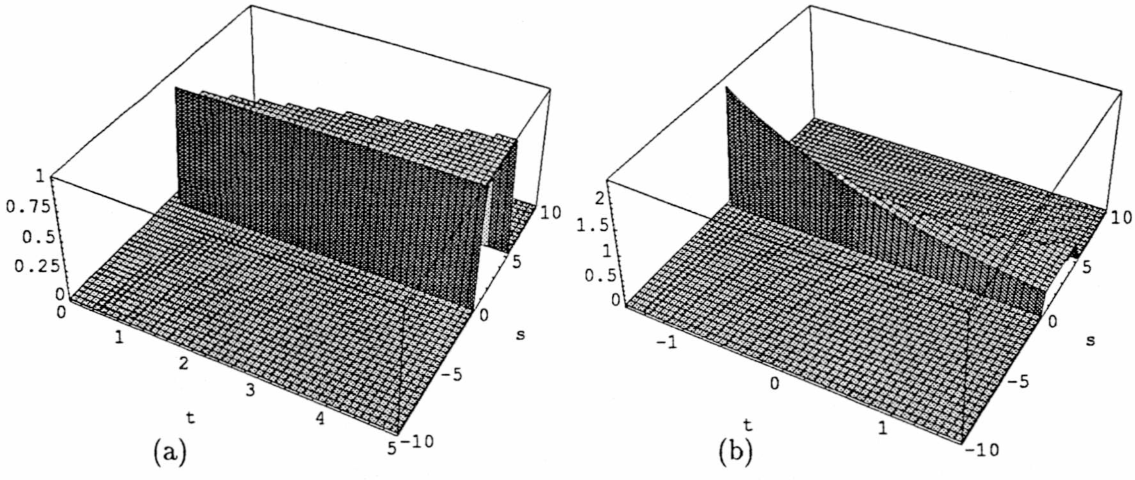

Figure 3.1: (a) The kernel of the spectral representation of Lévy motion, (b) the kernel of the corresponding stationary process through the Lamperti transformation for $H = 1/1.8$ (i.e. Ornstein–Uhlenbeck process).

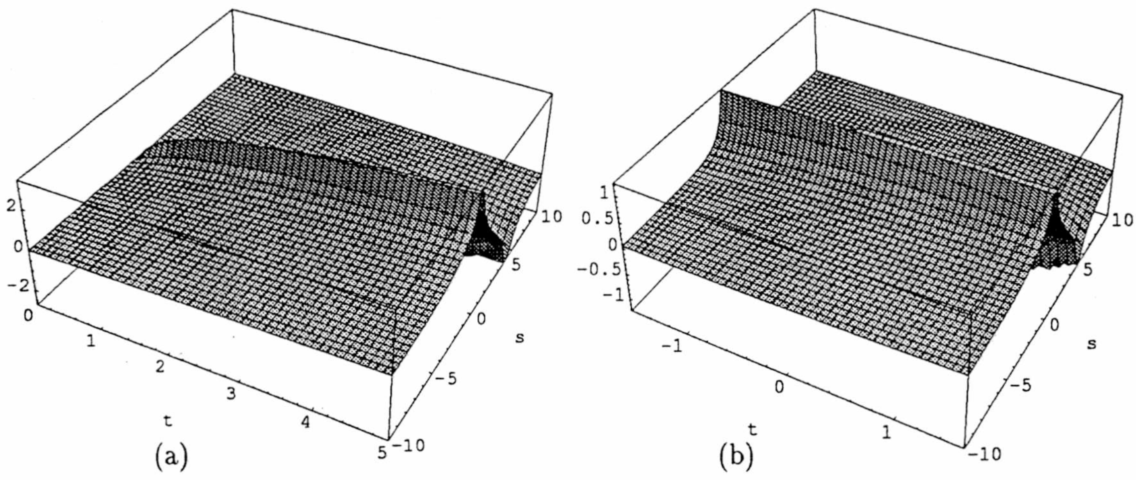

Figure 3.2: (a) The kernel of the spectral representation of log-fractional motion, (b) the kernel of the corresponding stationary process for $H = 1/1.8$.

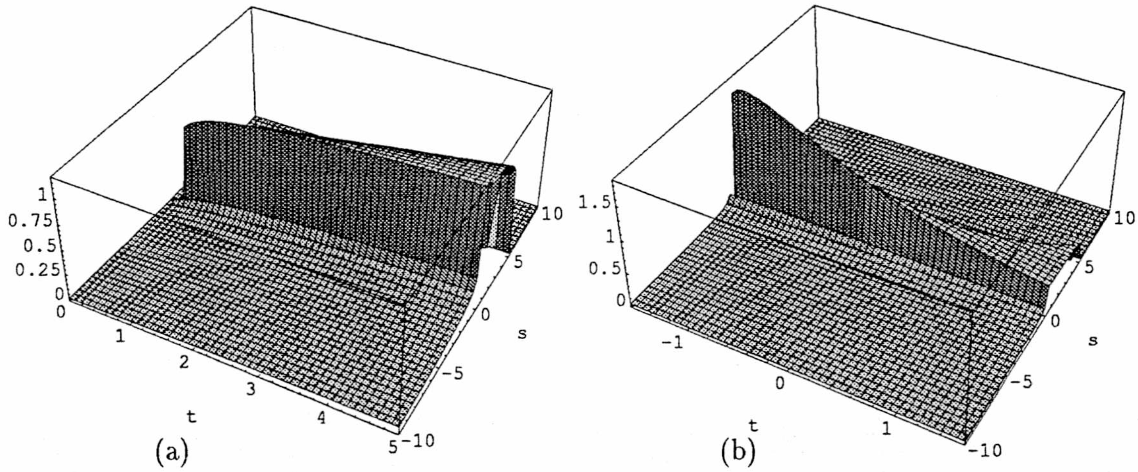

Figure 3.3: (a) The kernel of the spectral representation of linear fractional stable motion for $H - 1/\alpha = 0.1$, (b) the kernel of the corresponding stationary process for $H = 0.1 + 1/1.8$.

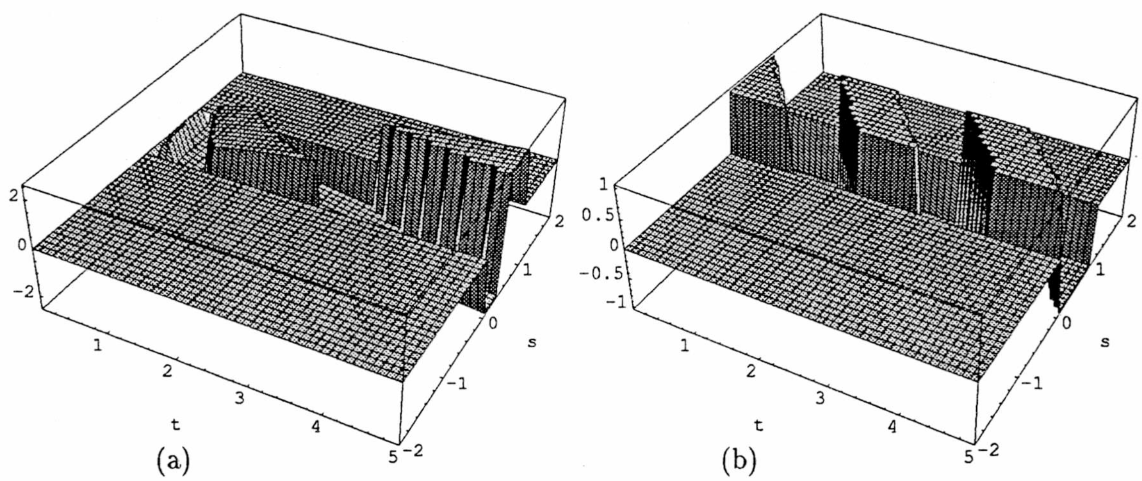

Figure 3.4: (a) The kernel of the spectral representation of the evanescent process, (b) the kernel of the corresponding stationary process for $H = 1/1.8$.

Chapter 4

Weak convergence of the risk process to H-ss si processes

The traditional approach in the *collective risk theory* is to consider a model of the risk business of an insurance company, and to study the probability of ruin, i.e. the probability that the risk business ever will be below some specific (negative) value. The classical risk process R is defined by

$$R(t) = u + ct - \sum_{k=1}^{N(t)} Y_k, \tag{4.1}$$

where $u \geq 0$ denotes the initial capital, c is a positive real constant, $N = (N(t))_{t\geq 0}$ is a point process independent of (Y_k) and $(Y_k)_{k=1}^{\infty}$ forms a stationary sequence of independent random variables, having the common distribution function F, with $F(0) = 0$, mean value μ, and variance σ^2. $N(t)$ is to be interpreted as the number of claims on the company during the interval $(0, t]$. At each point of N the company has to pay out a stochastic amount of money, and the company receives (deterministically) c units of money per unit time. The constant c is called the *premium income rate*.

However, in reality claims are mostly modelled by heavy-tailed distributions like e.g. Pareto. Moreover, the independence of Y_k's seems unrealistic since a correlation between claims is being observed. Therefore, in our approach we do not restrict ourselves to independent Y_k's with $EY_k^2 < \infty$. We merely assume that $\mu = E|Y_k| < \infty$.

Already in 1940 Hadwiger compared a discrete-time risk process with diffusion. This can be viewed, though theoretically not comparable with modern approach, as the first treatment of diffusion approximations in the risk theory. A more modern version, based on weak convergence, is due to Iglehart [18]. The idea is to let the number of claims grow in a unit time interval and to make the claim sizes smaller in such way that the risk process converges weakly to a diffusion. We shall consider weak approximations where the idea is to approximate the risk process with a self-similar si process with drift. While the classical theory of diffusion approximation requires either short-tailed or independent claims, these assumptions can be dropped in our approach.

4.1 Preliminaries and definitions

Let us specify in detail the assumptions in our model. We assume that the claims occur at jumps of a point process $(N(t))_{t\geq 0}$. While most work in the collective risk theory has assumed that

$N(t)$ is a Poisson process, this restrictive assumption plays no role in our analysis. The successive claims Y_k are supposed to form a stationary sequence, strongly dependent in general, with $EY_k = \mu > 0$. Furthermore, we assume that the initial risk reserve of the company is $u > 0$ and that the policyholders pay a gross risk premium of $c > 0$ per unit time. Thus the risk process is of the form (4.1).

One of the key problems of the collective risk theory concerns calculating the ruin probability, i.e. the probability that the risk process becomes negative.

Definition 4.1.1 *The ruin probability $\Phi(u,T)$ in finite time (or within finite horizon) of a company facing the risk process (4.1) is given by*

$$\Phi(u,T) = P(R(t) < 0 \text{ for some } t \leq T), \quad 0 < T < \infty, \quad u \geq 0.$$

Consequently, the ruin probability $\Phi(t)$ in infinite time can be defined as

$$\Phi(u) = \Phi(u,\infty).$$

We also assume that the net profit condition

$$\lim_{t\to\infty} \frac{ER(t)}{t} > 0$$

holds.

4.1.1 Weak convergence of stochastic processes

Let $D = D[0,\infty)$ be the space of *cadlag* functions, i.e. all real-valued functions that are right-continuous and have left-hand limits, on $[0,\infty)$. Endowed with the Skorokhod J_1 topology, D is a Polish space, i.e. separable and metrizable with a complete metric. A stochastic process $X = (X(t))_{t\geq 0}$ is said to be in D if all its realizations are in D.

Definition 4.1.2 *A sequence $(X^{(n)})_{n\in\mathbb{N}}$ of stochastic processes is said to converge weakly in the Skorokhod topology to a stochastic process X if for every bounded continuous functional f on D it follows that*

$$\lim_{n\to\infty} Ef(X^{(n)}) = Ef(X).$$

In this case one writes $X^{(n)} \Rightarrow X$.

Weak convergence implies, for example, convergence of the finite-dimensional distributions provided that the limit process X is continuous in probability.

Hereafter through this chapter we shall only consider processes in D and continuous in probability.

4.2 General results

The main aim of this section is to show the following.

Statement 4.2.1 *The only processes that emerge in a "natural way" as weak approximations of the risk reserve process are H-self-similar processes with stationary increments with $0 < H \leq 1$. Conversely, every H-self-similar si process X with $EX(t) = 0$, in D and $\frac{1}{2} < H \leq 1$ can serve as the weak approximation of some risk process.*

In order to justify this statement first we need Proposition 1.1.6. Combining that with Proposition 1.1.1 and the fact that weak convergence in the Skorokhod topology implies convergence with respect to finite-dimensional distributions we may assert the following.

Corollary 4.2.1 *Let $(Y_k)_{k\in N}$ be stationary sequence with common distribution function F and mean 0 such that*

$$\frac{1}{\phi(n)}\sum_{k=1}^{[nt]} Y_k \Rightarrow X(t) \text{ as } n \to \infty$$

for some reals $(\phi(n))_{n\geq 0}$, $\phi(n) > 0$, $lim_{n\to\infty}\phi(n) = \infty$ and X is a non-degenerate stochastic process, then for some $0 < H \leq 1$ X is H-ss, si and ϕ is of the form $\phi(n) = n^H L(n)$ for L being a slowly varying function. Conversely, every H-ss si process X in D, of the mean $EX(t) = 0$ can be obtained this way.

Proof. The last part of the thesis follows from the fact that the convergence in the converse part of Proposition 1.1.6 is in fact weak provided that X is in D. □

Now we can state the theorem that yields our statement.

Theorem 4.2.1 *Let $(Y_k)_{k\in N}$ be a stationary sequence with common distribution function F and mean $\mu > 0$ and let $(N^{(n)})_{n\in N}$ be a sequence of point processes such that*

$$\frac{N^{(n)}(t) - \lambda nt}{\phi(n)} \to 0 \tag{4.2}$$

in probability in the Skorokhod topology for some positive constant λ. Assume also that

$$\lim_{n\to\infty}\left(c^{(n)} - \lambda n\frac{\mu}{\phi(n)}\right) = c \tag{4.3}$$

and

$$\lim_{n\to\infty} u^{(n)} = u.$$

If

$$\frac{1}{\phi(n)}\sum_{k=1}^{[nt]}(Y_k - \mu) \Rightarrow X(t) \quad \text{as } n \to \infty \tag{4.4}$$

for some non-degenerate process X, then

(i) there exists an $0 < H \leq 1$, that X is H-ss, si, ϕ is of the form $\phi(n) = n^H L(n)$ for L being a slowly varying function, and

(ii)

$$Q^{(n)}(t) = u^{(n)} + c^{(n)}t - \frac{1}{\phi(n)}\sum_{k=1}^{N^{(n)}(t)} Y_k \Rightarrow Q(t) = u + ct - \lambda^H X(t) \tag{4.5}$$

in the Skorokhod topology as $n \to \infty$.

Conversely, every H-ss si process X in D, with $EX(t) = 0$ and $\frac{1}{2} < H \leq 1$ can be obtained via (4.5).

Proof. The (i) part of the thesis is obvious by Corollary 4.2.1. In order to prove the (ii) part let us recall the following Whitt theorem on random time change. Let $(Z_n)_{n\in\mathbb{N}}$, Z be processes in $D[0,\infty)$ and suppose that $Z_n \Rightarrow Z$. Let $(N_n)_{n\in\mathbb{N}}$ be a sequence of processes with nondecreasing sample paths starting from 0 such that $N_n \Rightarrow \lambda I$, $\lambda > 0$. For each $n \in \mathbb{N}$, Z_n and N_n are assumed to be defined on the same probability space. Then

$$Z_n(N_n) \Rightarrow Z(\lambda I). \tag{4.6}$$

Now let us rewrite the process $Q^{(n)}(t)$ in the following form

$$Q^{(n)}(t) = u^{(n)} + c^{(n)}(t) - \frac{1}{\phi(n)} \sum_{k=1}^{N^{(n)}(t)} Y_k = \tag{4.7}$$

$$u^{(n)} + t\left(c^{(n)} - \lambda n \frac{\mu}{\phi(n)}\right) - \mu\left(\frac{N^{(n)}(t) - \lambda n t}{\phi(n)}\right) - \frac{1}{\phi(n)} \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu). \tag{4.8}$$

From assumptions (4.2), (4.4) and the Whitt theorem (4.6) we obtain that

$$\frac{1}{\phi(n)} \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu) \Rightarrow \lambda^H X(t)$$

as $n \to \infty$. Since

$$u^{(n)} + t\left(c^{(n)} - \lambda n \frac{\mu}{\phi(n)}\right) - \mu\left(\frac{N^{(n)}(t) - \lambda n t}{\phi(n)}\right)$$

converges to $u + ct$ in probability in the Skorokhod topology, the proof is of the (ii) part is complete.

By Corollary 4.2.1 setting $Y_k = X(k) - X(k-1) + \mu$, $k = 1, 2, \ldots$, where $\mu > 0$ in order to conclude the converse part we merely have to construct a sequence $(N^{(n)})_{n\in\mathbb{N}}$ that fulfills the condition (4.2). To this end we consider the case where the occurrence of the claims is described by a renewal process N:

$$N(t) = \max\{n : \sum_{k=1}^{n} T_k \le t\}.$$

The inter-occurrence times $(T_k)_{k\in\mathbb{N}}$ are assumed to be independent, positive random variables with mean $\frac{1}{\lambda}$ and variance σ^2. We define

$$N^{(n)}(t) = N(nt).$$

Then for $\frac{1}{2} < H \le 1$ and $\phi(n) = n^H$ the condition (4.2) is fulfilled (see Furrer et al. [15]). This completes the proof. □

Remarks.

1. We could omit the point (4.8) and use just the previous relation (4.7) with slightly modified assumptions to state a more general result on the weak convergence to a self-similar si process. That is, it is enough to assume that instead of (4.3) we have $\lim_{n\to\infty} c^{(n)} = c$ and change (4.2) to the condition $\frac{N^{(n)}(t)}{n} \Rightarrow \lambda t$. Then, if we do not restrict Y_k's to variables with the finite mean, the resulting H-self similar si process X may be quite general with infinite mean. Nevertheless, this as a consequence would lead us to an artificial collective risk model interpretation of the final process Q (cf. Section 4.3). Thus we do not intend to generalize this theorem.

2. $H = 1$ corresponds to the case when X is trivial, see Proposition 1.1.2.

4.3 Approximation of ruin probability

Collective risk theory has paid considerable attention to the ruin functional in infinite and finite time. The weak convergence of $Q^{(n)}$ to Q implies, for example

$$\inf_{0\le t\le t_0} Q^{(n)}(t) \xrightarrow{d} \inf_{0\le t\le t_0} Q(t)$$

for any $t_0 < \infty$, and thus

$$\lim_{n\to\infty} P\left\{\inf_{0\le t\le t_0} Q^{(n)}(t) < 0\right\} = P\left\{\inf_{0\le t\le t_0} Q(t) < 0\right\}. \tag{4.9}$$

Therefore we may approximate the finite-time ruin probability of a risk process by the ruin probability in finite time of the corresponding weak approximation.

Theorem 4.3.1 *Consider a risk process $R(t) = u + ct - \sum_{k=1}^{N(t)} Y_k$. Denote the corresponding finite-time ruin probability by $\Psi(u,T)$. If the assumptions from Theorem 4.2.1 are satisfied for Y_k's, the sequence $N^{(n)}(t) = N(nt)$, $\phi(n) = n^H$ and $0 < H < 1$, and the relative safety loading $\theta = \frac{c}{\lambda\mu} - 1 > 0$, then*

$$\Psi(u,T) \sim_{n\to\infty} P\left\{\inf_{0\le s\le T} (u + \theta\lambda\mu s - \lambda^H X_H(s)) < 0\right\}.$$

Proof. For each $n \in \mathbf{N}$, we have

$$\begin{aligned}
\Psi(u,T) &= P\left\{\inf_{0\le s\le T}\left(u + cs - \sum_{k=1}^{N(s)} Y_k\right) < 0\right\} \\
&= P\left\{\inf_{0\le s\le T}\left(\frac{u}{\phi(n)} + \frac{cs}{\phi(n)} - \frac{1}{\phi(n)}\sum_{k=1}^{N(s)} Y_k\right) < 0\right\} \\
&= P\left\{\inf_{0\le s\le T/n}\left(\frac{u}{\phi(n)} + \frac{cns}{\phi(n)} - \frac{1}{\phi(n)}\sum_{k=1}^{N(ns)} Y_k\right) < 0\right\}
\end{aligned}$$

$$= P\left\{\inf_{0\le s\le T/n}\left(\frac{u}{\phi(n)} + s\left(\frac{cn}{\phi(n)} - \frac{\lambda\mu n}{\phi(n)}\right) - \mu\left(\frac{N(ns) - \lambda ns}{\phi(n)}\right) - \frac{1}{\phi(n)}\sum_{k=1}^{N(ns)}(Y_k - \mu)\right) < 0\right\}.$$

Now assume that $T_0 = T/n$, $\theta_0 = \frac{\theta\lambda\mu n}{\phi(n)}$ and $u_0 = u/\phi(n)$ are constants, i.e. we increase T and u with n, and decrease at the same time the safety loading θ with n (as $H < 1$). This means that a small safety loading is compensated by a large initial capital. Then we obtain

$$\Psi(u,T) = P\left\{\inf_{0\le s\le T_0}\left(u_0 + \theta_0 s - \mu\left(\frac{N(ns) - \lambda nt}{\phi(n)}\right) - \frac{1}{\phi(n)}\sum_{k=1}^{N(ns)}(Y_k - \mu)\right) < 0\right\}.$$

Applying Theorem 4.2.1 and (4.9) we obtain that

$$\Psi(u,T) \to P\left\{\inf_{0\le s\le T_0}(u_0 + \theta_0 s - \lambda^H X_H(s)) < 0\right\}.$$

By self-similarity this concludes the proof. □

4.4 Ruin probabilities for general self-similar processes

In the previous sections we showed that the process Q defined in (4.5) can be looked as an approximation of a risk process. Our aim in this section is to investigate the probability that the process Q reaches the level 0 before time t. In the Brownian case the probability can be calculated explicitly (see for instance Asmussen [2]):

$$P\left\{\sup_{0\le s\le t}(\lambda^{1/2}B(s)-cs)>u\right\}=\bar{\Phi}\left(\frac{u+ct}{\sqrt{\lambda t}}\right)+e^{-\frac{2uc}{\lambda}}\Phi\left(\frac{-u+ct}{\sqrt{\lambda t}}\right),$$

where Φ is the standard normal distribution and $\bar{\Phi}=1-\Phi$. The following theorems from Furrer et al. [15] and Michna [27] provide upper bounds where $X_H(t)$ is a standard symmetric α-stable Lévy motion Z_α or a standard fractional Brownian motion B_H, respectively.

Proposition 4.4.1 (Furrer et al. [15]) *Let Z_α be a standard $S\alpha S$ Lévy motion. For positive numbers u, c and λ we have*

$$P\left\{\sup_{0\le s\le t}(\lambda^{1/\alpha}Z_\alpha(s)-cs)>u\right\}\le 2\bar{G}\left(\frac{u}{(\lambda t)^{\frac{1}{\alpha}}}\right),$$

where $\bar{G}=1-G$ and G denotes the cumulative distribution function of a standard $S\alpha S$ variable.

Proposition 4.4.2 (Michna [27]) *Let B_H be a standard fractional Brownian motion with $\frac{1}{2}<H<1$. Then*

$$P\left\{\sup_{0\le s\le t}(\lambda^{H}B_H(s)-cs)>u\right\}\le\bar{\Phi}\left(\frac{u+ct}{(\lambda t)^H}\right)+\exp\left(\frac{-2uct}{(\lambda t)^{2H}}\right)\bar{\Phi}\left(\frac{u-ct}{(\lambda t)^H}\right).$$

Now let us state a theorem which yields a lower bound for the ruin probability of the process Q for an arbitrary self-similar process X_H with $H>0$.

Theorem 4.4.1 *Let $(X_H(t))_{t\ge 0}$ be an arbitrary self-similar process with the exponent $H>0$. If $0<H<1$ and t is sufficiently large, namely $\frac{uH}{ct(1-H)}<1$, then*

$$P\left\{\sup_{0\le s\le t}(\lambda^{H}X_H(s)-cs)>u\right\}\ge\bar{G}\left[\left(\frac{u}{1-H}\right)^{1-H}\left(\frac{c}{\lambda H}\right)^{H}\right], \tag{4.10}$$

otherwise

$$P\left\{\sup_{0\le s\le t}(\lambda^{H}X_H(s)-cs)>u\right\}\ge\bar{G}\left(\frac{u+ct}{(\lambda t)^H}\right), \tag{4.11}$$

where $\bar{G}=1-G$ and G denotes the distribution function of $X_H(1)$.

Proof. Since the process X_H is H-self-similar we have

$$P\left\{\sup_{0\le s\le t}(\lambda^{H}X_H(s)-cs)>u\right\}=P\left\{\sup_{0\le s\le 1}(\lambda^{H}t^{H}X_H(s)-cts)>u\right\}.$$

Furthermore, it is obvious that

$$P\left\{\sup_{0\le s\le 1}(\lambda^{H}t^{H}X_H(s)-cts)>u\right\}\ge P\left\{\lambda^{H}t^{H}X_H(\tau)-ct\tau>u\right\},$$

for all $\tau \in (0,1]$.

Eventually, applying one more time the definition of self-similarity we obtain

$$\begin{aligned} P\left\{\sup_{0\le s\le t}(\lambda^H X_H(s) - cs) > u\right\} &\ge P\left\{X_H(1) > \frac{u+ct\tau}{\lambda^H t^H \tau^H}\right\} \\ &= 1 - G\left(\frac{u+ct\tau}{(\lambda t\tau)^H}\right), \end{aligned} \tag{4.12}$$

for all $\tau \in (0,1]$, where G stands for the distribution function of $X_H(1)$.

In order to find the best possible lower bound for the ruin probability in finite time we are to find minimum of the function $f(\tau) = \frac{u+ct\tau}{(\lambda t\tau)^H}$ on the interval $(0,1]$. To this end we calculate the derivative of the expression: $(u\tau^{-H} + ct\tau^{1-H})$ and find out that it is equal to 0 for

$$\tau = \tau_0 = \begin{cases} \frac{uH}{ct(1-H)} & \text{if } H < 1, \\ \infty & \text{if } H \ge 1. \end{cases}$$

Hence, if $\tau_0 < 1$, then the minimum of the function f on $(0,1]$ is

$$f(\tau_0) = \left(\frac{u}{1-H}\right)^{1-H}\left(\frac{c}{\lambda H}\right)^H$$

otherwise

$$f(1) = \frac{u+ct}{(\lambda t)^H}.$$

This proves the theorem. □

Remark. The condition $0 < H < 1$ corresponds to the case when $X_H(t)$ is non-trivial, has stationary increments and finite first moment for each t.

Since the lower bound (4.10) does not depend explicitly on t, it can serve as well as a bound for the ruin probability for Q in infinite time. Furthermore, the bound defined in (4.11) tends to $\bar{G}(c/\lambda)$ when $H = 1$ and to $\bar{G}(0)$ when $H > 1$ as $t \to \infty$. Therefore, we may claim the following.

Corollary 4.4.1 *Let $(X_H(t))_{t\ge 0}$ be an arbitrary self-similar process with the exponent $H > 0$. Then we have*

$$P\left\{\sup_{s\ge 0}(\lambda^H X_H(s) - cs) > u\right\} \ge \begin{cases} \bar{G}\left[\left(\frac{u}{1-H}\right)^{1-H}\left(\frac{c}{\lambda H}\right)^H\right] & \text{if } H < 1, \\ \bar{G}\left(\frac{c}{\lambda}\right) & \text{if } H = 1, \\ \bar{G}(0) & \text{if } H > 1, \end{cases} \tag{4.13}$$

where $\bar{G} = 1 - G$ and G denotes the distribution function of $X_H(1)$.

Remarks.

1. The lower bound (4.13) was already obtained by Norros [28] for a special case when X is a FBM ($H < 1$), in the storage model setting.
2. Duffield and O'Connell [13] using the result from Norros [28] showed the bound is in fact accurate in the logarithmic sense (the case when X is a FBM).

Considering specific cases when X_H is a standard Gaussian or a standard $S\alpha S$ process, and letting the initial risk reserve u become large we obtain the following results.

Corollary 4.4.2 *If X_H is Gaussian with $X_H(1)$ being a standard normal variable and $0 < H < 1$, then*

$$P\left\{\sup_{s\geq 0}(\lambda^H X_H(s) - cs) > u\right\} \geq \exp\left\{-\frac{1}{2}\left(\frac{u}{1-H}\right)^{2-2H}\left(\frac{c}{\lambda H}\right)^{2H}\right\}.$$

Proof. Recall the elementary relation

$$1 - \Phi(x) \sim \frac{1}{x} f(x) \sim \exp(-x^2/2), \quad \textit{for } x \to \infty,$$

where ϕ, f stand for a distribution and density function of the standard normal distribution, respectively. □

Corollary 4.4.3 *If X_H is standard $S\alpha S$and $0 < H < 1$, then*

$$P\left\{\sup_{s\geq 0}(\lambda^H X_H(s) - cs) > u\right\} \geq \left(\frac{1-H}{u}\right)^{\alpha(1-H)}\left(\frac{\lambda H}{c}\right)^{\alpha H}.$$

Proof. This stems from the fact that the tail probabilities of a standard $S\alpha S$ distribution behave like $C_\alpha x^{-\alpha}$, where C_α is constant.

The construction of the lower bound (4.10) in the proof of Theorem 4.4.1 suggests that the bound should be quite a good estimate. For instance, the bound in Corollary 4.4.2 for $H = \frac{1}{2}$ gives in fact exact result for the Brownian motion. Michna [27] shows the lower bound (4.11) yields a good approximation of the ruin probability for the fractional Brownian motion, when u is large.

Chapter 5

H-ss processes in financial modelling

A "self-similar" structure is one that looks the same on a small or a large scale. For example, share prices of stock when plotted against time have very much the same shape on a yearly, monthly, weekly and even on a daily basis. Brownian motion ($\frac{1}{2}$-ss process) as a limit process is an unavoidable tool in finance. In his famous paper, Bachelier [3] proposed Brownian motion as an appropriate model for pricing. More recently, in the traditional approach to contingent pricing, in the Black–Scholes model, the log-Brownian model for the movement of share prices was used. However it has been empirically demonstrated to be incorrect in a number of ways. Certain attempts have been made to replace Brownian motion by another self-similar process – α-stable Lévy motion; see Rachev and Samorodnitsky [32] and Janicki and Weron [20]. It is believed that, to some extent, such model would explain the large jumps which evidently occur in prices and which are caused by dramatic political or economic events (see Embrechts et al. [14]). Moreover, various alternatives have been suggested to account for empirically observed defiances, among them the fractional Brownian motion which displays dependence between returns on different days, in stark contrast to Brownian motion (cf. Peters [30] and Bouchaud and Sornette [5]). However, FBM is not a semimartingale (except in the Brownian case), and therefore there can be no equivalent martingale measure. Hence, by general results (cf. Rogers [33]) this leads to a conclusion that there must be arbitrage. This practically disqualifies the FBM model. Nonetheless, FBM has recently attracted some attention in mathematical finance (see, e.g. Cutland et al. [10] and Dai and Heyde [11]).

In Section 5.1 we present a test on a DJIA index data which justifies using self-similar models as asset price processes. In Section 5.2 a modification of the Black–Scholes model is presented. The idea is to change, in the stochastic differential equation describing discounted stock prices process Z_t with respect to the reference measure Q, the differential $d\tilde{B}_t$ to dM_t, where M_t is a martingale generating the same filtration as B_t^H and is well defined for $\frac{1}{2} < H < 1$. As a result of the investigation we obtain an option pricing formula which appears to be distinct from the Black–Scholes one. The differences are illustrated graphically.

5.1 Variance–time plots

We are going to apply a method from Willinger et al. [42], which was called *variance–time plots*, for the DJIA index process. The method can be summarized as follows. Let $(X_t)_{t\geq 0}$ be an H-self-similar process with stationary increments. It is well known that if $EX_t^2 < \infty$ and $H \in \left(\frac{1}{2}, 1\right)$ then the increment process $(Y_k = (X_{k+1} - X_k) : k = 0, 1, \ldots)$ exhibits long-range dependence. This means the time series Y_k has the autocovariance function of the form

$$r(k) = Cov(Y_0, Y_k) \sim_{k\to\infty} L_1(k)k^{2H-2}, \quad H \in \left(\frac{1}{2}, 1\right), \tag{5.1}$$

where $L_1(k)$ is a slowly varying function as $k \to \infty$. Property (5.1) implies that the correlations are not summable and the spectral density has a pole at zero. More specifically, under suitable conditions on $L_1(\cdot)$, the spectral density has the property

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r(k) \exp(-ikx) \sim_{|x|\to 0} L_2(x)|x|^{1-2H} \tag{5.2}$$

for some $L_2(\cdot)$ that is slowly varying at the origin. The best known models with (5.1) and (5.2) are the fractional Gaussian noise model and the fractional autoregressive integrated moving-average model (FARIMA). The parameter H describes the long-memory behaviour of the series. Now, for each $m = 1, 2, \ldots$, let $\left(Y^{(m)} = \left(Y_k^{(m)}\right) : k = 1, 2, \ldots\right)$ denote a new time series obtained by averaging the original series Y over nonoverlapping blocks of size m; that is, for each $m = 1, 2, \ldots$, $Y^{(m)}$ is given by

$$Y_k^{(m)} = 1/m(Y_{km-m+1} + \cdots + Y_{km}), \quad k = 1, 2, \ldots$$

From a statistical point of view, the most salient feature of the process Y_k is that the variance of the arithmetic mean decreases more slowly than the reciprocal of the sample size; that is it behaves like n^{2H-2} for some $H \in \left(\frac{1}{2}, 1\right)$ instead of like n^{-1} for the processes whose aggregated series converge to a second-order pure noise. Cox [9] showed that a specification of the autocovariance function satisfying (5.1) (or equivalently of the spectral density function satisfying (5.2)) is the same as a specification of the sequence $(Var(Y^{(m)} : m \geq 1)$ with the property

$$Var(Y^{(m)}) \sim_{m\to\infty} am^{2H-2},$$

where a is a finite positive constant independent of m, and $H \in \left(\frac{1}{2}, 1\right)$. On the other hand, for covariance stationary processes whose aggregate series $Y^{(m)}$ tend to second-order pure noise it is easy to see that the sequence $(Var(Y^{(m)} : m \geq 1)$ satisfies

$$Var(Y^{(m)}) \sim_{m\to\infty} bm^{-1},$$

where b is a finite positive constant independent of m. Thus, for self-similar processes with stationary increments the variances of the aggregated processes $Y^{(m)}, m = 1, 2, \ldots$, decrease lineary (for large m) in log–log plots against m with slopes arbitrary flatter than -1. The so-called *variance-time* plots are obtained by plotting $\log(Var(Y^{(m)})$ against $\log(m)$ ("time") and by fitting a line through the resulting points in the plane, ignoring the small values for m. Values of the estimate $\hat{H}$ of the asymptotic slope between -1 and 0 suggest self-similarity.

Example 5.1.1 *Let us consider the DJIA index analysed from January 2, 1901 to May 17, 1996. We define Y_k's as log-returns of the index. We normalize the data in order to set the variance of the process Y_k to 10. Figure 5.1 shows an asymptotic slope that is clearly different from -1 and is estimated to be about -0.92, resulting in an estimate of the parameter H of about* 0.54.

5.2 Alternative approach to contingent pricing

First let us recall assumptions in the Black–Scholes model.

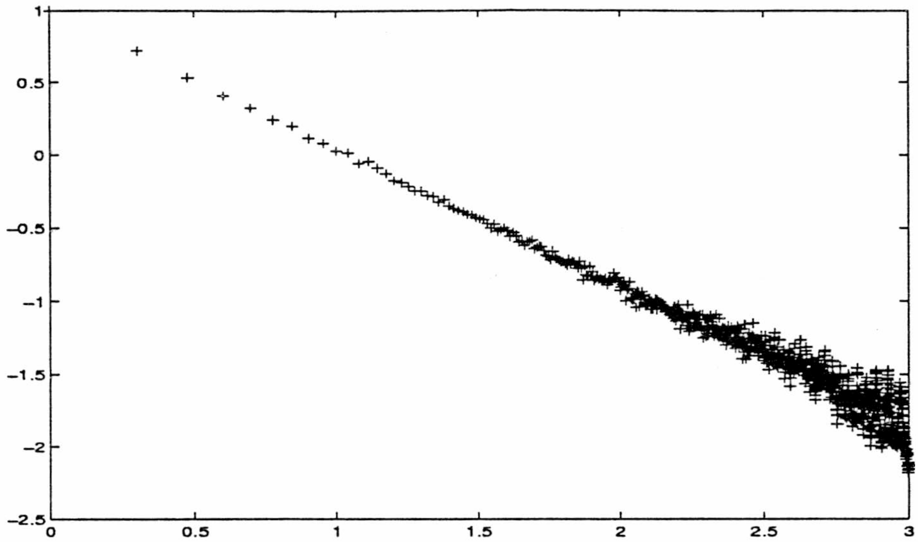

Figure 5.1: Variance–time plot of the sequence of log-returns of DJIA index from January 2, 1901 to May 17, 1996.

5.2.1 Black–Scholes model for arbitrary contingent pricing

Let B_t be, as usually, a standard (zero drift and unit variance) Brownian motion on some probability space (Ω, F, P). Let r, μ and σ be real constants with $\sigma > 0$. A market in the classical Black–Scholes model is defined as a pair $(\Lambda_t,\ S_t)$, where

$$\Lambda_t = \exp(rt),$$
$$S_t = S_0 \exp(\sigma B_t + \mu t). \tag{5.3}$$

Interpret Λ_t as the price at time t of a riskless bond and S_t as the price, in dollars per share, of a stock which pays no dividents. Furthermore r is called a fixed (riskless) interest rate, σ the volatility of the stock price process S_t and μ is his drift. Moreover, in the model, we assume a frictionless market with continuous trading, namely we demand that the two fundamental securities are traded continuously with no transaction costs with publically announced prices. Now we consider a ticket which entitles its bearer to buy one share of stock at the terminal date T, if he wishes, for a specified price of K dollars. This is a *European call option* on the stock, with *exercise price* K and *expiration date* T. It is easy to see that the call option is equivalent to a ticket which entitles a bearer to a payment of $X = (S_T - K)^+$ dollars at time T. Black and Scholes [4] asserted that there exist a unique rational value V for the option, namely

$$V = S_0 \Phi\left(\frac{\log\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

Originally Black and Scholes obtained the valuation formula by solving a differential equation. Our approach to option pricing is based on a martingale method presented by Harrison and Pliska [17], which generalizes the ideas to arbitrary contingent pricing. The Black–Scholes formula in this approach is proved by considering so called *completeness* of the market, finding the *reference measure* Q via Girsanov theorem, asserting the measure is unique, due to the representation theorem for martingales, and computing

$$V = \exp(-rT)E^Q(X),$$

where $X = (S_T - K)^+$.
The latter valuation formula is true for an arbitrary contingent claim X.

Now we are going to modify the assumptions of this model.

5.2.2 Martingale approach through the fractional Brownian motion

A modification of the Black–Scholes model which is going to be consider throughout this section is defined as follows. In (5.3) model stock price fluctuations are modelled by a stochastic differential equation with respect to the white noise dB_t, i.e.

$$dS_t = S_t \left(\sigma dB_t + \left(\mu + \frac{1}{2}\sigma^2 \right) dt \right).$$

If we introduce discounted stock price process

$$Z_t = \Lambda_t^{-1} S_t,$$

then applying Itô formula and Girsanov theorem for the Brownian motion we may claim the existance of a measure Q, such that

$$dZ_t = \sigma Z_t d\tilde{B}_t \quad \text{under } Q, \tag{5.4}$$

where $\tilde{B}_t$ is a standard BM with respect to that measure.

Now the idea is to substitute the process $\tilde{B}_t$ in the stochastic differential equation (5.4) for a process M_t, namely

$$M_t = \int_0^t c_1 s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} dB_s^H,$$

$$c_1 = \left[H(2H-1)\mathcal{B}\left(\frac{3}{2} - H, H - \frac{1}{2} \right) \right]^{-1},$$

where $\mathcal{B}$ stands for the Beta function and $\frac{1}{2} < H < 1$.

First let us take a closer look at the properties of such defined process M_t. It turns out (see Norros et al. [29]) that M_t is

- a martingale which generates the same filtration as B_t^H,
- Gaussian with $EM_t = 0$ and the second moment $EM_t^2 = c_2^2 t^{2-2H}$, where

$$c_2 = \left(H(2H-1)(2-2H)\mathcal{B}(H - \frac{1}{2}, 2-2H) \right)^{-1},$$

- $(1-H)$-ss,
- a process of independent but not stationary increments,
- a process of continuous paths.

Thus we obtain

$$dZ_t = \sigma Z_t dM_t.$$

Since M_t is a martingale, the equation has a unique solution given by a stochastic exponential (cf. Protter [31])

$$Z_t = Z_0 \exp\left\{\sigma M_t - \frac{1}{2}c_2^2\sigma^2 t^{2-2H}\right\},$$

which is a martingale with continuous paths.

Now let us define

$$E_t = E(\Lambda_T^{-1}X|F_t).$$

It is a martingale with respect to F_t (the filtration generated by M_t, so B_t^H). Nonetheless, since the martingale Z_t does not satisfy the representation theorem for martingales we can not guarantee the existence of a unique predictable process H_t such that

$$E_t = E_0 + \int_0^t H_s dZ_s$$

for an arbitrary contingent claim X (the model is not complete). Thereby we can not construct an appropriate self-financing strategy. Nevertheless we may compute a non-arbitrage price $E(\Lambda_T^{-1}X)$ for a specific contingent claim X. Let us take as an example $X = (S_T - K)^+$.

Example 5.2.1 (M price) *An European call option value in the model, driven by the martingale of FBM M_t, is given by*

$$V^M = S_0\Phi\left(\frac{\log\frac{S_0}{K} + rT + \frac{1}{2}c_2^2\sigma^2T^{2-2H}}{c_2\sigma T^{1-H}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + rT - \frac{1}{2}c_2^2\sigma^2T^{2-2H}}{c_2\sigma T^{1-H}}\right).$$

Proof. We are to compute

$$V^M = e^{-rT}E(S_T - K)^+.$$

Since we have

$$Z_t = e^{-rt}S_t,$$

the process S_t can be expressed as

$$S_t = S_0 \exp\left\{\sigma M_t + rt - \frac{1}{2}c_2^2\sigma^2 t^{2-2H}\right\}.$$

Hence, it is enough to calculate

$$V^M = e^{-rT}E(S_0\exp(Z + rT) - K)^+,$$

where $Z \sim N(-\frac{1}{2}c_2^2\sigma^2T^{2-2H}, c_2^2\sigma^2T^{2-2H})$.

This concludes the proof. □

The formula we obtained is different from the Black–Scholes one. It is not surprising as we are aware that the model we use in modelling stock prices has changed, i.e we incorporated an additional parameter H – index of self-similarity.

We will compare the two formulas using the data from Example 5.1.1 in order to compute DJIA index options. We take into consideration values of the DJIA index from February 2, 1901 to May 17, 1996. Analysing the data we obtain that the estimated standard deviation $\hat{\sigma} = 0.010644$.

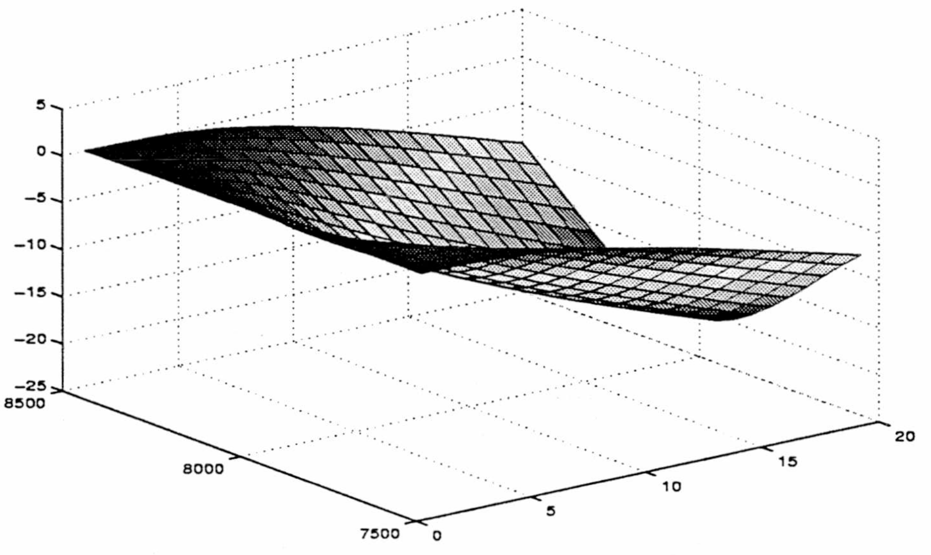

Figure 5.2: M minus BS price for the DJIA index option.

We assume that $S_0 = 8000$ and the striking price $K = 7500 \ldots 8500$. We consider index options on the interval [0, 20 days] and set the interest rate r to 0.05/365. Furthermore, Example 5.1.1 justifies the parameter H equal to 0.54. Figure 5.2 depicts the difference between M (obtained by the martingale M_t) and BS price.

To summarize, we have just presented a martingale model based on a fractional Brownian motion, the model which stems from the classical Black–Scholes one. We may claim that despite of its disadvantages (further we can add an inevitable Gaussianity of the model) it possesses interesting features and the new parameter H provides additional information allowing to improve adjustment to the real-world phenomena.

Chapter 6

Conclusions

Self-similar processes form exactly the class of possible "asymptotes" which can be obtained by taking some fixed random process and expanding indefinitely the units in which space and time are measured. This is the basic reason why such processes play a very large role in many aspects of probability theory and its applications. The limit theorems (see Propositions 1.1.5 and 1.1.6) generalize the role played by stable laws and processes within the limiting theory of sums of independent random variables. Due to that fact self-similar processes were originally called "semi-stable" (see Lamperti [23]).

Self-similar processes are closely related to stationary processes through the Lamperti transformation. In Chapter 2 we describe the classes of transformations leading from self-similar to stationary processes, and conversely. The relationship is used in Chapter 3 to characterize stable symmetric self-similar processes via their minimal integral representation. This leads to a unique decomposition of a symmetric stable self-similar process into three independent parts which are then characterized. The class of such processes appears to be quite broad and can stand as a basis of different risk models.

In Chapters 4 and 5 we give examples of applications of self-similar processes in risk modelling. We come to the conclusion that they appear naturally as weak limits of risk reserve processes in insurance (Chapter 4). There is much to be done in investigating different self-similar models as the weak approximations. We already know that quite a vast class of self-similar processes can serve as the examples. Applying the approximations we can try to cope with the most important functionals of the risk process, namely the ruin probability in finite and infinite time. In Chapter 5 we intend to justify that self-similar processes can be used as risk models in finance. We demonstrate this on a DJIA index data. As the classical Black–Scholes model is the log-Brownian model we try to look for an alternative model incorporating some of the features of the fractional Brownian motion. This as a result provides an option pricing formula distinct from the Black–Scholes one.

Summarizing, self-similar models, as we gain inside the structure of self-similar processes, can be widely used in the risk theory.

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