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Applications of Mathematics and Statistics in Economics

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NONELEMENTARY ANTIDERIVATIVES

1. The basic notions of the differential algebra

First some basic algebraic notions should be remembered. They can be found in any textbook of algebra, for instance [1]. A *ring* is an (additive) *Abelian group* $(R,+)$ together with a second binary operation $*$ (multiplication, the symbol is usually omitted) such that for all elements $a,b,c\in R$ $a(b+c)=ab+ac$, $(a+b)c=ac+bc$ (the distributive laws). One can assume moreover the multiplication is associative or commutative or there exists a unity 1 (identity of the ring) such that $a1=1a=a$ for any $a\in R$.

An associative and commutative ring with identity having no proper divisors of zero, that is, where the product of nonzero elements cannot be zero, is called the *integral domain*. The set Z of all integers is the typical integral domain.

A *field* is an integral domain of which every nonzero element is invertible. Over a field, we can perform addition, subtraction, multiplication and division. The set of nonzero elements of a field is an Abelian group (the multiplicative group of the field).

Example 1.1. The number domains of all rational (Q), real (R) and complex (C) numbers are typical fields.

Convention 1.2. In the following text we will denote the fields R and C by the common symbol T.

Example 1.3. The field Q is the smallest field over the integral domain Z (so called quotient field). More generally: if F is a field, then $F[x]$ denotes the integral domain of all polynomials over F, i.e. the functions $p(x)=c_0+c_1x+\ldots+c_nx^n$, where n is nonnegative integer and the elements $c_0, \dots, c_n$ (the coefficients of the polynomial) are from F and $F(x)$ is the field of all rational functions over F, i.e. the quotient field over $F[x]$.

The *characteristic of the ring* R is the smallest positive integer n such that $na = 0$ for all $a \in R$; here na stands for n summands $a + a + \dots + a$. If no such n exists, we say the characteristic is zero. If R is a field, a general element a can be replaced by the unity: $n1 = 1 + 1 + \dots + 1 = 0$ for some n or is nonzero for all n respectively.

Convention 1.4. Further we will assume only rings (integral domains, fields) of the characteristic 0.

A *subring (subfield) of a ring (field)* S is a subset R of S $(R \subset S)$ which is closed under the field operation + and * of R and which, with these operations, forms itself a ring (field).

If R is a subring (subfield) of a ring (field) S then S is a *ring (field) extension of the ring (field)* R. Suppose a field extension $F \subset E$ and $\alpha \in E$ is the root of a polynomial in $F[x]$. We say α is the *algebraic element over F*. Among the polynomials in $F[x]$ having α as a root there is one (unique) monic of the smallest degree (it is irreducible), so called the *minimal polynomial of α over F*. If α is not algebraic it is said to be a *transcendental over F*. If every element of E is algebraic over F, then E is an *algebraic extension of F*. A field is *algebraically closed* if and only if it has no proper algebraic extension. An *algebraic closure of a field F* is an algebraic extension of F that is algebraically closed.

Example 1.5. The fields Q, R, C are of characteristic 0. R is not algebraic extension of Q, for instance e or π are transcendental over Q. On the other hand C is an algebraic extension of R. The imaginary unit i is a root of the polynomial $p(x) = 1 + x^2$ and this is its minimal polynomial over R. The fundamental theorem of algebra states that the algebraic closure of the field R is the field C. The algebraic closure of the field Q is the field of algebraic numbers.

A ring R is said to be a *differential ring* if there is defined a mapping $\delta : R \to R$, which satisfies:

$$\text{(i) } \delta(r+s) = \delta(r) + \delta(s),$$
$$\text{(ii) } \delta(rs) = \delta(r)s + r\delta(s)$$

for all $r, s \in R$. The mapping δ is called the *derivation on R* so we adopt the usual notation from mathematical analysis, $\delta(r) = r'$. Any solution of the equation $x' = s$ is called a *primitive* of s. The *differential integral domain (field)* is thus an integral domain (field), which is equipped by a derivation. Using (ii) and the induction one can derive that

$$(r^n)' = nr^{n-1}r'$$

for any positive integer n.

Example 1.6. The mapping $o: R \to R: r \mapsto o(r) = 0$ is a derivation, so called *trivial derivation*. This is the only derivation on a finite field.

Suppose a ring R and its corresponding ring $R[x]$ of all polynomials over R. We define the mapping $\partial_x : R[x] \to R[x]: \sum_{j=0}^{n} r_j x^j \mapsto \sum_{j=0}^{n} j r_j x^{j-1}$. It is obvious we have obtained a derivation on $R[x]$. ∂_x is the "usual" derivation (in the sense of mathematical analysis) on the integral domain $T[x]$ (see Convention 1.2).

Suppose $R \subset S$ is a ring extension and both R and S are differential rings with derivations δ_R and δ_S, respectively. If the restriction $\delta_S \mid R = \delta_R$ then R is said to be a *differential subring of the ring S* or S is a *differential ring extension of the ring R* and δ_S is *the extension of the derivation* δ_R. Of course "ring" can be replaced by "field" ("integral domain") when R and S have that structure.

Theorem 1.7. When $F \subset E$ is an extension of fields and $\delta_F : F \to F$ is a derivation then the following statements hold:

(i) δ_F extends to a derivation $\delta_E : E \to E$.
(ii) When $F \subset E$ is an algebraic field extension then the derivation extension of (i) is unique.

Example 1.8. The set $R_C = \{z \in R : z' = 0\}$ is a differential subring (subfield) of a ring (field) R. R_C contains, for instance, all elements of the form $n1$, where $n \in Z$ if R has a unity. R_C is called the *ring (field) of constants of R*. Suppose an open non-empty set $G \subset T$. The function h is said to be a *meromorphic function on G* when it is analytic on G (it can be expanded into a power series in a neighbourhood of any point of G) up to a set $P(f)$ with no accumulation points in G where it has poles. It means for any point $x_0 \in P(f)$ the function $f(x) = (x - x_0)^n h(x)$ is analytic for a suitable positive integer n. The set of all meromorphic functions $M(G)$ becomes a field assuming usual addition and multiplication of functions. $R = T(x) \subset S = M(G)$ is a field extension. The structure of the differential fields is given as follows: δ_R as the derivation of the rational functions is the extension of the derivation on $T[x]$, i.e. the derivation of polynomials (see Example 1.6) by the rule $\delta_R\left(\frac{r}{s}\right)=\frac{\partial_x(r)s-r\partial_x(s)}{s^2}$, and δ_S with respect to the same formula, where ∂_x is the derivation of analytic function (it is again the derivation as in Example 1.6 but we sum up to infinity). This extension is not algebraic. For instance the function exp is transcendental over $T(x)$.

2. No new constants extensions of differential fields

The following text is inspired by [2] and [3]. If $R\subset S$ is a differential ring extension then it obviously holds $R_C\subset S_C$ (see Example 1.8). If $R_C=S_C$ we say this extension is a *no new constant differential extension of rings*.

Example 2.1. Suppose the differential field extension $R=T(x)\subset S=M(G)$ (see Example 1.8). If the set G is connected this is a no new constant extension. In the other case the sequentially constant functions (the functions constant on each of the components of G) are new constants.

Proposition 2.2. Suppose $F\subset E$ is a no new constant differential extension of fields and $h\in E\setminus F$ satisfies
(i) If $h'\in F$ then h is transcendental over F.
(ii) If $h'/h\in F$ then h is algebraic over F if and only if $h^n\in F$ for some positive integer n.

Proof: (i) If h is algebraic over F let $p(x)=c_0+c_1x+\ldots+c_{n-1}x^{n-1}+1x^n$, $c_j\in F$ be its minimal polynomial. Thus $p(h)=0$ and hence $q(h)=(p(h))'=0$. It means h is a root of the polynomial $q(x)=(c_0'+c_1h')+(c_1'+2c_2h')x+\ldots+(c_{n-1}'+nh')x^{n-1}$. The coefficient $d_{n-1}=c_{n-1}'+nh'\neq 0$. If it was not true then the element $c_{n-1}+nh\in E_C=F_C$ and $nh\neq 0$. It would imply $h\in F$. Contradiction. Furthermore h is a root of polynomial $q\in F[x]$ which is of lower degree than p but p is minimal. (ii) similarly.

Corollary 2.3. Suppose the differential field extension $F=T(x)\subset E=M(G)$. We can easily see with respect to Proposition 2.2 (i) neither arctan nor ln are algebraic over $F=T(x)$. Due to (ii) exp cannot be algebraic over $F=T(x)$.

3. Integration under elementary extensions

Definition 3.1. Let F be a differential field and $f, g \in F$ be such that $g' = f'/f$. Then
(i) the element g is a *logarithm* of an element f and we write $g = \ln f$,
(ii) the element f is an *exponential* of g and we write $f = \exp g$.

Definition 3.2. Let $F \subset E$ be a nontrivial extension of differential fields. We say that this extension is an *elementary extension* if there is a finite sequence of field extensions $F = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = E$ where each of the extensions $F_j \subset F_{j+1}$ has the form $F_{j+1} = F_j(g_j)$ where $g_j \in E$ is either algebraic over F_j or logarithm or exponential of an element in F_j.

Theorem 3.3 (Liouville). Let F be a differential field and suppose $f \in F$ has no primitive in F. The element f has a primitive within an elementary no new constant differential field extension $F \subset E$ if and only if there is a positive integer m, a collection of constants $c_1, \ldots, c_m \in F_C$ and elements $h_1, \ldots, h_m, h \in F$ such that

$$f = \sum_{j=1}^{m} c_j \frac{h_j'}{h_j} + h'.$$

Corollary 3.4. Suppose $F \subset F_1 = F(\exp g)$ is a no new constant differential extension, $g \in F$. Let $\exp g$ be transcendental over F and let $f \in F$. Then $f \exp g \in F_1$ has a primitive within some elementary no new constant differential field extension of F_1 if and only if there is an element $a \in F$ such that

$$f \exp g = (a \exp g)' \text{ or equivalently } f = a' + ag'.$$

Corollary 3.5. The function $\varphi(x) = \exp\frac{-x^2}{2}$, $x \in T$ has no elementary primitive.

Proof: By Corollary 3.4 the function $\varphi(x) = \exp\frac{-x^2}{2}$ has an elementary primitive if and only if there is a function $a \in T(x)$ such that

$$1 = a' - ax. \qquad (\oplus)$$

We set $F = T(x), g = -x^2/2, f = 1$ in the Corollary 3.4. We show that such a does not exist. Write $a = p/q$, where the polynomials $p, q \in T[x]$ have no common dividers. When we substitute we get $q(q+px-p'q)=-pq'$. Hence $q \mid pq'$ and thus $q \mid q'$. It is possible if and only if q is constant. It implies that $a \in T[x]$ but it is impossible with respect to ($\oplus$).

4. Summary

Some functions have no elementary primitive functions. Among them there is for instance the density of the normal distributions. We tried to show the algebraic reasons for this fact. Notice that the function is elementary in more narrow sense here than it is usual. This result does not prove there is another way to express the normal distribution function by such functions as arctan, arcsin for instance.

References

[1] Procházka L. et al., *Algebra*, Academia, Praha 1990.
[2] Ritt J.F., *Integration in Finite Terms*, Columbia University Press, New York 1948.
[3] Rosenlicht M., "Integration in Finite Terms", *The American Mathematical Monthly* 1972, 79 (9).

NIEELEMENTARNE PRZECIWPOCHODNE

Streszczenie

Wiadomo, że pewne funkcje nie mają elementarnych przeciwpochodnych. Taką funkcją jest na przykład gęstość rozkładu normalnego. Ale nie jest oczywistym spojrzenie w głąb tych faktów. Aparat dowodu przedstawionych propozycji składa się z metod algebry różniczkowej. W ten sposób można pokazać, że nieskończona całka z gęstości rozkładu normalnego nie wyraża się w postaci funkcji elementarnych.

Słowa kluczowe: ciało różniczkowe, pochodność, rozszerzenie ciała różniczkowego, rozszerzenie ciała algebraicznego, rozszerzenie elementarne.