PRACE NAUKOWE AKADEMII EKONOMICZNEJ WE WROCŁAWIU
Nr 1162 — 2007
Applications of Mathematics and Statistics in Economics

Marta Król

Wrocław University of Economics, Poland

CALCULATION OF PROBABILITY OF RUIN IN MODEL WITH DEPENDENT CLASSES

1. Introduction

The theory of ruin is one of the problems of actuarial science. The information about insurance company financial risk is extremely essential to preserving the insurer's solvency. The main goal of this paper was calculating the probability of the insurer ruin, in the case of using the model with dependent classes.

The theoretical part of this paper is mainly based on the paper by Cossette and Marceau [1]. At the beginning we describe the discrete-time model, next we define the probability of ruin. Usually it is impossible to calculate the exact value of the probability of ruin. In order to make possible to calculate this probability we approximate the survival function using the total probability distribution claims. This distribution is obtained by Fast Fourier Transform method (FFT).

In this paper we also present the model which introduces a dependence relation between classes. In this model we assume that the common component has an impact on classes, causing the dependence between number of classes for the claims.

In the end the author presents the example in which it is assumed that the number of claims in every class has a Poisson distribution and claim amounts have different distributions.

2. Discrete, finite-time ruin probability

Let us consider the discrete-time risk process (surplus process) $\{U_n, n = 0, 1, 2, ...\}$ generated by the number of claims process, the intensity of premium and the initial capital [1, p. 134; 5, p. 177; 6, p. 215; 3, p. 83]. The surplus of insurer at time n is defined as

$$U_n = u + cn - S_n, \tag{1}$$

where u is the initial capital (the initial surplus of insurer), c is the premium income received during each period, S_n is the total claim amount of the whole portfolio over the first n periods. The components of S_n are the total claim amounts W_i in period i:

$$S_n = W_1 + W_2 + W_3 + \dots + W_n. \tag{2}$$

We assume that W_1, W_2, ... are independent and identically distributed random variables with $E[W_i] = \mu_W < c$. Substituting the formula (2) to formula (1) we obtain

$$U_n = u + (c - W_1) + (c - W_2) + \dots + (c - W_n). \tag{3}$$

The time T in which insurer's ruin happens (it means the loss of solvency of insurance company) is defined as [1; 3; 5; 6]:

$$T = \min\{n: U_n < 0\}, n = 1, 2, \dots$$

If insurer's surplus is non-negative at each moment of time ($U_n \geqslant 0$), then T equals infinity. We defined the ruin probability in finite-time as follows:

$$\psi(u,1,n) = P[T \leqslant n]$$

If $n \to \infty$, we have $\psi(u) = P[T < \infty]$.

We sometimes consider non-ruin probability and respectively non-ruin probability in finite- and infinite-time. This is:

$$\varphi(u,1,n) = 1 - \psi(u,1,n), \tag{4}$$
$$\varphi(u) = 1 - \psi(u).$$

Assuming the risk process is given by (3) the non-ruin probability in finite time has the form:

$$\varphi(u,1,n) = P(U_1 \geqslant 0, U_2 \geqslant 0, \dots, U_n \geqslant 0) =$$
$$P(W_1 \leqslant u + c, W_1 + W_2 \leqslant u + 2c, \dots, W_1 + W_2 + \dots + W_n \leqslant u + nc).$$

In order to calculate the exact values of non-ruin probability $\varphi(u,1,n)$ we can use Theorem 1.

Theorem 1 [1]

Let $\{W_i, i = 1, 2, \dots\}$ be a sequence of independent and identically distributed random variables and c the annual premium income, constant over each period. Then

$$\varphi(u,1,n)=\int_0^{u+c}\varphi(u+c-W,1,n-1)dF_W(x). \tag{5}$$

The calculation of exact value of $\varphi(u,1,n)$ using (5) is rarely possible. Therefore we can use an approximation of the non-ruin probability.

The approximation presented in Theorem 2 assumes that $\tilde{W}$ is the discrete random variable with discrete distribution function $F_{\tilde{W}}$. It is necessary to do the discretization[1] of the distribution of random variable W. If $P(\tilde{W}=k)=f_k$, k = 1, 2, ..., M, then $F_{\tilde{W}}(k)=P(\tilde{W}\leqslant k)=\sum_{j=0}^{k}f_j$, where f_k are the mass of probabilities.

Theorem 2 [1; 2]

Let k (the initial surplus), p (an initial premium income), j be integers. Then

$$\varphi_{k,1,n}=\sum_{j=0}^{\min(k+p,M)}\varphi_{k+p-j,1,n-1}f_j,\ n=2,3,\ldots, \tag{6}$$

where

$$\varphi_{k,1,1}=F_{\min(k+p,M)}=\sum_{j=0}^{\min(k+p,M)}f_j. \tag{7}$$

The non-ruin probability can be approximated using the distribution function $F_{\tilde{W}}$ which can be obtained by Fast Fourier Transform (FFT) applied to $\phi_{\tilde{W}}(t)$ [8]. The first step is to transform the claim amounts probability distribution from a continuous to a discrete one and apply FFT to this probability vector. Then the join characteristic function of aggregate claims must be found. The resulted transforms are inverted with FFT method which gives the vector of mass probabilities defining the probability distribution function $F_{\tilde{W}}$.

3. Aggregation of dependent classes

Let us assume that the portfolio consists of two dependent classes [1]. The total claim amounts for the portfolio in period i in model presented at the beginning of this paper (p. 110) has the following form:

$$W_i=W_{i,1}+W_{i,2},\ i=1,2,\ldots,$$

---

[1] Methods of the discretization are presented in [4; 6; 8].

where $W_{i,j}$ is the total claim amount for the j-th class (j = 1, 2) in the i-th period. $W_i, W_{i'}$ are independently and identically distributed random variables for $i \neq i'$ but for fixed i classes are dependent. The total claim amounts for the j-th class in the i-th period is given by:

$$W_{i,j} = \sum_{k=0}^{N_{i,j}} X_{i,j,k},$$

where $X_{i,j,k}$ is the k-th (k = 1, 2, ..., $N_{i,j}$) individual claim amounts for the j-th (j = 1, 2) class in the i-th (i = 1, 2, ...) period, $N_{i,j}$ is the number of claims for the j-th class in the i-th period.

$X_{i,j,k}$ (i = 1, 2, ...; k = 1, 2, ..., $N_{i,j}$) are independently and identically distributed random variables for both classes. $X^{(j)}$denotes random variable with a common cumulative distribution function $F_{X^{(j)}}$. Let $N_{i,j}$ be identically distributed random variables for both classes. $N^{(j)}$ denotes random variable with a common distribution function.

Assuming these denotations the total claim amounts $W^{(j)}$ for the j-th class is equal to the following sum:

$$W^{(j)} = \sum_{i=1}^{N^{(j)}} X_i^{(j)}, \text{ for } j = 1, 2.$$

For the total claim amounts for the j-th (j = 1, 2) class, it is assumed that:

$X^{(j)}$ are independent,

$X^{(j)}$ i $N^{(j)}$ are independent,

$N^{(j)}$ are dependent.

This means that we consider the portfolio in which there is dependence between number of claims generated by each class.

Poisson model with common shock

We assume that factors (characteristic for the first and the second class, respectively) generate the number of claims $N^{(11)}$ and $N^{(22)}$, and other factors (common shock) have an impact on number of claims $N^{(12)}$ for both classes[2]. Moreover, let the claims have a Poisson frequency. The total number of claims for the j-th class is defined as follows [1; 8]:

$$N^{(1)} = N^{(11)} + N^{(12)},$$

[2] To illustrate this situation we give some example. Let us consider homeowners of two parts of a city with different living standards. We take into account the local and global conditions. The local conditions generate the two different claims number of homeowners in poor part of city and another in the rich one. But in the case of global conditions (for example flood or very strong wind) the damages cover both parts of the city.

$$N^{(2)} = N^{(22)} + N^{(12)},$$

where

$N^{(uv)} \sim \text{Poisson}(\lambda_{uv})$ $(u, v = 1, 2)$ and $N^{(1)} \sim \text{Poisson}(\lambda_1)$, $N^{(2)} \sim \text{Poisson}(\lambda_2)$ with $\lambda_1 = \lambda_{11} + \lambda_{12}$, $\lambda_2 = \lambda_{22} + \lambda_{12}$.

We can generalize this model to any number k of dependent classes ($j = k$, where k is natural number), then:

$$N^{(1)} = N^{(11)} + N^{(12)} + N^{(13)} + \ldots + N^{(1k)} + N^{(123)} + \ldots + N^{(1\,k-1\,k)} + \ldots + N^{(123\ldots k)},$$
$$N^{(2)} = N^{(22)} + N^{(12)} + N^{(23)} + \ldots + N^{(2k)} + N^{(123)} + \ldots + N^{(2\,k-1\,k)} + \ldots + N^{(123\ldots k)},$$
$$N^{(3)} = N^{(33)} + N^{(13)} + N^{(23)} + \ldots + N^{(3k)} + N^{(123)} + \ldots + N^{(3\,k-1\,k)} + \ldots + N^{(123\ldots k)},$$
$$\vdots$$
$$N^{(k)} = N^{(kk)} + N^{(1k)} + N^{(2k)} + \ldots + N^{(k-1\,k)} + N^{(12k)} + \ldots + N^{(k-2\,k-1\,k)} + \ldots + N^{(123\ldots k)},$$

where

$N^{(1)} \sim \text{Poisson}(\lambda_1)$, ..., $N^{(k)} \sim \text{Poisson}(\lambda_k)$, with

$$\lambda_1 = \lambda_{11} + \lambda_{12} + \lambda_{13} + \ldots + \lambda_{1k} + \lambda_{123} + \ldots + \lambda_{1\,k-1\,k} + \ldots + \lambda_{123\ldots k},$$
$$\vdots$$
$$\lambda_k = \lambda_{kk} + \lambda_{1k} + \lambda_{2k} + \lambda_{3k} + \ldots + \lambda_{k-1\,k} + \lambda_{123} + \ldots + \lambda_{k-2\,k-1\,k} + \ldots + \lambda_{123\ldots k}.$$

This means that it is possible to consider the portfolio divided in k dependent classes. In this case there are any factors: characteristic only for one class factors generate the number of claims $N^{(11)}$, $N^{(22)}$, ..., $N^{(kk)}$; the factors have an impact on numbers of claims for the pairs of classes, other factors have influance on three of classes, etc. and one for all classes.

For the model (for two classes but we can generalize to any number k of dependent classes) presented above, the joint characteristic function of the aggregate claims $(W^{(1)}, W^{(2)})$ is [1; 4; 8]:

$$\phi_{W^{(1)},W^{(2)}}(t_1,t_2) = P_{N^{(1)},N^{(2)}}(\phi_{X^{(1)}}(t_1),\phi_{X^{(2)}}(t_2)). \quad (8)$$

The join probability generating function $(N^{(1)}, N^{(2)})$ is

$$P_{N^{(1)},N^{(2)}}(t_1,t_2) = E[t_1^{N^{(11)}+N^{(12)}} t_2^{N^{(22)}+N^{(12)}}] = E[t_1^{N^{(11)}}]E[t_2^{N^{(22)}}]E[t_{12}^{N^{(12)}}] =$$
$$= P_{N^{(11)}}(t_1)P_{N^{(22)}}(t_2)P_{N^{(12)}}(t_1t_2).$$

From the form of probability generating function for the Poisson random variable $P_N(t) = e^{\lambda(t-1)}$, we obtain:

$$P_{N^{(1)},N^{(2)}}(t_1,t_2) = e^{\lambda_{11}(t_1-1)}e^{\lambda_{22}(t_2-1)}e^{\lambda_{12}(t_1t_2-1)}. \quad (9)$$

Given (8) and (9), we have

$$\phi_{W^{(1)},W^{(2)}}(t_1,t_2) = P_{N^{(1)},N^{(2)}}(\phi_{X^{(1)}}(t_1),\phi_{X^{(2)}}(t_2)) =$$
$$= e^{\lambda_{11}(\phi_{X^{(1)}}(t_1)-1)} e^{\lambda_{22}(\phi_{X^{(2)}}(t_2)-1)} e^{\lambda_{12}(\phi_{X^{(1)}}(t_1)\phi_{X^{(2)}}(t_2)-1)}.$$

For $t_1 = t_2 = t$, the characteristic function of total claim amounts W has the form:

$$\phi_W(t) = \phi_{W^{(1)},W^{(2)}}(t,t) = e^{\lambda(\phi_X(t)-1)}, \tag{10}$$

where $\lambda = \lambda_{11} + \lambda_{22} + \lambda_{12}$ and

$$\phi_X(t) = \frac{\lambda_{11}}{\lambda}\phi_{X^{(1)}}(t) + \frac{\lambda_{22}}{\lambda}\phi_{X^{(2)}}(t) + \frac{\lambda_{12}}{\lambda}\phi_{X^{(1)}}(t)\phi_{X^{(2)}}(t). \tag{11}$$

Example 1

Let us consider the following characteristic of the model presented above. The claim amounts generated from the first class have a Weibull distribution with parameters $\alpha = 2$, $\beta = 3$, the second class has lognormal distribution with parameters $\mu = 1.5$ and $\delta = 0.5$, and the number of claims have a Poisson distribution with parameters $\lambda_1 = 3$, $\lambda_2 = 3$, respectively.

Microsoft Excel was used for the calculations of the following steps. All algorithms needed for these calculations were studied by the author of this paper.

The first step is discretization of the distribution function of the claim amounts for each classes $F_{X^{(j)}}$ (j = 1, 2) with the method of rounding [4; 8], and taking their Fourier transform. Then we calculate the characteristic function of the total claim amounts W. Therefore, these transforms are put on the formulas (10) and (11). The received transforms are inverted with FFT method. We obtain the vector of probabilities which defines the probability distribution function $F_{\hat{W}}$. The approximation of F_W is used in the algorithm which approximates the non-ruin probability $\varphi_{k,1,n}$ described in the second theorem. Finally, we calculate the probability of the ruin using formula (4). The numerical results of the ruin probability are shown in Table 1.

Table 1. Ruin probability in Poisson models

u	$n = 10$	$n = 15$	$n = 20$	$n = 30$
10	0.0192	0.0923	0.3392	0.8292
20	0.0115	0.0803	0.3045	0.7834
50	0.0064	0.0402	0.1944	0.6465
100	0.0020	0.0167	0.1063	0.4762
150	0.0000	0.0050	0.0273	0.2989

Source: own computations

We observe that the ruin probability decreases if the initial capital is getting bigger. Let us compare the probability of ruin for the various initial capitals for a fixed period. For example, for n = 15 and growing initial capitals 10, 50, 150 the ruin probability increases to: $\psi(10,1,15) = 0.0923$, $\psi(50,1,15) = 0.0402$, $\psi(150,1,15) = 0.0050$.

The differences between the ruin probabilities for the various initial capitals are getting smaller: for n = 15 and for the initial capitals 10 and 20 the difference between ruin probability is equal to 0.012 ($\psi(10,1,15) - \psi(20,1,15) = 0.0923 - 0.0803 = 0.012$), in the case of the initial capitals are equal to 100 and 150 the difference is smaller, it equals 0.0117.

We know that the problem of the ruin probability in the model with dependent classes has not been fully described. Therefore, a further study will focus on the impact on the ruin probability of a relation of dependence between classes in an insurance portfolio and on the adjustment coefficient.

References

[1] Cossette H., Marceau E., "The Discrete-time Risk Model with Correlated Classes of Business", *Insurance: Mathematics and Economics* 2000, 26, 133-149.

[2] Dickson D.C.M., Waters H.R., "Recursive Calculation of Survival Probabilities, *ASTIN Bulletin* 1991, 21, 199-221.

[3] Kaas R., Goovaerts M., Dhaene J., Denuit M., *Modern Actuarial Risk Theory*, Kluwer Academic Publishers, Boston 2001.

[4] Klugman S.A., Panjer H.H., Willmot G.E., *Loss Models. From Data to Decision*, John Wiley & Sons, New York 1998.

[5] *Modele aktuarialne*, red. W. Ostasiewicz, AE, Wrocław 2000.

[6] Otto W., *Ubezpieczenia majątkowe. Część I. Teoria ryzyka*, WNT, Warszawa 2004.

[7] Panjer H.H., Willmot G.E., *Insurance Risk Models*, Society of Actuaries, Schaumburg 1992.

[8] Wang S.S., "Aggregation of Correlated Risk Portfolios: Models and Algorithms", *Proceedings of the Casualty Actuarial Society* 1998, 848-939.

WYZNACZENIE PRAWDOPODOBIEŃSTWA RUINY DLA MODELU Z ZALEŻNYMI KLASAMI

Streszczenie

Za główny cel artykułu uznaje się wyznaczenie prawdopodobieństwa ruiny, a więc niewypłacalności ubezpieczyciela w przypadku portfela składającego się z zależnych klas.

Przedstawiono opis dyskretnego modelu ryzyka ubezpieczyciela. Następnie zdefiniowano prawdopodobieństwo ruiny. Należy w tym miejscu zaznaczyć, że bezpośrednie wyliczenie prawdopodobieństwa ruiny z definicji często nie jest możliwe, dlatego w celu jego obliczenia stosuje się aproksymację. W dalszej części artykułu podano więc aproksymację funkcji przeżycia, umożliwiającą wyznaczenie prawdopodobieństwa ruiny.

Na potrzeby aproksymacji wykorzystano łączny rozkład prawdopodobieństwa szkód (wypłat) ubezpieczyciela, uzyskany metodą szybkiej transformaty Fouriera.

Przedstawiono również model ilustrujący zależność między klasami. Model zakłada wpływ dodatkowego czynnika zewnętrznego, działającego równocześnie na różne klasy, powodującego zależności między ilością szkód poszczególnych klas.

Na zakończenie podano przykład, w którym zakłada się, że ilość szkód w każdej klasie ma rozkład Poissona, natomiast szkody mają różne rozkłady.

Słowa kluczowe: zagregowane szkody zależne, model Poissona ze wspólnych czynnikiem zewnętrznym, prawdopodobieństwo ruiny.