Supplementary Financial Security for the Silver Generation

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Preface

The monograph is devoted to the issues of older people's financial security – in the literature this group is called the silver generation. This topic is increasingly important in the face of the global ageing of society. It touches on many important issues, such as retirement security, management of assets available to retirees, and broadly understood health and social policy. The authors were prompted to work on this topic by the belief that appropriate financial security for the elderly is the foundation of social stability and quality of life not only for seniors themselves, but also for the entire society.

The observed demographic changes result mainly from the increase in life expectancy and the significant decline in fertility rates, causing a rise in the number of older people in all European countries. The ever-larger group of older people (*silver generation*) causes significant economic changes. This will mainly affect social security, the pension system and the labour market. These areas influence each other, an example of which is the shortage of manpower, with the increasing outlay for the care of older people who are often unable to finance it privately due to low pensions. Note that the silver generation is a group of consumers whose needs are so specific, unique and simultaneously broad that they created a branch of the economy called the *silver economy*, a set of different products and services coming from other economic sectors. It combines various fields, including gerontology, health services, long-term care, finance and insurance.

The increasing percentage share of the silver generation in the whole society requires a thoughtful approach to the issue of their financial security to enjoy a peaceful and prosperous old age. This book focuses on two selected aspects, namely maintaining the standard of living of seniors in retirement and, at the same time, securing financial resources for possible treatment. Therefore, the study concentrated on insurance and financial solutions that enable older people to take into account their specific needs and possibilities. The authors tried to comprehensively present this issue from the theoretical side supported by examples illustrating the possibility of applying the presented solutions in practice.

In particular, the health-related needs of older people are increasing dramatically while their income is decreasing. The book aims to propose financial and insurance contracts that are intended, on the one hand, to protect seniors against a decline in their quality of life due to reduced income and, at the same time, to secure finances for possible treatment. The authors focus on the private sector because this is the only area where interested individuals can make autonomous decisions, and examined the situation of people who are retired and making decisions during this time. The study excluded the whole area of the most effective financial planning and taking appropriate actions at an early stage of life such as saving, diversifying investments, planning health expenses, continuing to work after retirement age, etc.

There is a wide range of insurance products offered on the market. In securing financial resources for treatment, the insurance market offers, among others, health insurance and critical illness insurance. However, the economic situation of pensioners may not allow them to purchase such insurance. For people with a life insurance contract, in the case of serious illness, it is possible to resell it on the secondary insurance market (viatical settlement). Yet, this solution is a one-off possibility and cannot be used again in the event of another illness. On the other hand, a life insurance contract concluded on the secondary insurance market by a healthy insured person (life settlement, senior settlement) allows for obtaining some capital to improve the standard of living, but this is a temporary solution, not a constant inflow of cash for the rest of life.

The financial market also includes products offers dedicated to seniors. Equity release agreements allow property owners to release part of the value of their home without having to sell it. These agreements are dedicated to seniors who own a home but have limited access to cash. This solution is not perfect, but despite its disadvantages (high costs, reduced value of property transferred to heirs), it has its advantages, such as staying at home and increasing financial means for current needs. Especially in Poland, this solution has potential because, as research from 2021 shows, approximately 80% of Poles own real estate. In the case of spouses, they are usually co-owners of the apartment or house where they live.

This monograph puts forward a *Comprehensive Marriage Contract with Health Protection* as a combination of a reverse annuity contract and critical insurance for married couples who are co-owners of real estate to finance the premiums for their health insurance and obtain the monthly capital (annuity) for increasing the standard of living. The authors focused on the actuarial model of such a contract and calculated the net benefits based on available data.

The monograph consists of five chapters. Chapter 1 characterises the 'silver generation' in European countries, in particular emphasising the Polish population. The demographic, economic and health situation of older people and their needs impacting on the silver economy's development are discussed. As older people are more exposed to the risk of poverty, health loss and disability, the need to create one's own (individual) financial security is emphasised (due to the state's stable financing of health care systems and pension systems are burdened with many types of risk). In the context of seniors' health and financial security, solutions such as co-payments for medical services, the possibility of additional pension and private financial security are reviewed.

The first part of Chapter 2 is devoted to constructing a multistate model for a specific insurance contract covering many types of risk and actuarial methods for analysing cash flows resulting from the implementation of such agreements. In particular, the probabilistic and financial structure of the multistate model is described. The basic assumptions used in the work for the valuation of insurance contracts are also formulated. In the second part of the chapter, the method of analysing cash flows based on the modified multistate model is described because this approach allows the introduction of matrix notation in the actuarial values, which is a useful tool facilitating the application of theory in practice.

Chapter 3 concerns individual insurance and annuity contracts related to the life and health of the insured and equity real estate agreement. In particular, the realised equity contracts are discussed in detail, with numerical examples illustrating the amount of benefits for a person reselling rights to their property. In the context of the risk of health loss, health insurance was classified. In addition, a detailed description of three types of insurance contracts for the risk of serious illness is made. Appropriate multistate models are presented, describing their probabilistic structure and cash flows in a form that enables matrix formulas to determine the premium.

Chapter 4 addresses marital contracts resulting from the combination of two individual contracts of a husband and wife. Possible variants of these contracts (Joint Life Status and Life Surviving Status) and modified multistate models, which are closely related not only to the variant of the agreement chosen by the insured but also to the type of benefits (annuities, lump sum benefits), are discussed in detail. This chapter focused on modelling the probabilistic structure of the contract's model. The assumptions that the future lifetimes of the spouses are independent and dependent are considered because they significantly impact the valuation of these contracts. This impact is illustrated in many numerical examples in which life annuities and reverse annuities are determined.

In Chapter 5, the Comprehensive Marriage Contract with Health Protection was proposed based on the combination of two individual critical health insurance contracts (for wife and husband) and a marital reverse annuity contract. In particular, a multistate model and its probabilistic structure for the proposed agreement are presented. Different scenarios depending on the type of health benefit chosen by the spouses and applied modified multistate model are given to value the benefits resulting from the contract using matrix formulas. The chapter also contains numerical examples demonstrating the practical applicability of the proposed solution. Finally, conclusions with a summary and a discussion of potential directions for further research and possibilities for expanding the proposed solutions are given.

The book mainly presents standardised numerical examples due to the available health data, using multistate life tables constructed for Poles with lung cancer. When creating the tables, 2008 was adopted as the reference year (as one of the middle periods), which allowed for taking into consideration the history of hospitalisation of these patients in 2006-2011, hence the value of the interest rate was estimated based on data from 2008. The numerical examples illustrate the possibilities of applying the proposed solutions in practice and are intended to indicate the directions of possible comparative analyses.

This monograph is the result of cooperation and combines areas of science from various fields: insurance, finance, demography and health protection, the authors making a valuable contribution, allowing to create a comprehensive study. This work should contribute to a better understanding of the issue of financial security of the silver generation and become an inspiration for further research and practical activities.

Chapter 1

INTRODUCTION

1.1. Characteristics of the elderly population

Currently, in Poland, as in other European countries, there is an increasing percentage of older people, mainly due to three factors (Palenik, 2012):

- a decline in fertility,
- increasing life expectancy of citizens,
- the baby boom after World War II.

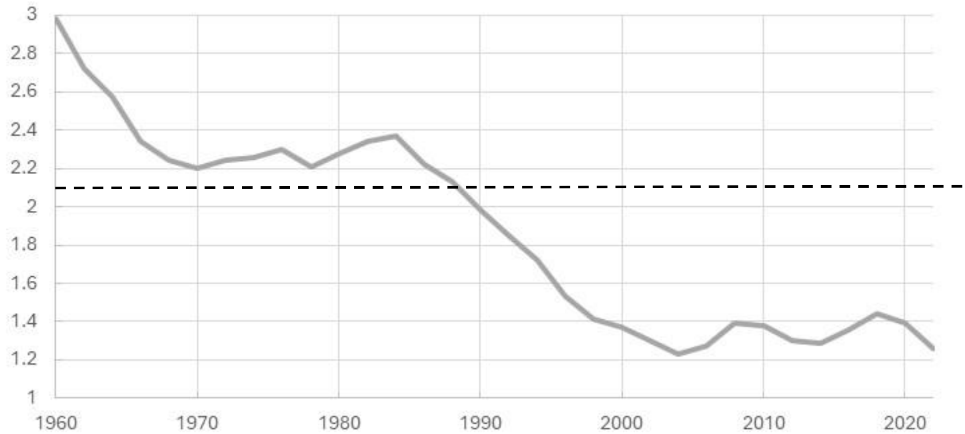


Figure 1.1. The fertility rate in 1960-2022

Source: based on (OECD, 2023).

In Poland, the fertility rate, which measures the average number of children per woman, dropped significantly before 1970 and between 1983 and 2003 (OECD, 2023). Until 1989, its value still provided a simple replacement for generations, later unfortunately, it fell below this limit, i.e. the value of 2.1 (see Figure 1.1 and Table 1.1). At present, a slight increase in this ratio could be observed for several years, however it does not exceed 1.5, which means it is not enough to ensure simple generation replacement, and in 2020 this rate fell again, due to the impact of the COVID-19 pandemic. One can see that the government's 500+ programme is not bringing the expected results. The low fertility rate caused a decrease in the share of young people and middle-aged people in society. This fact will definitely worsen the situation of older people, affecting the Polish pension system even harder, and will also shake up the Polish labour market in the future.

The fertility rate in Poland is lower than the EU average (1.56). In European countries, this indicator is less than 2, with the highest in Turkey at 1.99, which means one cannot speak of simple generation replacement (Eurostat Data Browser, 2023). The situation is different in non-European countries, especially in Africa (Figure 1.2). There the fertility rate clearly exceeds the limit value of 2.1, e.g. for Niger, it is up to 7. The average value calculated for all countries in the world is equal to 2.4.

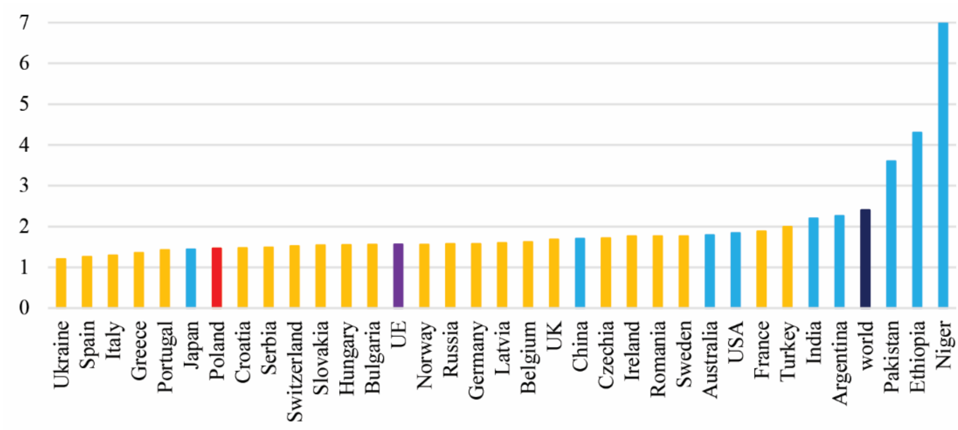


Figure 1.2. The fertility rate in selected countries

Source: based on (Eurostat Data Browser, 2023).

Life expectancy at birth is defined as how long, on average, a newborn can expect to live if current death rates do not change. The life expectancy of both women and men is steadily growing, i.e. people are living longer (Figure 1.3, Table 1.1) (OECD, 2023). Since 1991, this increase has been particularly significant, however in 2018 a slight decrease was recorded, followed by a larger one in 2020, and 2021. This is the impact of the COVID-19 pandemic, but in 2022 there was also significant growth. Moreover, an almost constant difference could be observed between the values of life expectancy for men and women. The increase in life expectancy, combined with a significant decline in the fertility rate in 1983-2003 caused an increase in the share of older people in modern society (OECD, 2023).

Table 1.2 shows the life expectancy at different ages for total, male and female, and older people – one can see that it is longer for women, but the difference decreases with age. On average, a man aged 60 will live to 78.7 years, and a woman will reach the age of 83.6. However, at the age of 85, a man will live to 90.2 and a woman to 91.2 (Eurostat Data Browser, 2023). In 2022 there was a slight decline compared to 2017, thus the impact of COVID-19 is visible.

Table 1.1. The fertility rate and life expectancy by sex in 1960-2022

Year	Fertility rate	Life expectancy		
		males	females	total
1960	2.98	64.9	70.6	67.8
1970	2.20	66.2	73.3	70,0
1980	2.28	66.0	74.4	70.2
1990	1.99	66.2	75.2	70.8
1995	1.55	67.7	76.4	72.1
2000	1.37	69.7	78.0	73.8
2005	1.24	70.8	79.3	75.1
2010	1.38	72.1	80.6	76.5
2015	1.29	73.0	81.6	77.6
2020	1.39	72.6	80.7	76.5
2022	1.26	73.4	81.1	77.4

Source: based on (OECD, 2023).

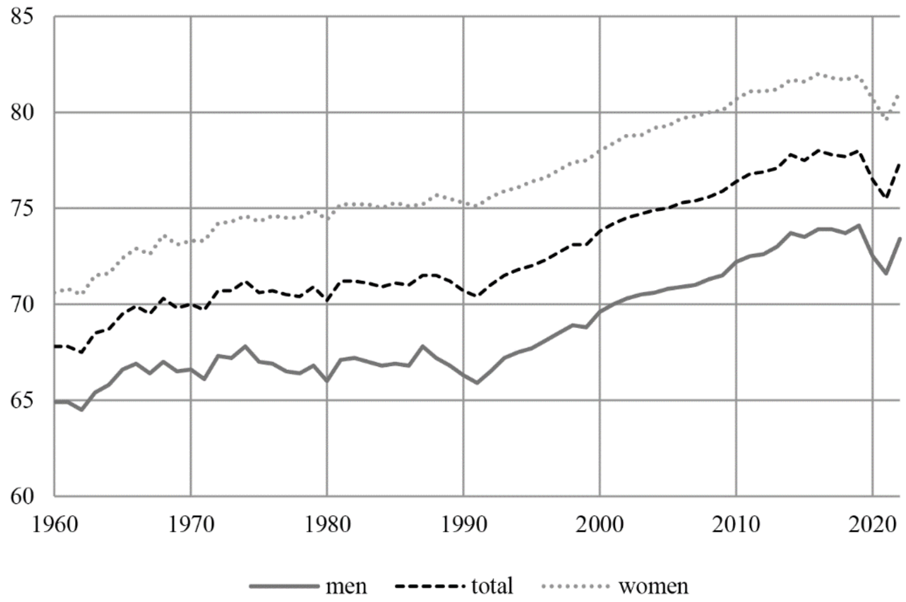


Figure 1.3. Life expectancy by sex in 1960-2022

Source: based on (OECD, 2023).

Table 1.2. Life expectancy at different ages in 2017 and 2022

Age	Total		Males		Females	
	2017	2022	2017	2022	2017	2022
60	22.0	21.2	19.2	18.7	24.3	23.6
65	18.3	17.5	15.9	15.3	20.2	19.5
70	14.9	14.2	12.9	12.4	16.4	15.6
75	11.8	11.0	10.2	9.7	12.8	12.0
80	8.9	8.2	7.8	7.2	9.5	8.8
85	6.6	5.8	5.9	5.2	6.9	6.2

Source: based on (Eurostat Data Browser, 2023).

The ever-larger percentage of older people causes significant changes in the economy. This mainly affects social security, the pension system, and the labour market, because the number of post-working people increases and the number of people of working age decreases. There is also a shortage of people able to work and the expenditure for the care of elderly people is increasing.

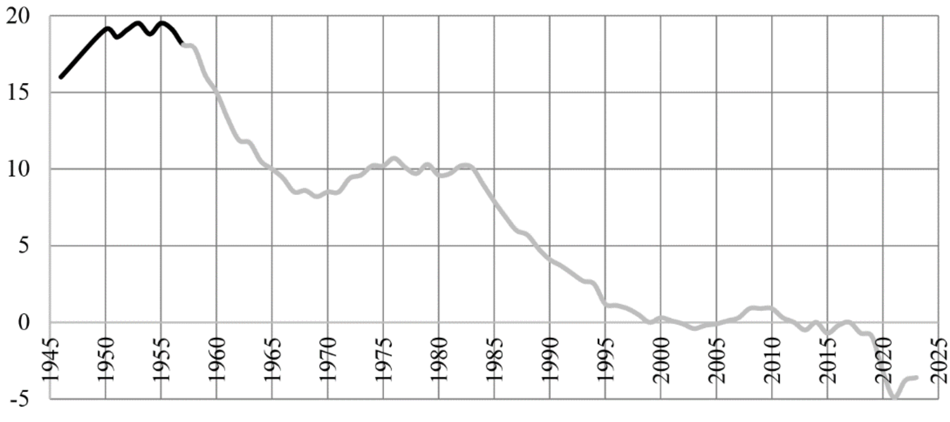


Figure 1.4. The natural increase in 1946-2023

Source: based on (GUS, 2023b, 2024).

The group of elderly people includes, among others, people from the post-war Baby Boomers generation. Figure 1.4 shows the natural increase per 1000 population. The red colour indicates the post-war baby boom in 1946-1958. In 2023, they were aged 65-77, if, of course, they lived to that year; the natural increase is gradually decreasing. In the period 1970-1983, another baby boom was recorded, i.e. the children of Baby Boomers. However, in the 21st century some stabilisation arrived at 0. This was caused by the appearance of the grandchildren of Baby Boomers (Główny Urząd Statystyczny [GUS], 2023b, 2024). In 2017-2021, a significant decline in natural growth was observed, especially in 2020-2021, due to the pandemic. However, in 2022 it increased once more.

The economic emigration of Polish citizens also had some impact on the larger share of older people in the population as it is usually younger people who leave. For example, in 2022, only about 6% of persons leaving for permanent stay were seniors, namely people over 60 years of age, whilst 25.9% of the whole population were elderly people (GUS, 2023b).

There are, of course, some positive sides. A large, specific sales market has been created for older customers. This forces the emergence of new branches of the economy providing access to specific goods and services for seniors that improve their quality of life. New jobs are created, and significant development of innovative products targeted at older people can also be observed. This issue is discussed more broadly in Section 1.2.

It is difficult to define clearly the threshold from which people can be considered old. Statistics Poland (SP) defines older people as those aged 60 and older, see the study Situation of older people in Poland in 2022 published by the Statistics Poland (GUS, 2023a,2023b). This definition is also included in the Act of 11 September 2015 on the elderly (Ustawa z dnia 11 września 2015…, Art. 4). Similarly, old people are defined by WHO, which also distinguishes three periods of old age: young-old (60-74 years old), old-old (75-84), and oldest-old (85 and older) (WHO, 2004), whereas Kowalczyk-Rólczyńska (2018) writes that Dzienio indicates four stages of old age: initial old age (60-69), transition age (70-74), advanced old age (75-84) and infirm old age (85 and older). Pędich (2000) divides old age into two stages: early old age (60--75) and late old age (75 and older). In this section, the authors define old people as 60 years old and older, due to the available data that allows reliable analysis of this group and the definition adopted by SP. However, in the next section, this limit is repositioned to 50 years, which involves the concept of the silver generation that does not fully coincide with the grouping of seniors.

One can also encounter the division into working age and post-working age. This is a fairly natural, intuitive division, but its disadvantage is the different retirement age for men (65 years) and women (60 years). Similarly, old people are defined by Eurostat (cf. European Commission, 2022), setting the boundary at 65.

As mentioned before, older people make up an increasing percentage of society. The change in age structure by sex is shown in Table 1.3, showing that society is clearly ageing. The share of older people in the population was increasing and in 2022 it amounted to 25.9. Before 2000, the percentage share in particular age groups decreased with age. The 20-39 class was the largest in the following years, and in 2022, the 40-59 class. A similar situation occurs for men and women. Women make up the majority of older people. The share of older women was 29.2% among all women in 2022, whilst only 22.5% for older men.

Table 1.3. Share of persons by age groups and sex (in %)

Age	1980			1990			2000		
	Total	M	F	Total	M	F	Total	M	F
0-19	32.1	33.7	30.5	32.6	34.2	31.1	28.2	29.7	26.7
20-39	31.8	33.0	30.6	30.6	31.8	29.5	28.3	29.5	27.0
40-59	22.9	22.5	23.4	22.0	21.9	22.0	26.9	27.0	26.7
60+	13.2	10.8	15.5	14.9	12.1	17.5	16.7	13.7	19.5
Age	**2010**			**2020**			**2022**		
	Total	**M**	**F**	**Total**	**M**	**F**	**Total**	**M**	**F**
0-19	21.7	22.9	20.5	20.0	21.2	18.9	20.2	21.5	19.1
20-39	31.1	32.6	29.7	28.8	30.2	27.5	25.9	27.2	24.6
40-59	27.9	28.4	27.4	26.7	27.5	26.0	28.8	28.8	27.1
60+	19.3	16.0	22.3	24.5	21.1	27.6	25.9	22.5	29.2

Source: (GUS, 2023b).

Table 1.4 presents the distribution of older people by age class of people over 60. The age class 65-69 is the largest, with 25.7% of older people, then the percentage decreases with age. A similar situation occurs in the case of old women, only the 85 years and over class is larger than the previous one; for older men, their share decreases with age. The feminisation ratio, the number of females per 100 males, rises with age – those aged 85 and over, it reached even 261.7 (GUS, 2023b).

Table 1.4. Share of older persons by age group and sex (in %)

Age	Total	Males	Females	Feminization ratio
60-64	24.9	28.1	22.7	112
65-69	25.7	27.5	24.3	123
70-74	21.0	21.1	20.8	137
75-80	12.1	11.3	12.7	157
80-84	8.2	6.6	9.3	194
85 +	8.2	5.4	10.1	261

Source: (GUS, 2023c).

A similar and even more drastic situation is evident all over Europe (Figure 1.5), becoming more and more a continent of old people. The average percentage of older people in the European Union is 26.3%, and it is a little higher than in Poland with 25.9%. It has the highest value in Europe in Italy at 30.2%, and in the world – in Japan (34.7%). It should also be noted that the two countries with the largest populations, namely China and India, have a relatively small percentage of older people, respectively 13.8 and 8.5. The fertility rate of India is 2.03, which means that it ensures simple generation replacement, whereas in China. this is equal 1.16, i.e. below the limit of replaceability (OECD, 2023).

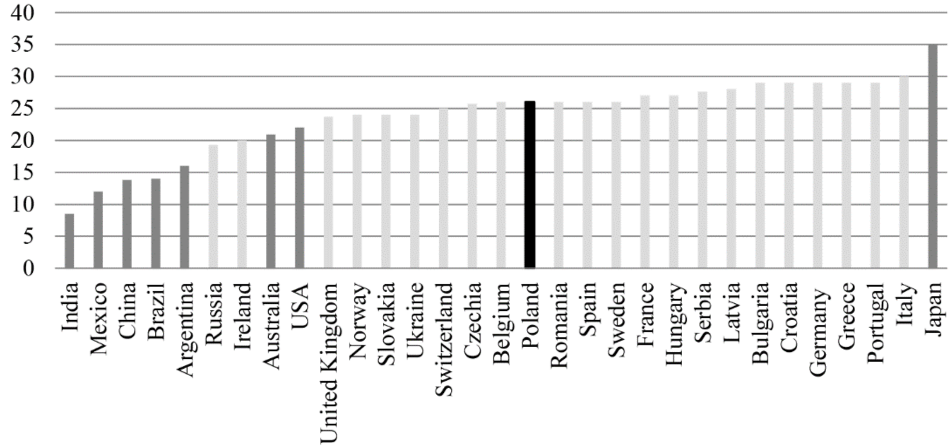


Figure 1.5. Share of older persons in selected countries (in %)

Source: based on (GUS, 2023b)

According to recent forecasts, in Poland there will be an even larger share of older people in the population (Table 1.5); in 2050, it can be expected to account for 37.4%. The feminisation ratio will also decrease in this group, from 139 in 2022 to 129 in 2050; the numerical advantage of older women will not be as great as it is today (GUS, 2023c).

Table 1.5. The forecasts for the share of older people in Poland (in %)

Item	2030	2040	2050
Total	27.8	32.5	37.4
Males	24.3	29.0	33.9
Females	31.0	35.7	40.6
Feminization ratio	138	133	129

Source: (GUS, 2023c).

Analysing the distribution of the share of older people in individual voivodeships, one can notice its rather average diversity (Figure 1.6). The differences between individual regions are not too big, with the highest percentage of older people in the Świętokrzyskie voivodeship: 28.9% and the lowest in Podkarpackie, at 23.9% (GUS, 2023c).

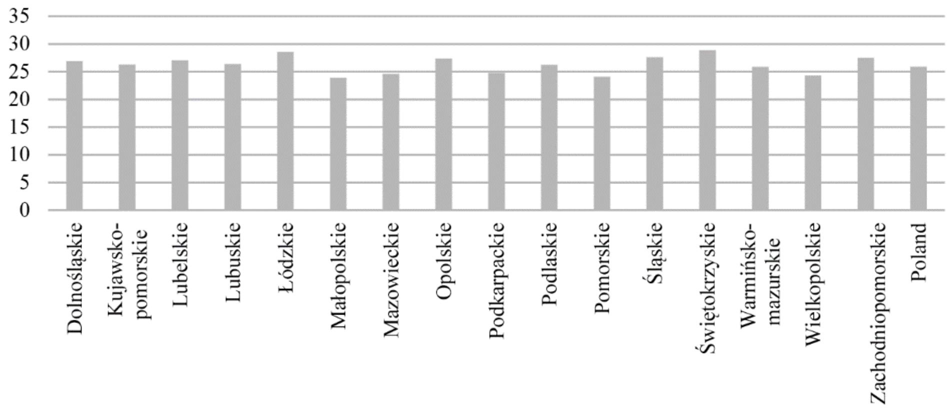


Figure 1.6. The share of older people by voivodeship (in %)

Source: (GUS, 2023c).

In 2022, 86.5% of the deceased in Poland were older people, aged 60 and over. Among men of this age, there were 80.9% of deaths, and among women much more – 92.5%. This, of course, results from the fact that women live longer on average. The mortality rate is presented in Table 1.6 in total and for men and women, showing the number of deaths in each age group per 1000 persons per year. This ratio increases with age and is lower for older men than for older women (GUS, 2023c).

Table 1.6. The mortality ratio by age groups

Age	**Mortality ratio**		
	total	**males**	**females**
60 and more	39.3	45.5	35.7
60-64	13.6	19.6	8.4
65-69	20.6	30.0	12.9
70-74	29.5	41.6	20.7
75-79	43.7	59.3	33.8
80-84	73.4	95.2	62.3
85 and more	168.9	189.8	161.0

Source: (GUS, 2023c).

Next the causes of death for older people are examined, based on the data provided in Table 1.7, the first with data from 2021 and the second from 2022. This division shows the impact of the COVID-19 pandemic on the mortality of older people. Unfortunately, Statistics Poland did not publish the data from 2020 as these data would present the examined problem better and more precisely. The situation for total and older people was compared. The causes of death were examined taking into account all people, male and female. The most common cause of death in 2021 was the diseases of the circulatory system. The share of this type of disease as the cause of death is slightly higher for older people (37.6%) than in the total population (34.8%), and is much higher for males than for females. Neoplasms are the second cause of death. There is no significant difference in the share of the cause of death between old people and the total, it is equal to 19.6 % and is slightly higher for men. The percentage of deaths caused by diseases of the respiratory system is noticeably lower at about 6%, yet this percentage for COVID-19 is slightly lower than for the neoplasms and is insignificantly higher for older people and for men. However, COVID-19 as the cause of death dropped significantly in 2022, whilst such figures are close to those obtained in the case of diseases of the respiratory system. This decline resulted in an increase in the percentages of other causes of death; there are similar reasons for them as in 2021.

Table 1.7. Deaths by causes in 2021and 2022 (in %)

Deaths by causes in 2021	**Total**			**Persons aged 60 and more**		
	total	**male**	**female**	**total**	**male**	**female**
Diseases of the circulatory system	34.8	30.8	39.1	37.6	33.7	41.2
Neoplasms	19.6	20.4	18.8	19.6	21.6	17.6
Diseases of the respiratory system	5.4	5.7	5.1	5.7	6.2	5.2
COVID-19	17.9	18.5	17.2	18.6	19.9	17.3
Others	22.3	24.6	19.8	18.5	18.6	18.7
Deaths by causes in 2022	**Total**			**Persons aged 60 and more**		
	total	**male**	**female**	**total**	**male**	**female**
Diseases of the circulatory system	36.0	32.3	39.8	38.9	35.5	41.9
Neoplasms	23.6	24.8	21.2	23.8	26.6	23.1
Diseases of the respiratory system	6.7	7.1	6.3	7.0	7.6	6.5
COVID-19	6.7	6.6	6.9	7.2	7.4	7.0
Others	27.0	29.2	25.8	23.1	22.9	21.5

Source: based on (GUS, 2022, 2023b).

In Table 1.8 the issue of the causes of death of older people is considered by age group. One can observe similar trends in both 2021 (a) and in 2022 (b). First of all, it should be noted that the percentage of deaths caused by diseases of the circulatory system increases dramatically with the increase in the age of people, from 24.3 to 51.2% in 2021 and from 25.9 to 50.1% in 2022. The reverse situation applies for neoplasms. The percentage decreased from 28.8 to 9.2% in 2021 and from 34.6 to 11% in 2022. Moreover, among people in the first class (60-69), neoplasms caused significantly more deaths than diseases of the circulatory system in 2021, and in class 70-74 in 2022 (GUS, 2023b). The share of COVID-19 increased with age to 79 in 2021, and to 84 in 2022.

Table 1.8. Deaths by causes and age groups in 2021 and 2022 (in %)

Deaths by causes and age groups in 2021	**Age groups**					
	60-64	**65-69**	**70-74**	**75-79**	**80-84**	**85+**
Diseases of the circulatory system	24.3	25.9	28.9	32.1	39.7	51.2
Neoplasms	28.8	28.5	27.5	23.2	17.5	9.2
Diseases of the respiratory system	4.4	5.1	5.7	6.1	6.2	5.8
COVID-19	17.5	20.2	21.1	22.2	19.5	15.2
Others	25.0	20.3	16.8	16.4	17.1	18.6
Deaths by causes and age groups in 2022	**Age groups**					
	60-64	**65-69**	**70-74**	**75-79**	**80-84**	**85+**
Diseases of the circulatory system	25.9	28.2	31.7	35.4	40.3	50.1
Neoplasms	34.6	34.8	33.9	29.4	22.0	11.0
Diseases of the respiratory system	5.3	6.4	7.3	7.5	7.7	7.1
COVID-19	5.0	5.9	6.6	7.5	8.6	7.8
Others	29.2	24.7	20.5	20.2	21.4	24.0

Source: based on (GUS, 2022, 2023b).

Next, the authors analysed the structure of the elderly group by marital status, but were constricted to use data regarding the marital status of persons only from 2002, 2011 and 2021, as the National Censuses of Population and Housing took place in these years. Statistics Poland provides data on marital status only based on National Censuses, and Eurostat does not take Poland into account when analysing such issues, therefore, for instance, accurate data on the marital status of older people in 2020 could not be included.

Table 1.9 shows the structure of the elderly group by marital status. Most are married (61.1%), followed by widowers (24.9%), whilst the other states were not relevant as their share does not exceed 10%. There is a significant difference between the sexes here. Among men, married (75.9%) predominate, and there are much fewer widowers (9.8%). Among women, the disparities are much smaller; married people are the largest group (50.4%), and widowed smallest (35.8%). The share of other states also does not exceed 10%. There is a fifth state, unknown, however its share is minimal at less than 0.5%. The differences between 2002 and 2011 are not large. The relations are similar to 2021, only among women there are more widows than married women (GUS, 2013, 2023a).

Table 1.9. The share of older people by marital status and sex (in %)

	Year	Single	Married	Widowed	Divorced	Unknown
Total	2002	4.4	55.8	35.5	3.8	0.5
	2011	4.7	56.8	33.7	4.5	0.3
	2021	5.5	61.1	24.9	8.2	0.3
Male	2002	3.7	79.2	13.1	3.5	0.5
	2011	4.8	78.3	12.5	4.1	0.3
	2021	6.6	75.9	9.8	7.2	0.4
Female	2002	4.9	40.4	50.2	4.0	0.5
	2011	4.6	42.2	48.0	4.8	0.4
	2021	4.7	50.4	35.8	8.9	0.3

Source: (GUS, 2013, 2023a).

A more detailed analysis of the above issues is presented in Table 1.10. Here, the group of older people in 2021 was divided into five smaller age categories. Two states dominate in all categories: widowed and married. Other states are not significant, their share decreases with age. In the 60-64 class, they constitute 13.2%, and in the last, those over 80 years, only 6.1%. The shares of these two main states also change with age. In the first class, there is a clear advantage of married, 71.9% to 14.9%, in the penultimate class 75-79 there is almost a balance: 46.6% to 46.4%, whereas in the last, there is a clear advantage of the widowed class: 27.7% to 66.1%. This fact is due to greater mortality, especially for men of this age (GUS, 2023a).

Married men predominate in every age class. In the first four classes, there are over 75% married men, whereas only in the last one 61.9% – however the share of widows in each age class is greater than that of all citizens. Only in the first two classes are there more married women than widows, and in the last class, this ratio is 13.1% to 79.9%. The predominance of the share of widows over widowers is overwhelming in every age category, 79.9% of all women over 80 are widows, and only 33.8% of men of this age are widowed (GUS, 2023a).

The changing fertility rate and the extension of life expectancy also contribute to the increase in the number of people in post-productive age and the decrease in the number of people in productive and pre-productive age. There is also an increase in the old-age dependency ratio, i.e. the quotient of the number of people in post--working age to the number of people in productive age. The size of this proportion has a large impact on the functioning of the pension system. Table 1.11 shows the values of this ratio in the years 2002-2021 (OECD, 2023).

One can observe the stabilisation of the value of the old-age dependency ratio until 2006. After 2005, this ratio increased from 24.1 to 38.1 (OECD, 2023).

Table 1.10. The share of older people by marital status, age, and sex in 2021 (in %)

Age	Sex	Single	Married	Widowed	Divorced	Unknown
60-64	total	6.1	71.9	14.9	6.8	0.3
	male	6.6	81.7	5.4	5.9	0.4
	female	5.6	63.5	23.1	7.5	0.3
65-69	total	4.8	66.5	23.3	5.1	0.3
	male	4.9	82.4	7.9	4.4	0.4
	female	4.7	54.1	35.3	5.7	0.2
70-74	total	4.1	57.8	34.1	3.8	0.3
	male	4.2	80.5	11.9	3.2	0.3
	female	4.0	42.4	49.2	4.2	0.3
75-79	total	3.5	46.9	46.4	2.9	0.3
	male	3.2	76.2	18.0	2.3	0.3
	female	3.7	29.8	63.1	3.1	0.3
80 and more	total	3.9	27.7	66.1	1.8	0.4
	male	2.4	61.9	33.8	1.6	0.3
	female	4.6	13.1	79.9	1.9	0.5

Source: (GUS, 2023a).

Table 1.11. The old-age dependency ratio (in %)

Year	Old-age dependency ratio	Year	Old-age dependency ratio
2002	24.2	2012	27.9
2003	24.1	2013	29.0
2004	24.1	2014	30.2
2005	24.1	2015	31.4
2006	24.4	2016	32.7
2007	24.8	2017	34.0
2008	25.2	2018	35.3
2009	25.6	2019	36.6
2010	26.0	2020	37.4
2011	26.9	2021	38.1

Source: based on (OECD, 2023)

The forecast presented in Table 1.12 shows an increase in the value of the old-age dependency ratio, but it is getting weaker. In 2032, this ratio will reach the value of 44.1%, which is a disturbingly large value.

Table 1.12. The forecast of the old-age dependency ratio (in %)

Year	Old-age dependency ratio
2022	39.0
2023	39.8
2024	40.4
2025	40.9
2026	41.3
2027	41.7
2028	42.0
2029	42.5
2030	42.9
2031	43.5
2032	44.1

Source: based on (Mrugała & Tomczyk, 2023).

The above factors significantly worsen the situation of older people. They increase the proportion of older people to others and weaken the pension system. In contrast, immigration has a positive effect on this situation, as recently there has been a significant increase in the number of immigrants declared for pension insurance (Figure 1.7).

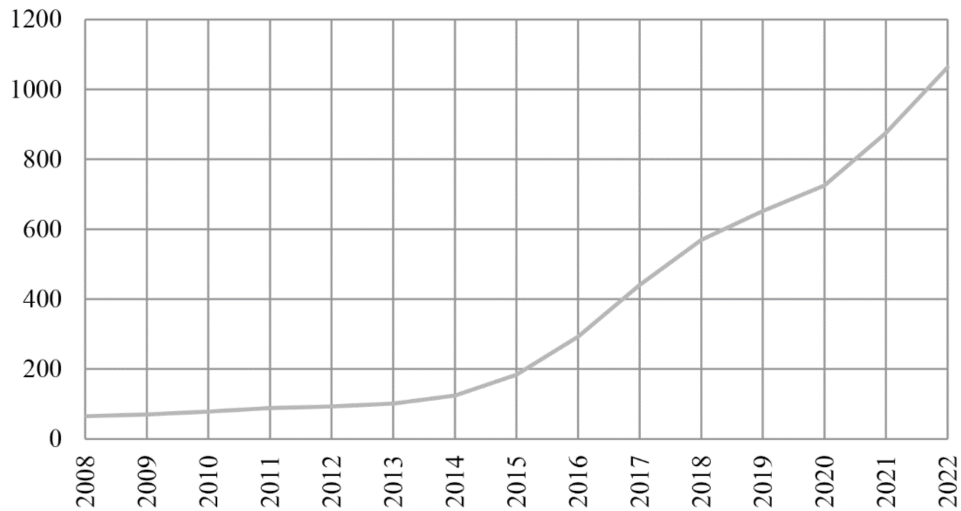


Figure 1.7. The number of immigrants declared for pension insurance (in thousands)

Source: based on (Mrugała & Tomczyk, 2023; Zakład Ubezpieczeń Społecznych [ZUS], 2023).

Note that in the period 2008-2022, there was a more than 16 times increase in the number of immigrants declared for pension insurance. Since 2013, we have seen a sharp increase in the number of these immigrants. The largest, almost 60% was in 2016, in 2018 the growth rate slightly slowed, but in 2020 there was an increase again. Among the immigrants, predominate Ukrainian citizens, 70.4% in 2022, men at 59.5% (Mrugała & Tomczyk, 2023).

Unfortunately, the share of immigrants in premium allocation for retirement and pension insurance is still small, but is constantly increasing, see Table 1.13.

Table 1.13. Share of foreigners in the premium allocation for retirement and pension insurance (in %)

Year	2017	2018	2019	2020	2021	2022
Foreigners	2.0	2.6	3.1	3.3	3.9	5.0

Source: (Mrugała et al., 2023).

Immigration is now an important factor in population growth in Europe. Natural growth among immigrants is significantly higher than among the permanent population. A similar situation may arise in Poland, especially since in Poland immigration concerns European countries, mainly Eastern Europe, Ukraine, and Belarus, which will greatly facilitate integration.

However, Poland has a worse demographic situation than most European countries. To have an increase in the population, mainly of working age, a significant increase in the number of immigrants is needed. Based on forecasts prepared in 2022 by the Ministry of Finance (see Table 1.12), two simulations were conducted on the size of the old-age dependency ratio. The first showed by how much the working-age population should increase to keep this rate at 2022 levels (39.0%), and the second indicates halving it in each year considered. The results of these simulations are presented in Tables 1.14 and 1.15 (Mrugała & Tomczyk, 2023).

Unfortunately, to achieve a steady level of the old-age dependency ratio of 36.6%, one needs a large increase in the number of foreigners of working age, especially in the initial period. By 2032, more than 2.7 million people should arrive, which would represent about 12% of all people of working age, i.e. almost one in eight people of working age would be foreigners. Unfortunately, this is impossible to achieve (Mrugała & Tomczyk, 2023).

A similar situation, perhaps a little less drastic, is the case with the second simulation. In order only to halve the annual growth of the old-age dependency ratio, more than 2 million foreigners of working age are needed by 2029, i.e. one in ten people of this age. This case is also unlikely.

In conclusion, immigrants can only slightly improve the demographic situation or the pension system, but they are not able to eliminate the disastrous effect of the ageing Polish population.

Table 1.14. Increase in the number of foreigners of working age would ensure that the old-age dependency ratio is stopped at 2022 levels

Year	Increase in the number of foreigners of working age (in thousands)	
	compared to the population from the demographic forecast	compared to the previous year
2023	422.2	422.2
2024	788.3	366.1
2025	1070.7	282.4
2026	1283.4	212.8
2027	1469.1	185.7
2028	1677.1	207.9
2029	1906.6	229.5
2030	2154.9	248.3
2031	2442.1	287.2
2032	2751.1	309.0

Source: (Mrugała & Tomczyk, 2023).

Table 1.15. Increase in the number of foreigners of working age would ensure halving the rate of increase in the old-age dependency ratio

Year	Increase in the number of foreigners of working age (in thousands)		The old-age dependency ratio (%)
	compared to the population from the demographic forecast	compared to the previous year	
2023	209.1	209.1	39.4
2024	387.2	178.1	39.7
2025	522.5	135.3	40.0
2026	623.3	100.8	40.2
2027	710.5	87.2	40.3
2028	807.2	96.7	40.5
2029	912.8	105.6	40.7
2030	1025.8	113.0	41.0
2031	1154.7	129.0	41.3
2032	1291.5	136.7	41.6

Source: (Mrugała & Tomczyk, 2023).

1.2. Silver generation

Recently, the group of older people is referred to as 'silver generation', due to their hair colour. The concept of silver generation is closely related to the branch of the economy now called the silver economy, aimed at meeting the needs of older people. However, it does not constitute a single sector of the economy – there is a collection of different products and services from different branches of the economy, and combines various sectors such as (Zsarnoczky, 2016):

- media,
- fashion, design,
- gerontology, health services, long-term care,
- real estate, smart homes, architecture,
- education system,
- tourism, medical tourism,
- nursing home, assisted living,
- finance, insurance,
- cosmetics,
- mobility, public transport,
- culture, recreation fitness,
- IT, innovative technology, robotics, telecommunication,
- energy,
- home delivery,
- local markets (e.g. food), local services.

Thus, the silver generation is a consumer group associated with the silver economy, which provides products and services that are friendly to older people and adapted to their state of health, physical condition and cognitive abilities. It facilitates work and learning for older people, enabling better social inclusion and preventative healthcare. The silver economy also implements new technologies and artificial intelligence, e.g. health monitoring, care robots, intelligent houses, as autonomous vehicles. As mentioned in Section 1.1, the silver economy helps to a large extent to revitalise the economy, reduces the cost of the ageing process and provides more jobs. It creates additional sectors of the economy aimed at meeting the needs of older people concerning health, food, cosmetics, clothing, tourism, recreation, and IT gadgets. Moreover, seniors turned out to be the only consumer group with stable and sufficient purchasing power during the financial crisis of the early 21st century (Eatock, 2015; Van Der Gaag et al., 2015).

Three lifestyles of the elderly can be distinguished. The first category of seniors is characterised by high independence, and live in independent apartments, while the second also live in apartments, but they are largely dependent on help, mainly from their families. The last group of seniors live in old-age housing provided by different types of health services. These groups require diverse services, and they are associated with different sectors of the silver economy market. They also require specific local and regional solutions, as well as government support.

It was assumed that in the context of the silver economy, the term 'silver generation' refers to the age group of potential consumers aged 50+, whereas seniors are understood as people over 65 years of age. This group of older people can be divided into the following three groups (Palenik, 2012):

50-64 years – young-old,
65-79 years – old-old (middle old),
more than 80 years – oldest-old (very old).

A slightly different division was presented (Zsarnoczky, 2016). It divided seniors into five classes taking into account marketing aspects, their preferences and purchasing power:

51-64 years – mature,
65-74 years – young old,
75-84 years – middle-old,
85-94 years – old-old,
more than 95 years – very old.

It should be noted that this classification may vary from country to country, reflecting differences in social classes and the ability to work.

In addition to age, other factors that differentiate the group of seniors can be distinguished, such as sex, acquired skills, life experience, cultural background, and, of course, health status (Zsarnoczky, 2016). The author pointed out that there is a huge segment of the market with untapped opportunities. In this segment, called the silver economy, the right to well-being is just as important as in other segments.

One can also encounter the term 'silver market'. It was used earlier, originating in the 1970s in Japan, where the earliest phenomenon of an ageing population occurred. It is a concept often used interchangeably with the silver economy, but in a slightly narrower scope. By contrast, the term 'silver economy' itself was created by researchers at Oxford University to articulate the economic activity of people over the age of 50.

Issues related to the silver economy and the silver generation have been discussed at numerous conferences and committees organized by the European Union. In the 21st century, four such silver economy conferences were already held. Previously, however, silver economy issues were raised in 2001 at the Stockholm European Council and the Second World Assembly on Ageing. This second conference took place in Madrid in 2002 and was organized by the UN. The Political Declaration and Madrid International Plan of Action on Ageing were published, highlighting advancing health and well-being into old age and ensuring supportive environments (Eatock, 2015; United Nations, 2002).

The first conference connected with the silver economy was organized in 2005 by the Government of North Rhine-Westphalia in Bonn. This conference highlighted the importance of the silver economy, anchoring it in the economic strategy at national and EU levels. It was proposed to include it in the Lisbon Strategy and 7th framework programme of the European Commission. The Silver Economy Network of European Regions (SEN@ER) was founded and the Bonn Declaration for Silver Economy was issued. SEN@ER was set up at the request of ten European regions (North Rhine--Westphalia, Andalusia, Burgenland, Extremadura, Gelderland, Limburg, Middle East Ireland, Middle Region of Ireland, Scotland and the Central West Counties of England) (SEN@ER, 2005).

The Bonn Declaration states that "the older people expect new and innovative products and services for greater quality of life in their old age (Silver Economy), while an appropriate innovative drive results in growth and new jobs and in a global context increases Europe's competitiveness and that of the companies operating here."

In addition, an agreement was reached on Europe-wide cooperation, which focused on the following issues (SEN@ER, 2005):

- "Create a general greater awareness of the opportunities of an ageing society.
- Support regional companies and service providers in the Silver Economy.
- Initiate and stimulate an interchange of know-how on the creation and development of regional networks for the silver economy.
- Create a Europe-wide pool of knowledge on the attendant economic, social, and political topics.
- Support network members in the regional development initiatives and global marketing in relation to silver economy products and services.
- Promote political initiatives at the European level and draw up a program to create a worldwide leading European silver economy.
- Provide mutual support at all political levels: regional, national and European."

The second European Conference on the Silver Economy took place under the auspices of the province of Limburg in Kerkrad (Netherlands) in 2006, and touched on five main themes related to the silver economy and silver generation, namely:

- health and lifestyle,
- housing and the home environment,
- work and income,
- tourism and culture,
- integration and communication.

The conference highlighted that an ageing population could have a major impact on the country's economy as it can bring a vast range of new products and services to emerging and existing businesses. One should take advantage of this situation as well as take risks by developing those sectors of the economy that will be targeted at the silver generation in the future (Palenik, 2012).

The Third European Conference on the Silver Economy focused mainly on the following topics:

- new technologies versus independent living,
- tourism and culture,

- financial services,
- catering.

This conference took place in Sevilla in 2007, and presented the results of the 6th SOPRANO Framework Programme focused on services for intelligent environments for older people, healthy ageing and independent living of this group of people. The aim of this project was to provide greater consistency and safety for older people in their daily lives.

In 2010, the Fourth European Conference on the Silver Economy was held in Limoges, France. A round table was held there, where issues related to the regional silver economy and strategy, and problems related to the ageing of the population were discussed. It was pointed out that an ageing population, global warming and globalisation are the most important challenges that EU society will face in the next 20-50 years (Palenik, 2012).

In the following years, a number of conferences and symposiums were held on issues related to the silver generation and silver economy.

1.3. The need for financial security for silver generation

Financial security is a very broad concept. In macroeconomic terms, it means the stability of the market in order to ensure long-term economic growth. In the aspects of microeconomics, it determines the safety of individuals, enterprises and households. In the case of social science, financial security is considered from the viewpoint of individuals or families, and is defined as the ability to meet basic needs in the economic dimension. The state of poverty results from a situation of its absence.

In modern economies, the state tries to ensure the financial security of citizens by supporting, managing and financing health care and pension systems. Ensuring the stable financing of both systems involves many risks, therefore the state uses incentives to trigger activities related to the participation of citizens in forming their personal financial security. This is particularly important for the elderly, who are at a greater risk of poverty, as well as the loss of health and disability risks.

1.3.1. Co-payment for medical services

Contemporary societies in many countries are affected by the problems of an ageing society and civilization (lifestyle) diseases associated with older age. Even the richest countries have problems with financing health care due to the growing demand for health services and deficit in health care expenditure. Additionally, in many countries the problem with financing of the pension system is observed. In Poland, this phenomenon is caused by unfavourable changes in the population structure in terms of age, due to the large percentage of people in non-working age, including retirement age, in relation to the percentage of people in working age. This shows that retirees should take measures to ensure financial security by providing additional financial support, as resources from public aid systems may turn out to be insufficient. These activities should be undertaken in advance in order to ensure a decent future and a comfortable standard of living after the end of one's professional activity.

Healthcare is a socially sensitive area of great importance for the economy. Financing health care has become increasingly difficult due to limited resources compared to the demand for medical services. The growth in demand for medical services is observed all over the world and is determined by at least four factors: demographic, cultural, income and technological. The demographic factor is associated with the phenomenon of the ageing of society, which causes an increase of the incidence of various diseases and of the demand for geriatric care. The cultural factor determines the eating habits and lifestyle which affect the spread of civilization diseases. The income factor is related to the increase in society's wealth. All over the world, the phenomenon of increased health care expenditure caused by raised income is observed. In addition, these expenses are rising faster than income, which indicates that health is perceived as a luxury good. The technological factor is related to the development of modern medical technologies and knowledge which makes treatment more expensive (Rudawska, 2007).

The growth of demand for medical services and the development of medical technology causes an increase of expenditure on health in all countries. Outlays on health care from public funds per capita for selected countries, including Poland, are presented in Figure 1.8. The increase in healthcare spending is visible in all countries. Government budgetary resources are limited and are proving insufficient to meet the growing demand.

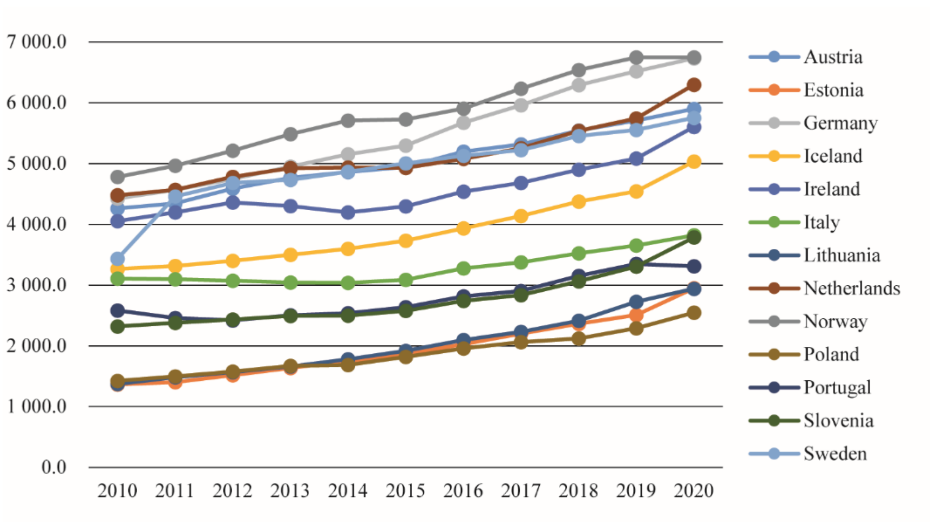


Figure 1.8. The total health care expenditure *per capita*

Source: (OECD, 2021).

Thus, the rationing of resources is common practice, which adversely affects patients' health. The tools to deal with the deficit should be considered within the framework of social and health politics, and three ways to deal with deficit were suggested (Jezierska, 2016; Łanda, 2012):

1) the increase of a basic health premium size in the public system,
2) removing many medical technologies from the basket of guaranteed benefits,
3) the growth of patients' contribution to the cost of healthcare by introducing a co-payment system.

Raising the health insurance premium and limiting the range of benefits would require systemic changes and could limit access to healthcare. The third element, i.e. patient co-payment, is commonly used in many countries.

The co-payment system can be an effective instrument for balancing and improving the healthcare system in Poland. Firstly, it could be a way of dealing with the excessive consumption of healthcare services. The consumption on the medical services market is greater than an insurer assumes, where in case of public healthcare an insurer is identified with a state or a public payer. The phenomenon of excessive consumption is a result of information asymmetry, in which patients have more knowledge about their health than the insurer (the state or the public payer), and a patient has a possibility to produce a moral hazard. Economic studies have proven that excessive consumption will always occur if three conditions are met. The first condition concerns information asymmetry present on the health services market, the second one means that a patient decides alone about the choice of treatment methods, and the third one that the state as the public insurer covers total treatment cost. The effect of the above-described phenomena is an inefficient allocation of resources, which are already limited (Rudawska, 2007). The insurer cannot influence and control the first two factors. It has the ability to reduce excessive consumption only by limiting finance of medical services by introducing a co-payment as a cost-sharing concept into the healthcare system.

The scope and financial framework for the participation of a beneficiary of a medical service in the costs incurred by the payer must be determined on the basis of an agreement between the patient and the provider of the medical service. Two types of cost sharing mechanisms are singled out in theoretical considerations and practice. The first one is related to various forms of co-payment by a patient, the second one to the introduction of the possibility of taking out additional private insurance. Each type of mechanism has some advantages, but also significant limitations.

Co-payment is a mechanism consisting in a necessity of covering some costs incurred as a result of a medical service by a patient, regardless of the beneficiary's participation in the public health insurance system. Three main forms of co-payment are distinguished in the health service, namely co-insurance, co-payment and the deductible. Co-insurance consists in covering a fixed part of a medical service cost in contrast to co-payment, which relies on incurring a fixed payment for medical service. The last form, the deductible, occurs if a patient covers the entire cost to a fixed limit beyond which the service becomes free. The primary disadvantage of co-payment is the ability to limit not only the excessive but also necessary (needed) consumption of medical services, which can have negative social effects. From a theoretical point of view, it is assumed that increasing a co-payment rate reduces demand for medical services but, in practice, it is manifested by shortening of queues for the services. The demand for medical services is most effectively limited by the deductible.

The second type of mechanism concerns the private health insurance market (Jezierska, 2016; Łanda, 2012), where three types of insurance can be distinguished, namely supplementary, complementary and substitute insurances. Substitutive insurance is recommended to people not covered by public insurance, therefore it has a limited impact on public insurance sector financing. Supplementary insurance poses an extra aspect to public insurance because it covers benefits that are guaranteed in the public system but allows the insured to choose a broader range of providers of healthcare and faster access than in the public system. Supplementary insurance provides higher quality services for patients who are able to pay for them. Complementary insurance includes benefits which are not reimbursed in the public system. This kind of insurance can be used to cover non-refunded or only partially reimbursed by the public payer benefits.

The experience of the countries that have introduced the co-financing rules shows that the applied mechanisms are not able to provide full cost control. The occurrence of certain phenomena limits the effectiveness of the impact of co-payment mechanisms on excessive consumption on the health services market. The first phenomenon occurs when patients use the substitutability of different medical services. For example, instead of using paid specialist services, patients decide to go to an emergency ward that is free for patients, thereby avoid paying. Additionally, the cost-sharing mechanism includes a wide range of protective instruments that prevent adverse effects of co-payment by safeguarding the interests of weaker groups of patients (e.g. children or the elderly). Furthermore, the introduction of a cost-sharing mechanism generates additional management and control costs. Thus, the income from this mechanism is reduced. In Poland, the co-payment occurs in the case of dental treatment, the purchase of medicines, and some surgeries e.g. cataract treatment. Despite numerous disadvantages, the co-payment system can bring many positive effects for both the healthcare system and patients. From the point of view of the public payer, it enables obtaining additional funds for the health system and the more optimal use of public funds although it concerns narrow areas. Furthermore, it limits excessive consumption of health services (reduction of the moral hazard phenomenon) in some aspects. From the patient’s point of view, co-payment can protect against the need for out-of-pocket expenses, which patients actually pay even when the performed medical procedures are refunded. In addition, it reduces queues for guaranteed health benefits, increases the provider’s choice, and brings a sense of security for patients and their families (Rudawska, 2007; Suchecka, 2010).

The share of private expenditure in financing health care is common in most countries. Two main sources of financing for health care can be distinguished, namely government and private expenditure, the latter being an important element of the financing health care system. The structure of financing according to its source for selected European countries is presented in Figure 1.9. In analysing the structure of expenditure according to the source of financing, it can be noted that a significant part of funds comes from non-governmental sources. In some countries they range from a dozen or more, to over 40%, whilst in Poland it is approximately 30% (OECD, 2021).

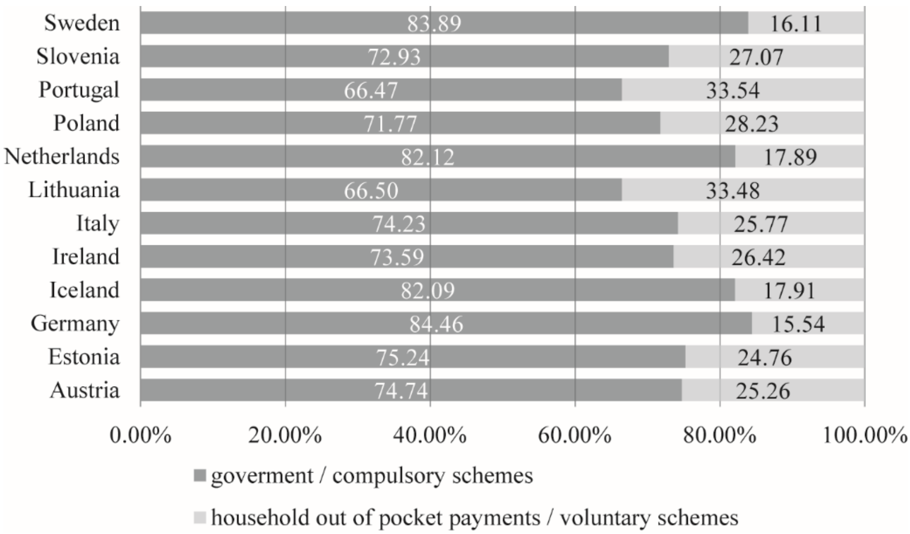


Figure 1.9. The structure of financing in 2018

Source: (OECD, 2021).

A detailed specification of health care expenditure in Poland in 2018 is presented in Table 1.16; out-of-pocket payments constitute the largest share of private spending.

Table 1.16. Specification of expenditures on healthcare in Poland in 2018

Specification	**The amount in billion PLN**	**% GDP**
1	*2*	*3*
Total Current Health Expenditures	*134 244*	*6.33*
Government schemes and compulsory contributory healthcare financing schemes	*95 977.10*	*4.53*
of which:		
government schemes	13 381.76	0.63
central government schemes	7 891.05	0.37

Table 1.16, cont.

1	*2*	*3*
state/regional/local government schemes	5 490.71	0.26
compulsory contributory health insurance schemes	82 595.30	3.90
Voluntary health care payment schemes	*38 267.34*	*1.80*
of which:		
household out-of-pocket payment	27 413.15	1.29
other private healthcare expenditure	10 854.20	0.51

Source: (Statistics Poland, 2020).

The financing structure, taking into account the type of medical service, is presented in Table 1.17. The possibility of using several sources simultaneously, means that the value of 100% can be exceeded.

Table 1.17. The structure of healthcare services according to the financing source (in %)

Financed:	**Stationary care (hospitals)**	**Primary healthcare (family doctor)**	**Outpatient specialist care**	**Dentistry**	**Laboratory tests**
by National Health Fund	96.50	97	63.90	26.80	79.60
by out of pocket	*2.30*	*2.20*	*38.30*	*74*	*9.90*
by additional health fund (private insurances)	*1.20*	*2.90*	*5.50*	*3*	*3*
under occupational medicine					7.50

Source: (GUS, 2018).

The largest share of out-of-pocket expenses is in the case of dental care, next the outpatient specialist care and laboratory testing. The share of private insurance in financing healthcare in Poland is not large. On the one hand, taking into account the fact that this type of financing accounts for almost 30% of total healthcare expenditure, patients in Poland contribute significantly to their healthcare, and on the other, funding from private health insurance is minimal. The share of this type of insurance should be increased as it enables the costs to be shared between a patient and an insurer.

This book focuses on contracts with the possibility of using private health insurance to ensure financial protection. The funds obtained from insurance will be used to improve the quality of treatment for severe or chronic diseases. Additional funds can also be used to pay for long-term care.

1.3.2. Additional pension security

Pension systems in different countries rely on providing some type of basic benefits. Many types of basic benefits can be distinguished such as a minimum retirement pension, a citizens' pension (in the same amount for all insured persons), and an income-related pension (Golinowska, 1994; Rutecka, 2014). The difference between the desired level of coverage of needs and the level offered by the base system is known as the pension gap.

In Poland the public pension system relies on benefits connected with income. Additionally, for historical reasons it has a mixed form, i.e. repartitions-capital. The reparative system (redistributive) means that the employee's contributions go to a common pool, from which they are paid out for current benefits for entitled groups. Upon retirement, benefits are paid from the contributions of those who pay them at that time. After the death of the insured, their payment is withheld, and the sum of contribution paid by the employee continues to circulate in the system. The funded system means that the employee pays contributions throughout the entire period of his/her professional activity, which are deposited in interest-bearing bank accounts. After retirement, the employee may dispose of the entire capital with interest, or only the monthly benefits derived from interest on capital, paid for life. After death, the capital may be inherited by the employee's family.

The Polish pension system consists of three pillars. The basis of the system is the first pillar of the National Defined Contribution. Contributions from this pillar go to the account of the Social Insurance Institution (in Polish: Zakład Ubezpieczeń Społecznych) and are used for the current payment of benefits for all retirees.

The system is replenished with two more pillars of a capital nature. The second pillar comprises the accounts at Voluntary Open Pension Funds (OFE), the third pillar is voluntary individual accounts: Occupational Pension Programmes (PPE), Individual Pension Accounts (IKE/KZE) and Employee Capital Plans (PPK). Reforms are planned to reduce the system to two pillars, through the liquidation of OFE.

The repartition dimension of the pension system is very susceptible to population ageing and the age structure changes in society. The decreasing number of people of working age in relation to the number of pensioners makes the system ineffective and requires an increase in the capital element of the system.

The pension gap can be defined by the gross replacement rate, which is a commonly used measure of capital adequacy of the pension system. The gross replacement rate is calculated as the ratio of the value of the first gross pension to the last gross salary and is presented in percentage form. The difference between the expected (target) and actual replacement rates is defined as the pension gap. Theoretical analyses and empirical studies indicate that for the majority of retirees, the optimal target replacement rate is in the range of 60-80%. The results of the analyses are described in (Jedynak, 2017).

In Poland, the increase in pensions has been observable for many years. However, the increase in salaries showed greater dynamics, which means that the replacement rate is systematically decreasing. Data about the values of average salary in the enterprise sector, the average amount of the newly appointed pension and the replacement rate in Poland are presented in Table 1.18. The declining trend of the replacement rate has been visible for years; the average rate level in 2014-2020 decreased by over 10% (from 52.5 to 42.4).

Table 1.18. Replacement rate in Poland in 2014-2020

Year	Average amount of newly appointed pension	Average salary in the enterprise sector	The replacement rate
2014	2174.63	4139.42	52.5
2015	2154.62	4280.80	50.3
2016	2167.11	4404.17	49.2
2017	2156.90	4739.91	45.5
2018	2185.87	5071.41	43.1
2019	2338.38	5368.01	43.6
2020	2400.00	5656.51	42.4

Source: based on data from (GUS, 2021a; ZUS, 2024).

In the case of the repatriation system, the amount of the replacement rate depends on the structure of the population, and more precisely, on the burden coefficient of the working population by the non-working one. Changes of the values of indicators from 1980 to 2020 and the forecasts until 2050 are shown in Table 1.19. The forecasts show that a system based largely on contributions deposited in the Social Insurance Institution (ZUS) accounts may be insolvent.

Table 1.19. Population at non-working age per 100

Year	Total	Pre-working age	Post-working age
1980	69	49	*20*
1990	72	50	*22*
2000	64	40	*24*
2010	55	29	*26*
2019	67	30	*37*
2020	68	31	*38*
2030 (forecast)	73	30	*43*
2040 (forecast)	83	27	*49*
2050 (forecast)	105	30	*75*

Source: (GUS, 2014, 2021b).

The role of the capital part of the pension system will grow in the coming years. The introduction of incentives for individual enhancement of financial security by pensioners should be an important part of social policy.

1.3.3. Financial security of pensioners

According to demographic data, on average one in five people in the world may live to their nineties. Population ageing is becoming a global problem. The authorities in many countries face enormous challenges in terms of increasing costs associated with maintaining pension and health care systems. Opinions about life in retirement are also changing – many people expect the possibility of active participation in social life when retired. Increasingly, attention is also paid to good health in old age. Thus, healthy old age and long-term financial security are increasingly being promoted. A successful retirement pension is the result of long-term cooperation between the person, the employer and the government. The participation of future pensioners is extremely important (AEGON, 2017).

The need for an independent assurance of financial security seems indispensable to guarantee a decent life in retirement. Various forms of additional pension security are proposed.

Investing in the family, and especially in their offspring, may be the only way to protect themselves for some people. Despite the controversy surrounding the definition of children as an investment, support from adult children can be a way to improve their quality of life (Kwiatkowska, 2016).

Another form of additional security is saving either on one's own or through the pension system. In Poland, this is possible thanks to the third pillar of the system or participation in Employee Capital Plans (PPK) (Rutecka, 2014).

The forms of retirement security could be also based on the real estate ownership. Under this form of collateral, it is possible to sell or rent real estate or use equity release contracts (there are two kinds of contracts – reverse mortgage or reverse annuity) (Marciniuk, 2017a). More details connected with these contracts are described in Section 2.2. In Poland, over 84% of the population lived in owner-occupied properties in 2019 (the EU average is 69.8%). Only 12.2% of Poles lived in real estate charged with a mortgage or a loan. Only 15.8% of Poles lived in rented properties, while the EU average is as much as 30.2%. Therefore, owning real estate is a very convenient form of financial security in Poland (Eurostat, 2019).

In this book, the authors focus on contracts based on equity release. Due to the fact that a large increase in spending on health services in old age has been observed, contracts combining equity release with private health insurance are being proposed. The financial resources necessary to pay the health insurance premium come from the mortgage annuity. Such solutions can be addressed to seniors who have not implemented a pre-retirement saving plan and do not have to leave the property as inheritance (Dębicka & Zmyślona, 2019; Zmyślona & Marciniuk, 2020).

1.4. Private financial security

The population is ageing, the proportion of people of post-productive age is increasing and the level of pensions relative to salary is decreasing year on year. People who have not yet entered retirement age are becoming increasingly aware of the need to save for their future retirement.

The financial market offers more and more options for saving: bank deposits, Treasury bonds and shares. The money you have can also be invested in buying rental property or gold bullion. However, nowadays, with very high inflation, low returns on bank deposits, at the same time as high mortgage rates and ever-increasing property prices, these options seem to be severely limited.

Current retirees often have life and endowment insurance which was widely offered at the end of the 1990s. Insurance was, and still is, offered as a form of security for mortgages, hence in 2022 almost 24 million Poles had a life insurance policy in place, more than half of which was taken out under group insurance (Machniewski, 2023).

The term of life and endowment insurance often coincides with one's retirement, or a little earlier. In this case, the benefit may be received in the form of a monthly annuity, which may be in addition to the pension. Alternatively, the benefit may be received as a lump sum and then invested, for example, in the purchase of a marital reversionary annuity, the benefit of which continues to be paid even after the death of one of the spouses, depending on the option chosen at a fixed or lower amount (Denuit et al., 2001; Marciniuk, 2017b).

According to research conducted (Biernacki et al., 2021), more than 80% of Poles own real estate. They can sell such property, rent it, or give it to a company that is interested in such an acquisition, in exchange for a monthly benefit. The latter form is called an equity release contract.

Equity release contracts have existed for a very long time and come in various forms (Shao et al., 2015). Californian banker Nelson Haynes invented the reverse mortgage back in 1961. The most developed and largest market for this type of contract is in the United States of America, due to its very well-functioning legal system. These contracts also exist in many Western European countries. There is a well-known example of a French woman, Jeanne Calmet, who was offered an equity release contract by her lawyer when she was 90 years old. The woman died at the age of 122, having outlived her lawyer. His heirs had to continue to pay her an annuity, and the total benefits paid out doubled the value of the property (Marciniuk et al., 2020).

The largest European market for equity release contracts is in the United Kingdom, where these contracts were introduced over 25 years ago (Charupat et al., 2016). The agreements in both the sale and loan model are mainly sold in Spain, Ireland, France, Germany. In France, Germany, and Italy, they can also be concluded between natural persons (Shao et al., 2015).

Figure 1.10 summarises the pensioner's options for obtaining additional benefits.

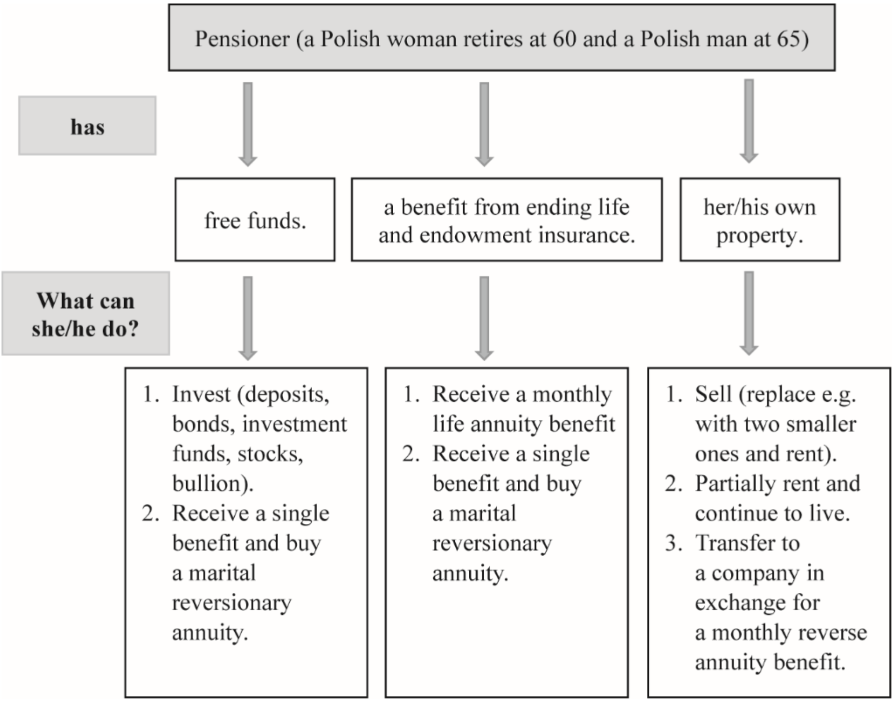


Figure 1.10. The possibility of obtaining additional financial resources

Source: own elaboration.

Health insurance, or critical illness insurance, is widely known. Considering the health situation of the elderly, they can be offered to take out an insurance policy against critical illnesses, which are also extensively discussed in this book. However, the financial situation of pensioners may not allow them to purchase such insurance. Therefore, this monograph proposes a combination of dread disease insurance and a reverse annuity contract to finance the premium that would have to be paid for the above-mentioned contract.

Chapter 2

INSURANCE PRICING

2.1. Overview

In this book, a contract understood as a written agreement between the parties to establish the mutual rights and obligations that result in cash flows, and it is assumed that the contract is between an individual (one person or a married couple) and an institution (insurance or financial).

Consider a contract issued at time 0 (defined as the time of issue of the contract) and, according to the plan, terminating at a later time n (n is the contract term); t ($t \geq 0$) denotes the time that has elapsed since the beginning of the contract. Due to the application in practice, the authors mainly focus on the discrete-time model. This means that the analysis concerns contracts whose duration has been divided into separate time periods, e.g. days, months, or years. If the term of the contract is divided into years, then for the k-th year of the contract, the cash flows paid in advance are realised at time $k-1$ (at the beginning of year k), and the cash flows paid from below are realised at time k of the contract (at the end of year k); compare Figure 2.1.

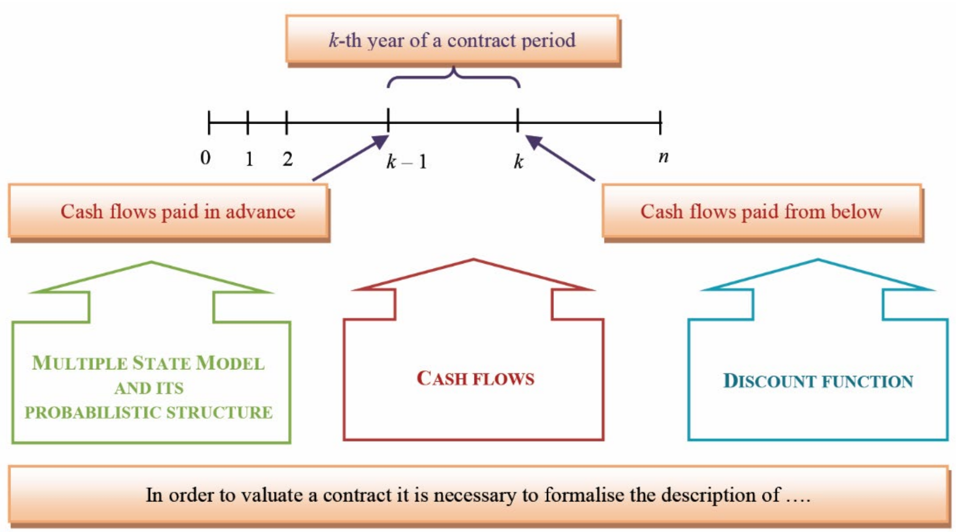


Figure 2.1. Scheme of the discrete-time contract and its valuation components

Source: own elaboration.

In order to valuate a contract, it is necessary to formalise the three related areas (see Figure 2.1). If the performance of the contract depends on random events (e.g. if the subject of the contract is the life or health of a person, such as in a life insurance contract or health insurance), it is necessary to describe all possible individual risk events as far as its evolution is concerned until the end of the contract. For this purpose, the so-called multistate model (Section 2.2) and its probabilistic structure (Section 2.3) are defined. Then, for each random event, a specific type of cash flow is assigned (Section 2.4). As the cash flows are implemented throughout the contract period, to determine their value at the contract's inception, it is necessary to decide on the discount rate (Section 2.5). Detailed principles of contract valuation are presented in Section 2.6, and a matrix representation of actuarial values is in Section 2.7.

2.2. Multistate model

Multistate models (MSM) are the most classical approach to model longitudinal, time-to-event data that result from a stochastic process. They describe stochastic processes that progress through various states associated with competing risks and recurring events. In other words, life history is described as a subject moving from state to state and multistate models are applied to describe the events (i.e. the transitions between states) for an individual (less often for a group of persons), which only occupies one of a few possible states at any time. The analysis usually focuses on modelling the total waiting time between two states of interest or determining possible paths and their probability distribution over a given period for individuals.

Multistate models have a wide application for describing different kinds of problems, among others, in demography (survival of individuals or married couples and families), medicine (time to the outbreak of disease, complications and death and recurrent episodes of diseases), economics, finance and insurance (analysing cash flows arising from different kind of contracts, venture and systems or economic events dynamics). In particular, in recent years, such issues as the projection of elderly disability (Van Der Gaag et al., 2015), kinship models (Caswell, 2020), the process of return and re-entry migration (Vega & Brazil, 2015) and labour-migration dynamics (Bijwaard, 2014), unemployment dynamics (Dębicka & Mazurek, 2001; Spierdijk & Koning, 2011) are being considered. From the point of view of this monograph, the most important are the applications of multistate models in medicine, finance and insurance.

Multiple-state modelling has been used primarily in medical research because it is a flexible tool for analysing complex disease processes. The states may be defined for the stage of the disease, incidents of clinical symptoms, occurring complications or different causes of death. The states and the possible transitions between these states fully characterise the disease process. Fix and Neyman (1951) describe one of the earliest tools used in medical statistics to model patients' recovery, relapse and death.

The most commonly applied multistate model in biostatistics is the active-illness-death model, e.g. (Asanjarani et al., 2022; Bijwaard, 2014; Eulenburg et al., 2015; Pitacco, 2014; Putter et al., 2007). There is a vast literature devoted to developing this model in the context of the analysis of specific diseases, e.g. colorectal cancer (Alafchi et al., 2021), lung cancer (Dębicka & Zmyślona, 2018, 2019), breast cancer (Meier-Hirmer & Schumacher, 2013), diabetes (Andersen, 1988; Ramezankhani et al., 2018), bone marrow transplantation (Andersen & Pohar Perme, 2008; Klein & Shu, 2002), AIDS (Sweeting et al., 2005; Tapak et al., 2018), illness including risk factor obesity (Van Der Gaag et al., 2015) and in recent years the COVID-19 pandemic (Rieg et al., 2020; Ursino et al., 2021).

In the realm of insurance and finance, the inception of multistate models occurred with Hoem's work in 1969 (Hoem, 1969), with their initial applications to actuarial issues appearing in 1972 (Hoem, 1972). Subsequently, in 2014, Pitacco demonstrated the utility of multistate models (MSM) in actuarial modelling for health insurance policies (Pitacco, 2014), whilst the monograph by Haberman and Pitacco (1999) offered a comprehensive review of the utilisation of MSM in disability insurance and various forms of care insurance. The general methodology of modern life insurance mathematics in the framework of a MSM can be found among others in (Dickson et al., 2019; Norberg, 2008; Spierdijk & Koning, 2011). Recently, MSM was also used for analysis: mortgage and reverse loan contracts (Dębicka et al., 2015, 2020; Dębicka & Marciniuk, 2014; Marciniuk, 2014, 2017a, 2021; Marciniuk et al., 2020; Zmyślona & Marciniuk, 2020), life insurances contracts with accelerated death benefits (Dębicka & Zmyślona, 2018, 2019), contracts in the secondary insurance market (Dębicka & Heilpern, 2018, 2020; Heilpern & Dębicka, 2020) and other areas.

Many articles (Andersen & Keiding, 2002; Hoem et al., 1976; Hougaard, 1999; Putter et al., 2007) and monographs (Cook & Lawless, 2018; Haberman & Pitacco, 2018; Hougaard, 2000; Huzurbazar, 2019) have the character of a review and provide a lot of detail on the application and use of multistate models. These works show the potential of using this type of modelling and are a significant source of knowledge on this subject.

Suppose a contract is concluded between two parties: an individual and an institution (insurance or financial), and it is modelled by the multistate model. In formal terms, MSM is a pair of two sets, the first one is set space S. At any time, the individual risks are in one of a finite number of states labelled 1, 2, …, N. The second denotes a set of direct transitions between states of state space T. The pair (i, j) denotes a direct transition from state i to state j ($i \neq j; i, j \in S$). The pair (S, T) describes all possible individual risk events as far as its evolution is concerned (until the end of the contract).

The simplest multistate model consists of two states and one direct transition between them:

$$(S, T) = (\{1,2\}, \{(1,2)\}). \tag{2.1}$$

This MSM models life insurance contracts, life annuities, or selected equity release contracts (cf. Section 3.3). In this model, state 1 means the person is alive, and state 2 means the person is dead. The pair (1, 2) means the insured person's death (transition from alive to dead). Each multistate model can be presented graphically as a graph, with nodes representing states and arrows representing direct transitions between them. The simplest model is presented graphically in Figure 2.2.

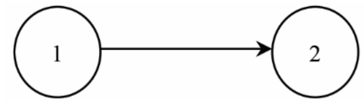


Figure 2.2. The simplest multistate model

Source: e.g. (Haberman & Pitacco, 1999).

The model (2.1) is insufficient in the framework of study and analysing the detailed life history data which frequently occur in practice, for example, to model more complex contracts taking into account the different events in a person's life (such as illness, disability, hospital stay, unemployment) and death from various causes (due to a specific disease, accident, complications after illness). An example of this type of contract is multistate insurance, i.e. insurance covering various life cases. This type of insurance consists of a basic (usually a life insurance contract) and supplemental contracts such as insurance against the risk invalidity or accident risk. One way to construct a multistate model for this type of contract is the appropriate division of states of the model 2.1. Example 2.1 presents the construction of the multistate model for the selected multistate insurance contract.

Example 2.1. *Multistate insurance contract*

Consider an endowment insurance contract to be a basic insurance contract. According to this type of contract, an endowed person obtains a benefit if the insured person's death occurred before the end of the insurance contract. Moreover, an insured person receives the pure endowment benefit payable at the end of the contract if he/she is still alive. Let us assume that the insured has purchased the following supplementary insurance contracts to the basic contract:

- *Unfortunate Accident Insurance* (UA), for example the agreement consists of the fact that if the death occurs by unfortunate accident, then the insurer pays the double death benefit,
- *Insurance against the Risk of Job Loss* (unemployment) (RJL), for example a contract where the insurer takes over the payment of premiums when the insured lost his/her job for reasons beyond his/her control (e.g. as a result of restructuring or bankruptcy of the employer),

- *Permanent Disability Insurance* (PD), for example the agreement is based on the fact that if the insured suffered permanent health impairment due to an accident, then the insurer pays a lump sum benefit and disability annuity.

The construction of the multistate model for an endowment insurance contract with three supplementary contracts (UA, RJL, PD) has four stages.

STAGE 1. *Endowment Insurance*

The model of the basic insurance contract, i.e. endowment insurance, is given in formula (2.1) and is graphically presented in Figure 2.2.

STAGE 2. *Endowment Insurance* + UA

State 2 (dead) in the model (2.1) was divided into two states 2a – natural death of the insured, and state 2b – death of the insured due to an accident. Increasing the number of states also increases the number of direct transitions between them. The resulting multistate model is illustrated in Figure 2.3A. After renumbering the states (Figure 2.3B), a three-state multistate model has the following form:

$$(S, \mathrm{T}) = (\{1,2,3\}, \{(1,2), (1,3)\}). \tag{2.2}$$

The states and direct transitions between them, added to the model (2.1), are marked in green in Figure 2.3.

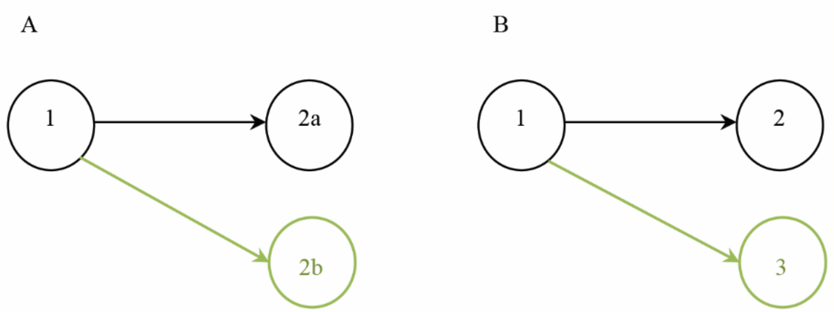


Figure 2.3. The multistate model for endowment insurance + UA

Source: own elaboration.

STAGE 3. *Endowment Insurance* + UA + RJL

State 1 in the model (2.2) has been divided into 1a – the insured is alive and working, and 1b – the insured is alive and not working. The model is illustrated in Figure 2.4A. After renumbering the states (Figure 2.4B), a four-state multistate model of the form is created:

$$(S, T) = (\{1,2,3,4\}, \{(1,2), (1,3), (1,4), (4,1), (4,2), (4,3)\}). \tag{2.3}$$

Additional states and related direct transitions are marked in blue in Figure 2.4.

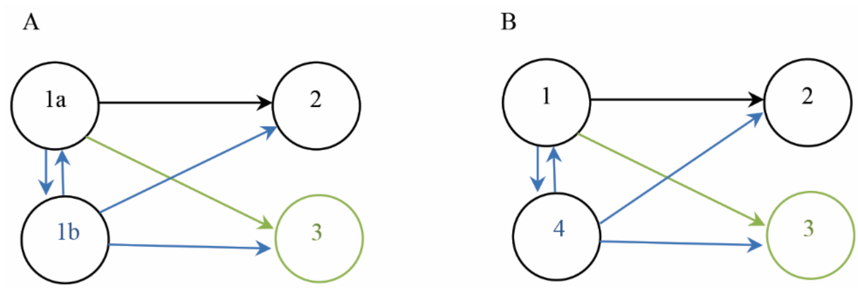


Figure 2.4. The multistate model for endowment insurance + UA + RJL

Source: own elaboration.

STAGE 4. *Endowment Insurance* + UA + RJL + PD

To include supplementary PD insurance in the model, states 1 and 4 in the model (2.3) were divided into two states (compare the scheme in Figure 2.5A):

1a – the insured person lives, works and is healthy,
1b – the insured person lives, works and is disabled,
4a – the insured person is alive and not working and is healthy,
4b – the insured person lives, does not work, and is disabled.

Note that becoming disabled typically significantly limits or eliminates the possibility of continuing work for a long time, therefore, in realistic models, state 1b is usually eliminated. For simplification, it is assumed that when the insured becomes disabled, they cannot fully perform their work due to reasons beyond their control. With this assumption, the new model is illustrated in Figure 2.5B. After renumbering the states (Figure 2.5C), the following five-state and ten-direct transition MSM is obtained:

$$(S,T) = (\{1,2,3,4,5\}, \{(1,2),(1,3),(1,4),(1,5),(4,1),\dots,(5,3)\}). \quad (2.4)$$

In addition to the model (2.3), states and direct transitions between them are marked in red in Figure 2.5.

The alive state (state 1) can be split into more than two states which, when applied to illness insurance, typically correspond to occurrences of various medical complications or courses of illness because, in medical survival analysis, the development of a patient's illness can be considered as a process that progresses through several stages, where the first stage is labelled as the diagnosis of disease, the second stage is the advanced stage of the disease, and the last stage represents the event of patient's death. Additionally, if a person's remaining in a certain state of illness is associated with an increasing or decreasing risk of leaving it, then the division of this

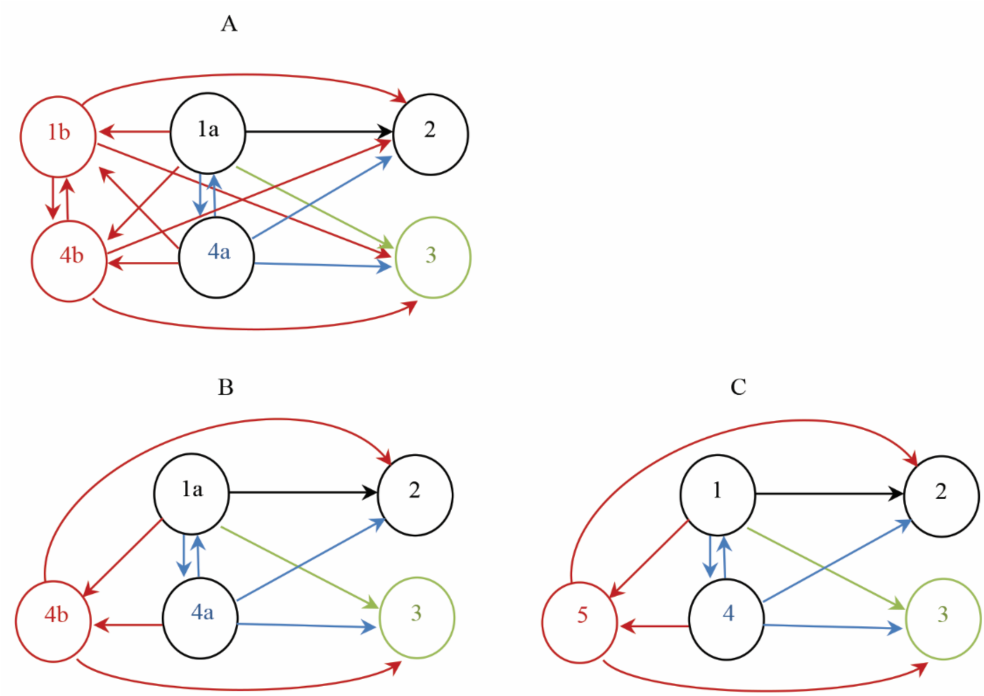


Figure 2.5. The multistate model for endowment insurance + UA + RJL + PD

Source: own elaboration.

state is also justified. The death state (state 2) can also be split into more than two states, which in applications typically correspond to analysing causes of death and the corresponding survival risks. The example of constructing an MSM for health insurance is described in Section 3.2 in detail.

In the same way as in Example 2.1, MSM is created when the contract covers the life of more than one person, such as in the case of marriage insurance (see Section 4.1), covers more types of health risks (cf. Section 3.3), or a combination of different forms of insurance and financial contracts (see Chapter 5).

2.3. Model's probabilistic structure

The analysis of changes of state (evolution of the states of the contract) from the moment of concluding the contract until its end is one of the basic elements influencing its valuation.

If the subject of the contract is the life or health of a person (life insurance contract or health insurance), x denotes the age of the individual concluding the contract, i.e. the so-called *age at entry*.

For a given contract, described by a multistate model (S, T), function $X(t) \in S$ where $X(x,t) = i$ (for $i \in S, t \geq 0$) means that at time t (meaning the time that has elapsed from the beginning of the contract), the person is affected by the life situation to which state i has been assigned. If this does not lead to inaccuracies, the age at entry is omitted, and a simplified notation is used, i.e. $X(x,t) \equiv X(t)$. The life cases covered by the contract (especially the insurance contract) are random in nature. Hence, assuming that $\{X(t); t \geq 0\}$ is a stochastic process, taking values from finite set space S is natural.

This monograph considers both cases where the process $\{X(t)\}$ is continuous (as in Section 4.2 and Section 4.3) and a discrete process, and more so, due to the application in practice, the authors focus on the discrete-time model, i.e. $\{X(t);\ t = 0,1,2,\ \ldots, n\}$. It is assumed that a process may change state once in a one-time unit (only one random event may occur). In addition, the state marked with the number 1 represents the initial state (that is $X(0) = 1$) and starting from state 1, any state can be reached.

The basic quantities describing the evolution of $\{X(t); t = 0,1,2, \ldots\}$ process are finite-dimensional distributions, which for every $k \in \{1,2, \ldots, n\}$ and $t_1, t_2, \ldots, t_k$ (where $0 \leq t_1 < t_2 < \cdots < t_k$) and $s_1, s_2, \ldots, s_k \in S$ require a probability of

$$P(X(t_1) = s_1, X(t_2) = s_2, \ldots, X(t_k) = s_k) \tag{2.5}$$

to be calculated. The knowledge of the above (2.5) allows for the determination of the conditional distributions:

$$P(X(t_k) = s_k | X(t_1) = s_1, X(t_2) = s_2, \ldots, X(t_{k-1}) = s_{k-1}), \tag{2.6}$$

which are necessary in the valuation of each contract. Finite dimensional distributions depend directly on the type of contract. For example, in health insurance, the distribution is influenced by random factors such as the insured's further lifetime as well as the risk of falling ill and the chance of recovery from the disease.

In practice, determining the probabilities (2.5) and (2.6) is often very difficult due to the limited access to data, therefore process $\{X(t); t = 0,1,2, \ldots\}$ is assumed to be a Markov chain. This means that for every $k \in N$ and $0 \leq t_1 < t_2 < \cdots < t_k$ and $s_1, s_2, \ldots, s_k \in S$, the following condition is fulfilled (c.f. (Billingsley, 1995)):

$$\begin{aligned} P(X(t_k) = s_k | X(t_1) = s_1, X(t_2) = s_2, \ldots, X(t_{k-1}) = s_{k-1}) \\ = P(X(t_k) = s_k | X(t_{k-1}) = s_{k-1}) \end{aligned}$$

if only $P(X(t_1) = s_1, X(t_2) = s_2, \ldots, X(t_k) = s_k) > 0$.

Let $q_{ij}(t,u) = P(X(u) = j | X(t) = i)$ be the transition probability of process $\{X(t)\}$ going to state j under condition that $X(t) = i$ (for $0 \leq t < u$). For $t = u$

$$q_{ij}(t,t) = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j. \end{cases} \tag{2.7}$$

Moreover, let $q_{\underline{ii}}(t,u) = P(X(z) = i \text{ for } t < z \leq u | X(t) = i)$ be the probability of process $\{X(t)\}$ remaining in state *i* from moment *t* to moment *u*.

According to the classification useful in the analysis of MSM (Haberman & Pitacco, 1999), state $i \in S$ is called state ($0 \leq t \leq u$):

- transient, if $\lim_{u\to\infty} q_{ii}(t,u) = 0$,
- strictly transient, if $q_{ii}(t,u) = q_{\underline{ii}}(t,u) < 1$,
- absorbing, if $q_{\underline{ii}}(t,u) = 1$.

Note that it is not possible for process $\{X(t)\}$ to be permanently in a transient state, although there is a possibility of returning after exiting it. On the other hand, a strictly transient state is characterised by the fact that process $\{X(t)\}$ cannot return to it after leaving – also, if $\{X(t)\}$ enters the absorbing state, it cannot exit it anymore. Note that a transient or strictly transient state for which $q_{ii}(t, t+1) = 0$ is called a *reflex state* (that is after one unit of time, the insured risk leaves this state). Other classifications of the state of the Markov chain can be found, for example, in (Billingsley, 1995).

Table 2.1 contains the classification of states for all MSMs described in Section 2.2.2.

Table 2.1. The classification of states for multistate models for selected insurance contracts

Contract	Formula of MSM	Type of state		
		transient	strictly transient	absorbing
Life insurance	(2.1)		1	2
Endowment insurance + UA	(2.2)		1	2, 3
Endowment insurance + UA + RJL	(2.3)	1, 4		2, 3
Endowment insurance + UA + RJL + PD	(2.4)	1, 4, 5		2, 3

Source: own elaboration.

If process $\{X(t)\}$ is observed at times $t = 0, 1, 2, \ldots$, then the introduction of the matrix notation is a convenient and frequently used procedure. Hence, one--dimensional distributions for *N*-state MSM have the following form:

$$\mathbf{P}(t) = \left(p_1(t), p_2(t), p_3(t), \ldots, p_N(t)\right)^T \in \mathrm{R}^N, \tag{2.8}$$

where $p_i(t) = \mathrm{P}(X(t) = i)$.

In particular, the initial distribution $\mathbf{P}(0) = (p_1(0), p_2(0), \ldots, p_N(0))^T \in R^N$, assuming that X(0) = 1 has the form $\mathbf{P}(0) = (1, 0, 0, \ldots,)^T \in R^N$.

Under the assumption that $\{X(t)\}$ is a nonhomogeneous Markov chain (Dębicka, 2012, 2013; Hoem, 1969, 1988; Waters, 1984; Wolthuis, 1994), vector P(t) can be determined from the sequence of the matrix transition $\{\mathbf{Q}(t)\}_{t=0}^{n-1}$ (where $q_{ij}(t) = P(X(t+1) = j|X(t) = i)$ for time interval $t, t+1$)) according to the formula:

$$\mathbf{P}^T(t) = \mathbf{P}^T(0)\prod_{k=0}^{t-1}\mathbf{Q}(k) . \tag{2.9}$$

Note that $\mathbf{Q}(t)$ is a transition matrix, therefore it has the following properties:

- for each $i \in S$it holds that $\sum_{j=1}^{N} q_{ij}(t) = 1$ (elements of each row sum to one),
- $q_{ij}(t) = 0$ if there is no direct transition from state i to state j in MSM (i.e. $(i,j) \notin T$).

Before defining $\mathbf{P}(k)$ and $\mathbf{Q}(k)$ matrices for MSM given by the model (0.1), the necessary determinations should be introduced. The age at entry x is related to random variable T_x, the future lifetime of an x-year-old person. Then, random quantity $x + T_x$ determines the age at which death occurred. Moreover, let K_x be a random variable representing the total number of years left to live for a person aged x, that is $K_x = \lfloor T_x \rfloor$ where $\lfloor a \rfloor$ represents the integer part of a. The probability distribution of random variable $K_x$is defined in Table 2.2.

Table 2.2. The probability distribution of the total number of years left to live for a person aged x

k	0	1	2	…	k	…	$\omega - x$
${}_{k\|}q_x = \mathrm{P}(K_x = k)$	${}_{0\|}q_x$	${}_{1\|}q_x$	${}_{2\|}q_x$	…	${}_{k\|}q_x$	…	${}_{\omega-x\|}q_x$

Source: (Dębicka, 2014).

In Table 2.2 ${}_{k|}q_x$ denotes the probability that a person aged x will die between $x + k$ and $x + k + 1$ year of life (i.e. the probability that the x-year-old will live another k years and die within a year), whereas ω means the maximum number of years that a person can live, i.e. the limit age (depending on the country, it is assumed to be 100 or 110 years).

For a person aged x and $k = 0,1,2,\ldots$, the following symbols are associated with the future life time (Bowers et al., 1986):

- the conditional probability of surviving the next year provided that x-year-old earlier survives at least k years

$$p_{x+k} = \mathrm{P}(K_x \geq k+1|K_x \geq k), \tag{2.10}$$

- the conditional probability of death within the next year provided that x-year-old earlier survives at least k years

$$q_{x+k} = 1 - p_{x+k}, \tag{2.11}$$

- the probability that x-year-old will survive at least k years

$$ {}_{k}p_x = \mathrm{P}(K_x > k). \tag{2.12} $$

Note that for MSM given by (2.1) for each k, the probability distribution of random variable $X(x,k)$ can be determined based on the distribution of random variable $K(x)$. Hence, we can rewrite probabilities (2.10) and (2.11) as follows:

$$ p_{x+k} = \mathrm{P}(X(k+1) = 1 | X(t) = 1), \tag{2.13} $$

$$ q_{x+k} = \mathrm{P}(X(k+1) = 2 | X(t) = 1), \tag{2.14} $$

$$ {}_{k}p_x = \mathrm{P}(X(k) = 1), \tag{2.15} $$

$$ \sum_{t=0}^{k-1} {}_{t|}q_x = \mathrm{P}(X(k) = 2). \tag{2.16} $$

Using (2.10) and (2.11), we can determine the matrix transition (for $k = 0, 1, 2, \ \dots, n-1$):

$$ \mathbf{Q}(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) \\ q_{21}(k) & q_{22}(k) \end{pmatrix} = \begin{pmatrix} p_{x+k} & q_{x+k} \\ 0 & 1 \end{pmatrix}. \tag{2.17} $$

Then, substituting (2.17) in formula (2.9), we obtain (2.8):

$$ \mathbf{P}(k) = (p_1(k), p_2(k))^T = \begin{cases} (1{,}0)^T & \text{for} \quad k = 0 \\ \left({}_{k}p_x, \sum_{t=0}^{k-1} {}_{t|}q_x \right)^T & \text{for} \quad k = 1{,}2, \dots, n \end{cases}. \tag{2.18} $$

Using the properties of a transition matrix, matrix $\mathbf{Q}(k)$ for multistate insurance contracts from Example 2.1 is presented in Table 2.3.

Table 2.3. The transition matrices for multistate models for selected insurance contracts

Contract	Formula of MSM	$\mathbf{Q}(k)$
Endowment Insurance + UA	(2.2)	$\begin{pmatrix} q_{11}(k) & q_{12}(k) & q_{13}(k) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Endowment Insurance + UA + RJL	(2.3)	$\begin{pmatrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q_{41}(k) & q_{42}(k) & q_{43}(k) & q_{44}(k) \end{pmatrix}$
Endowment Insurance + UA + RJL + PD	(2.4)	$\begin{pmatrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) & q_{15}(k) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ q_{41}(k) & q_{42}(k) & q_{43}(k) & q_{44}(k) & q_{45}(k) \\ 0 & q_{52}(k) & q_{53}(k) & 0 & q_{55}(k) \end{pmatrix}$

Source: own elaboration.

In actuarial practice, the values of individual elements of matrices **P**(t) and **Q**(t) for a particular contract are determined based on life tables (in the case of the model (2.1)) and appropriate for particular MSM multiple increment-decrement tables (or multiple state life tables).

In order to empirically determine the multiple-state life table, it is advisable, in addition to life tables, to have tables of morbidity, handicap, mortality of invalids and incapacity for work. National statistical offices publish mortality data by cause. On their basis, the probability of death by cause can be calculated, which makes it possible to determine the impact of particular diseases on the risk of death. However, they do not make it possible to estimate the probability and duration of the disease depending on age. The literature lacks reliable prevalence and disability tables that could be used to calculate actuarial quantities. Currently, most insurers build their tables considering different criteria for recognising invalidity and the country's economic and social conditions.

More about the rules for creating multiple state life tables and determining transition probabilities based on them can be found in (Bowers et al., 1986; Haberman, 1983, 1984; Jordan, 1982; Mattsson, 1977). In this monograph, multiple state life tables for people with lung cancer found in (Dębicka & Zmyślona, 2016) were used in the analysis in Section 3.3 and in Chapter 5.

2.4. Types of cash flows

The individual's presence in a given state or movement (transition) from one state to another may have some financial impact. Thus, as a result of the agreement, financial streams arise, consisting of cash flows between the parties over time. In particular, as a result of concluding an insurance contract, two streams of cash flows emerge:

- a stream of premiums (directed from the insured to the insurer),
- a stream of insurance benefits (directed from the insurer to the insured).

One can distinguish between the following forms of cash flows related to multistate insurances ($j \in \{1,2,\dots,N\}, k = 0,1,2,\dots,n$):

$p_j(k)$ – a period premium amount at time k if $X(k) = j$,

$\pi_j(k)$ – a premium amount at some fixed time k if $X(k) = j$,

$\ddot{b}_j(k)$ – an annuity due benefit at time k if $X(k) = j$,

$b_j(k)$ – an intermediate annuity benefit at time k if $X(k) = j$.

$d_j(k)$ – a lump sum at some fixed time k if $X(k) = j$ (for instance, pure endowment),

$c_{ij}(k)$ – a lump sum at time k if a transition occurs from state i to state j at that time (for a discrete-time model it means that $X(k-1) = i \neq j$, $X(k) = j$ and $X(k+1) \neq j$).

When describing each cash flow arising from the insurance contract, its nature (benefit/premium), frequency (single/periodic) and form of realisation (in advance/from below) are important. Such characteristics, which allow the different types of cash flows, are presented in Table 2.4.

Table 2.4. The classification of type of cash flows arising from insurance contracts

Symbol	Description of the type of cash flow	Frequency		Moment of realisation	
		single	periodic	in advance	from below
p	period premium		+	+	
π	premium paid at a fixed time of the contract	+		+	
$\ddot{b}$	annuity due		+	+	
b	immediate annuity		+		+
d	lump sum benefit paid at a fixed time of the contract	+			+
c_1	lump sum benefit related to leaving state 1	+			+
…	…	…			…
c_N	lump sum benefit related to leaving state N	+			+

Source: own elaboration.

Let $\wp_j(t)$ (where $\wp \in \{p, \pi, \ddot{b}, b, d, c_1, \ldots, c_N\}$) denote the cash flow realised at time t if $X(t) = j$ (Dębicka, 2012). Then one can say that each of premiums $p_j(k)$ is cash flow type p. Similarly, $\pi_j(k)$, $\ddot{b}_j(k)$, $b_j(k)$, $d_j(k)$ are cash flows type $\pi, \ddot{b}, b, d$ respectively. The $c_{ij}(k)$ benefit is c_i type of cash flow because the amount of the single benefit paid in j may depend on the state of the process $\{X(t)\}$ at the time preceding the transition to state j. The introduction of $\wp_j(t)$ allows for the standardised formulas of actuarial quantities in Section 2.6.

Note that $\pi, \ddot{b}, b, d$ are the cash flows connected with the staying of the process $\{X(t)\}$ in a considered state, whereas cs correspond to cash flows associated with transitions between states. The qualitative type of cash flows and the differences between them are described in Table 2.4. This also means that set direct transition T contains two kinds of transitions. Hence, pair $(i, j) \in T$ is of *cf* (cash flow) type if k ($k \in \{1,2, \ldots, n\}$) exists for which $c_{ij}(k) \neq 0$; T^{cf} $\left(T^{cf} \subseteq T\right)$ denotes the subset of direct transitions of type *cf*. The transitions not associated with any flows of type c_is are called *not-cf*.

Example 2.1. *Multistate insurance contract – continuation*

Consider the stream of actuarial payment functions arising from n-year endowment insurance contract with supplementary insurances, i.e. UA, RJL, PD. Assume that the insured pays constant annual premium p for the whole insurance period when he/she lives, works and is healthy:

$$p_j(k) = \begin{cases} p & \text{for } j = 1 \text{ and } k = 0,1,2, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.19}$$

For an endowment insurance contract, let $c(k)$ denote the benefit payable at time k if the insured person's death occurred in the time interval $[k-1, k)$ before the end of the insurance contract. Moreover, let d be the pure endowment benefit payable at time n if the insured person is still alive. For such a basic insurance contract, illustrated by Figure 2.2, the stream of benefits consists of the following type of cash flows:

$$d_j(k) = \begin{cases} d & \text{for } j = 1 \text{ and } k = n \\ 0 & \text{otherwise} \end{cases}, \tag{2.20}$$

$$c_{ij}(k) = \begin{cases} c(k) & \text{for } (i,j) = (1,2) \text{ and } k = 1,2, \dots, n \\ 0 & \text{otherwise} \end{cases}. \tag{2.21}$$

If we take into consideration endowment with supplementary insurances, then the benefit stream for insurance contract (illustrated by Figure 2.5C) consists of seven types of benefits $\wp \in \{\ddot{b}, b, d, c_1, c_4, c_5\}$, namely:

- endowment benefit

$$d_j(k) = \begin{cases} d & \text{for } j = 1,4,5 \text{ and } k = n \\ 0 & \text{otherwise} \end{cases}, \tag{2.22}$$

- annuity benefit payable under the terms of supplementary insurance RJL

$$\ddot{b}_j(k) = \begin{cases} \ddot{b} & \text{for } j = 4,5 \text{ and } k = 1,2,3, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}, \tag{2.23}$$

- annuity benefit payable per the conditions of supplementary insurance PD

$$b_j(k) = \begin{cases} b & \text{for } j = 5 \text{ and } k = 1,2,3, \dots, n \\ 0 & \text{otherwise} \end{cases}, \tag{2.24}$$

- lump sums payable under the terms of the endowment insurance contract and supplementary insurances UA and PD

$$c_{ij}(k) = \begin{cases} c(k) & \text{for } (i,j) = (1,2) \text{ and } k = 1,2,3 \dots, n \\ & \text{and for } (i,j) \in \{(4,2);(5,2)\} \text{ and } k = 2,3 \dots, n \\ 2 \cdot c(k) & \text{for } (i,j) = (1,3) \text{ and } k = 1,2,3, \dots, n \\ & \text{and for } (i,j) \in \{(4,3);(5,3)\} \text{ and } k = 2,3 \dots, n, \\ \bar{c} & \text{for } (i,j) = (1,5) \text{ and } k = 1,2,3, \dots, n \\ & \text{and for } (i,j) = (4,5) \text{ and } k = 2,3 \dots, n \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

where $\bar{c}$ is a lump sum disability benefit (as per the terms of the supplementary insurance PD).

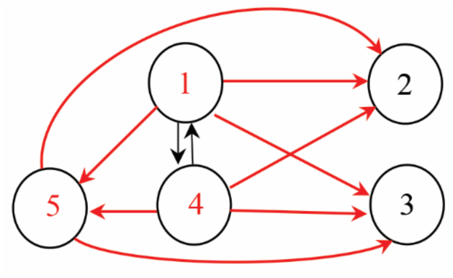


Figure 2.6. Cash flows on the multistate model for endowment insurance + UA + RJL + PD

Source: own elaboration.

In Figure 2.6, the direct transitions of *cf* type (associated with cash flows of c_i type) are bolded and coloured in red, and a number of states in which cash flows of type $\ddot{b}, b, d$ are realised, are marked in red.

2.5. Interest rate

In the context of insurance, the discounting function is mainly used to calculate the present value of future insurance payments. Let $v(t, k)$ denote the discounting function for time interval $[t, k]$, which means how much one unit realised at time k is worth at time t.

Since insurance often involves payments of future benefits, the discounting function is most often used to convert these future payments into their equivalent at the beginning of the insurance period (for $t = 0$), which allows for the valuation of the insurance contract (in particular, the determination of the insurance premium). Assuming that we have a time-discrete model, it is necessary to determine the discounting function $v(0, k)$ over the entire period of the insurance contract, i.e. for $k = 0, 1, 2, ..., n$. Let vector

$$\mathbf{V} = (v_0, v_1, \ldots v_n)^{\mathrm{T}} \in R^{n+1}, \tag{2.26}$$

where $v_k = v(0, k)$ consists of all such discounting functions. Note that v_k is the amount at time 0 that is equivalent to one unit of currency payable at time k.

Generally, discount factor $v(0, t)$ is calculated based on the adopted interest rate for period $[0, t]$, when considering interest in continuous time. Let $Y(t) := Y(0, t)$ be denoted by the *interest rate for period* $[0, t]$. Assume $Y(0) = 1$ and that the interest rate evolves according to the following differential equation $dY(t) = r(t)Y(t)dt$, where $r(t)$ is the instantaneous rate usually referred to as a *short rate* (or the *interest rate at time t*). So we obtain straightforwardly $Y(t) = e^{\int_0^t r(s)ds}$, and the discount factor takes the form:

$$v_t = \frac{Y(0)}{Y(t)} = Y(t)^{-1} = e^{-\int_0^t r(s)ds}. \tag{2.27}$$

In many pricing applications, $r(t)$ is assumed to be a deterministic function of time, so both $Y(t)$ and the discount factors (2.27) at any future time are deterministic time functions. However, when dealing with interest-rate products, the variability that matters is that of the interest rates themselves. Therefore, sometimes it is desirable to drop the deterministic setup and to start modelling the evolution of interest rate in time through a stochastic process. Interest rate modelling is a key element of insurance analysis because it primarily directly impacts insurance contract pricing (see, for example, comparative analysis in (Dębicka et al., 2022)). $Y(t)$ and $r(t)$ can be modelled in various ways, depending on the needs and specificity of the analysed issue. An overview of interest rate modelling methods used in the valuation of insurance contracts can be found, among others, in (Gibson et al., 2010; Jajuga, 2006; Mehalla, 2021).

Generally, in the case of the interest rate, the following four approaches can be distinguished.

1. *Fixed interest rate*. In the simplest case, the interest rate is constant over time and does not depend on any external factors (Bowers et al., 1986). This model is used when changes in interest rates are negligible, or the insurance valuation analysis focuses on the impact of other risk factors (e.g. demographic or health risk). In such a situation, the discount factors take the form $v(t, k) = (1 + r)^{-(k-t)}$ (for $0 \leq t \leq k$), and one directly obtains the element of vector $\mathbf{V}$

$$v_k = (0, k) = (1 + r)^{-k} = \left(\tfrac{1}{1+r}\right)^k = v_k, \tag{2.28}$$

where r is a constant interest rate throughout the whole insurance period.

2. *Stochastic model*. Stochastic models are used in advanced analyses, where $r(t)$ is treated as a random process. Consequently, $Y(t)$ and the discount factors (2.27) will be stochastic processes, too. This approach was pioneered in the 1970s by Boyle (1976). In the financial and actuarial literature, Gaussian stochastic processes with stationary increments (satisfying the Markov property) are used to model these processes (Beekman & Fuelling, 1990, 1993; Dhaene, 2000; Garrido, 1988; Parker, 1994a, 1994b, 1994d; Vasicek, 1977). In particular, it is assumed that

- $Y(t) = \delta W(t) + \mu t$, where $\{W(t); t \geq 0\}$ is the Wiener process $(\mu, \delta > 0)$,
- $r(t) = \delta U(t) + \mu$, where $\{U(t); t \geq 0\}$ is the Ornstein-Uhlenbeck process $(\mu, \delta > 0)$ with the covariance function $\mathrm{Cov}(t) = e^{(-\alpha t)}$ $(\alpha, t > 0)$.

Note that assuming that for a discrete model of cash flows (i.e. $\{X(t): t = 0, 1, 2, ...\}$), the interest rate is modelled by a continuous time model as a stochastic process, which is sampled at moments $k = 0, 1, 2, ..., n$, is consistent with, among others (Bruno et al., 2000; Parker, 1994a, 1994b, 1994c, 1997, 1998).

3. *Historical data modelling*. These models rely on analysing historical interest rate data and its variability over time to forecast future interest rate values. One popular approach is to use time series analysis methods to model interest rates. Panjer and Bellhouse developed a general theory including continuous and discrete models (Bellhouse & Panjer, 1981; Panjer & Bellhouse, 1980). The theory was further worked out for unconditional and conditional autoregressive processes of orders one and two, such as autoregressive models, moving average models or their combinations. In addition, more advanced econometric models such as vector autoregressive models, general autoregressive conditionally heteroskedastic models and stochastic models can also be used to model interest rates based on historical data.

It is important to select the appropriate model for a given problem, taking into account the available data and the purpose of the study. It turns out that the forecasts obtained using the class of models proposed in (Cox et al., 1985; Dai & Singleton, 2000; Duffie & Kan, 1996; Vasicek, 1977) do not outperform random forecasts, as shown in e.g. (Duffee, 2002). Since time functions aim to describe dynamic properties, they are better suited for forecasting (see also (Dębicka et al., 2022)). Hence the subject literature again draws attention to the Nelson and Siegel model (1987) and its generalisation, the Svensson model (1994), for modelling the yield curve over time (Diebold et al., 2006; Diebold & Li, 2006; Koopman et al., 2010).

Due to the long-term nature of life insurance, the econometric Svenson model, which is a generalised Nelson-Siegel model, is selected to model the interest rate. In the Svenson model, the *spot interest rate* $R(0, t) := R(t)$ is described as follows (Anderson et al., 1996; Marciniuk et al., 2020; Yallup, 2012):

$$\begin{aligned} R(t) = \beta_0 + \beta_1 \frac{\tau_1}{t}\left(1 - e^{-\frac{t}{\tau_1}}\right) + \beta_2 \left(\frac{\tau_1}{t}\left(1 - e^{-\frac{t}{\tau_1}}\right) - e^{-\frac{t}{\tau_1}}\right) \\ + \beta_3 \left(\frac{\tau_2}{t}\left(1 - e^{-\frac{t}{\tau_2}}\right) - e^{-\frac{t}{\tau_2}}\right), \end{aligned} \tag{2.29}$$

where $\beta_0, \beta_1, \beta_2, \beta_3$ are model parameters, and τ_1, τ_2 are the time parameters of the model. Note that $R(t) := \frac{\ln Y(t)}{t}$, therefore $Y(t) = e^{R(t) \cdot t}$. Based on (2.27), the discounting function has the following form: $v(t) = e^{-R(t) \cdot t}$.

Note that parameter β_0 (2.29) corresponds to the long-term (infinite maturity) spot rate because $\lim_{t\to\infty} R(t) = \beta_0$ and represents the long-term interest rate. This means that β_0 determines the level towards which the interest rate converges over the long term in response to changes in economic conditions and monetary policy. Fixed interest rate r can be assumed at the level of $\beta_0 \cdot 100\%$, because contracts related to human life are long-term contracts. When one considers the discrete-time model, then the annually-compounded interest rate prevailing at time 0 for maturity k is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $v(k)$ units of currency at time 0, when reinvesting the obtained amounts once a year. Based on (Brigo & Mercurio, 2006), we can show that then $v(k) = \left(\frac{1}{1+\beta_0}\right)^k$.

The advantage of the Svenson model lies in its ability to flexibly adjust the interest rate curve to historical data by altering the values of the β_s and τ_s parameters. Enriching the Nelson-Siegel model with additional parameters allows for considering more extremes and provides greater flexibility in modelling the curve.

4. *Fuzzy interest rate* refers to an interest rate that is not clearly defined but is modelled using fuzzy set theory. In the case of a fuzzy interest rate, the value of the interest rate can be ambiguous and allows for more flexible modelling of financial conditions and taking into account uncertainty and subjectivity in the financial decision-making process.

To put it simply, assume having the imprecise information that the interest rate is "about b", we can model such interest rates as triangular fuzzy number $r = (a, b, d)$. The parameters of this fuzzy number are determined based on assessments of additional information experts. The mean value of this fuzzy number is equal to $M(r) = 0.25(a + 2b + d)$ and imprecision $\text{Imp}(r) = 0.5(d - a)$. Fuzzy number $1 + r$, where 1 is treated as a degenerate triangular fuzzy number (1, 1, 1), is triangular too. We have $1 + r = (1 + a, 1 + b; 1 + d)$, but the fuzzy discount factor $v = \frac{1}{(1 + r)}$ is not a triangular fuzzy number. Every α-cut of it takes the form $v_a = [(1 + d - (d - b)\alpha)^{-1}, (1 + a - (b - a)\alpha)^{-1}]$, and the fuzzy discount factor v can be approximated as triangular fuzzy number $v = (v_a, v_b, v_d)$. The α-cut of fuzzy power $(\boldsymbol{v}^c)_\alpha$, where $c > 0$, takes the form $(\boldsymbol{v}^c)_\alpha = [(v_a + (v_b - v_a)\alpha)^c, (v_d + (v_d - v_b)\alpha)^c]$. Note that by substituting $\alpha = 0$ we obtain $v_a = (1 + d)^{-1}$, $v_d = (1 + a)^{-1}$, and substituting $\alpha = 1$ gives $v_b = (1 + b)^{-1}$. Therefore, we obtain the elements of vector $\boldsymbol{v}$.

In insurance, this method of modelling the interest rate was used in the valuation of contracts – see for example (Anzilli et al., 2018; Anzilli & Facchinetti, 2017; De Andrés-Sánchez et al., 2020; De Andrés-Sánchez & González-Vila Puchades, 2012, 2017a, 2017b, 2017c, 2023; De Andres & Terceno, 2003; Dębicka et al., 2022; Heilpern, 2014; Lemaire, 2005).

The Svenson model with spot interest rate $R(t)$ given by the above (2.29) was used to model the interest rate in this monograph. In particular, the values of the estimated parameters were presented in Section 3.2, and it was also used to determine actuarial values in Chapters 4 and 5. The parameters were estimated based on actual data. Since the analysed the discrete-time model and life insurances are long-term contracts, the elements of vector $\mathbf{V}$ are given by the above (2.28) for a constant interest rate throughout the whole insurance period as $r = \beta_0$. On the one hand, in numerical analysis, modelling based on historical data can better assess the risk associated with the valuation of insurance contracts, whilst on the other, this approach allows for highlighting the impact of non-financial types of risk (thanks to a constant interest rate).

2.6. Insurance premiums

Let $\Upsilon_t^{\wp,j}(k)$ be the *discounted value* at time t of cash flow $\wp$ realised at time k when $X(t) = j(t \leq k)$. Note that $\Upsilon_t^{\wp,j}(k)$ is a random variable whose distribution depends on the distribution of process $\{X(t)\}$ and stochastic interest rate $\{Y(t)\}$. If $\wp$ is one of the cash flows associated with the state of process $\{X(t)\}$, i.e. $\wp \in \{p, \pi, \ddot{b}, b, d\}$, then

$$\Upsilon_t^{\wp,j}(k) = \begin{cases} v(t,k) 1_{\{X(k)=j\}} \wp_j(k) & \text{for} \quad 0 \leq t < k \\ 1_{\{X(k)=j\}} \wp_j(k) & \text{for} \quad 0 \leq t = k \end{cases}, \tag{2.30}$$

where 1_A denotes the indicator of set A, and $\wp_j(k)$ is the cash flow realised at time k when $X(k) = j$. If $\wp$ is a cash flow associated with a state change by process $\{X(t)\}$, i.e. $\wp \in \{c_1, c_2, \ldots, c_N\}$, then

$$\Upsilon_t^{\wp,j}(k) = \begin{cases} v(t,k) 1_{\{X(k-1)=i \wedge X(k)=j\}} c_{ij}(k) & \text{for} \quad 0 \leq t < k \quad \text{and} \quad i \in S\{j\} \\ 1_{\{X(k-1)=i \wedge X(k)=j\}} c_{ij}(k) & \text{for} \quad 0 \leq t = k \quad \text{and} \quad i \in S\{j\}. \\ 0 & \text{for} \quad 0 \leq t \leq k \quad \text{and} \quad i = j \end{cases} \tag{2.31}$$

Since $\Upsilon_t^{\wp,j}(k)$ is a random variable with a dual stochastic nature (random interest rate $\{Y(t)\}$ and process $\{X(t)\}$), determining its moments in a general form is complex. Further considerations were conducted under the following assumptions (Frees, 1990; Parker, 1994c):

Assumption A1

Random variable $X(t)$ is independent of $Y(t)$.

Assumption A2

All moments of random discounting function $e^{-Y(t)}$ are finite.

Note that A1 implies that the interest rate does not influence the realisation of random variable $X(t)$. Furthermore, A2 guarantees that determining the conditional expected value $\Upsilon_t^{\wp,j}(k)$ is feasible.

In further consideration, it is always assumed that assumptions A1 and A2 are fulfilled.

The *actuarial value* at time t of cash flow $\wp$ realised at time k, when $X(k) = j$ and $X(t) = i$, is called the *conditional expected value* $\mathrm{E}\left(\Upsilon_t^{\wp,j}(k)|X(t)=i\right)$, which for $\wp \in \left\{p, \pi, \ddot{b}, b, d\right\}$, takes the following form:

$$\mathrm{E}\left(\Upsilon_t^{\wp,j}(k)|X(t)=i\right) = \begin{cases} \mathrm{E}\big(v(t,k)\big)q_{ij}(t,k)\wp_j(k) & \text{for} \quad 0 \le t < k \\ \wp_j(k) & \text{for} \quad 0 \le t = k \quad \text{and} \quad i = j \\ 0 & \text{for} \quad 0 \le t = k \quad \text{and} \quad i \neq j \end{cases} \tag{2.32}$$

and for $\wp \in \{c_1, c_2, \dots, c_N\}$ is as follows:

$$\mathrm{E}\left(\Upsilon_t^{c_h,j}(k)|X(t)=i\right) = \begin{cases} \mathrm{E}\big(v(t,k)\big)q_{ih}(t,k-1)q_{hj}(k-1,k)c_{hj}(k) & \text{for} \quad 0 \le t < k \quad \text{and} \quad h \in S\{j\} \\ q_{hi}(t-1,t)c_{hi}(k) & \text{for} \quad 0 \le t = k \quad \text{and} \quad h \in S\backslash\{i\}, \\ 0 & \text{for} \quad i = j \end{cases} \tag{2.33}$$

where $\mathrm{E}(v(t,k)) = \mathrm{E}\left(e^{-\left(Y(k)-Y(t)\right)}\right)$.

Knowledge of actuarial values of future cash flows at the beginning of the insurance contract (for $t = 0$) is particularly useful in calculating insurance premiums. The sum of all payments of a given type, realised in given state j, throughout the entire insurance period is as follows:

$$\mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)|X(0)=1\right) = \begin{cases} \sum_{k=0}^{n-1} \mathrm{E}\big(v(0,k)\big)q_{1j}(0,k)\wp_j(k) & \text{for} \quad \wp \in \left\{p, \pi, \ddot{b}\right\} \\ \sum_{k=1}^{n} \mathrm{E}\big(v(0,k)\big)q_{1j}(0,k)\wp_j(k) & \text{for} \quad \wp \in \{b, d\} \\ \sum_{k=1}^{n} \mathrm{E}\big(v(0,k)\big)q_{1h}(0,k-1)q_{hj}(k-1,k)c_{hj}(k) & \text{for} \quad \wp \in \{c_1, \dots, c_N\}\backslash\left\{c_j\right\} \\ 0 & \text{for} \quad \wp \in \left\{c_j\right\} \end{cases}. \tag{2.34}$$

In particular, for annuity-type benefits (benefits with periodic frequency) equal to one, the following notation is used:

- for payments equal to 1, which are paid in advance ($\wp \in \left\{p, \ddot{b}\right\}$)

$$\ddot{a}_{1j}(0,n) = \sum_{k=0}^{n-1} \mathrm{E}\big(v(0,k)\big)q_{1j}(0,k), \tag{2.35}$$

- for payments equal to 1, which are paid from below ($\wp \in \{b\}$)

$$a_{1j}(0,n) = \sum_{k=1}^{n} \mathrm{E}\big(v(0,k)\big)q_{1j}(0,k). \tag{2.36}$$

The realisation of a life insurance contract gives rise to two cash flow streams: the premium stream (flowing from the insured to the insurer) and the benefit stream (flowing in the opposite direction). From the perspective of the insurer's total loss fund, premiums represent an outflow (having a negative value as they decrease the fund's value), while benefits constitute an inflow (having a positive value as they increase the fund's value). Let Z denote the insurer's total loss, defined as the difference between the discounted value of all payments incurred under the contract (i.e. benefits) and the current value of the premiums paid by the insured during the contract term. Therefore, the insurer's total loss can be expressed as

$$Z = \sum_{\wp\in\{\ddot{b},b,d,c_1,\ldots,c_N\}} \sum_{j\in S} \sum_{k=0}^{n} \Upsilon_0^{\wp,j}(k) - \sum_{\wp\in\{p,\pi\}} \sum_{j\in S} \sum_{k=0}^{n-1} \Upsilon_0^{\wp,j}(k). \tag{2.37}$$

Note that Z is a function of random variables $\Upsilon_t^{\wp,j}(k)$, thus it also has a dual stochastic nature (it depends on random interest rate $\{Y(t)\}$ and process $\{X(t)\}$). Assuming that assumptions A1 and A2 are fulfilled and $P(X(0) = 1) = 1$ then

$$\mathrm{E}(Z) = \sum_{\wp\in\{\ddot{b},b,d,c_1,\ldots,c_N\}} \sum_{j\in S} \sum_{k=0}^{n} \mathrm{E}\left(\Upsilon_0^{\wp,j}(k)\right) - \sum_{\wp\in\{p,\pi\}} \sum_{j\in S} \sum_{k=0}^{n-1} \mathrm{E}\left(\Upsilon_0^{\wp,j}(k)\right). \tag{2.38}$$

Using the above (2.34), we obtain

$$\mathrm{E}(Z) = \sum_{\wp\in\{\ddot{b},b,d,c_1,\ldots,c_N\}} \sum_{j\in S} \mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)\right) - \sum_{\wp\in\{p,\pi\}} \sum_{j\in S} \mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)\right). \tag{2.39}$$

Insurance premiums are called *net premiums* if the *equivalence principle* is satisfied, that is $\mathrm{E}(Z) = 0$ (Bowers et al., 1986; Dickson et al., 2019; Gerber, 1995). Based on (2.39) the equivalence principle for the multistate insurance may be written in the following form:

$$\sum_{\wp\in\{\ddot{b},b,d,c_1,\ldots,c_N\}} \sum_{j\in S} \mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)\right) = \sum_{\wp\in\{p,\pi\}} \sum_{j\in S} \mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)\right). \tag{2.40}$$

The net premium can be determined directly from the formula (2.40). In particular:

- *net single premium* π payable at the time of concluding the insurance contract (when $X(0) = 1$);

$$\pi = \pi_1(0) = \sum_{\wp\in\{\ddot{b},b,d,c_1,\ldots,c_N\}} \sum_{j\in S} \mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)\right), \tag{2.41}$$

- *net period premium* with a fixed amount p, payable for the first m units of the insurance contract if $X(t) = 1$ for $t = 0, 1, .., m - 1$

$$p = \frac{\sum_{\wp \in \{\ddot{b},b,d,c_1,\dots,c_N\}} \sum_{j \in S} \mathrm{E}\left(\Upsilon_0^{\wp,j}(0,n)\right)}{\sum_{k=0}^{m-1} \mathrm{E}\big(v(0,k)\big) q_{1j}(0,k)} = \frac{\pi}{\ddot{a}_{11}(0,m)}. \tag{2.42}$$

Since multistate models are usually quite complex, and the probabilistic structure of the model and the current values related to the interest rate can be expressed in a matrix form, the idea of expressing the net premiums (2.41) and (2.42) in the form of a matrix formula seems natural, which would make determining them much easier. In Section 2.7, these premiums are presented in matrix form.

2.7. Actuarial values in matrix form

The objective of this section was to accommodate the classical multiple-state model (introduced in Section 2.2) in a manner where the state of the insured risk at time k uniquely determines the cash flow value at that time. This pursuit is motivated by the distinct role played by type $c_{ij}(k)$ cash flows resulting from state transitions. Specifically, a new state (denoted as j^+) associated with state j is introduced, such that the insured risk is in state j^+ and $c_{ij}(k)$ is realised at time k. Subsequently, at succeeding time $k+1$, the insured risk transitions to state $X(k+1)$. Adjusting state space S entails appropriate alterations in the set of direct transitions T and process $\{X(t)\}$. The formal framework for this construction is delineated through a recursive procedure, first introduced in (Dębicka, 2006) and subsequently in (Dębicka, 2012, 2013).

Let $cf_j(k)$ be the future cash flow payable at time k if $X(k) = j$ $(k = 0,1,\dots,n)$, because at each time k, all of the types of cash flows may occur, then

$$\begin{aligned} cf_j(k) = p_j(k) + \pi_j(k) + \ddot{b}_j(k) + b_j(k) + d_j(k) \\ + \sum_{i \in S\{j\}} c_{ij}(k) 1_{\{X(k-1)=i\}}, \end{aligned} \tag{2.43}$$

where 1_A denotes the indicator of set A. Observe that information $X(k) = j$ is insufficient to determine the future cash flow at time k (2.43) arising from a multistate insurance contract in the discrete-time model (because of $\sum_{i \in S\{j\}} c_{ij}(k) 1_{\{X(k-1)=i\}}$). For the lump sum, the information that the insured risk is in a particular state at moment k is not enough to determine the benefit at moment k because one needs additional information about the state of the insured risk at time $k-1$. For this reason, if lump sums result from the contract's terms, it is impossible to represent cash flows in matrix form, which would consequently enable obtaining formulas for premiums in this form. Following (Dębicka, 2012, 2013) in this section, this problem is addressed by properly extending the MSM. The idea of using matrix notation for actuarial values is presented in Figure 2.7.

Assumptions B1 and B2 are related to the type of direct transitions between states (*cf* and *not-cf*) in MSM; they are as follows:

Assumption B1

For each i, all pairs $(i, j) \in T$ are of the same type, *cf* or *not-cf*.

Assumption B2

For each i, such that $(i, j) \in T^{cf}$ and each $k \in \{1,2, \dots, n\}$, then $c_{ij}(k) = c_j(k)$. For each state in the MSM, three possible situations are shown in Figure 2.8. MSM satisfies assumption B1 if, for each state j, all transitions to that state are of the same type (case I and II in Figure 2.8). However, MSM meets assumption B2 when, for case II, the amounts of cash flows resulting from the transition to state j are identical ($c_{i_1 j}(k) = c_{i_2 j}(k)$ for each moment k).

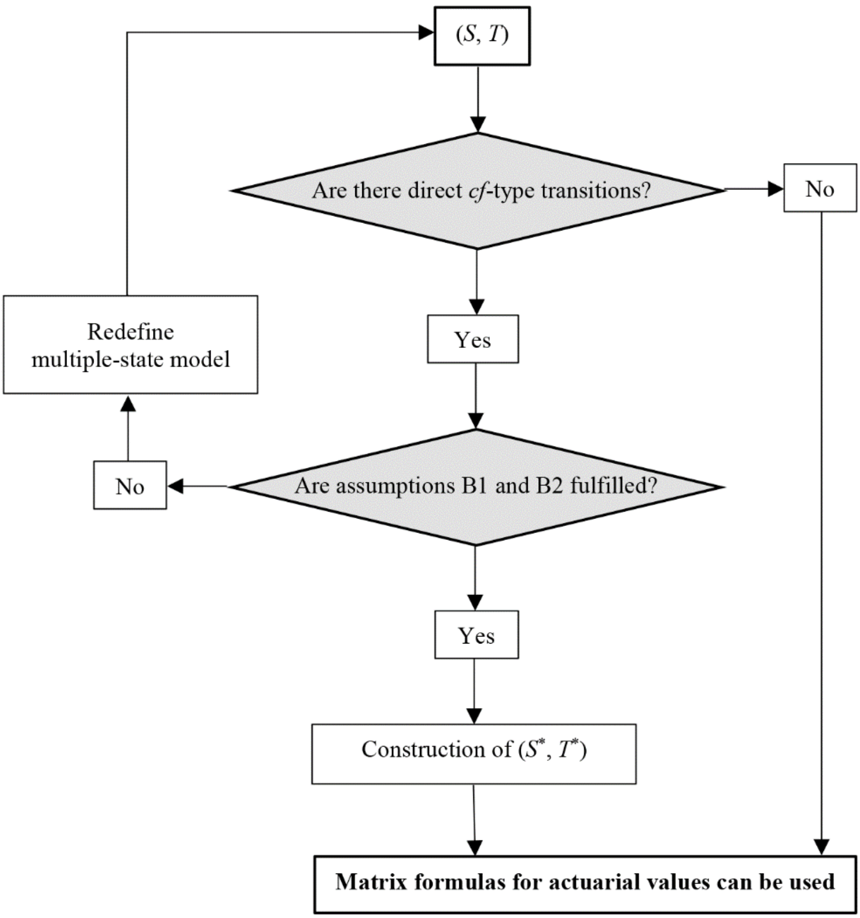


Figure 2.7. The idea of using matrix notation for actuarial values

Source: own elaboration.

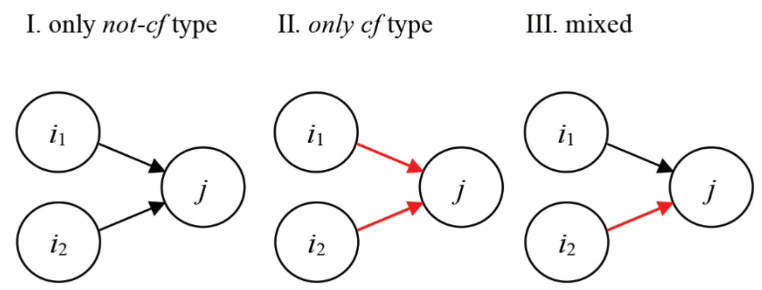


Figure 2.8. Possible situations of direct transitions to state j

Source: own elaboration.

The simplest way to verify whether assumptions B1 and B2 are fulfilled is to construct a sequence of a control array $\boldsymbol{W}(k) = \left(w_{ij}(k)\right)_{i,j=1}^{N}$ at moment k ($k \in \{1,2, \dots, n\}$). The components of control array $\boldsymbol{W}(k)$ for MSM are defined as follows:

$$w_{ij}(k) = \begin{cases} + & \text{for} \quad (i,j) \in T \backslash T^{cf} \quad \text{and} \quad p_i(k-1) \cdot q_{ij}(k) \neq 0 \\ c_{ij}(k) & \text{for} \quad (i,j) \in T^{cf} \quad \text{and} \quad p_i(k-1) \cdot q_{ij}(k) \neq 0 \cdot \\ - & \quad \text{otherwise} \end{cases} \tag{2.44}$$

Assumption B1 implies that in each column of $\boldsymbol{W}(k)$, the presence of symbol + precludes the presence of $c_{ij}(k)$ and *vice versa*. Assumption B2 indicates that within any given column of $\boldsymbol{W}(k)$, all elements other than the symbols + or – share identical values.

Consider that MSM, where Assumptions B1 and B2 are not applicable, can be converted into one where both Assumptions B1 and B2 are fulfilled. For instance, if a lump sum pertains to the death of an insured individual and its amount varies based on the insured's health condition – whether healthy or ill – one can establish an equivalent MSM with two distinct states: death due to illness or death due to other causes. In such instances, the sequence of control arrays indicates the states to be distinguished.

If assumptions B1 and B2 are met, then we can assign the so-called *Modified Multiple-State Model* (MMSM) denoted by a pair of sets (S^*, T^*). The recursive procedure presented in (Dębicka, 2013) allows to construct the (S^*, T^*). In the following $N+1$ steps procedure, MMSM was constructed:

STEP 0

$$\begin{matrix} S^* & := & S \\ T^* & := & T \\ T^{cf*} & := & T^{cf} \end{matrix},$$

STEP j ($j = 1,2, \dots, N$).

- If there exists $i \in S^*$ such that $(i,j) \in T^{cf*}$, then associate new state j^+ with state j, and

$$\begin{aligned} S^* &:= S^* \cup \{j^+\}, \\ T^* &:= \left(T^* \backslash \{(i,j): i \in S^* \wedge (i,j) \in T^{cf*}\}\right) \\ &\quad \cup \{(i,j^+): i \in S^* \wedge (i,j) \in T^{cf*}\} \\ &\quad \cup (j^+,j) \cup \{(j^+,i): i \in S^* \wedge (j,i) \in T^*\}, \\ T^{cf*} &:= \{(i_1,i_2): i_1,i_2 \in S^*\{j\} \wedge (i_1,i_2) \in T^{cf*}\} \cup \{(j,i): (j,i) \in T^{cf*}\} \\ &\quad \cup \{(i,j^+): (i,j) \in T^{cf*}\} \cup \{(j^+,i): (j,i) \in T^{cf*}\}. \end{aligned}$$

- If $j < N$, then go directly to STEP $j+1$. If $j = N$, the procedure is completed.

Observe that the new set space consists of all states $j \in S$ and the set of new states $j^+ \in S^+$ ($S^* = S \cup S^+ = \{1,2,\dots,N^*\}$). After changing (S,T) into (S^*,T^*), it is necessary to introduce a new discrete-time stochastic process $\{X^*(k); k = 0,1,2,\dots,n\}$ with values in finite set S^*which describes the evolution of the insured risk:

$$X^*(k) = \begin{cases} j^+ & \text{for} \quad X(k) = j \wedge X(k-1) \neq j \wedge \exists_{i \in S}(i,j) \in T^{cf} \\ j & \text{otherwise} \end{cases}. \quad (2.45)$$

Note that states j^+ and j deal with the same event, but the difference is connected with realisation $c_j(k)$. This means that if $X^*(k) = j$, then $c_j(k) = 0$, but if $X^*(k) = j^+$, then $c_j(k)$ occurs at time k. For MMSM, all types of cash flows are connected with the process $\{X^*(k); k = 0,1,2,\dots,n\}$ staying in a considered state, although cs correspond to cash flows associated with transitions between states. This situation occurs because of the procedure of the modification of the multistate model and the assumption that lump sum $c_j(k)$ does not depend on state i and $q^*_{j^+j^+}(k) = \mathrm{P}\left(X^*(k+1) = \frac{j^+}{X^*(k)} = j^+\right) = 0$.

As in (2.9), in order to describe the probabilistic structure of $\{X^*(k)\}$, for any moment $k \in \{0,1,2,\dots,n\}$, introduce

$$\mathbf{P}^*(k) = \left(p_1(k), p_2(k), p_3(k), \dots, p_{N^*}(k)\right)^T \in \mathrm{R}^{N^*}, \quad (2.46)$$

where $p_i(k) = \mathrm{P}(X^*(k) = i)$. The values of individual elements of matrices $\mathbf{P}^*(t)$ are based on transition probability matrix $\mathbf{Q}^*(k) = \left(q^*_{ij}(k)\right)_{i,j \in S^*}$ which can be calculated based on the probabilities from transition probability matrices $\mathbf{Q}(k) = \left(q_{ij}(k)\right)_{i,j \in S}$ determined for classical model (S,T) before modification. To describe the probabilistic structure of $\{X^*(k)\}$ in the whole insurance period that is for any moment $k = 0, 1, 2, \dots, n$, introduce

$$\mathbf{D} = \begin{pmatrix} \mathbf{P}^*(0)^T \\ \mathbf{P}^*(1)^T \\ \vdots \\ \mathbf{P}^*(n)^T \end{pmatrix} \in R^{(n+1)\times N^*}. \tag{2.47}$$

Moreover, $cf_j^*(k)$ (cash flows arising from MMSM) are based on future cash flows for MSM as follows:

$$cf_j^*(k) = \begin{cases} p_j(k) + \pi_j(k) + \ddot{b}_j(k) + b_j(k) + d_j(k) + c_j(k) & \text{for} \quad j \in S^+ \\ p_j(k) + \pi_j(k) + \ddot{b}_j(k) + b_j(k) + d_j(k) & \text{for} \quad j \in S \end{cases}. \tag{2.48}$$

Since in the modified multiple-state model (S^*, T^*), all cash flows depend only on given moment k and state j in which process $\{X^*(t)\}$ is found, that is, on information $X^*(t) = j$, the cash flow stream during the duration of the insurance contract can be expressed using a matrix.

Let $\mathbf{C} = \left(cf_i^*(k)\right)_{\substack{i = 1,2,\ldots,N^* \\ k = 0,1,\ldots,n}}$ denote $(n+1) \times N^*$ *cash flows matrix*. Cash flow $cf_i^*(k)$ is the sum of inflows and outflows to a particular fund. Then for the insurer's total loss fund $\mathbf{C} = \mathbf{C}_{in} + \mathbf{C}_{out}$, where $\mathbf{C}_{in}$ consists only of an income to this fund (benefits of type $b, \ddot{b}, d, c$) and $\mathbf{C}_{out}$ consists only of an outgo from this fund (premiums of type π, p).

Based on (Dębicka, 2013), if A1 and A2 are satisfied, then for (S^*, T^*), the equivalence principle can be rewritten in the following form:

$$\mathrm{E}(Z) = \mathbf{M}^T Diag\left(\mathbf{C}\mathbf{D}^T\right)\mathbf{S} = 0, \tag{2.49}$$

where $\mathbf{M} = (m_0, m_1, \ldots, m_n)^T \in R^{n+1}$ and $m_k = \mathrm{E}(v_k)$. Moreover $\mathbf{S} = (1, 1, \ldots, 1)^T \in R^{N^*}$ and for any $\mathbf{A} = \left(a_{ij}\right)_{i,j=1}^{n+1}$ matrix $Diag(\mathbf{A})$ is defined as follows:

$$Diag(\mathbf{A}) = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_{n+1n+1} \end{pmatrix}. \tag{2.50}$$

Based on (2.49), equation (2.40) has the matrix form as follows:

$$\mathbf{M}^T Diag\left(-\mathbf{C}_{out}\mathbf{D}^T\right)\mathbf{S} = \mathbf{M}^T Diag\left(\mathbf{C}_{in}\mathbf{D}^T\right)\mathbf{S}. \tag{2.51}$$

From (2.51) we obtain matrix formulas for the following actuarial values:

- *net single premium* π payable at the time of concluding the insurance contract (when $X(0) = 1$) (cf. (2.41))

$$\pi = \pi_1(0) = \mathbf{M}^T Diag\left(\mathbf{C}_{in}\mathbf{D}^T\right)\mathbf{S}, \tag{2.52}$$

- *net period premium* with fixed amount p, payable for the first m units of the insurance contract if $X(t) = 1$ for $t = 0,1, \dots, m-1$ (cf. (2.42))

$$p = \frac{\pi}{\ddot{a}_{11}(0,m)} = \frac{\mathbf{M}^T Diag\left(\mathbf{C}_{in}\mathbf{D}^T\right)\mathbf{S}}{\mathbf{M}^T\left(\mathbf{I} - \sum_{k=m+1}^{n+1}\mathbf{I}_k\mathbf{I}_k^T\right)\mathbf{D}\mathbf{J}_1}, \tag{2.53}$$

- n-year *temporary life annuity due* – annuity instalment equal to 1 unit (cf. (2.35))

$$\ddot{a}_{11}(0,n) = \mathbf{M}^T\left(\mathbf{I} - \mathbf{I}_{n+1}\mathbf{I}_{n+1}^T\right)\mathbf{D}\mathbf{J}_1, \tag{2.54}$$

- n-year *immediate life annuity* – annuity instalment equal to 1 unit (cf. (2.36))

$$a_{11}(0,n) = \mathbf{M}^T\left(\mathbf{I} - \mathbf{I}_1\mathbf{I}_1^T\right)\mathbf{D}\mathbf{J}_1, \tag{2.55}$$

where

$$\mathbf{I}_k = \Big(0, \dots, 0, \underset{k}{1}, 0, \dots 0\Big)^T \in R^{n+1} \text{ for } k = 1,2, \dots, n+1, \tag{2.56}$$

$$\mathbf{J}_i = \Big(0, \dots, 0, \underset{i}{1}, 0, \dots 0\Big)^T \in R^{N^*} \text{ for } i = 1,2, \dots, N^*, \tag{2.57}$$

and $\mathbf{I}$ is identity matrix of size $(n+1) \times (n+1)$.

The matrix representation of actuarial values, as expressed in equations (2.52) to (2.55), not only facilitates the calculation of net premiums and life annuities, but also decomposes the doubly stochastic nature of Z. Matrix $\mathbf{D}$ is contingent solely upon the distribution of process $\{X^*(t)\}$, while $\mathbf{M}$ is contingent solely upon the interest rate. Furthermore, matrix $\mathbf{C}$ depends on cash flows and describes the type of insurance contract.

Note that if it is assumed that the interest rate is constant throughout the insurance period, then $\mathbf{M} = \mathbf{V}$ in all equations from (2.49) to (2.55). In other cases, matrix $\mathbf{M}$ elements must be calculated appropriately; for example, for the interest rate modelled by a stochastic process, the explicit forms of the components of matrix $\mathbf{M}$ can be found, among others, in (Dębicka, 2003, 2012, 2013; Dębicka et al., 2022; Marciniuk, 2004, 2009).

Example 2.1. *Multistate insurance contract – continuation*

Section 2.4 considers the stream of actuarial payment functions arising from n-year endowment insurance contract with supplementary insurances, i.e. UA, RJL, PD. Since this insurance contract has cash flows of type c (symbolically indicated by bolded red arrows in Figure 2.6), a determination of MMSM is needed. First of all, it is necessary to check whether assumptions B1 and B2 are met. For this purpose, control array $W(k)$ is defined

$$W(k)=\begin{bmatrix} - & c & 2c & + & \bar{c} \\ - & - & - & - & - \\ - & - & - & - & - \\ + & c & 2c & - & \bar{c} \\ - & c & 2c & - & - \end{bmatrix}. \tag{2.58}$$

Note that assumption B2 is fulfilled because the values of non-zero benefits are the same in each column of the verification table. Additionally, assumption B1 is met because, in each of the columns of the verification table, there are non-zero benefit values or symbol + (no column contains both + and non-zero benefit values simultaneously). Hence, according to Figure 2.7, the modified multistate model had to be constructed using the recursive procedure described in (Dębicka, 2012, 2013). After applying this procedure, the modified multistate model of MSM given by the above (2.4) has eight states and fifteen direct transitions between them:

$$(S^*,T^*)=\left(\left\{1,2^+,2,3^+,3,4,5^+,5\right\},\left\{\left(1,2^+\right),\left(1,3^+\right),(1,4),\left(1,5^+\right),\ldots,(5,3)\right\}\right). \tag{2.59}$$

The modified MMSM (2.59) is illustrated in Figure 2.9, where states are associated with the health-work situation of the insured person, which means that the insured:

1 – is healthy and works,
2^+ – died of natural causes,
2 – has been dead of natural causes for at least a year,
3^+ – died due to an accident,
3 – has been dead due to an accident for at least a year,
4 – is healthy and does not work,
5^+ – has become disabled (does not work),
5 – is disabled (does not work).

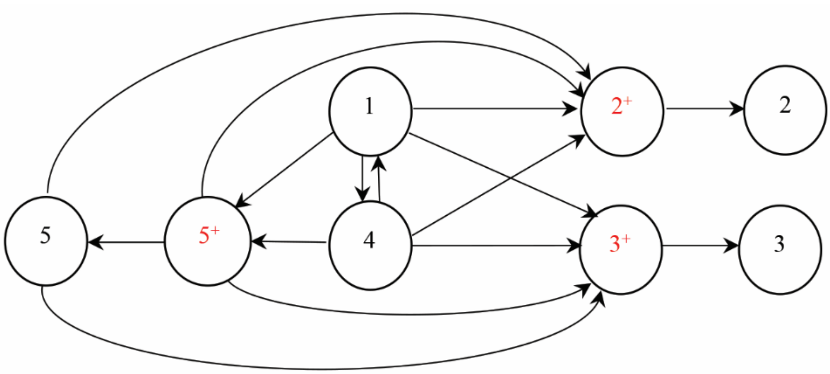


Figure 2.9. Modified multistate model for endowment insurance + UA + RJL + PD

Source: own elaboration.

Note that the cash flows associated with state changes have been converted to cash flows related to the presence of process $\{X^*(t)\}$ in states (marked with a + symbol in

(2.59)). Therefore, in Figure 2.9, none of the direct transitions are of type *cf* (none of the arrows are red as in Figure 2.6) and lump sums payable under the terms of the endowment insurance contract and supplementary insurances UA and PD are defined as follows (cf. (2.25)):

$$c_j(k) = \begin{cases} c(k) & \text{for } j = 2^+ \text{ and } k = 1,2,3 \dots, n \\ 2 \cdot c(k) & \text{for } j = 3^+ \text{ and } k = 1,2,3 \dots, n \\ \bar{c} & \text{for } j = 5^+ \text{ and } k = 1,2,3 \dots, n \\ 0 & \text{otherwise} \end{cases}, \tag{2.60}$$

where $c(k)$ denotes the benefit payable at time k in the event of the insured person's death in time interval $[k-1, k)$ and $\bar{c}$ is a lump sum disability benefit according to the terms of the supplementary insurance PD. The other benefits in this model are as follows (cf. (2.22)-(2.24))

- endowment benefit

$$d_j(k) = \begin{cases} d & \text{for } j = 1,4,5^+,5 \text{ and } k = n \\ 0 & \text{otherwise} \end{cases}, \tag{2.61}$$

- annuity benefit payable under the terms of supplementary insurance RJL

$$\ddot{b}_j(k) = \begin{cases} \ddot{b} & \text{for } j = 4, 5^+, 5 \text{ and } k = 1,2,3, \dots, n-1 \\ & \text{and } j = 5 \text{ and } k = 2,3,4, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}, \tag{2.62}$$

- annuity benefit payable per the conditions of supplementary insurance PD

$$b_j(k) = \begin{cases} b & \text{for } j = 5^+ \text{and } k = 1,2,3, \dots, n-1 \\ & \text{and } j = 5 \text{ and } k = 2,3,4, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.63}$$

Based on (2.60) to (2.63), the cash flow matrices resulting from the implementation of this contract can be determined as follows:

$$\boldsymbol{C} = \begin{pmatrix} -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p & c(1) & 0 & 2 \cdot c(1) & 0 & \ddot{b} & \bar{c} + \ddot{b} + b & 0 \\ -p & c(2) & 0 & 2 \cdot c(2) & 0 & \ddot{b} & \bar{c} + \ddot{b} + b & \ddot{b} + b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -p & c(n-1) & 0 & 2 \cdot c(n-1) & 0 & \ddot{b} & \bar{c} + \ddot{b} + b & \ddot{b} + b \\ d & c(n) & 0 & 2 \cdot c(n) & 0 & d & d + \bar{c} + b & d + b \end{pmatrix}, \tag{2.64}$$

$$\boldsymbol{C}_{in} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c(1) & 0 & 2 \cdot c(1) & 0 & \ddot{b} & \bar{c} + \ddot{b} + b & 0 \\ 0 & c(2) & 0 & 2 \cdot c(2) & 0 & \ddot{b} & \bar{c} + \ddot{b} + b & \ddot{b} + b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c(n-1) & 0 & 2 \cdot c(n-1) & 0 & \ddot{b} & \bar{c} + \ddot{b} + b & \ddot{b} + b \\ d & c(n) & 0 & 2 \cdot c(n) & 0 & d & d + \bar{c} + b & d + b \end{pmatrix}, \tag{2.65}$$

$$\boldsymbol{C}_{out} = \begin{pmatrix} -p & 0 & 0 & \cdots & 0 \\ -p & 0 & 0 & \cdots & 0 \\ -p & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.66}$$

For n-year endowment insurance contract with supplementary insurances (i.e. UA, RJL, PD), the transition probability matrix $\boldsymbol{Q}^*(k) = \left(q^*_{ij}(k)\right)_{i,j \in S^*}$ for the MMSM given by (2.59) takes the following form (compare matrix $\boldsymbol{Q}(k)$ defined in the last row of Table 2.3):

$$\boldsymbol{Q}^*(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) & 0 & q_{13}(k) & 0 & q_{14}(k) & q_{15}(k) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ q_{41}(k) & q_{42}(k) & 0 & q_{43}(k) & 0 & q_{44}(k) & q_{45}(k) & 0 \\ 0 & q_{52}(k) & 0 & q_{53}(k) & 0 & 0 & 0 & q_{55}(k) \\ 0 & q_{52}(k) & 0 & q_{53}(k) & 0 & 0 & 0 & q_{55}(k) \end{pmatrix}. \tag{2.67}$$

Utilising the matrices specified for MMSM given by (2.59), insurance premiums for the contract described in Example 2.1 can be determined by using formulas (2.52) and (2.53).

Note that matrix notation can be used not only to determine net premiums, but also for other actuarial quantities essential in valuing insurance contracts. In particular, in the case of individual contracts, it can be employed to (Dębicka, 2012):

- determine the safety loading on the gross premium,
- allocate the net premium into savings and risk-specific components,
- quantify insurance reserves,
- assess policy profitability, e.g. expected cash flow for year k, profit expected to emerge at the end of year k (Dębicka et al., 2016).

Matrix notation is also useful in issues related to group insurance, indexing insurance policies and optimising the selection of insurance packages.

Moreover, in (Dębicka & Zmyślona, 2018), we can find the matrix formula for specifying the premiums paid in more than one state. A similar issue to that of determining premiums is calculating insurance benefits based on the equivalence principle. Matrix formulas for annuities can be found in (Dębicka et al., 2022), as well as in Section 3.2 for equity release contracts and in Section 4.2 for marriage contracts. Matrix formulas for benefits are also presented in Section 5.2.

Chapter 3

INDIVIDUAL CONTRACTS

3.1. Life insurance and annuities

In the actuarial literature, life insurance and annuities are classified in various ways depending on the objectives and specifics of the contracts, e.g. (Booth et al., 2020; Bowers et al., 1986; Gerber, 1995; Haberman & Pitacco, 2018; Pitacco et al., 2009). Generally, these classifications can be divided into several categories that facilitate the modelling and analysis of these financial products. This section presents a selected classification that is helpful in the context of modelling to introduce matrix formulas for actuarial quantities, whilst it also provides a background to illustrate how seniors can benefit from owning life insurance policies.

Insurance companies currently operating in the life insurance market offer a wide range of diverse individual contracts. Life insurance policies differ based on their purpose, duration, and type of benefit and are generally classified according to these criteria.

When dividing life insurance policies based on contracts that meet different financial and protective needs (purpose), the following types of policies can be distinguished:

- *Term Life Insurance:* the insurer guarantees a payout only in the event of the insured's death within the policy period, i.e. within *n*-years after the insurance contract starts. This type of insurance usually has the lowest premiums compared to other types of life insurance.
- *Whole Life Insurance*: this provides coverage for the insured's entire life. These policies also include an investment component that allows cash value accumulation. Premiums for this type of insurance are typically paid by the insured throughout their lifetime and are higher than those for term life insurance due to the investment component.
- *Universal Life Insurance*: similarly to whole life insurance, this type of policy includes both savings and insurance elements. However, during the policy term, the insured can change the amount of premiums and benefits according to their current needs, offering unlimited flexibility in managing the saved capital. Due to the flexible premiums and the ability to adjust the insurance amount, this type of policy is also known as *Flexible Whole Life* or, due to the insurance period, *Lifetime Cover Life*.

- *Variable Life Insurance*: this type of policy provides lifetime coverage with the possibility of investing part of the premium in various investment funds. As a result, the value of the benefits can increase or decrease depending on investment performance. Therefore, it is characterised by higher risk and potentially higher returns.
- *Variable Universal Life Insurance*: this policy combines universal and variable insurance features, offering flexible premiums and the ability to invest in various funds.
- *Group Life Insurance:* these are individual policies offered by employers or organisations as part of benefits for employees or members. Such policies are usually easier to obtain than private individual policies because they do not require a detailed insurance application form. However, insurance coverage is often limited to the period of employment with a given company or membership in an organisation.
- *Credit Life Insurance:* under this policy, the insured ensures the repayment of the remaining loan balance in the event of their death. This insurance contract is carried out with mortgage, car, or consumer loans.
- *Life Insurance with Long-Term Care Rider*: this policy includes benefits related to the insured's health and combines traditional life insurance with protection in the event of a need for long-term medical care. The benefits usually cover care costs in a nursing home, assisted living facility, or home care.

Another key classification criterion for life insurance is the division based on the length of the insurance contract (i.e. the insurance period), allowing for better adjustment of the type of coverage to individual needs and life situations. The following groups of insurance can be distinguished:

- *Short-term Life Insurance*: provides coverage for a specified period, usually from 1 year to 30 years. After the coverage period ends, the policy expires unless it is renewed. An example of this type of contract is Term Life Insurance.
- *Long-term Life Insurance* (also known as *Whole Life Insurance*): provides coverage for the insured's entire life. The policy accumulates cash value, which can be used as savings or investments. Examples of this type of contract include *Whole Life Insurance*, *Universal Life Insurance*, and their variants that allow for investing the savings component of the premium in investment funds, such as *Variable Life Insurance* and *Variable Universal Life Insurance*.
- *Specialised Life Insurance*: this can be either short-term or long-term, depending on the policy conditions, such as *Group Life Insurance* or *Life Insurance with Long-Term Care Rider*.

The division of life insurance based on the type of benefit allows for distinguishing two basic groups:

- *Life Insurance*: the insurer commits to paying a lump-sum benefit if the insured dies during the insurance period or a benefit for survival if the insured lives for a specified number of years.

- *Annuities*: an insurance contract under which, in exchange for premiums received from the insured, the insurer commits to paying a series of benefits (called annuity payments), which typically cease upon the annuitant's death or after a specified period stated in the contract. They may also be paid to the beneficiaries after the insured's death.

It is important to note that life insurance and annuities serve different purposes and provide various forms of financial protection. Life insurance typically provides a lump-sum benefit in the event of the insured's death, while an annuity provides a regular income. Both types of contracts can be either short-term or long-term. They are described in more detail further in this section.

According to the basic actuarial classification, life insurance includes contracts for the risk of loss of life (temporary or whole life insurance) and survival of the insured to a specified age (pure endowment insurance). Endowment insurance is a combination of temporary life insurance and pure endowment insurance.

Annuities paid to the insured who is alive are called *Life Annuities* and include *Life Annuity Due* (annuity rates are paid from below, which means at the end of the time interval into which the insurance period has been divided) and *Immediate Life Annuity* (annuity paid in advance it means at the beginning of each period), both in term and lifetime forms. In actuarial science, other types of these annuities are also distinguished, such as:

- *Deferred Annuities*: annuity payments begin after a specified period following the premium payment, allowing for capital accumulation before payouts start.
- *Fixed Annuities*: annuity with fixed payment amounts paid throughout the contract period.
- *Increasing/Decreasing Annuities*: the insured receives regular payments that increase/decrease at a predetermined rate throughout the contract period.
- *Variable Annuities*: the payment amount varies based on the insurer's investment performance related to the savings portion. If the payment amount is tied to market indices, annuity growth is allowed based on market performance while protecting against losses. Such an annuity is called *Indexed Annuities*.

However, annuities paid out after the death of the insured person, known as *death benefit annuities*, include contracts such as:

- *Survivor's Annuity*: paid to the deceased's family members, such as spouse, children, or other dependent relatives.
- *Widow's/Widower's Pension*: intended for the deceased's spouse. It can be paid for the lifetime of the widow/widower or a specified period.
- *Orphan's Pension*: paid to the deceased's children who have not yet reached the age of majority or are still in full-time education. It can last until the children reach a certain age or complete their education.
- *Spousal Continuance*: part of the options available when entering into a pension agreement, where after the death of the main retiree, their spouse receives continued payments in full or reduced amounts.

- *Dependent's Pension*: intended for the spouse who supports other dependents, such as children or disabled relatives.

In addition to typical life annuities and death benefit annuities, there are *mixed annuities*, examples of which are:

- *Joint and Survivor Annuities*: paid during the insured's life and continued to a spouse or other beneficiary after his or her death.
- *Guaranteed Annuities*: paid for a specified period regardless of whether the insured is alive, with the option to continue payments to the beneficiary if the insured dies before the guaranteed period ends.

Each type of the aforementioned annuities has its own specific conditions and payment rules specified in the insurance contract.

Matrix notation can be used to model insurance contracts and annuities, which allows for a clearer and more efficient representation of actuarial values. Section 2.7 introduces the basic concepts and techniques for this notation, which will be applied to this group of contracts for valuation purposes.

The multistate model for life insurance and annuities is presented in Figure 2.2. However, it should be noted that each of the mentioned in this section groups of contracts is characterised by different types of benefits connected with the state of process $\{X(t)\}$ in a considered state, i.e. in state 1 (the insured lives) and/or state 2 (the insured dies), or as a result of transition process $\{X(t)\}$ from state 1 to state 2. In Figure 3.1, the direct transitions of *cf* type (associated with cash flows of c_1 type) are bolded and coloured in red, and the number of states in which cash flows of type *b*, *b*, *d* are realised, is marked in red. The different kinds of benefits allowed for the distinction of five life insurance and annuities subgroups.

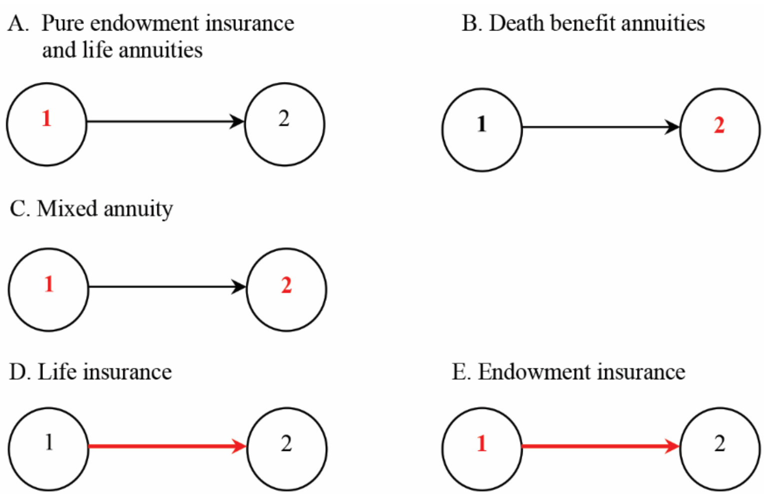


Figure 3.1. Cash flows on the multistate model life insurances and annuities

Source: own elaboration.

The MSM for the groups of contracts at points A-C in Figure 3.1 allow for the direct use of matrix notation to calculate the actuarial value because the only direct transition (1,2) is the *not-cf* type. In this situation, transitions matrix $\mathbf{Q}^*(k) = \mathbf{Q}(k)$ takes the form (2.17) and $\mathbf{P}^*(k) = \mathbf{P}(k)$ by (2.18). The appropriate definition of inflow matrix $\mathbf{C}_{in}$ for each contract allows for determining the single and periodic premiums using formulas (2.52) and (2.53), respectively.

However, in the case of life insurance (point D) and endowment insurance (point E), direct transition (1,2) is of the *cf* type. Conditions B1 and B2 (defined in Section 2.7) are satisfied because control array $W(k)$ is of the form:

$$W(k) = \begin{bmatrix} - & c_{12}(k) \\ - & - \end{bmatrix}.$$

Therefore, the MSM can be modified. According to Figure 2.7, the MMSM had to be constructed using the recursive procedure. After applying this procedure, the modified multistate model of MSM given by (3.1) has three states and two direct transitions between them:

$$(S^*, T^*) = \left(\left\{1, 2^+, 2\right\}, \left\{\left(1, 2^+\right), \left(2^+, 2\right)\right\}\right). \tag{3.1}$$

MMSM (3.1) is illustrated in Figure 3.2, where states mean that the insured:
1 – is alive,
2^+ – has died,
2 – has been dead for at least a year.

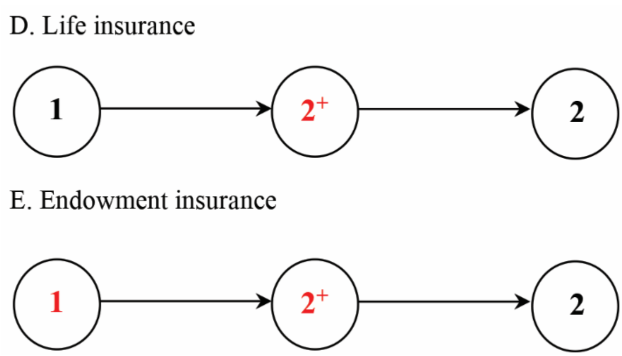


Figure 3.2. The modified multistate model for life and endowment insurance

Source: own elaboration.

For n-year temporary life and endowment insurance contract, transition probability matrix $\mathbf{Q}^*(k)$ for the modified multistate model given by (3.1) takes the following form (compare matrix $\mathbf{Q}(k)$ described by (2.17)):

$$\begin{aligned}\mathbf{Q}^*(k) = &\begin{matrix} q^*_{11}(k) & q^*_{12^+}(k) & q^*_{12}(k) \\ q^*_{2^+1}(k) & q^*_{2^+2^+}(k) & q^*_{2^+2}(k) \\ q^*_{21}(k) & q^*_{22^+}(k) & q^*_{22}(k) \end{matrix} = \begin{pmatrix} q_{11}(k) & q_{12}(k) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p_{x+k} & q_{x+k} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned} \tag{3.2}$$

The cash flow matrices resulting from the implementation of contracts can be determined for:

- life insurance contract

$$\boldsymbol{C}_{in} = \begin{matrix} 0 & 0 & 0 \\ 0 & c_{12}(1) & 0 \\ 0 & c_{12}(2) & 0 \\ \vdots & \vdots & \vdots \\ 0 & c_{12}(n-1) & 0 \\ 0 & c_{12}(n) & 0 \end{matrix}, \tag{3.3}$$

- endowment insurance contract

$$\boldsymbol{C}_{in} = \begin{matrix} 0 & 0 & 0 \\ 0 & c_{12}(1) & 0 \\ 0 & c_{12}(2) & 0 \\ \vdots & \vdots & \vdots \\ 0 & c_{12}(n-1) & 0 \\ d_1(n) & c_{12}(n) & 0 \end{matrix}. \tag{3.4}$$

Utilising the matrices (3.2)-(3.4) specified for MMSM given by (3.1), insurance premiums for life and endowment insurances can be determined using formulas (2.52) and (2.53).

In the case of seniors, their financial situation can drastically change after retirement. Studies show that in this age group, the needs related to maintaining good health increase while the resources available to meet these needs drastically decrease (see Section 1.3). This can lead to two primary problems: keeping the current lifestyle and securing funds for healthcare expenses. A natural solution in such situations is to use the capital accumulated so far. These are not always directly saved financial resources but also assets in the form of life insurance policies or real estate (see Section 1.4).

Note that seniors with a life insurance policy can obtain funds to improve their standard of living (daily or healthcare-related) in several ways, depending on the specific terms of their policy and their financial needs. Here are some potential options:

- *Policy Surrender*: seniors can choose to surrender their life insurance policy. In this case, they receive a lump-sum payment known as the cash surrender value, the insurance expires, and the insured loses the insurance coverage. The surrender value corresponds to the reserve of premiums minus the costs associated with policy termination, and depending on how long the insurance has been in force, it may be lower or higher than the total premiums paid by the insured. The closer to the end of the insurance period the surrender occurs, the closer the surrender amount is to the insurance sum (Bowers et al., 1986; Cole & Fier, 2021; Gerber, 1995). Surrender options are particularly beneficial in the case of so-called participating (or with-profits) life insurance policies (Grosen & Løchte Jørgensen, 2000).

- *Loan on Life Insurance Policy*: if the insurance policy terms allow for a loan, the senior can take out a loan using the accumulated cash value of the policy. It allows for obtaining funds without giving up the insurance coverage completely (Dawson, 1895; Gephart, 1914; Liebenberg et al., 2010; Wood, 1964).
- *Conversion to Annuity*: sometimes the terms of a life insurance policy offer the option to convert the policy's value into a life annuity. In this case, the senior is provided with regular monetary benefits (lifelong or term), which can help improve their standard of living (Millossovich & Biffis, 2006; Pla-Porcel et al., 2017). Such annuities can be fixed or variable/flexible (Direr, 2010; Feng et al., 2022).
- *Accelerated Death Benefits* (ADBs): life insurance coverage can be extended by purchasing an additional option that allows the insured to receive part of the death benefit while still alive, for example, if the insured is terminally ill. Beneficiaries can still receive a benefit after the insured's death, reduced by the amount received earlier. Sometimes insurance companies allow the purchase of an option for the accelerated payout of the death benefit upon diagnosis of a disease (Carannante et al., 2022; Dash & Grimshaw, 1993; Dębicka & Zmyślona, 2019; Haberman & Pitacco, 2018).
- *Selling the Policy on the Secondary Market*: the insured person can sell the rights to the benefit to third parties (an individual investor or a company specialising in buying insurance policies) for an amount greater than the surrender value but less than the death benefit called *Settlement Payment*. The investor then takes over the premium payments for the insurance and receives the full insurance benefit upon the insured's death. In the secondary insurance market, contracts for the resale of death benefit rights are offered to individuals diagnosed with a serious/terminal illness (*viatical settlement*) or elderly in good health (*life settlement or senior settlement*). Viatical settlement contracts are short-term (Hanewald et al., 2016), and the insured's decision is related to their health condition. In contrast, life settlement contracts are long-term, and the insured's decision is based on their financial situation (Bhuyan, 2011; Braun et al., 2016; Dębicka & Heilpern, 2018, 2020; Gatzert et al., 2009; Heilpern & Dębicka, 2020; Sood, 2003).

In these situations, it is important for the senior to fully understand the policy terms, financial consequences, and available alternative options before deciding on the use of the policy's value.

Note that each of the mentioned options involves the partial or complete loss of insurance coverage. One solution to avoid this situation is for seniors to use the ownership rights to their real estate. Considerations regarding the valuation of such possibilities are presented in Section 3.2. Another solution, when the senior is primarily concerned with securing funds for potential medical treatment, is to take advantage of the wide range of health insurance options detailed in Section 3.3.

3.2. Equity release

3.2.1. Sales and credit model in Poland

Equity release contracts are based on transferring a property to a company interested in acquiring it in exchange for a benefit (term or whole life, term or single) while ensuring that the owner can live in the property for whole life. These contracts are primarily designed for older people who often do not have enough money to live on but own property. However, most people do not want to sell their property and have even less desire to move from their homes. This is why equity release contracts are offered to retirees in the markets of many countries (Hanewald et al., 2016).

There are two main models for such contracts:

- loan (credit) model (reverse mortgage scheme),
- sales model (home reversion scheme).

In the loan model, the financial entity offering the contract undertakes to pay out a benefit and receive the property's right upon the owner's death. In the sales model, property ownership is transferred to the lender when signing the contract.

In Poland, both types of contracts theoretically function in the form of a *Reverse Mortgage* (credit model) and *Reverse Annuity Contract* (sales model) (Marciniuk, 2014; Kowalczyk-Rólczyńska, 2018). The reverse annuity contract was introduced in 2005 and is offered by several mortgage funds. The sales model contract is irreversible, and the heirs cannot recover the property. In this market the firms compete in offering various additions to the basic reverse annuity contracts. They take over maintenance fees and their price increases, pay the real estate tax and perpetual usufruct tax, and add free health insurance, property insurance, free legal advice, training and trips, etc. (Dębicka & Marciniuk, 2014). Reverse annuity contracts carry high risks (Marciniuk, 2017a, 2017b). The mortgage funds are not subject to the supervision of the Financial Supervision Authority. They are not obliged to maintain reserves and meet financial requirements. If the company taking over the property goes bankrupt, the older person, who formally does not own the flat, could be evicted by the company's creditors from the flat. The Civil Code (CC) governs clients' security, and if the company goes bankrupt, the only thing left to do is enforce their rights in court. This could be a problem that elderly, sick, and often lonely people will not be able to face.

A competing contract for mortgage annuities was to be the so-called reverse mortgage. In 2014, the Reverse Mortgage Act came into force (Ustawa z dnia

23 października 2014…). The contract is reversible. The agreement can be broken within 30 days of signing. The contract can also be terminated at any time with 30 days' notice by repaying the amount of the loan paid plus interest. When the bank defaults, the contract can be terminated too by repaying the disbursed portion of the loan. The heirs have the possibility of recovering the property after repayment of the loan.

The Act on Reverse Mortgage Loans regulates the safety of the transaction and the customers, according to which a Guarantee Fund (GF) must be established, subject to the Polish Financial Supervision Authority (PFSA). Currently, no financial institution has introduced this type of contract in its product offer.

A comparison of both contracts is presented in Table 3.1.

Table 3.1. The differences between the reverse annuity contract and reverse mortgage

Characteristic	The reverse annuity contract	The reverse mortgage
Nature of product	selling	credit
Age of customer	elderly people	young and elderly people
Instalment	period or whole life	single or period
Who can offer	mortgage funds	the state and the international financial institutions, subject to the supervisory authorities in the European Union Member States
Right to the property	passes after the signing of a notarial act	passes 12 months after the death of the owner
Security	governed by the CC	regulated by the Act – ensure the GF is subject to the PFSA
Breach of contract	impossible – the contract is irreversible	possible in specific cases
Bonus	yes	no

Source: (Biernacki et al., 2021; Dębicka & Marciniuk, 2014).

3.2.2. Benefit valuation

In order to calculate the benefits of the equity release contracts, it is necessary to introduce some notations and derive actuarial value formulas.

Consider two variants of equity release, i.e. the whole life reverse annuity contract and the term reverse mortgage. Life Tables are restricted by limit age ω, which can be either 100 or 110 depending on the used tables. Hence, a contract duration can be written in the form:

$$n = \begin{cases} n & \text{in case of reverse mortgage,} \\ \omega - x & \text{in case of reverse annuity contract.} \end{cases} \tag{3.5}$$

Divide the year into m ($m > 0$) equal parts. The division is conventional, for example, $m = 2$ means a division into half-years, $m = 4$ into quarters and $m = 12$ into months, which are not exactly equal in reality. This means that an n-year insurance period is divided into $n \cdot m$ equal time subperiods and in further considerations $k = 0, 1, 2, \ldots, n \cdot m$.

Let W denote the real value of a property, where α is the reduction factor of W. It is obvious that $\alpha \in$ 0%, 100%, but usually $\alpha \leq 50\%$. Let $b^{(m)}$ denote the benefit paid at the beginning of the subperiod of the year when the property's owner is alive. As a result of each of the analysed contracts, a cash flow stream is generated consisting of two types of payments (cf. Section 2.4):

- premium $\pi = \pi_1(0)$ for this contract is the real value of the property. It is a single fee paid immediately, hence

$$\pi_j(k) = \begin{cases} \alpha W & \text{for } j = 1 \text{ and } k = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{3.6}$$

- the only benefit arising from the terms of the mentioned contracts is the annuity payable in advance, thus

$$b_j(k) = \begin{cases} b^{(m)} & \text{for } j = 1 \text{ and } k = 1,2,3,\ldots,n \cdot m \\ 0 & \text{otherwise} \end{cases}. \tag{3.7}$$

Individual equity release contract with payments at the beginning of the subperiod of the year can be represented by the use of the two-states model. It is irrelevant whether there is a term contract (a reverse mortgage) or a whole life contract (a reverse annuity contract). In both cases, the model looks identical. It does not matter how often the benefit is paid. If $m = 1$, the payment is only once a year in the amount of 1 financial unit. If $m > 1$, there are m payments of $1/m$ per year. MSM for such contracts is presented in Figure 3.1 (scheme A).

To determine the probabilistic structure that accounts for the division of the year into smaller sub-periods, it is necessary to specify the survival and death probabilities within these sub-periods. Assume that x is the age of the property owner who has decided to enter into the contract. Therefore, let $K_x^{(m)}$ denote the future lifetime of the person at age x, where $K_x^{(m)}$ is defined in sub-periods of the year, i.e. $K_x^{(m)} \in \{0,1,\ldots,m \cdot \omega_x\}$. Probability ${}_{k/m}p_x$ is calculated in the generalised case from the formula (Marciniuk, 2016):

$${}_{k/m}p_x = P\left(\frac{K_x^{(m)}}{m} \geq \frac{k}{m}\right) = P\left(K_x^{(m)} \geq k\right) = {}_{[k/m]}p_x \cdot {}_{(k \div m)}p_{x+[k/m]}, \tag{3.8}$$

where $k = 0,1,2,\ldots$, $[a]$ denotes the integer part of the number a, while $(a \div b)$ represents the fractional part from dividing the numbers a and b.

If the distribution of deaths during the year is uniform (Bowers et al., 1986), then the formula (3.8) takes the following form:

$$ {}_{k/m}p_x = P\left(\frac{K_x^{(m)}}{m} \geq \frac{k}{m}\right) = {}_{[k/m]}p_x \cdot \left(1 - (k \div m) \cdot \left(1 - p_{x+[k/m]}\right)\right), \tag{3.9} $$

where $k = 0, 1, 2, \ldots$

Therefore, the probabilistic structure of the MSM is defined based on a sequence of transition matrices $\mathbf{Q}(k)$, which have the following form:

$$ \mathbf{Q}(k) = \begin{pmatrix} q_{11}^{(m)}(k) & q_{12}^{(m)}(k) \\ 0 & 1 \end{pmatrix}, \tag{3.10} $$

where $q_{11}^{(m)}(k) = {}_{k/m}p_x$, $q_{12}^{(m)}(k) = 1 - {}_{k/m}p_x$ and ${}_{k/m}p_x$ is denoted in the generalised case by formula (3.8).

Analogously to the methodology introduced in Section 2.6, formulas for the amount of annuity benefits payable m times a year for the discussed contracts are derived. In their considerations the authors focus on the assumption that the interest rate is constant throughout the insurance period.

For $m = 1$, annuity value $b^{(1)}$ can be directly determined based on the methodology developed in Section 2.6. Note that b, π are cash flows associated with the state of process $\{X(t)\}$ in a particular state, then based on (2.30) the discounted value at the beginning of contract ($t = 0$) of cash flow b, π realised at time k are as follows:

$$ \Upsilon_0^{\pi,j}(k) = \begin{cases} \alpha W & \text{for} \quad k = 0 \quad \text{and} \quad j = 1 \\ 0 & \text{otherwise} \end{cases}, \tag{3.11} $$

$$ \Upsilon_0^{b,j}(k) = \begin{cases} v^k 1_{\{X(k)=1\}} b^{(1)} & \text{for} \quad k = 0,1,2,\ldots,n-1 \quad \text{and} \quad j = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{3.12} $$

From (2.37) we obtain

$$ Z = \sum_{k=0}^{n-1} \Upsilon_0^{b,1}(k) - \alpha W. \tag{3.13} $$

Assuming that assumptions A1 and A2 are fulfilled and $P(X(0) = 1) = 1$, from (2.38) we have

$$ \mathrm{E}(Z) = \sum_{k=0}^{n-1} E\left(\Upsilon_0^{b,1}(k)\right) - \alpha W. \tag{3.14} $$

By applying (2.32) and (2.34) to (3.14), we obtain

$$ \mathrm{E}(Z) = \sum_{k=0}^{n-1} v^k q_{11}(0,k) b^{(1)} - \alpha W = b^{(1)} a_{11}(0,n) - \alpha W. \tag{3.15} $$

The insurance premium is a net premium if the equivalence principle is satisfied, that is $\mathrm{E}(Z) = 0$. Therefore, assuming that $\pi = \alpha W$ is the net premium for equity release contract, from (3.15) we have $b^{(1)}a_{11}(0,n) - \alpha W = 0$ and hence, directly, arrive at the formula for the annuity payment amount:

$$b^{(1)} = \frac{\alpha W}{a_{11}(0,n)}, \tag{3.16}$$

where $a_{11}(0,n)$ is given by (2.34).

In general (for $m \geq 1$), benefit $b^{(m)}$ is also determined using the equivalent principle. Note that in formula (3.16), only $a_{11}(0,n)$ depends on the period of insurance, and consequently on the future lifetime of owner $K_x^{(m)}$. Note that if $m = 1$ (as for (3.16)), the payment takes place only once a year in the amount of 1 unit. If $m \geq 2$, there are m payments of $1/m$ per year. Then an actuarial value of the individual life annuity paid m times during a year is specified as follows (Bowers et al., 1986):

$$a_{11}^{(m)}(0,n) = \begin{cases} a_{x:n|}^{(m)} = \dfrac{1}{m}\displaystyle\sum_{k=0}^{m\cdot n-1} v^{\frac{k}{m}}\ {}_{\frac{k}{m}}p_x & \text{in case of term life annuity} \\ a_x^{(m)} = \dfrac{1}{m}\displaystyle\sum_{k=0}^{m\cdot(\omega-x)-1} v^{\frac{k}{m}}\ {}_{\frac{k}{m}}p_x & \text{in case of whole life annuity} \end{cases}, \tag{3.17}$$

where v is yearly discount factor, ${}_{k/m}p_x$ is probability that the beneficiary at age x is alive for at least k sub-periods of a year. Note that when $m = 1$, the traditional actuarial notation for unit life annuities payable annually is as follows:

$$a_{11}^{(1)}(0,n) = a_{11}(0,n) = \begin{cases} a_{x:n|} = a_{x:n|}^{(1)} = \displaystyle\sum_{k=0}^{n-1} v^k\ {}_kp_x & \text{in case of term life annuity} \\ a_x = a_x^{(1)} = \displaystyle\sum_{k=0}^{\omega-x-1} v^k\ {}_kp_x & \text{in case of whole life annuity} \end{cases}. \tag{3.18}$$

Thus, formula (3.16) for general case ($m \geq 1$), i.e. for the annuity at m times during a year with the instalment equal to $1/m$ can be rewritten in the following form:

$$b^{(m)} = \frac{\alpha W}{a_{11}^{(m)}(0,n)}, \tag{3.19}$$

where $a_{11}^{(m)}(0,n)$ is given by (3.17).

It should be noted that it follows directly from (3.17) that

$$a_{11}^{(m)}(0,n) = \frac{1}{m}a_{11}(0,n \cdot m), \tag{3.20}$$

where $a_{11}(0,n \cdot m)$ is a unit annuity payable in advance over $n \cdot m$ units of time, assuming that the discount factor for a unit of time is equal to $v^{1/m}$ and survival probabilities are defined for sub-periods of a year.

The reverse mortgage is paid termly and the benefit is denoted by $b_{x:n|}^{(m)}$. The reverse annuity contract is a whole life benefit, adequately indicated by $b_x^{(m)}$. Using formulas (3.19) and (3.17), $b^{(m)}$can be written in the generalised form (see Lemma 3.1).

Lemma 3.1

Assume that the owner is at age x, the real value of property equals W, and the reduction factor of W is α. Let $b_{x:n|}^{(m)}$ denote the reverse mortgage's payment and $b_x^{(m)}$ the reverse annuity contract's payment, which both are paid m ($m > 0$) times yearly at the beginning of a sub-period of a year as long as the beneficiary is alive. The benefits are calculated according to the formula:

$$b^{(m)} = \begin{cases} b_{x:n|}^{(m)} = \frac{\alpha \cdot W}{a_{x:n|}^{(m)}} & \text{in case of reverse mortgage} \\ b_x^{(m)} = \frac{\alpha \cdot W}{a_x^{(m)}} & \text{in case of reverse annuity contract} \end{cases}, \tag{3.21}$$

where $a_{x:n|}^{(m)}$ and $a_x^{(m)}$are determined from (3.17).

Since equity release contracts do not result in *cf*-type cash flows, the matrix notation can be used directly to determine $b^{(m)}$ as formulated in Lemma 3.2.

Lemma 3.2

Let x be the owner aged at entry, the real value of property equals W, and the reduction factor of W is α. The benefits paid m ($m > 0$) times yearly at the beginning of a sub-period of a year as long as the beneficiary is alive are calculated as follows:

$$b^{(m)} = \frac{\alpha \cdot W}{m} \cdot \left(\mathbf{V}^T (\mathbf{I} - \mathbf{I}_{n \cdot m+1} \mathbf{I}_{n \cdot m+1}^T) \mathbf{D} \mathbf{J}_1 \right), \tag{3.22}$$

where $\mathbf{V} \in R^{n \cdot m+1}$is determined for a discount factor equal to $v^{1/m}$, where v is the annual discount factor. It means that

$$v_k = \left(v^{\frac{1}{m}} \right)^k. \tag{3.23}$$

Furthermore, matrix $\mathbf{D} \in R^{2 \times (n \cdot m+1)}$ is determined from the sequence of transient matrices $\mathbf{Q}(k)$ defined by (3.10).

The proof of Lemma 3.2 is based on applying formulas (2.54) to (3.20), and then applying the matrix variant of formula (3.20) directly to formula (3.19).

Benefit $b_x^{(m)}$ for reverse annuity contract is obtained by considering in formula (3.21) that $n = \omega - x$.

3.2.3. Empirical examples of benefits

The calculations presented in this section are made by using the authors' own programmes written in MATLAB. It is assumed that the value of the property $W = 100\,000$ euros and $\alpha = 50\%$. Probability ${}_{\frac{k}{m}}p_x$ was calculated based on the Life Table for Lower Silesia from 2011, obtained from Statistics Poland, assuming that the distribution of deaths during the year is uniform.

To determine the fixed rate of interest, the Svenson interest rate model was used. The parameters of function $R(t)$ were estimated using the least-squares method based on real Polish market data (Bossa.pl, 2024; Money.pl, 2024), related to the yield to maturity on fixed interest bonds and Treasury bills from 2008 (Dębicka et al., 2015). The estimation was made with the use of the Solver in Microsoft Excel, then obtaining (cf. (2.29)):

$$R(t) = \begin{array}{l} 0.0579 + 0.002\frac{0.667}{t}\left(1 - e^{-\frac{t}{0.667}}\right) - 0.002\left(\frac{0.667}{t}\left(1 - e^{-\frac{t}{0.667}}\right) - e^{-\frac{t}{0.667}}\right) \\ -0.0034\left(\frac{1.323}{t}\left(1 - e^{-\frac{t}{1.323}}\right) - e^{-\frac{t}{1.323}}\right) \end{array}. \quad (3.24)$$

Note that parameter β_0 corresponds to the long-term (infinite maturity) spot rate, and fixed interest rate r was chosen at the level of 5.79% (because long-term rate β_0 was equal to 0.0579). If interest rate r is higher, the value of benefit payment will also be higher, and *vice versa* (Marciniuk, 2014). Finally the elements of vector **V** are as follows:

$$v_k = v(0, k) = (1 + 0.0579)^{-k} = 0.9453^k. \quad (3.25)$$

Note that the annual discount factor amounted to 0.9453. However, when the annuity is paid more than once a year, then for the example sub-periods the discount factors (3.23) are shown in Table 3.2.

Table 3.2. The discount factors for sub-periods

m	1	4	6	12
$v^{1/m}$	0.945269	0.986027	0.990663	0.995320

Source: own elaboration.

This period was chosen to align the numerical analyses of the financial structure with the probabilistic structure of the health model, as both these structures are used in Chapter 5.

Example 3.1

Assume that the man and the woman are x years old ($x \in \{65,70,75,80,85\}$). The reverse annuity contract's benefit is paid m ($m \in \{1,4,6,12\}$) times yearly in the constant amount of $1/m$ financial unit at the beginning of a sub-period of a year as long as a person is alive. The benefit is calculated from (3.21) using Table 3.2. The results are presented in Table 3.3.

Table 3.3. The reverse annuity contract's benefit payment for woman and man at age x

x	m	Benefit payment		Yearly sum of benefit payment	
		woman	man	woman	man
65	1	4322.5	5085.6	4322.5	5085.6
	4	1117.3	1322.7	4469.2	5290.8
	6	747.7	885.7	4486.1	5314.2
	12	375.3	444.8	4503.0	5337.8
70	1	4908.1	5828.0	4908.1	5828.0
	4	1274.5	1524.8	5098.0	6099.2
	6	853.3	1021.7	5120.0	6130.2
	12	428.5	513.5	5141.9	6161.9
75	1	5809.8	6847.7	5809.8	6847.7
	4	1519.5	1806.2	6078.0	7224.8
	6	1018.1	1211.5	6108.6	7269.0
	12	511.7	609.4	6140.0	7313.0
80	1	7207.3	8272.5	7207.3	8272.5
	4	1905.9	2207.1	7623.6	8828.4
	6	1278.7	1482.3	7672.2	8893.8
	12	643.4	746.7	7721.3	8960.2
85	1	9285.9	10319.0	9285.9	10319.0
	4	2495.9	2798.4	9983.6	11193.6
	6	1677.7	1883.1	10066.2	11298.6
	12	845.9	950.4	10150.4	11405.2

Source: own elaboration.

It can be observed that the benefit is lower for women than for men. The yearly benefit increases with the increasing frequency of payments and the owner's age. The increase of m causes the highest difference between annuities for $m = 1$ and $m = 4$. For higher m the rise is not so significant.

Example 3.2

Assume that the man and the woman are x years old ($x \in \{65,70,75,80,85\}$). The reverse mortgage's benefit is paid m ($m \in \{1,4,6,12\}$) times yearly in the constant amount of $1/m$ financial unit at the beginning of a sub-period of a year as by n =10 years. The benefit is calculated based on Lemma 3.1. The results are presented in Table 3.4.

Table 3.4. The reverse mortgage's benefit payment for woman and man at age x

x	m	Benefit payment		Yearly sum of benefit payment	
		woman	man	woman	man
65	1	6726.4	7184.3	6726.4	7184.3
	4	1727.5	1857.8	6910.0	7431.2
	6	1155.2	1243.2	6931.2	7459.2
	12	579.3	624.0	6951.6	7487.6
70	1	6937.2	7587.2	6937.2	7587.2
	4	1788.2	1973.4	7152.8	7893.6
	6	1106.2	1321.5	6637.2	7929.0
	12	600.2	663.7	7202.4	7964.4
75	1	7384.2	8212.0	7384.2	8212.0
	4	1917.2	2154.3	7668.8	8617.2
	6	1283.6	1444.0	7701.6	8664.0
	12	644.5	726.0	7734.0	8711.5
80	1	8287.9	9201.8	8287.9	9201.8
	4	2179.2	2444.6	8716.8	9778.4
	6	1461.1	1641.0	8766.6	9846.0
	12	734.7	826.2	8816.6	9914.4
85	1	9864.1	10816.0	9864.1	10816.0
	4	2643.0	2926.4	10572.0	11705.6
	6	1776.0	1968.7	10656.0	11812.2
	12	895.1	993.3	10741.2	11919.6

Source: own elaboration.

In this case, the benefit is also lower for women than for men. The yearly pension increases with the increasing frequency of payments and the owner's age. The payout of a reverse mortgage is higher than that of a reverse annuity contract, with the differences for older people getting smaller.

Example 3.3

Assume that the man and the woman are x years old ($x \in \{65,70,75,80,85\}$). The reverse mortgage's benefit is paid in the constant amount of one financial unit at the beginning of a sub-period of a year as by n ($n \in \{5,10,15,20\}$) years. The benefit was calculated based on (3.21). The results are presented in Table 3.5.

Table 3.5. The reverse mortgage's benefit payment for woman and man at age x and different n

x	n	Benefit payment	
		woman	man
65	5	11418.0	11745.0
	10	6726.4	7184.3
	15	5282.8	5855.3
	20	4679.1	5345.0
70	5	11551.0	12022.0
	10	6937.2	7587.2
	15	5594.0	6373.6
	20	5102.7	5967.8
75	5	11824.0	12451.0
	10	7384.2	8212.0
	15	6222.4	7173.8
	20	5883.3	6900.4
80	5	12409.0	13113.0
	10	8287.9	9201.8
	15	7388.9	8415.3
	20	7207.3	8272.5
85	5	13478.0	14205.0
	10	9864.1	10816.0
	15	9285.9	1019.0
	20	–	–

Source: own elaboration.

It can be seen that benefits decrease as the length of the contract n increases, and the shorter the contract length, the significantly higher the benefit differences.

3.3. Health insurance

3.3.1. Classification of health insurance

Health insurance is the most complex form of insurance, consisting of (Suchecka, 2010):

1) universal medical (health) care – a broadly understood health insurance programme in which the whole society is entitled to benefits in the area of healthcare services,

2) contract for the provision of services – a type of insurance cover in which the provision of a service is guaranteed (not a cash benefit),
3) contract for the coverage of medical services – a type of insurance cover equated with health expenditure insurance,
4) sickness insurance, which means receiving sickness benefit.

Health insurance should fulfil the following (Suchecka, 2010):

1) a compensation function resulting from insurance cover,
2) a financial function of capital accumulation,
3) a preventive function aimed at limiting the occurrence of adverse events,
4) a function that stimulates technological progress.

Compulsory health insurance is not able to perform all the functions, so private health insurance can be used as a supplement, which is voluntary and allows patients to have faster access to tests, treatments and visits to specialists, and to receive treatment which is not financed by compulsory insurance. Private health insurance is especially useful for long-term treatment, long-term care and/or palliative care. In these situations, financial assets from health insurance policy also allow to raise the patient's quality of life. They play an important role especially in the case of elderly people, whose budget is exposed to large expenses generated by health care. Health insurance belongs to the group of personal insurance in which the subject of protection is life, health and the ability to work, and all the related financial consequences. This can be an alternative to financing health-related expenses from other sources, such as out-of-pocket expenses and savings.

Insurance products cover a range of risks which may be caused in the case of emergencies such as illness, accident, loss of income due to inability to work, and due to expenses related to hospital stay, long-term treatment and care and/or rehabilitation. The contract provides various types of benefits. The basic classification distinguishes the monetary benefits and service benefits, where the amount of monetary benefits amount in case of monetary ones can be predefined (fixed or related to health status) or expense-related (reimbursement benefits are designed to meet health cost), whilst the service benefits provide healthcare, e.g. the long-term service benefits. The duration of coverage by the health insurance may be one-year, multi-year or lifelong.

The following basic products can be distinguished in health insurance: sickness insurance, accident insurance, disability insurance, income protection, critical illness insurance and long-term insurance (Pitacco, 2014).

Accident Insurance ensures protection against a wide range of risk caused by an accident, first of all the risk of permanent disability or death and the others. The insurance policy can provide various types of benefits like death benefits, permanent disability benefits, reimbursement of medical expenses, a daily benefit during disability period (in case of temporary disability).

Sickness Insurance provides benefits in the event of illness. The most important benefit which is provided in this kind of insurance is reimbursement of medical expenses connected with hospitalisation, treatment in physician's surgery, or permanent disability caused by sickness. The amount of the benefit coverage depends on the terms and conditions agreed in the contract. There are three ways to estimate benefit amount: the fixed-amount deductible, the proportional deductible and stop--loss. The fixed-amount deductible is an amount that the insured has to pay out-of--pocket before the insurer will cover the remaining expenses. In cases of the proportional deductible amount, the insurer covers only a fraction of eligible medical expenses paid by the insured. The last kind, namely the stop loss, is determined as the maximum amount which is out-of-pocket for medical expenses by the insured.

Disability Insurance enables obtaining various types of cover in cases of temporary or permanent disability. Disability benefits are paid by individual disability insurance, group insurance or pension plans. There are other comprehensive names for these types of insurance, namely loss-of-income insurance, disability income insurance, and permanent sickness insurance. In an individual policy, the permanent or not necessarily permanent disability, is taken into account. Benefits can be paid as a fixed-amount annuity or a lump sum. Disability group insurance provides short-term disability benefits (it protects against loss of income during a short disability period) or the long--term disability benefits (it protects against long-term disabilities and those possibly permanent or lasting to retirement age). In cases of insurance included in pension plans, the benefits are paid in the form of an annuity to a disabled employee or a deferred annuity to a (permanently) disabled employee beginning at retirement age.

Long-term Care Insurance ensures financial protection for the insured in the case of nursing and/or medical care due to chronic health conditions or other ailments. Three types of long-term insurance products can be distinguished. Benefits in the first group of products have a predefined amount. The fixed-amount annuities or the degree-related benefits belong to this group. The second group benefits provide reimbursement of care expenses. The third group are care services benefits, which cover nursing and medical benefits.

Critical Illness or *Dread Disease Insurance* ensures financial benefits in cases of a severe illness such as heart attack, coronary artery diseases requiring surgery, cancer and stroke, which are paid on diagnosis and could be used in covering medical expenses or providing protection against possible loss of income. This type of insurance could be also arranged in the scope of life insurance as an additional benefit.

Modelling financial flows related to health insurance requires the introduction of multistate models. The form of the model depends on the type of contract affected by the benefits offered in the policy, the duration of the contract and whether the benefits are related to the health of the insured person. A broad overview of the models for health insurance is presented in (Dash & Grimshaw, 1993; Haberman & Pitacco, 1999; Pitacco, 2014). This section only describes those models that are used for the construction of critical illness insurance.

3.3.2. Multistate model for serious illness insurance

This section presents three types of multistate models that describe health insurance contracts associated with serious illnesses, known in the literature as dread diseases or critical illnesses.

Note that, in line with Chapter 2, the form of MSM (S, T) is not only defined by set space $S = \{1,2,\ldots,N\}$, which describes the individual risks connected with the insured and the set of direct transition between states $T = \{(i,j)\}$, but depends on the types of benefits paid during the insurance contract. Life events such as death, risk of illness, and accidents affect the health of the insured and are random in nature, so state changes are described by stochastic process $\{X(t);\ t \geq 0\}$. The risk of illness (accident) is measured by the population morbidity risk (probability of accident). The risk of illness (accident) is measured by the population morbidity risk (probability of accident). The study considered discrete-time model $\{X(t);\ t = 0,1,2,\ldots,n\}$ with initial state 1.

Model I. *Health and Accident Insurance with Lump Sum Benefit*

In the case of accident or health insurance, in which the benefit is paid out once in the event of an adverse occurrence. The multiple model takes the form:

$$(S, T) = (\{1,2,3\}, \{(1,2),(1,3),(2,3)\}), \tag{3.26}$$

where the individual states denote:
1 – the insured person is healthy or alive (A – alive),
2 – the insured person is ill / suffered an accident (I – ill),
3 – the insured person is dead (D – dead).

The model (3.26) is shown in Figure 3.3.

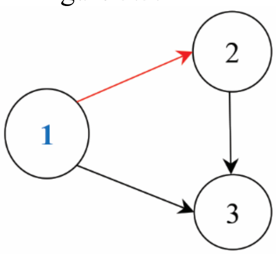


Figure 3.3. Multistate model for health and accident insurance with lump sum benefit

Source: own elaboration.

In Figure 3.3, the transition of process $\{X(t)\}$ from state 1 to state 2 is coloured red, which is associated with the lump sum benefit paid after an accident or illness; state 1 is coloured blue because the stay of process $\{X(t)\}$ in this state is associated with the fact that the insured pays premiums. The probabilistic structure of the analysed model is described by the transition matrix:

$$\mathbf{Q}(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) & q_{[13}(k) \\ 0 & q_{22}(k) & q_{23}(k) \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.27}$$

where state 3 is the absorbing state in that model. Transition probabilities (3.27) are obtained using the methodology of a multistate life table (increment-decrement table) (Bowers et al., 1986; Dębicka, 2012; Haberman, 1983; Jordan, 1982; Mattsson, 1977).

The values of probabilities depend on some characteristics connected with the whole population. The probabilities associated with staying in state 1 and transitions from state 1 to state 2 and 3 depend on the probability of death in population q_{x+k} and the morbidity (incidence) rate in population (or the probability of an accident) ζ_{x+k} for the insured at age $x + k$.

The *probability of death* in the general population is defined in demographic terminology as the all-cause mortality in the population and is defined as the number of deaths that occur in a given period, usually taken as the period of a calendar year. The all-cause mortality is called death rate or mortality rate, and is estimated for a person at age x as

$$q_x = \frac{\text{the number of deaths in the population of } x\text{-year-olds}}{\text{the number of the population of } x\text{-year-olds}}. \tag{3.28}$$

In actuarial science, this value is derived from life tables.

The *morbidity rate* measures the percentage of new cases in a given period in the studied population is given by the following formula:

$$\zeta_x = \frac{\text{the number of new cases in the population of } x \text{ years-olds}}{\text{the number of the population of } x \text{ years-olds}}. \tag{3.29}$$

The *disease-specific mortality rate* is defined as percentage of deaths caused by specific diseases or disease groups in a given period in the studied population as

$$\varpi_x = \frac{\text{the number of deaths caused by specific diseaes in the population of } x \text{ years-olds}}{\text{the number of the population of } x \text{ years-olds}}. \tag{3.30}$$

Then based on probabilities obtained from the theory of multiple increment-decrement table (or multiple state life table), the transition matrix (3.27) can be described as follows:

$$\mathbf{Q}(k) = \begin{pmatrix} 1 - \left(q_{x+k} - \varpi_{x+k}\right) - \zeta_{x+k} & \zeta_{x+k} & q_{x+k} - \varpi_{x+k} \\ 0 & 1 - q_{x+k} & q_{x+k} \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.31}$$

where x is the age of entry. The procedure for determining the formulas for transition probabilities is described in detail in (Dębicka & Zmyślona, 2016, 2019).

In the model presented in Figure 3.3, there is a cash flow associated with the transition from state 1 to state 2. Thus, in order to introduce the matrix notation to premium calculation, a modified model with 4 states should be introduced (according to procedure described in Section 2.7), where:
1 – the insured person is healthy or alive (A – alive),
2 – the insured person fell ill/suffered an accident (I + – fell ill),
3 – the insured person is ill for at least 1 unit of time (I),
4 – the insured person is dead (D – dead).
The MMSM is presented in Figure 3.4.

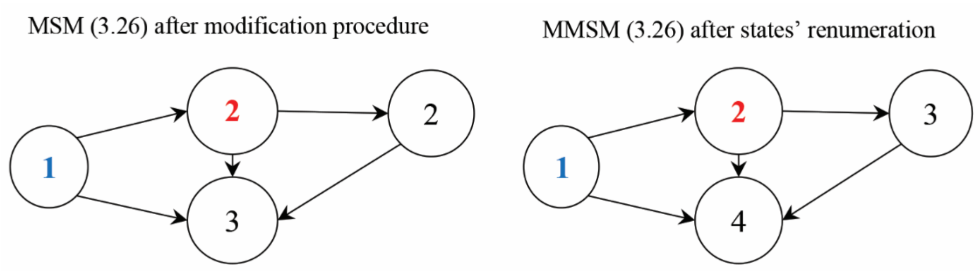


Figure 3.4. Modified multistate model for health and accident insurance with lump sum benefit

Source: own elaboration.

Note that the benefit is paid in state 2 (denoted in Figure 3.4 scheme B), which is coloured in red (instead of the red arrow between states 1 and 2 in Figure 3.3). The transition matrix for MMSM presented in Figure. 3.5 has the following form (cf. (3.31)):

$$\mathbf{Q}^*(k) = \begin{matrix} q^*_{11}(k) & q^*_{12}(k) & 0 & q^*_{14}(k) \\ 0 & 0 & q^*_{23}(k) & q^*_{24}(k) \\ 0 & 0 & q^*_{33}(k) & q^*_{34}(k) \\ 0 & 0 & 0 & 1 \end{matrix}$$
$$= \begin{matrix} 1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k} & \zeta_{x+k} & 0 & q_{x+k}-\varpi_{x+k} \\ 0 & 0 & 1-q_{x+k} & q_{x+k} \\ 0 & 0 & 1-q_{x+k} & q_{x+k} \\ 0 & 0 & 0 & 1 \end{matrix}, \tag{3.32}$$

where death probability q^*_{24} is defined as the mortality in the population of patients who suffer from given diseases and $q^*_{23} = 1 - q^*_{24}$ (analogously for state 3). The form of the transition matrixes $\mathbf{Q}^*(k)$ differs only in the 3rd and 4th columns, thus differentiating only the two last components $p_3(k)$ and $p_4(k)$ of the vector $P(k) = \left(p_1(k), p_2(k), p_3(k), p_4(k)\right)$.

The MMSM provides the following cash flows:

- a constant period premium, which is paid only when the insured is healthy (alive)

$$p_j(k) = \begin{cases} p & \text{for } j = 1; \;\; k = 0, 1, \dots n-1 \\ 0 & \text{otherwise} \end{cases}, \tag{3.33}$$

- a lump sum benefit c, which is paid at the moment of diagnosis (i.e. when the insured risk is in state 2 at time k)

$$c_j(k) = \begin{cases} c & \text{for } j = 2; \;\; k = 1, \dots n \\ 0 & \text{otherwise} \end{cases}. \tag{3.34}$$

Then the cash flow matrices are defined as follows

$$\boldsymbol{C} = \begin{matrix} -p & 0 & 0 & 0 \\ -p & c & 0 & 0 \\ -p & c & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -p & c & 0 & 0 \\ 0 & c & 0 & 0 \end{matrix} = \begin{matrix} -p & 0 & 0 & 0 \\ -p & 0 & 0 & 0 \\ -p & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & c & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & c & 0 & 0 \\ 0 & c & 0 & 0 \end{matrix} \tag{3.35}$$

$$= (\boldsymbol{C}_{out} + \boldsymbol{C}_{in}) \in R^{4 \times (n+1)}.$$

Using matrices (3.32) and (3.35) formula (2.53), the period premium p for benefit c can be determined.

Model II. *Health Insurance with Lump Sum Benefits*

If one considers such an insurance contract which provides benefits paid not only in the case of diagnosis on covering of treatment cost, but also in cases of the deterioration of a patient's condition on covering end-of-life cost. For that reason, two kinds of health condition are distinguished, mild and critical. For example, in cases of cancer, the mild stage could mean diagnosis without distant metastases, and the critical stage – diagnosis with distant metastases to other organs or inoperable tumours.

The financial resources are obtained in two moments of the course of the disease. First, a patient receives benefits at the moment of diagnosis which he/she can spend on treatment. The second moment is the deterioration of the health condition to the critical state. This is important because patients in a terminal state often need funds for palliative care because end-of-life costs are often very high in cases of dread illness. The risk of morbidity and the course of the disease in the mild state are described using the MSM model:

$$(S, T) = (\{1,2,3,4\}, \{(1,2), (1,3), (1,4), (2,3), (2.4), (3,4)\}) \tag{3.36}$$

presented in Figure 3.5, where the states have the following meaning:
1 – the insured person is healthy/alive (A – alive),
2 – the insured person is ill in mild stage (I – ill),
3 – the insured person is ill in critical stage (C – ill),
4 – the insured person is dead (D – dead).

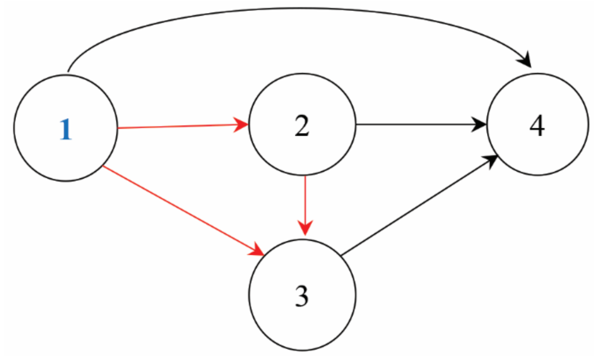


Figure 3.5. Multistate model for critical health insurance with lump sum benefits

Source: own elaboration.

The transition matrix for the MSM presented in Figure 3.5 has the following form:

$$\mathbf{Q}(k) = \begin{matrix} q_{11} & q_{12} & q_{13} & q_{14} \\ 0 & q_{22} & q_{23} & q_{24} \\ 0 & 0 & q_{33} & q_{34} \\ 0 & 0 & 0 & 1 \end{matrix} . \tag{3.37}$$

In this type of insurance contract, two types of lump sum benefits are considered. The first relates to the payment of a lump sum benefit at the time of diagnosis ($c_{12}(k) = c$), and the second relates to the payment of a lump sum benefit in the event of deterioration in health ($c_{13}(k) = c_{23}(k) = \hat{c}$), written as

$$c_{ij}(k) = \begin{cases} c & \text{for } (i,j) = (1{,}2); k = 1{,}2, \ldots, n \\ c & \text{for } (i,j) \in \{(1{,}3), (2{,}3)\}; k = 1{,}2, \ldots, n \\ 0 & \text{otherwise} \end{cases} . \tag{3.38}$$

The study considers contracts with a fixed premium paid in advance p throughout the insurance period (see blue coloured state 1 in Figure 3.6).

The considered model presented in Figure 3.5 was modified by adding two additional states which describe the situation that the insured fell ill in the mild state or in the critical state during a time period. After modification, we obtain a 6-state MMSM, in which the meaning of the states is as follows:

1 – the insured person is healthy/alive (A – alive),
2 – the insured person fell ill in mild stage during the last year (I+ – ill),
3 – the insured person is ill in mild stage for at least one year (I – ill in mild stage),
4 – the insured person fell ill in critical stage during the last year (C+ – ill in critical stage),
5 – the insured person is ill in critical stage for at least one year (C – ill in critical stage),
6 – the insured person is dead (D – dead).

The MMSM is presented in Figure 3.6 in scheme A, and after renumbering the states, in scheme B.

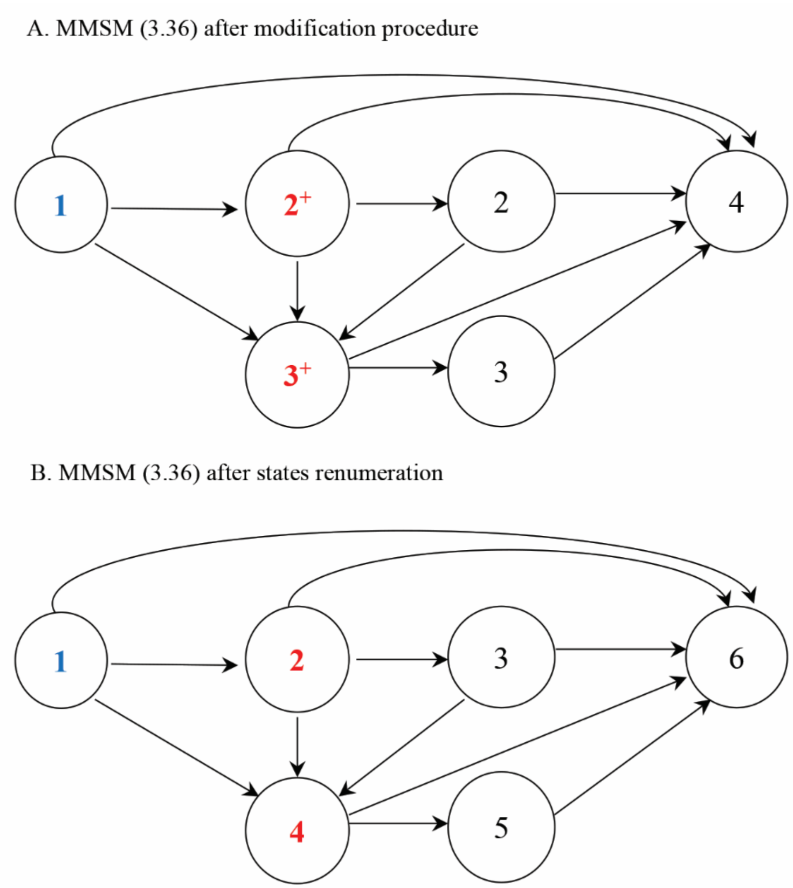


Figure 3.6. Modified multiple state model for health insurance with lump sum benefits

Source: own elaboration.

Besides the population parameters related to mortality, specific mortality and morbidity, the estimation of transition probability matrixes requires the inclusion of the parameters related to the patient population. This group of parameters depends on the characteristics connected with the population of the ill, namely the percentage of patients diagnosed in the critical stage, and the probability of health deterioration during the course of the disease. These parameters often depend on the patient's age and are therefore estimated by age groups or by a model that includes age as an independent variable. The following parameters are considered:

β_x – the percentage of x year-old patients who fell ill in a particular year with the first diagnosis in the critical stage,

ρ_x – the percentage of x year-old patients whose health status deteriorated in a given year.

The transition probability matrix for the MMSM presented in Figure 3.6 (scheme B) is given as (cf. (3.37))

$$\boldsymbol{Q}^*(k) = \begin{matrix} q_{11}(k) & q_{12}(k) & 0 & q_{14}(k) & 0 & q_{16}(k) \\ 0 & 0 & q_{23}(k) & q_{24}(k) & 0 & q_{26}(k) \\ 0 & 0 & q_{33}(k) & q_{24}(k) & 0 & q_{36}(k) \\ 0 & 0 & 0 & 0 & q_{45}(k) & q_{46}(k) \\ 0 & 0 & 0 & 0 & q_{55}(k) & q_{56}(k) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}. \tag{3.39}$$

Then the transition probabilities determined on the basis of the theory of multiple increment-decrement table (or multiple state life table) are given by the formulas (Dębicka & Zmyślona, 2016, 2019). Thus, for the first state

$$q_{11}(k) = 1 - \left(q_{x+k} - \varpi_{x+k}\right) - \zeta_{x+k}, \tag{3.40}$$

$$q_{12}(k) = \zeta_{x+k} \cdot \left(1 - \beta_{x+k}\right), \tag{3.41}$$

$$q_{14}(k) = \zeta_{x+k} \cdot \beta_{x+k}, \tag{3.42}$$

$$q_{16}(k) = q_{x+k} - \omega_{x+k}. \tag{3.43}$$

For the second and third state the probabilities are as follows:

$$q_{23}(k) = q_{33} = 1 - q_{x+k} - \rho_{x+k}, \tag{3.44}$$

$$q_{24}(k) = q_{34}(k) = \rho_{x+k}, \tag{3.45}$$

$$q_{26}(k) = q_{36}(k) = q_{x+k}. \tag{3.46}$$

In this contract, the mortality rate in the terminally ill population is estimated d_x (cf. (Dębicka & Zmyślona, 2019)). The transition probability connected with the fourth and fifth states are given by

$$q_{45}(k) = q_{55}(k) = 1 - d_{x+k}, \tag{3.47}$$

$$q_{46}(k) = q_{56}(k) = d_{x+k}. \tag{3.48}$$

Using (3.40)-(3.48), we can define the transition matrix (3.39) for the MMSM presented in Figure 3.6 (scheme B).

Two kinds of cash flow are distinguished in *Model II*. The period premium denoted by p is paid only when the insured is healthy and two types of lump sums. The cash flow matrix connected with this model is given by

$$\boldsymbol{C} = \begin{matrix} -p & 0 & 0 & 0 & 0 & 0 \\ -p & c & 0 & c & 0 & 0 \\ -p & c & 0 & c & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -p & c & 0 & c & 0 & 0 \\ 0 & c & 0 & c & 0 & 0 \end{matrix}. \tag{3.49}$$

Similarly to *Model I,* and using the formulas (3.49) and (2.53), it is possible to determine a fixed periodic premium p payable over the entire insurance period n.

Note that *Models I* and *Model II*, presented in this section, take into account one-off payments related to the diagnosis of a serious illness and deterioration of health. Funds from the benefit can be used to provide additional financial resources in cases of critical illness treatment costs and/or help improve living conditions and the quality of life. A one-off payment may not be sufficient for long-term treatment and/or palliative care. In this situation, it is necessary to develop a model that takes into account the duration of life during illness. Then, a disability benefit is more important than a lump sum, which is linked to the possibility of protecting the patient's financial situation due to the long-term illness condition, therefore *Model III* is introduced.

Model III. *Critical Health Insurance with Annuity Benefits*

The introduction of a model that takes into account multiple annuities (disability payments) for the seriously ill requires consideration of not only the future life expectancy, the risk of illness and the course of the disease. The patient's lifetime in a critical condition should also be modelled. The model which considers the above factors is presented in Figure 3.7.

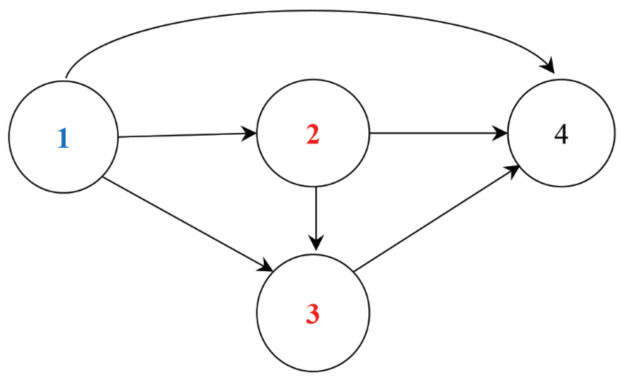


Figure 3.7. Multistate model for critical health insurance with annuity benefits

Source: own elaboration.

The model provides the following cash flows:

- constant period premium p, which is paid only when the insured is healthy (alive),

$$p_j(k) = \begin{cases} p & \text{for } j = 1; \ \ k = 0,1, \ldots n-1 \\ 0 & \text{otherwise} \end{cases}, \tag{3.50}$$

- disablement constant pension payable in advance in cases of the mild (state 2) or critical health condition (state 3),

$$b_j(k) = \begin{cases} b_2 & \text{for } j = 2, \quad k = 1,2, \ldots, n-1 \\ b_3 & \text{for } j = 3, \quad k = 1,2, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases}. \tag{3.51}$$

The application of Model III requires estimating the survival times for those insured under severe conditions. Since life expectancy decreases with disease duration in seriously ill patients, it is necessary to take into account this fact. To achieve this, an extended model divides state 3 into multiple states in order to describe changes in life expectancy during the critical stage. It is assumed that during the mild stage, a patient's life expectancy is similar to that of the general population at the same age, but in the critical stage, the patient may survive only g periods following diagnosis or deterioration of health. An example of such a model can be found in (Dębicka et al., 2015; Dębicka & Zmyślona, 2018).

The extended version of the MSM presented in Figure 3.7 is the scheme in Figure 3.8.

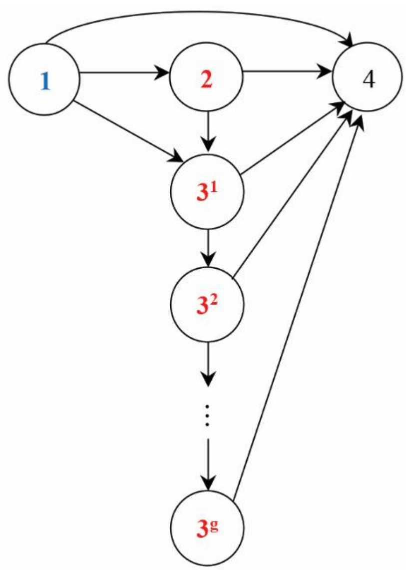


Figure 3.8. Extended multistate model for critical health insurance with annuities

Source: own elaboration.

Note that states $3^1, 3^2, \ldots, 3^g$ are termed reflex states, implying that the probability of process $\{X(t)\}$ remaining in these states is zero (the insured risk does not remain in these states for longer than one unit of time). Then the probabilistic structure of such an MSM model is described by

$$\boldsymbol{Q}^*(k) = \begin{matrix} q_{11}(k) & q_{12}(k) & q_{13^1}(k) & 0 & 0 & \ldots & 0 & q_{14}(k) \\ 0 & q_{22}(k) & q_{23^1}(k) & 0 & 0 & \ldots & 0 & q_{24}(k) \\ 0 & 0 & 0 & q_{3^13^2}(k) & 0 & \ldots & 0 & q_{3^14}(k) \\ 0 & 0 & 0 & 0 & q_{3^23^3}(k) & \ldots & 0 & q_{3^24}(k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_{3^{g-1}3^g}(k) & q_{3^{g-1}4}(k) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{matrix} . \tag{3.52}$$

The elements of the transition probability matrix (3.52) are determined analogously to *Model II*. *Model III* requires the estimation of the survival function determining the probability of death in the critical stage (the fatality rate) $d_x^{3^h}$ for $h = \{1, 2, \ldots, g\}$. The transition probabilities connected with the state 3 are given by

$$q_{3^h3^{h+1}}(k) = 1 - d_{x+k}^{3^h} , \tag{3.53}$$

$$q_{3^h4}(k) = d_{x+k}^{3^h} . \tag{3.54}$$

The modelling of the death probability in the critical stage (fatality rate) was based on the analysis of the population of patients in the critical stage. Typically, fatality rates increase with the duration of the disease, therefore $q_{3^14}(k) < q_{3^24}(k) < \ldots < q_{3^g4}(k)$.

The cash flow matrix connected with the MSM model presented in Figure 3.8 is given by

$$\mathbf{C} = \begin{matrix} -p & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -p & b_2 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ -p & b_2 & b_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -p & b_2 & b_3 & b_3 & \cdots & b_3 & 0 & 0 \\ -p & b_2 & b_3 & b_3 & \cdots & b_3 & b_3 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{matrix} . \tag{3.55}$$

Matrix $\mathbf{C}_{in}$ (i.e. the matrix given by equation (3.30), which has zeros in the first column) and $\mathbf{Q}(k)$ in the form (3.52) can be applied directly to equation (3.53) to determine constant period premium p and to equation (3.52) for net single premium π.

The MSM presented in Figure 3.8 can be used to create hybrids, together with other contracts such as life insurance (ADBs option), or the reverse annuity contract. Examples with numerical illustrations of such hybrids are shown in Chapter 5. The models described in this section are a very flexible group to characterise the cash flows associated with the one-off and the fixed benefits paid according to the health status.

Chapter 4

MARRIAGE CONTRACTS

4.1. Marital annuities and insurances

This chapter deals with marriage insurance contracts, a particular category of group insurance. They offer contracts that provide for a contingent payment in the event of the death of a spouse in exchange for a single or a series of periodic payments, the so-called premiums, which protect against serious financial consequences resulting from a person's death. There are several variants of marital life insurance, including life insurance (whole life or term life) and mixed life and investment policies. Each of these variants has its own features and benefits that can be tailored to the needs and preferences of the spouses. Life insurance for marriage can be important for couples who have a joint mortgage, children to raise, or other financial obligations. This is especially important for seniors when a single pension benefit is insufficient to maintain their current residence and cover the cost of living for a widow or widower. In the event of the death of one of the partners, the policy can help secure the remaining spouse against unforeseen financial difficulties.

Two categories of marriage insurance can be distinguished: *Last Surviving Status* (LSS) and *Joint Life Status* (JLS). In the first case, the benefit is paid to both partners during their lives and also to the surviving partner in the event of the death of the other one. In the second case, the lump sum or death annuity is paid till the death of each of the spouses. The multistate models for both variants of marital life insurance are presented in Figure 4.1.

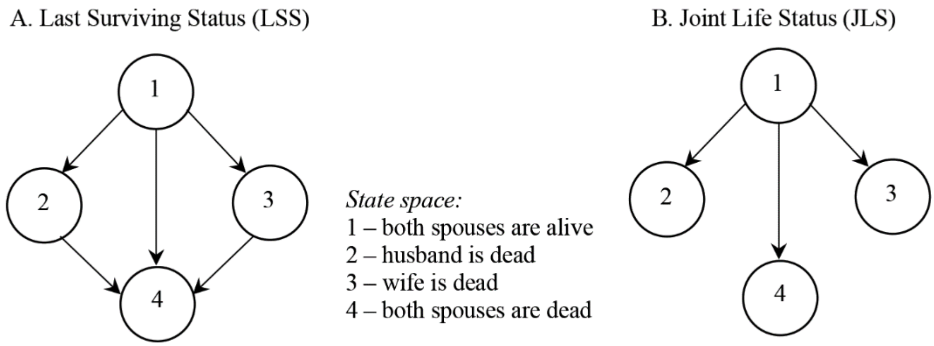


Figure 4.1. The multistate model for marriage insurance

Source: based on (Dickson et al., 2019).

Note that the multistate model for marriage insurance with LSS has the following form:

$$(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(2,4),(3,4)\}), \tag{4.1}$$

and the multistate model for marriage insurance with JLS:

$$(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4)\}). \tag{4.2}$$

Generally, for a n-year insurance contract, the transition probability matrix for the MSM described by (4.1) and the MSM given by (4.2), take the following forms, respectively:

$$\mathbf{Q}(k) = \begin{matrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) \\ 0 & q_{22}(k) & 0 & q_{24}(k) \\ 0 & 0 & q_{33}(k) & q_{34}(k) \\ 0 & 0 & 0 & 1 \end{matrix}, \tag{4.3}$$

$$\mathbf{Q}(k) = \begin{matrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}. \tag{4.4}$$

Note that (4.4) comes from (4.3) because $q_{22}(k) = q_{33}(k) = 1$ and $q_{24}(k) = q_{34}(k) = 0$ for JLS.

In order to define more precisely the probabilistic structure of the models, notations are introduced. Let x and y be the age at the policy issue of the wife and husband, respectively. Moreover, let T_x^W and T_y^M be the remaining lifetimes of the x-year-old woman (wife), and the y-year-old man (husband). These lifetimes take values in $[0, \omega_x^W]$ and $[0, \omega_y^M]$, where ω_x^W (respectively ω_y^M) denote the difference between border age ω (for example, 100 or 110 years old) of the man (respectively, woman) and y (respectively, x). Hence, the insurance period n depends on the status:

- $n = \max\{\omega_x^W, \omega_y^M\}$ for Last Surviving Status,
- $n = \min\{\omega_x^W, \omega_y^M\}$ for Joint Life Status.

Note that regardless of the status, we are dealing with whole life insurance contracts when equating the policy period with the upper limit of the insurance period. Firstly, the life of the insured person specified in the life tables is limited by the cut--off age. Secondly, a matrix notation is used to calculate actuarial values, and the matrix requires specifying its dimensions. This means that at the time of concluding the insurance contract, the theoretical length of the insurance period is determined based on the possible maximum age to which the husband and wife can live (included in life expectancy tables). For example, if the husband is 60 years old and the wife is 50 years old and $\omega = \omega_x^W = \omega_y^M = 110$, then

- $n = \max\{\omega_x^M, \omega_y^W\} = \max\{110 - 60;\ 110 - 50\} = \max\{50; 60] = 60$ for LSS,
- $n = \min\{\omega_x^M, \omega_y^W\} = \min\{110 - 60;\ 110 - 50\} = \min\{50; 60] = 50$ for JLS.

Therefore, for MSM given by (4.1) transition matrix $\mathbf{Q}(k)$ for $k \in \{0, 1, 2, \ldots n - 1\}$, $x + k < \omega$ and $y + k < \omega$ has the form (4.3). Hence, for k such that $x + k \geq \omega$ or $y + k \geq \omega$, matrices $\mathbf{Q}(k)$ consists of rows that have only zero elements. After the husband's death, model MSM degenerates into the model $(\{2{,}4\}, \{(2{,}4)\})$. Then, for $k \in \{1{,}2, \ldots n - 1\}$ such that $x + k \geq \omega$ and $y + k < \omega$, matrix (4.3) is reduced to a matrix of the form:

$$\mathbf{Q}_x(k) = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & q_{22}(k) & 0 & q_{24}(k) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \ . \qquad (4.5)$$

After the wife's death models $(\{3{,}4\}, \{(3{,}4)\})$ are analysed, and for $x + k < \omega$ and $y + k \geq \omega$ we obtain

$$\mathbf{Q}_y(k) = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33}(k) & q_{34}(k) \\ 0 & 0 & 0 & 0 \end{matrix} \ . \qquad (4.6)$$

Note that the matrices given by formulas (4.5) and (4.6) apply only if the spouses are not of the same age. In such a situation, if one of the spouses exceeds the maximum duration of life ω, the spouse theoretically dies because the Life Tables do not allow to determine the probability of survival for him/her. At this point, the probabilities of transition for the surviving spouse do not change and are defined at the time of concluding the insurance contract. If such a situation occurs in the k-th year of the insurance contract, then the transition probability matrix $\mathbf{Q}(k)$ automatically includes a few rows containing only zeros. The insurance contract for LLS is still in force. However, matrices $\mathbf{Q}(u)$ (for $u \geq k$) cannot be entirely regarded as transition probability matrices because the sum of elements is not equal to 1 in all rows. In this sense, the matrix 'degenerates', and the rows of $\mathbf{Q}(k)$ matrix whose elements add up to 1 correspond to the multistate model that would be constructed for a surviving spouse's life insurance contract.

In marital life insurance, the two primary types of benefits paid upon the death of the spouse can be realised: the death annuity paid in advance or paid from below ($b_i(k)$ or $b_i(k)$ – the not-*cf* type of cash flow) and the lump-sum benefit ($c_{ij}(k)$ – the *cf* type of cash flow). The possible variants taking into account the type of benefits are presented in Figure 4.2, where the number of states in which cash flows of the not-*cf* type are marked in red and the direct transitions associated with cash flows of *cf* type are bolded and also in red.

Annuities are contractual guarantees that promise to provide a periodic income over the individuals' lifetime(s), such as MSMs for the groups of contracts in schemes A-1 and B-1 (Figure 4.2). Note that these types of contracts allow the direct use of matrix notation to calculate the actuarial value. Cash flow matrix $\mathbf{C}_{in}$ resulting from the implementation of the last-survivor annuity (scheme A-1 in Figure 4.2) is determined as follows:

- for LSS annuity paid in advance

$$\mathbf{C}_{in} = \begin{matrix} b_1(0) & b_2(0) & b_3(0) & 0 \\ b_1(1) & b_2(1) & b_3(1) & 0 \\ b_1(2) & b_2(2) & b_3(2) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_1(n-1) & b_2(n-1) & b_3(n-1) & 0 \\ 0 & 0 & 0 & 0 \end{matrix}, \tag{4.7}$$

- for LSS annuity paid from below

$$\mathbf{C}_{in} = \begin{matrix} 0 & 0 & 0 & 0 \\ b_1(1) & b_2(1) & b_3(1) & 0 \\ b_1(2) & b_2(2) & b_3(2) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_1(n-1) & b_2(n-1) & b_3(n-1) & 0 \\ b_1(n) & b_2(n) & b_3(n) & 0 \end{matrix}. \tag{4.8}$$

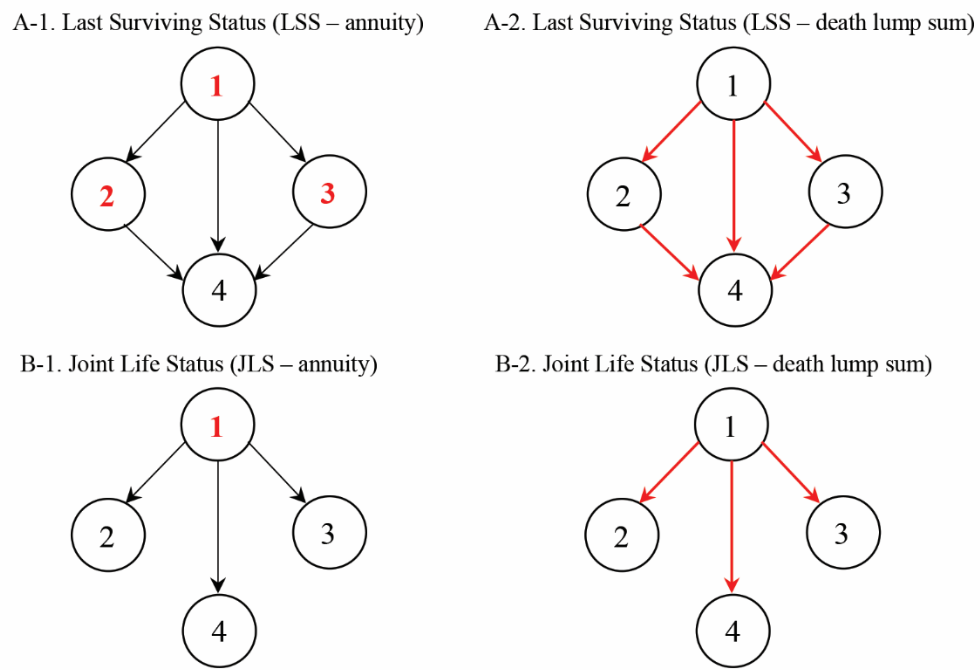


Figure 4.2. MSM for marriage insurance with selected benefit types

Source: own elaboration.

Hence, the cash flow matrices for JLS annuity (i.e. JLS – annuity presented as the scheme B-1 in Figure 4.2) look like the matrices (4.7) and (4.8), but the second and third columns contain only zeros.

Utilising the matrices (4.3) and (4.7) or (4.8) specified for the MSM given by (4.1) and respectively (4.4) and (4.7) or (4.8), with the second and third columns zeroed out for the MSMs given by (4.2), the net single premiums for marriage annuities can be determined using formulas (2.52). In such situations, transitions matrix $\mathbf{Q}^*(k) = \mathbf{Q}(k)$ takes the form (4.3) for the MSM described by (4.1) and the form (4.4) for the MSM described by (4.2).

In marital life insurance with lump-sum benefits (such as the MSM for the groups of contracts in schemes A-2 and B-2 in Figure 4.2), the direct transitions of the *cf* type make it impossible to apply the matrix notation directly. Conditions B1 and B2 (defined in Section 2.7) are satisfied for the marriage insurance contract with JLS and death lump sum (scheme B-2 in Figure 4.2) because the control array $W(k)$ is of the form:

$$W(k) = \begin{matrix} - & c_{12}(k) & c_{13}(k) & c_{14}(k) \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{matrix}. \tag{4.9}$$

Condition B1 is satisfied for a marriage insurance contract with LSS and death lump sum (scheme A-2 in Figure 4.2) only under the condition that $c_{14}(t) = c_{24}(t) = c_{34}(t) = c_4(t)$. Then, the control array $W(k)$ is as follows:

$$W(k) = \begin{matrix} - & c_{12}(k) & c_{13}(k) & c_{14}(k) \\ - & - & - & c_{24}(k) \\ - & - & - & c_{34}(k) \\ - & - & - & - \end{matrix} = \begin{matrix} - & c_{12}(k) & c_{13}(k) & c_{4}(k) \\ - & - & - & c_{4}(k) \\ - & - & - & c_{4}(k) \\ - & - & - & - \end{matrix}. \tag{4.10}$$

According to the scheme presented in Figure 2.7, the MMSM had to be constructed using the recursive procedure. After applying this procedure, the modified multistate model of MSM presented by scheme A-2 in Figure 4.2 takes the following form:

$$(S^*, T^*) = \begin{pmatrix} \{1, 2^+, 2, 3^+, 3, 4^+, 4\}, \\ \{(1,2^+), (1,3^+), (1,4^+), (2^+,2), (2^+,4^+), (2,4^+), (3^+,3), (3^+,4^+), (3,4^+), (4^+,4)\} \end{pmatrix}, \tag{4.11}$$

while the modified multistate model of MSM presented by scheme B-2 in Figure 4.2 has the following form:

$$(S^*, T^*) = \left(\{1, 2^+, 2, 3^+, 3, 4^+, 4\}, \{(1,2^+), (1,3^+), (1,4^+), (2^+,2), (3^+,3), (4^+,4)\}\right). \tag{4.12}$$

The MMSM (4.11) and MMSM (4.12) are illustrated in Figure 4.3 (schemes A-2 and B-2), where the states mean:

1 – both spouses are alive,
2^+ – death of the husband, wife is alive,
2 – husband has been dead for at least one year, wife is alive,
3^+ – death of the wife, husband is alive,
3 – wife has been dead for at least one year, husband is alive,
4^+ – death of two spouses or death of one spouse while the other is dead,
4 – both spouses have been dead for at least one year.

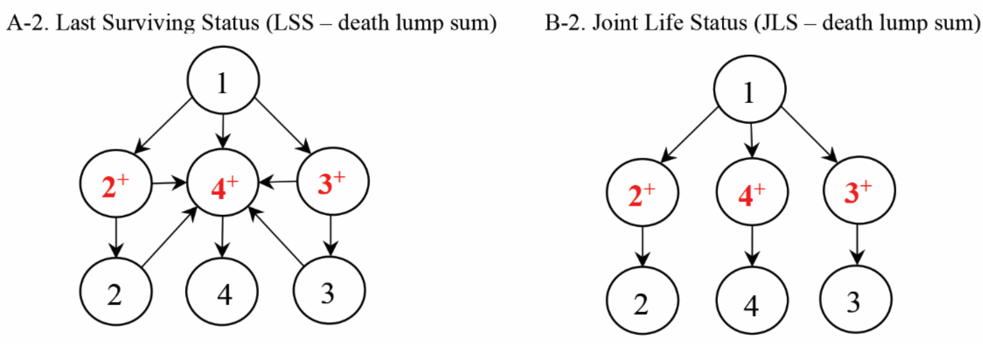


Figure 4.3. The modified multistate models for marriage insurance contracts with lump sum benefits

Source: own elaboration.

In marital life insurance with the lump-sum benefits and LLS (for these, MMSM is given in Figure 4.2 scheme A-2), the transition probability matrix $\mathbf{Q}^*(k)$ (compare matrix $\mathbf{Q}(k)$ describe by (4.3)) for $k \in \{0, 1, 2, \ \dots \ n-1\}$, $x + k < \omega$ and $y + k < \omega$ is as follows:

$$\mathbf{Q}^* \ (k) = \begin{matrix} q_{11}(k) & q_{12}(k) & 0 & q_{13}(k) & 0 & q_{14}(k) & 0 \\ 0 & 0 & q_{22}(k) & 0 & 0 & q_{24}(k) & 0 \\ 0 & 0 & q_{22}(k) & 0 & 0 & q_{24}(k) & 0 \\ 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 \\ 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} . \tag{4.13}$$

Then, for $k \in \{1,2, \dots n-1\}$ such that $x + k \geq \omega$ and $y + k < \omega$, matrix (4.13) is reduced to a matrix of the form:

$$\mathbf{Q}_x^*(k) = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{22}(k) & 0 & 0 & q_{24}(k) & 0 \\ 0 & 0 & q_{22}(k) & 0 & 0 & q_{24}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} , \tag{4.14}$$

whereas after the wife's death, i.e. for $x + k < \omega$ and $y + k \geq \omega$ we obtain

$$\mathbf{Q}_y^*(k) = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 \\ 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}. \quad (4.15)$$

Note that $\mathbf{Q}^*(k)$ ((4.13)-(4.15)) determined for LLS can be used in actuarial values calculations for both LSS and JLS if the matrix $\mathbf{C}_{in}$ for JLS is specified appropriately (taking into consideration that when the insured risk falls in one of the states 2^+–4^+, the insurance contract will expire), that is for

- Last Surviving Status (LLS)

$$\mathbf{C}_{in} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{12}(1) & 0 & c_{13}(1) & 0 & c_4(1) & 0 \\ 0 & c_{12}(2) & 0 & c_{13}(2) & 0 & c_4(2) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{12}(n) & 0 & c_{13}(n) & 0 & c_4(n) & 0 \end{matrix} \in \mathfrak{R}^{(n+1)\times 7}, \quad \longleftarrow \max\{\omega_x^M, \omega_y^W\} + 1 = n + 1 \quad (4.16)$$

- Joint Life Status (JLS)

$$\mathbf{C}_{in} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{12}(1) & 0 & c_{13}(1) & 0 & c_4(1) & 0 \\ 0 & c_{12}(2) & 0 & c_{13}(2) & 0 & c_4(2) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{12}(k) & 0 & c_{13}(k) & 0 & c_4(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \in \mathfrak{R}^{(n+1)\times 7}. \quad \begin{matrix} \longleftarrow \min\{\omega_x^M, \omega_y^W\} + 1 = k \\ \longleftarrow \max\{\omega_x^M, \omega_y^W\} + 1 = n + 1 \end{matrix} \quad (4.17)$$

Utilising the matrices (4.13) to (4.15) and (4.16) specified for the MMSM given by (4.11), i.e. for LSS, and respectively (4.17) for the MSMS given by (4.12), i.e. for JLS, insurance premiums for marriage insurances can be determined using formulas (2.52) and (2.53).

Another way to determine premiums for marital life insurance with lump-sum benefits and JLS (for which the MMSM is given in Figure 4.2, scheme B-2) is based straightforwardly on the transition probability matrix and the cash flow matrix, which are defined respectively as follows:

$$\mathbf{Q}^*(k) = \begin{matrix} q_{11}(k) & q_{12}(k) & 0 & q_{13}(k) & 0 & q_{14}(k) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}, \quad (4.18)$$

$$\mathbf{C}_{in} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{12}(1) & 0 & c_{13}(1) & 0 & c_{14}(1) & 0 \\ 0 & c_{12}(2) & 0 & c_{13}(2) & 0 & c_{14}(2) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{12}(n) & 0 & c_{13}(n) & 0 & c_{14}(n) & 0 \end{matrix} \in \mathfrak{R}^{(max\{\omega_x^M,\omega_y^W\}+1)\times 7}. \tag{4.19}$$

Note that conditions B1 and B2 (defined in Section 2.7) are satisfied for the marriage insurance contract with JLS and death lump sum (scheme B-2 in Figure 4.2), and there is no need to take the assumption that $c_{14}(t) = c_{24}(t) = c_{34}(t) = c_4(t)$ (compare (4.17) and (4.19)).

If one were to abandon the assumption that $c_{14}(t) = c_{24}(t) = c_{34}(t) = c_4(t)$ for the MSM illustrated by scheme A-2 in Figure 4.2, and assume instead that the benefits differ depending on whether the wife dies ($c_{13}(t) = c_{24}(t) = c^W$) or the husband dies ($c_{12}(t) = c_{34}(t) = c^M$), and that if both die at the same time, the benefit is paid as the sum of the benefits for the death of the husband and the wife ($c_{14}(t) = c^W + c^M = c$), then the original MSM must be appropriately expanded to meet condition B1. Subsequently, the modified MSM procedure results in an MMSM, as shown in Figure 4.4, where the benefits amounts were entered in addition to the states related to the payment of benefits marked in red.

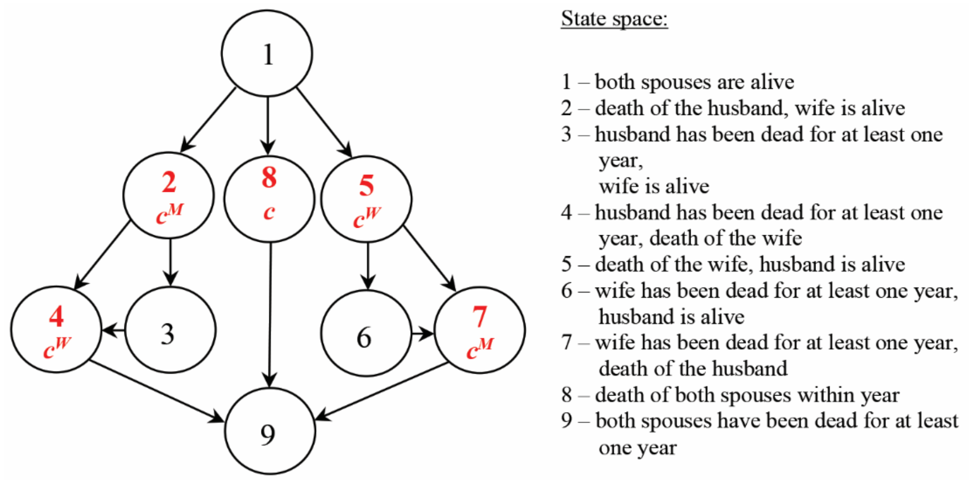


Figure 4.4. The modified multistate models for marriage insurance contracts with various amounts of lump sum benefits

Source: (Dębicka et al., 2023).

Note that for the MMSM presented in Figure 4.4, states 2 and state 3 correspond to the situation in which the wife lives and the husband is dead (compare state 2 in the scheme A-2 of Figure 4.2). The difference between these two states (2 and 3) is related to the period that elapses from the death of her husband, which directly affects the payment of benefits to his wife. Therefore, in the first year after the husband's death, the wife receives the lump sum benefit (state 2), and in the subsequent years of her widowhood, the lump sum benefit is not paid. Analogously, this also happens in the case of states 5 and 6.

The MMSM in Figure 4.4 can also be used in contracts that provide for the payment of annuities to a widow or widower. Then the annuity instalments are paid when the process describing the change of states is in states 2 and 3 (for wife) and 5 and 6 (for husband).

The methodology for valuing contracts modelled by the MMSM presented in Figure 4.4, along with numerical examples, is provided in (Dębicka et al., 2023).

This chapter focuses on the methods for determining the elements of the transition probability matrix for life insurance contracts for marriages. Specifically, Section 4.2 is dedicated to marriage life insurance in the context of standard net single premiums relating to annuities, while Section 4.3 covers equity release marriage contracts in the context of valuing annuity benefits.

The classic approach (Gerber, 1995) assumes that the future lifetimes of the spouses are independent; however, this study allows for dependencies between these lifetimes, which is a more realistic approach. Spouses are often affected by the same factors, risks or habits that affect the future lifetime of spouses. In practice, the so-called 'broken heart syndrome' can occur when the death of one spouse significantly impacts the surviving spouse's subsequent life. In addition, there is also the phenomenon of 'shock', where an external event drastically affects the further life expectancy of the spouses. For example, car or plane accidents, as well as natural disasters, often result in the deaths of both spouses (Denuit et al., 2001).

In this chapter, the authors introduce two methods for modelling the continued life of spouses, allowing for dependencies. The first is based on Markov chains and uses data published in generally available sources, e.g. Statistics Poland. The second method utilises copula functions, which require more detailed data and are more difficult to implement; however, it is possible to combine both methods.

4.2. Modelling the structure of spouses' future lifetime

4.2.1. General ideas

In the context of individual annuities and insurances, to determine the probabilistic structure of model (2.1), state probability vectors at a given point in time were used, i.e. $\mathbf{P}(t) = (p_1(t), p_2(t))^T$. Hence, for marital insurance modelled by the MSM given by (4.1) and the MSM described by (4.2), these vectors are more complex because they account for the states of both spouses, i.e. $\mathbf{P}(t) = (p_1(t), p_2(t), p_3(t), p_4(t))^T$. In this section, the authors focus on describing the probability vectors for marriage contracts under the assumption that process $\{X(t)\}$ is continuous.

T_y^M and T_x^W represent the remaining lifetimes (or time-until-death random variables) of a y-year-old man and his x-year-old wife, respectively.

Let ${}_tp_y^M$ and μ_y^M be a survival function of the lifetime T_y^M, i.e.

$$ {}_tp_y^M = S_y^M(t) = P(T_y^M > t) \tag{4.20}$$

and a force of mortality, which satisfies the following formula:

$$ {}_tp_y^M = exp\left(-\int_0^t \mu_{y+r}^M dr\right). \tag{4.21}$$

Thus, for an individual life insurance contract bought by y-year-old men, we have $\mathbf{P}^M(t) = (p_1(t), p_2(t))^T = \left({}_tp_y^M, 1 - {}_tp_y^M\right)^T$.

We define the survival function of T_x^W and the force of mortality for x-year-old women similarly and obtain $\mathbf{P}^W(t) = (p_1(t), p_2(t))^T = \left({}_tp_x^W, 1 - {}_tp_x^W\right)^T$.

The next step is to estimate the probabilities associated with the states, which are ${}_tp_y^M$ and ${}_tp_x^W$. We estimate ${}_tp_y^M$ for integer values t and x using the life tables. Hence

$$ {}_tp_y^M = \frac{l_{y+t}^M}{l_y^M}, \tag{4.22}$$

where l_y^M is the number of living x-year-old men from the initial l_0^M = 10 000 men (Bowers et al., 1986). Taking value ${}_tp_y^M$ for all, not just natural values, we can use the Makeham estimation (Denuit et al., 2001), and obtain

$$ {}_tp_y^M = s_M^t g_M^{c_M^{y+t} - c_M^y}, \tag{4.23}$$

$$ \mu_y^M = A_M + B_M c_M^y, \tag{4.24}$$

where $c_M > 1$, $s_M, g_M \in [0, 1]$, $s_M = \exp(-A_M)$, $B_M = -\ln(c_M)\ln(g_M)$. The estimation method of these parameters is presented in (Denuit et al., 2001; Heilpern, 2015).

Similarly, probability ${}_tp_x^W$ can be estimated.

Now, examine the joint distribution of the couple's lifetimes: husband T_y^M and wife T_x^W. The survival joint function takes the following form:

$$ S_{xy}(t, z) = P\left(T_x^W > t, T_y^M > z\right). \tag{4.25}$$

Probability (4.25) can be calculated using the joint survival function of the husband and wife, which considers their respective ages and the correlation between their lifetimes.

The probability that the spouses, the husband at the age of y and the wife aged x, will both survive for another t years is denoted by

$$ {}_t p_{xy} = P(T_x^W > t, T_y^M > t) = S_{xy}(t,t). \tag{4.26}$$

On the other hand, the probability that at least one of the spouses will survive is equal to

$$\begin{aligned} {}_t p_{\overline{xy}} &= P\left(\max\left\{T_x^W, T_y^M\right\} > t\right) = P\left(\left\{T_x^W > t\right\} \cup \left\{T_y^M > t\right\}\right) \\ &= S_{xy}(t,0) + S_{xy}(0,t) - S_{xy}(t,t) = {}_t p_x^W + {}_t p_y^M - {}_t p_{xy}. \end{aligned} \tag{4.27}$$

Based on (4.26) and (4.27), we obtain

$$\begin{aligned} \mathbf{P}(t) &= (p_1(t), p_2(t), p_3(t), p_4(t))^T \\ &= \left({}_t p_{xy}, {}_t p_x^W - {}_t p_{xy}, {}_t p_y^M - {}_t p_{xy}, 1 - {}_t p_{xy}\right)^T. \end{aligned} \tag{4.28}$$

Vector (4.28) enables to determine the basic actuarial pensions, i.e. unit immediate annuity, which pays the monetary unit at the end of years 1, 2, …, n as long as the spouses survive. The n-year immediate life annuity for JLS is defined as (compare (2.36))

$$a_{xy:\overline{n}|} = \sum_{k=1}^{n=\min\{\omega_x^M, \omega_y^W\}} v_k^k \, p_{xy} = \mathbf{V}^T \left(\mathbf{I} - \mathbf{I}_1 \mathbf{I}_1^T\right) \mathbf{D} \mathbf{J}_1 = a_{11}(n), \tag{4.29}$$

where vector **V** contains a discount factor connected with the annual effective rate r, i.e. $v^k = (1 + r)^{-k}$, and **D** is described based on (4.28) (compare (2.48)).

On the other hand, the n-year immediate life annuity for LSS (pays the monetary unit as either spouse survives) has the following form:

$$\begin{aligned} a_{\overline{xy}:\overline{n}|} &= \sum_{k=1}^{max\{\omega_x^M, \omega_y^W\}} v^k \, {}_k p_{\overline{xy}} \\ &= \mathbf{V}^T (\mathbf{I} - \mathbf{I}_1 \mathbf{I}_1^T) \mathbf{D} (\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3) \\ &= a_{11}(0,n) + a_{1(2)}(0,n) + a_{1(3)}(0,n) \end{aligned}, \tag{4.30}$$

where $a_{1(i)}(0,n)$ denote the actuarial value of the stream of unit benefits arising from an immediate life annuity contract payable in period $(0,n]$ if at that time $X(k) = i$. Actuarial value is calculated at the beginning of insurance period ($k = 0$); it is tacitly assumed that $X(0) = 1$ (Dębicka & Zmyślona, 2018).

Now examine the situation when one only knows the distribution of the joint lifetimes of spouses for reference age x_0 for husbands and y_0 for wives, namely the function of survival:

$$S_{x_0 y_0}(t,z) = P(T_{x_0}^W > t, T_{y_0}^M > z). \tag{4.31}$$

In addition, let $x = x_0 + s$ and $y = y_0 + w$, then the probability that x-year-old husband and y-year-old wife will survive another t years is

$$
\begin{aligned}
{}_tp_{xy} &= S_{xy}(t,t) = P(T^W_{x_0} > s+t, T^M_{y_0} > w+t \mid T^W_{x_0} > s, T^M_{y_0} > w) \\
&= \frac{S_{x_0,y_0}(s+t, w+t)}{S_{x_0,y_0}(s,w)}.
\end{aligned}
\tag{4.32}
$$

On the other hand, the probability that at least one of the spouses will survive takes the form:

$$
{}_tp_{\overline{xy}} = \frac{S_{x_0,y_0}(s+t,w) + S_{x_0,y_0}(s,w+t) - S_{x_0,y_0}(s+t,w+t)}{S_{x_0,y_0}(s,w)}. \tag{4.33}
$$

Probabilities (4.32) and (4.33) are determined in Section 4.2.2, assuming that the spouses' lifetimes are independent, and in Section 4.2.3, considering that the lifetimes of the husband and wife are dependent.

4.2.2. A case of independence

It is generally assumed – in the classic approach – that the lifetimes of spouses are independent. This assumption facilitates further analysis, e.g. determining the actuarial value of annuities. Many mathematical formulas are greatly simplified. The survival joint function takes the following form in this case:

$$
\begin{aligned}
S_{xy}(t,z) &= P\left(T^W_x > t, T^M_y > z\right) = P\left(T^W_x > t\right) \cdot P\left(T^M_y > z\right) \\
&= S^W_x(t) \cdot S^M_y(z)
\end{aligned}
\tag{4.34}
$$

and for $t = z$ we have

$$
= S_{xy}(t,t) = S^W_x(t) S^M_y(t) = {}_tp^W_x \; {}_tp^M_y. \tag{4.35}
$$

In this independent case, we can see that it is enough only to use life tables for men and women, whereas the formula connected with the LSS annuity is (compare (4.27))

$$
{}_tp_{\overline{xy}} = {}_tp^W_x + {}_tp^M_y - {}_tp^W_x \; {}_tp^M_y. \tag{4.36}
$$

Under the assumption of independence and formulas (4.35) and (4.36), we can express the state probability vectors using the survival functions defined separately for the wife and husband:

$$
\begin{aligned}
\mathbf{P}(t) &= (p_1(t), p_2(t), p_3(t), p_4(t))^T \\
&= \left({}_tp^W_x \cdot {}_tp^M_y,\ {}_tp^W_x\left(1 - {}_tp^M_y\right),\ {}_tp^M_y\left(1 - {}_tp^W_x\right), 1 - {}_tp^W_x - {}_tp^M_y + {}_tp^W_x \cdot {}_tp^M_y \right)^T.
\end{aligned}
\tag{4.37}
$$

Based on (4.37) we can efficiently compute the net value for n-year JLS and LSS annuities given by (4.29) and (4.30), respectively.

Example 4.1

Assume that husband and wife are $x = y = 60$ years old and determine the actuarial values of JLS and LSS annuities for different values of n. Assume that the lifetimes of the spouses are independent. The study used the Life Tables for Lower Silesia in 2011, obtained from Statistics Poland. These values are presented in Table 4.1.

Table 4.1. The actuarial values of JLS and LSS annuities for independent lifetimes

n	JLS	LSS
5	4.2110	4.5715
10	7.2251	8.4657
15	9.2043	11.7038
20	10.3298	14.2609
25	10.8298	16.0671
30	10.9770	17.0939
35	10.9999	17.5016
40	11.0013	17.5978

Source: own elaboration based on Life Tables for Lower Silesia in 2011.

Note that the Last Survival annuities provide greater actuarial values than the Joint Life annuities. In addition, for n greater than 25, we obtain certain stabilisation of actuarial values for Joint Life annuities and for n greater than 30 for Last Survival annuities.

These actuarial values are compared with those obtained in the case of dependent lifetimes of spouses.

4.2.3. A case of dependence

Where spouses' lifetimes are dependent on determining actuarial values or carrying out another analysis, the joint distribution of such lifetimes should be known, for example, the survival joint function $S_{xy}(t, z)$.

One can designate such joint distributions using two methods. The first is based on the Markov chain: the survival joint function is computed using the Markov model based on the stationary Markov chain (Denuit et al., 2001; Norberg, 1989; Wolthuis & Van Hoek, 1986) for fixed ages x and y. This model has four states presented in Figure 4.5. The authors used the transition probabilities $p_{ij}(t, s)$ and the Markovian forces of mortality $\mu_{ij}(t)$. The transition probability is the conditional probability that the couple is in state j at time s, given that it was in state i at time t. The Markovian forces of mortality from state i to state j at time t are defined by

$$\mu_{ij}(t) = \lim_{\Delta t \to 0} \frac{p_{ij}(t, t + \Delta t)}{\Delta t}. \tag{4.38}$$

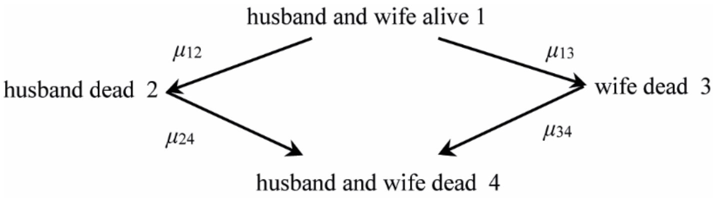


Figure 4.5. The space of states of the Markov model

Source: based on (Denuit et al., 2001).

Note that the analysed model is continuous, namely $\{X(t): t \geq 0\}$ is a continuous process, as opposed to the previous chapters, in which $\{X(t): t = 0,1,2,\dots\}$ is considered. Consequently, in Figure 4.5 there is no direct transition from state 1 to state 4. For the continuous time, it is extremely rare for both spouses to die at the exact moment, and thus, the transition intensity from state 1 to state 4 is not defined.

The independence of the lifetimes of spouses occurs in the classic approach in multiple life insurance. This assumption implies that $\mu_{12}(t) = \mu_{34}(t)$, $\mu_{13}(t) = \mu_{24}(t)$ (Norberg, 1989); however, these variables are often dependent in real life, and such equations do not occur. Denuit et al. (2001) proposed simple relations between Markovian $\mu_{ij}(t)$ and usual μ_x^M, μ_y^W forces of mortalities:

$$\mu_{12}(t) = (1 + \alpha_{12})\mu_{x+t}^M \qquad \mu_{34}(t) = (1 + \alpha_{34})\mu_{x+t}^M,$$

$$\mu_{13}(t) = (1 + \alpha_{13})\mu_{x+t}^W \qquad \mu_{24}(t) = (1 + \alpha_{24})\mu_{x+t}^W.$$

and the procedure of estimating parameters α_{12} (Heilpern, 2011). This procedure is based on the Nelson-Aalen estimator. Knowledge of the Markovian forces of mortalities allows for determining transition probabilities (Denuit et al., 2001), and obtaining the following relations:

$$p_{11}(t,z) = \exp\left(-\int_t^z (\mu_{12}(u) + \mu_{13}(u))du\right), \tag{4.39}$$

$$p_{ii}(t,z) = \exp\left(-\int_t^z \mu_{i4}(u)du\right), \tag{4.40}$$

$$p_{1i}(t,z) = \int_t^z p_{11}(t,u)\mu_{1i}(u)p_{ii}(u,z)du, \tag{4.41}$$

where $i = 2, 3$. We can present the explicit form of transition probabilities p_{ii}, where $i = 1, 2, 3$ using the Makeham estimation. For instance,

$$p_{22}(t,z) = \exp\left(-(1 + \alpha_{24})\left(A_W(z - t) + B_W c_W^x \frac{c_W^s - c_W^t}{\ln c_W}\right)\right). \tag{4.42}$$

We can derive the joint survival function using the transition probabilities in the following way:

$$S_{xy}(t,z) = P(T_x^W > t, T_y^M > z) = \begin{cases} p_{11}(0,z) + p_{11}(0,t)p_{13}(t,z) & \text{for } 0 \le t \le z \\ p_{11}(0,t) + p_{11}(0,z)p_{12}(z,t) & \text{for } 0 \le z \le t \end{cases}. \quad (4.43)$$

Knowing marginal survival functions ${}_tp_x^W$ and ${}_tp_y^M$, we can obtain the joint distribution function:

$$F_{xy}(t,z) = S_{xy}(t,s) + 1 - {}_tp_x^W - {}_tp_y^M. \quad (4.44)$$

Using joint survival function $FS_{xy}(t,z)$, we can compute the n-year Joint Life annuities and the Last Surviving annuities. Hence

$${}_tp_{xy} = p_{11}(0,t) = exp\left(-\int_0^t \left(\mu_{12}(u) + \mu_{13}(u)\right)du\right) = \left({}_tp_x^W\right)^{1+\alpha_{12}}\left({}_tp_y^M\right)^{1+\alpha_{13}} \quad (4.45)$$

in this case.

Example 4.2

Now calculate the actuarial values of JLS and LSS annuities for different values of n for husband and wife who are 60 years old, i.e. $x = y = 60$, using the dependent structure of their lifetimes (Heilpern, 2015). The author used data from Lower Silesia during 2011 obtained from Statistics Poland and obtained the following parameters: $\alpha_{12} = -0.0427$ and $\alpha_{13} = -0.0792$. These actuarial values are shown in Table 4.2.

Table 4.2. The actuarial values of Joint Life and Last Surviving annuities for Markov-dependent lifetimes

n	**JLS**		**LSS**	
	independent	**Markov**	**independent**	**Markov**
5	4.2110	4.2297	4.5715	4.5528
10	7.2251	7.2876	8.4657	8.4032
15	9.2043	9.3252	11.7038	11.5829
20	10.3298	10.5104	14.2609	14.0804
25	10.8298	11.0565	16.0671	15.8404
30	10.9770	11.2272	17.0939	16.8436
35	10.9999	11.2565	17.5016	17.2450
40	11.0013	11.2586	17.5978	17.3405

Source: own elaboration based on the life tables from (Heilpern, 2015).

Note that, as with the independent lifetimes of spouses, the actuarial value of the Last Survival annuity is greater than the value of the Joint Life annuity. In the case of the Joint Life annuity, the actuarial value of this annuity is greater in the case of dependents than the independent lifetimes of spouses, whereas for the Last Survival annuity the opposite relation occurs.

We can also determine the dependent structure of the lifetime of spouses using the copula – copula $C(u, v)$ is a link between the joint and marginal distributions (Genest & MacKay, 1986; Nelsen, 2006):

$$F_{xy}(t,z) = C(P(T_x^W \leq t), P(T_y^M \leq z)). \tag{4.46}$$

Knowing the copula and marginal distributions, we can determine the joint distribution and obtain for the independent random variables the following copula:

$$C(u,v) = uv. \tag{4.47}$$

The survival function can be also characterised by the copula. It is a survival copula defined by the formula (Nelsen, 2006):

$$C^*(u,v) = u + v - 1 + C(1-u, 1-v). \tag{4.48}$$

Then

$$S_{xy}(t,z) = C^*(P(T_x^W > t), P(T_y^M > z)) = C^*\left({}_tp_x^W, {}_zp_y^M \right). \tag{4.49}$$

If one knows copula C which characterises the structure of the dependency, then functions ${}_tp_{xy}$ and ${}_tp_{\overline{xy}}$ can be presented using the copulas in the following way:

$${}_tp_{xy} = S_{xy}(t,t) = C^*\left({}_tp_x^W, {}_tp_y^M \right), \tag{4.50}$$

$$\begin{aligned} {}_tp_{\overline{xy}} &= S_{xy}(t,0) + S_{xy}(0,t) - S_{xy}(t,t) \\ &= C^*\left({}_tp_x^W, 1\right) + C^*\left(1, {}_tp_y^M\right) - C^*\left({}_tp_x^W, {}_tp_y^M \right) \\ &= {}_tp_x^W + {}_tp_y^M - C^*\left({}_tp_x^W, {}_tp_y^M \right). \end{aligned} \tag{4.51}$$

Now consider the case that one only knows the distribution of the joint lifetimes of spouses for reference age x_0 for husbands and y_0 for wives, namely the function of survival:

$$S_{x_0y_0}(t,z) = P\left(T_{x_0}^W > t, T_{y_0}^M > s\right) = C_0^*\left({}_tp_{x_0}^W, {}_sp_{y_0}^M \right), \tag{4.52}$$

where C_0^* is the survival copula induced by survival function $S_{x_0y_0}$.

In addition, let $x = x_0 + s$ and $y = y_0 + w$, then the probability that y-year-old husband and x-year-old wife will survive another t years is

$$\begin{aligned} {}_tp_{xy} &= S_{xy}(t,t) = P(T_{x_0}^W > s+t, T_{y_0}^M > w+t | T_{x_0}^W > s, T_{y_0}^M > w) \\ &= \frac{S_{x_0,y_0}(s+t,w+t)}{S_{x_0,y_0}(s,w)} = \frac{C_0^*({}_{s+t}p_{x_0}^M, {}_{w+t}p_{y_0}^W)}{C_0^*({}_sp_{x_0}^M, {}_wp_{y_0}^W)}. \end{aligned} \tag{4.53}$$

On the other hand, the probability that at least one of the spouses will survive takes the form:

$$
{}_t p_{\overline{xy}} = \frac{S_{x_0,y_0}(s+t,w) + S_{x_0,y_0}(s,w+t) - S_{x_0,y_0}(s+t,w+t)}{S_{x_0,y_0}(s,w)}
$$
$$
= \frac{C_0^*\left({}_{s+t}p_{x_0}^W, {}_w p_{y_0}^M\right) + C_0^*\left({}_s p_{x_0}^W, {}_{w+t}p_{y_0}^M\right) - C_0^*\left({}_{s+t}p_{x_0}^W, {}_{w+t}p_{y_0}^M\right)}{C_0^*\left({}_s p_{x_0}^W, {}_w p_{y_0}^M\right)}. \tag{4.54}
$$

The Archimedean copula has a simple, quasi-additive form (Nelsen, 2006):

$$
C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)), \tag{4.55}
$$

where φ: $[0, 1] \rightarrow R_+$ is a decreasing function, satisfying condition $\varphi(1) = 0$. This function is called a generator. Archimedean copulas make families of copulas, characterised by some parameters describing the degree of dependence. Kendall's tau coefficient of correlation τ, the often-used measure of dependence, is univocally provided by its generator:

$$
\tau = 1 + 4\int_0^1 \frac{\phi(t)}{\phi\prime(t)}dt. \tag{4.56}
$$

There is an explicit formula describing the relation between Kendall's tau coefficient and the parameter in many cases of the families, e.g. $\tau = 1 - \frac{1}{\alpha}$ for the Gumbel family:

$$
C(u,v) = \exp\left(-((-\ln u)^\alpha + (-\ln v)^\alpha)^{1/\alpha}\right) \tag{4.57}
$$

and $\tau_\alpha = 1 - \frac{2}{3\alpha} - \frac{2(1-\alpha)^2}{3\alpha^2}\ln(1-\alpha)$ for the AMH family:

$$
C(u,v) = \frac{uv}{1-\alpha(1-u)(1-v)}. \tag{4.58}
$$

More information about copulas can be found in (Nelsen, 2006).

Now, the authors present the procedure of copula selection (Durrleman et al., 2000; Genest & MacKay, 1986). First, Kendall's tau coefficient of correlation is obtained using the formula (Nelsen, 2006):

$$
\tau = 4\int_x^\infty \int_y^\infty F_{xy}(t,s)dF_{xy}(t,s) - 1. \tag{4.59}
$$

Next, the distance is computed between the joint distribution function of lifetimes F_{xy} and theoretical distribution function F_C induced by copula C:

$$
d_2(F_{xy}, F_C) = \left(\int_0^\infty \int_0^\infty \left|F_{xy}(t,s) - C(F_x(t), F_y(s))\right|^2\right)^{1/2} dxdy \tag{4.60}
$$

for the representatives of different families, which correspond to Kendall's coefficient equal to τ. The best copula minimises this criterion.

In (Denuit et al., 2001), the authors conducted a survey based on data from cemeteries in Brussels. They obtained Kendall's tau coefficient τ = 0.092 and the Gumbel copula with parameter α = 1.1015, which is the optimal copula. Heilpern (2011) carried out a similar research based on cemeteries in Wrocław, and obtained the following results: τ = 0.156 and the AMH copula with α = 0.5879 as the optimal copula. In 2014, Heilpern repeated the study based on data from Lower Silesia for spouses of $x = y = 60$ years of age and received a coefficient of τ = 0.073. The best copula turned out to be Gumbela's copula with α = 1.0786. These data came from the Polish General Census 2011, and the results were obtained using the Markov methodology presented at the beginning of the chapter. Unfortunately, this study could not be repeated in the following years because the estimation of α_{ij} parameters requires, among others, the number of husbands and wives in individual years, and these values are given only in the years of the Census. The next Polish General Census took place in 2021, and even if we knew, for example, the number of husbands in 2011, it would have been necessary to know this number in the next year to obtain estimators of these parameters. In the article, this was only an estimation, hence no actual, accurate results could be presented. The authors were also unable to show how they changed in individual years and could not examine the dynamics of the phenomenon.

Example 4.3.

Now, we can compute the actuarial values of JLS and LSS annuities for different values of n for husband x = 70 years old and wife y = 65 years old. The study used the dependent structure produced by the data from Lower Silesia during 2011 when the reference ages were $x_0 = y_0 = 60$. Survival copula C_0^* resulting from this dependent structure is the Gumbel copula with parameter α = 1.0786. The Gumbel copula is defined by the folloing formula:

$$C_0^*(u,v) = \exp(-((-\ln u)^\alpha + (-\ln v)^\alpha)^{1/\alpha}), \tag{4.61}$$

where $\alpha > 1$. A fairly weak relation was obtained between the spouses' lifetimes because Kendall's correlation coefficient was τ = 0.073. However, there are differences in these actuarial values of annuities in the cases of the independence and dependence of spouses' lifetimes (see Table 4.2). The actuarial values of the Joint Life annuities and the Last Surviving annuities for different values of n for husband x = 70 years old and wife y = 65 years old are included in Table 4.3 and Figure 3.1.

The relation between the life expectancy of spouses is relatively weak at $\tau = 0.073$. Hence, the differences between the actuarial value of the annuities determined in the dependent and independent case are small, especially in the case of a Joint Life annuity, where the premium for a dependent lifetime is slightly higher. The opposite is true for a Last Surviving annuity. The case of independence makes a greater contribution. In addition, for $n \geq 20$, the actuarial value of the last-survival annuity shall be stabilised at 8.1.

Table 4.3. The actuarial values of Joint Life and Last Surviving annuities for $x = 70$, $y = 65$ and dependent lifetimes

n	JLS	LSS
5	3.9311	4.5538
10	6.3204	8.3385
15	7.5414	11.2843
20	8.0111	13.2945
25	8.1298	14.3794
30	8.1479	14.7910

Source: own elaboration based on the life tables from (Heilpern, 2015).

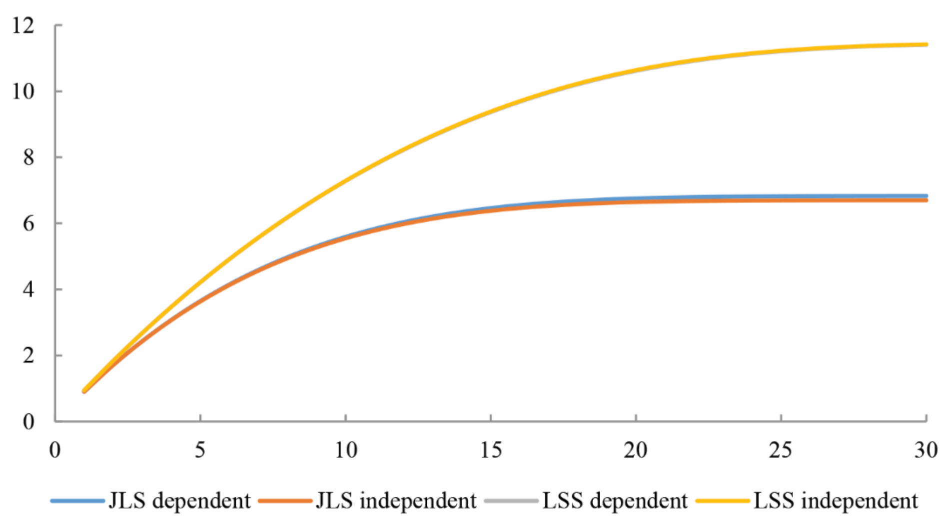


Figure 4.6. The actuarial values of Joint Life and Last Surviving annuities for $x = 70$, $y = 65$, dependent and independent lifetimes

Source: own elaboration based on the life tables from (Heilpern, 2014).

Based on Statistics Poland data, it was not possible, year on year, to accurately determine, using the methodology based on Markov chains, the structure of the relations between the lifetimes of the spouses. Only the years when the Census takes place enable to determine their exact structure.

For other countries this problem may not occur. In Eurostat, one can find accurate data to determine this degree of dependence for some countries, for example the Czech Statistical Office (Český Statistický Úřad, 2023) publishes such accurate data yearly. Based on the data from 2017, a dependency structure was obtained for which a correlation coefficient $\tau = 0.10638$ was set. The best fit for this dependency structure was the Gumbel copula with parameter $\alpha = 1.11904$, while in second place was copula AMH with parameter $\alpha = 0.4240$, showing that there is a slightly larger dependence between the lifetimes of spouses.

4.3. Marital equity release

4.3.1. Benefit valuation

This section concentrates on a marriage equity release contract, i.e. a marital reverse annuity contract and a marital reverse mortgage because spouses often jointly own property. However, one can also distinguish other indirect cases. In Section 4.2, the authors focused on determining the required capital (net single premium) for a specified annuity amount, whereas in contrast, Section 4.3 focused on determining the annuity instalment paid more than once a year based on a known net premium (capital owned by the spouses). The considerations are conducted under the assumption that the future lifetimes of the spouses are independent – in Section 4.3.2, and the future lifetimes of the husband and wife are dependent – in Section 4.3.3.

Under these contracts, annuity benefits are payable when both spouses are alive and sometimes after a spouse's death but in a different amount. Thus, there are two types of such contracts: the *JLS contract*, when the benefit is paid only until the death of the first spouse, and the *LSS contract* by which the benefit is paid until the death of the other spouse. Multistate models for this type of contract are presented in Figure 4.1. One can also distinguish an indirect case where the annuity is paid after the demise of one of the spouses, but at a lower reduction amount $R \left(R \in (0,1)\right)$.

The benefits of both contracts depend on the age of the spouses and their future lifetimes. There is the general case, when the benefit is paid m ($m > 0$) times a year, therefore the future lifetime of spouses are determined as multiple m sub-periods of a year. Note that (x, y) are the ages of entry of x-year-old wife and y-year-old husband. Let $K_x^{(m)}$ and $K_y^{(m)}$ be the future lifetimes of x-year-old woman and y-year-old man. $K_x^{(m)} \in \{0,1,\ldots,m \cdot \omega_x^W\}$ and $K_y^{(m)} \in \{0,1,\ldots,m \cdot \omega_x^M\}$, where ω_x^W (respectively, ω_y^M) denotes the difference between the age limit ω of the woman (man) and woman's (man's) age at entry x (respectively, y). The benefit of a reverse annuity contract is paid for the whole life ($max\{\omega_x^W, \omega_y^M\}$), and a reverse mortgage is paid only for
n-years (a period of contract is equal $max\{n, \omega_x^W, \omega_y^M\}$).

To calculate the benefit, the authors used the formula (3.19) and determined the expected value of the discounted value of all future benefits Z.

The future life of JLS is the minimum of the future lifetime of spouses analogously to the notation introduced in Section 4.2.1, and it is denoted by

$$K_{xy}^{(m)} = \min\left(K_x^{(m)}, K_y^{(m)}\right). \tag{4.62}$$

The probability that the spouses will survive for at least k sub-periods of a year is calculated by the following formula (Marciniuk, 2017b):

$${}_{\frac{k}{m}}p_{xy} = P\left(\frac{K_{xy}^{(m)}}{m} \geq \frac{k}{m}\right) = P\left(\frac{K_x^{(m)}}{m} \geq \frac{k}{m}, \frac{K_y^{(m)}}{m} \geq \frac{k}{m}\right), \tag{4.63}$$

where $k \in \{0,1,\ldots,m \cdot \omega_x^W\}$ or $k \in \{0,1,\ldots,m \cdot \omega_y^M\}$.

Analogously, the future lifetime of the LSS is the maximum of the future lifetimes of both spouses and corresponds to $K_{\overline{xy}}$ (Marciniuk, 2017b):

$$K_{\overline{xy}}^{(m)} = \max\left(K_x^{(m)}, K_y^{(m)}\right).$$

The probability that at least one of the spouses will survive for at least k sub-periods of a year is calculated as follows:

$$\begin{aligned} {}_{\frac{k}{m}}p_{\overline{xy}} &= P\left(K_{\overline{xy}}^{(m)} \geq k\right) = P\left(K_x^{(m)} \geq k \ \vee K_y^{(m)} \geq k\right) \\ &= P\left(K_x^{(m)} \geq k\right) + P\left(K_y^{(m)} \geq k\right) - P\left(K_x^{(m)} \geq k, K_y^{(m)} \geq k\right) \\ &= {}_{\frac{k}{m}}p_x^W + {}_{\frac{k}{m}}p_y^M - {}_{\frac{k}{m}}p_{\overline{xy}}, \end{aligned} \tag{4.64}$$

where $k \in \{0,1,\ldots,m \cdot \omega_x^W\}$ or $k \in \{0,1,\ldots,m \cdot \omega_y^M\}$.

To determine the value of all future benefits, the study introduced the definition of reversionary annuity (Luciano et al., 2016; Marciniuk, 2017b). This annuity is paid periodically $1/m$ ($m > 0$) financial unit as long as both parties are alive and a fraction R of it, when only one of the couple is alive, where $R \in [0,1]$. In this scheme, the special cases are (Marciniuk, 2017b):

- when $R = 1$, this means the LSS and then the benefit paid remains constant also after the first death,
- when $R = 0$, this corresponds to the JLS and means that nothing is paid to the last survivor.

The yearly actuarial value of this life annuity for spouses (x, y), which is paid $1/m$ at the beginning of the sub-period of a year for the whole life, is calculated as follows (Marciniuk, 2017b):

$$\begin{aligned} a_{(x,y)}^{(m)} &= \frac{1}{m} \sum_{k=0}^{max\{\omega_x^W,\omega_y^M\}-1} v^{\frac{k}{m}} \ R\left({}_{\frac{k}{m}}p_x^W - {}_{\frac{k}{m}}p_{xy}\right) + R\left({}_{\frac{k}{m}}p_y^M - {}_{\frac{k}{m}}p_{xy}\right) + {}_{\frac{k}{m}}p_{xy}\Big] \\ &= Ra_x^{(m)} + Ra_y^{(m)} + a_{xy}^{(m)}(1-2R), \end{aligned} \tag{4.65}$$

where $a_x^{(m)}$ and $a_y^{(m)}$ are calculated based on (3.17), respectively for wife and husband, and $a_{xy}^{(m)}$ means the actuarial value of the whole life annuity for the JLS, which pays $1/m$ at the beginning of a year, is determined as follows:

$$a_{xy}^{(m)} = \frac{1}{m}\sum_{k=0}^{max\{\omega_x^W,\, \omega_y^M\}-1} v^{k/m}\ {}_{k/m}p_{xy}. \tag{4.66}$$

Hence from formula (3.19) the following lemma follows (Marciniuk, 2017b).

Lemma 4.1

The yearly benefit of a marriage reverse annuity contract for spouses (x, y), which pays $1/m$ at the beginning of a sub-period of a year as long as both are alive and R/m ($m > 0$) when only one of the couple is alive, is calculated as follows:

$$b_{(x,y)}^{(m)} = \frac{\alpha \cdot W}{Ra_x^{(m)} + Ra_y^{(m)} + a_{xy}^{(m)}(1-2R)}, \tag{4.67}$$

where $a_x^{(m)}$ and $a_y^{(m)}$ are calculated based on (3.17) in the case of whole life annuity and $a_{x:y}^{(m)}$ on formula (4.66).

The reverse mortgage's benefit is paid for n years; therefore, the above case could be generalised for a marital reverse mortgage in the following lemma (Marciniuk, 2017b).

Lemma 4.2

The term yearly actuarial value of due life annuity for spouses (x, y), which pays $1/m$ ($m > 0$) financial unit at the beginning of a sub-period of a year as long as both are alive, and a fraction R of it when only one of the couple is alive, is calculated as follows:

$$\begin{aligned} a_{(x,y):n|}^{(m)} &= \frac{1}{m}\sum_{k=0}^{n\cdot m-1} v^{\frac{k}{m}}\ R\left({}_{\frac{k}{m}}p_x^W - {}_{\frac{k}{m}}p_{xy}\right) + R\left({}_{\frac{k}{m}}p_y^M - {}_{\frac{k}{m}}p_{xy}\right) + {}_{\frac{k}{m}}p_{xy}\Big] \\ &= Ra_{x:n|}^{(m)} + Ra_{y:n|}^{(m)} + a_{xy\,:n|}^{(m)}(1-2R). \end{aligned}$$

Hence, the due benefit of a marriage reverse mortgage is determined by the following formula:

$$b_{(x,y):n|}^{(m)} = \frac{\alpha \cdot W}{Ra_{x:n|}^{(m)} + Ra_{y:n|}^{(m)} + a_{xy\,:n|}^{(m)}(1-2R)}, \tag{4.68}$$

where $a_{x:n|}^{(m)}$ and $a_{y:n|}^{(m)}$ are calculated from (3.17) in the case of term life annuity and $a_{xy:n|}^{(m)}$ is calculated according to the formula:

$$a^{(m)}_{xy:n|} = \frac{1}{m} \sum_{k=0}^{m \cdot n-1} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_{xy}. \tag{4.69}$$

In both lemmas, LLS arises when $R = 1$, and JLS when $R = 0$.

4.3.2. Independent lifetimes of spouses

It is generally assumed in the classic approach, that the future lifetimes of spouses are independent, which means that variables $K_x^{(m)}$ and $K_y^{(m)}$ are independent. Then probability ${}_{k/m}p_{xy}$ is calculated as follows (compare (4.35)):

$${}_{\frac{k}{m}}p_{xy} = P\left(K_{xy}^{(m)} \geq k\right) = {}_{\frac{k}{m}}p_x^W \cdot {}_{\frac{k}{m}}p_y^M, \tag{4.70}$$

where ${}_{k/m}p_x$ and ${}_{k/m}p_y$ are determined by the use of the above formula (3.8).

In Sections 4.3.2 and Section 4.3.3 all numerical examples, which derived from the authors' own programs written in MATLAB, are based on the following conditions:
C1. The value of property $W = 100\,000$ euros.
C2. $\alpha = 50\%$.
C3. Probability ${}_{k/m}p_x$ is calculated based on the Life Table for Lower Silesia from 2011 from formula (3.9) under the assumption that the distribution of deaths during the year is uniform (Bowers et al., 1986).
C4. Fixed interest rate $r = 5.79\%$ (see Section 3.2.3).

Example 4.4

The study distinguished five situations of (x, y) for a different x and y. The marriage reverse annuity contract's benefit is paid m ($m \in \{1,4,6,12\}$) times yearly (i.e. annually, quarterly, every other month and monthly) in constant amount $1/m$ financial at the beginning of a sub-period of a year for the JLS ($R = 0$) and the LLS ($R = 1$). The benefit is calculated based on Lemma 4.1, i.e. formula (4.68), under conditions C1 to C4. The results are presented in Table 4.4.

The JLS contract's annuity is higher than the benefit of the LSS contract. The differences between the annuities increase with the age of the spouses, for example when (80,80) and $m = 1$ the benefit is higher by 80% and for (65,65) by 50%. The benefits increase with the rise of the spouses' age. The JLS contract's benefit is higher when the woman is older and *vice versa* for the LSS. As m increases, the yearly payout also increases. The JLS contract's benefit is higher when the woman is older and *vice versa* for the LSS. For the JLS, the benefit is higher when the woman is older and *vice versa* for the LSS. As m increases, the annual payout also increases. The differences between benefits for $m = 4$ and $m = 6$, and also for $m = 6$ and $m = 12$, are almost the same.

Table 4.4. The reverse annuity contract's benefit payment for marriage (x, y)

(x, y)	***m***	**Benefit payment**		**Yearly sum of benefit payments**	
		JLS (R = 0)	**LSS (R = 1)**	**JLS (R = 0)**	**LSS (R = 1)**
(65,65)	1	5836.9	3896.3	5836.9	3896.3
	4	1527.6	1003.6	6110.4	4014.4
	6	1023.7	671.3	6142.2	4027.8
	12	514.5	336.8	6174.0	4041.6
(70,65)	1	6280.8	4146.5	6280.8	4146.5
	4	1649.9	1070.0	6599.6	4280.0
	6	1106.1	715.9	6636.6	4295.4
	12	556.1	359.2	6673.2	4310.4
(65,70)	1	6495.3	4016.5	6495.3	4016.5
	4	1709.1	1035.5	6836.4	4142.0
	6	1146.0	692.7	6876.0	4156.2
	12	576.3	347.6	6915.6	4171.2
(70,70)	1	6897.8	4341.1	6897.8	4341.1
	4	1821.1	1121.9	7284.4	4487.6
	6	1221.6	750.8	7329.6	4504.5
	12	614.6	376.8	7375.2	4521.6
(80,80)	1	10811.0	5983.2	10811.0	5983.2
	4	2950.9	1565.2	11803.6	6260.8
	6	1987.1	1048.9	11922.6	6293.4
	12	1003.6	527.2	12043.2	6326.4

Source: own elaboration.

Example 4.5

Consider the example of a marriage reverse annuity contract under conditions C1 to C4. The benefit is paid yearly at the beginning of year ($m = 1$) for different R and the benefit is calculated based on Lemma 4.1. The results are presented in Table 4.5.

Note that the benefit decreases when reduction factor R increases. Again, it can be seen that the annuity payment is higher for smaller R when the wife is older than the husband. For larger R, the opposite relation occurs.

Table 4.5. The reverse annuity contract's benefit payment for marriage (*x, y*) and different *R*

R / (*x*, *y*)	**Benefit payment**						
	0	**1/4**	**1/3**	**1/2**	**2/3**	**3/4**	**1**
(65,65)	5836.9	5190.6	5005.8	4673.1	4381.9	4249.5	3896.3
(65,70)	6495.3	5627.1	5387.1	4963.6	4601.9	4440.1	4016.5
(65,75)	7439.1	6189.8	5861.6	5299.7	4836.1	4633.4	4116.0
(70,65)	6280.8	5564.8	5361.0	4995.2	4676.2	4531.5	4146.5
(70,70)	6897.8	6012.5	5765.8	5328.6	4953.1	4784.4	4341.1
(70,75)	7796.2	6597.2	6275.5	5717.9	5251.3	5045.4	4514.4
(75,65)	7051.9	6131.5	5875.8	5423.6	5036.0	4862.2	4406.2
(75,70)	7622.6	6599.7	6317.2	5818.9	5393.5	5203.3	4705.4
(75,75)	8465.0	7214.7	6876.1	6286.2	5789.5	5569.4	4999.4

Source: own elaboration.

Example 4.6

In this example the authors analysed the marriage reverse mortgage contract. The benefit is paid yearly at the beginning of year ($m = 1$) for the JLS ($R = 0$) and the LLS ($R = 1$) under the assumptions C1 to C4. The benefits are calculated using the above formula (4.68) in Lemma 4.2. The results are presented in Table 4.6.

Table 4.6. The reverse mortgage's benefit payment for marriage (*x*, *y*) and $n = 10$ or $n = 15$

(*x*, *y*)	**Benefit payment *n* = 10**		**Benefit payment *n* = 15**	
	JLS (*R* = 0)	**LSS (*R* = 1)**	**JLS (*R* = 0)**	**LSS (*R* = 1)**
(65,65)	7567.8	6421.6	6354.1	4933.4
(65,70)	7976.4	6447.5	6874.7	4981.8
(65,75)	8608.2	6482.0	7673.4	5041.1
(70,65)	7784.9	6456.2	6665.2	5012.2
(70,70)	8195.5	6496.4	7182.6	5090.7
(70,75)	8829.6	6550.2	7973.8	5188.1
(75,65)	8242.4	6523.4	7289.6	5146.4
(75,70)	8656.2	6591.9	7799.9	5279.8
(75,75)	9293.3	6684.8	8576.1	5449.5

Source: own elaboration.

Note that the benefit decreases when reduction factor R increases. The annuity grows when the spouses are older. Again, it can be seen that the annuity payment is higher for $R = 0$ when the wife is older than the husband; for $R = 1$ the opposite occurs. The marriage reverse mortgage benefit is higher than the reverse annuity contract payment. If the spouses are older and the reverse mortgage lasts longer, then the differences between the pensions are smaller (cf. Table 4.5 and Table 4.6).

4.3.3. Dependent lifetimes of spouses

Similarly to Section 4.2.3, the authors considered a scenario where the future lifetimes of the husband and wife are dependent. Two cases were examined: one where process $\{X(t)\}$ describing the evaluation of insured risks in the models shown in Figure 4.1 is a continuous-time Markov process, and the other where the probabilistic structure of the model is modelled by copula functions.

In the first case, under the assumption that $\{X(t): t \geq 0\}$ is the time-continuous process the intensity of transitions μ_{ij} from state i to state j $(i, j = 1,2,3,4)$ are presented in Figure 4.5. The problem of determining probability ${}_t p_{xy}$, which is necessary to calculate benefit payments of marital equity release, is reduced to assessing the probability p_{11} of the marriage remaining in state 1 by t. Hence (Marciniuk, 2016):

$$ {}_{\frac{k}{m}}p_{xy} = P\left(K_u^{(m)} \geq k\right) = p_{11}\left(0, \tfrac{k}{m}\right). \tag{4.71}$$

Note that by using (4.39) for $t = 0$ and $z = k/m$, we straightforwardly rewrite (4.71) in the following form:

$$ \begin{aligned} {}_{\frac{k}{m}}p_{xy} = p_{11}\left(0, \frac{k}{m}\right) &= \exp\left(- \int_0^{\frac{k}{m}} \left(\mu_{12}(\tau) + \mu_{13}(\tau)\right) d\tau \right) \\ &= \left({}_{\frac{t}{m}}p_x^W \right)^{1+\alpha_{13}} \left({}_{\frac{t}{m}}p_y^M \right)^{1+\alpha_{12}}. \end{aligned} \tag{4.72}$$

Parameters α_{12} and α_{13} in (4.72) were calculated based on the Nelson-Aalen estimator, proposed by (Denuit et al., 2001). They were estimated in (Heilpern, 2015), using data from Lower Silesia from 2011 obtained from Statistics Poland. The details are described in Section 4.2.3 before Example 4.2.

Example 4.7

Assume that conditions C1-C4 are fulfilled. The study distinguished five situations of (x, y) for different x and y. The marriage reverse annuity contract benefit is paid m $(m \in \{1,4,6,12\})$ times yearly in the constant amount of $1/m$ financial unit at the beginning of a sub-period of the year for the JLS ($R = 0$) and the LLS ($R = 1$). The benefit was calculated using (4.67); the results are presented in Table 4.7.

It is clear that the benefit increases with the age of the spouses and the frequency of payments. The largest differences in yearly benefits are between $m = 1$ and $m = 4$. A larger m does not result in a significant increase in the annuity. The JLS annuity is significantly higher than the benefit obtained in the LLS case. For 80-year-old people, this difference is almost 80%. The JLS benefit is higher when the woman is older than the husband, while the LLS benefit is the opposite. The benefit when future lifetimes are dependent random variables is slightly lower than for independent variables (cf. Table 4.4).

Table 4.7. The reverse annuity contract benefit payment for marriage (*x, y*), when the future life of spouses are dependent random variables (Markov model)

(*x*, *y*)	***m***	**Benefit payment**		**Yearly sum of benefit payments**	
		JLS (*R* = 0)	**LSS (*R* = 1)**	**JLS (*R* = 0)**	**LSS (*R* = 1)**
(65,65)	1	5721.7	3848.2	5721.7	3848.2
	4	1496.1	990.8	5984.4	3963.2
	6	1002.4	662.7	6014.4	3976.3
	12	503.8	332.5	6045.0	3989.4
(70,65)	1	6140.8	4089.9	6140.8	4089.9
	4	1611.2	1054.9	6444.8	4219.6
	6	1080.0	705.7	6480.0	4234.4
	12	542.9	354.1	6515.3	4249.3
(65,70)	1	6360.9	3961.7	6360.9	3961.7
	4	1671.8	1020.9	6687.2	4083.6
	6	1120.9	682.9	6725.4	4097.5
	12	563.6	342.6	6763.3	4111.4
(70,70)	1	6740.1	4273.9	6740.1	4273.9
	4	1777.1	1103.9	7108.4	4415.6
	6	1191.9	738.6	7151.4	4431.8
	12	599.5	370.7	7194.2	4448.2
(80,80)	1	10480.0	5841.5	10480.0	5841.5
	4	2852.1	1526.1	11408.4	6104.4
	6	1919.8	1022.5	11518.8	6135.0
	12	969.3	513.9	11631.5	6166.2

Source: own elaboration.

In the second case, the authors determined the dependent structure of the future lifetimes of spouses by using the copula. From Section 4.2.3 it is known that the problem of establishing probability ${}_tp_{xy}$ necessary to calculate benefit payments of marital equity release, is reduced to assessing joint survival function $S_{xy}(t,t)$ defined by the copula as given in (4.49).

Assume that the distribution of the joint lifetimes of spouses is known for reference age x_0 for wives and y_0 for husbands. In addition, let $x = x_0 + s$ and $y = y_0 + t$, then the probability that (x, y) will survive at least k years is equal (compare (4.32)):

$$
\begin{aligned}
{}_kp_{xy} &= P\left(K_x^W > k, K_y^M > k\right) = P\left(K_{x_0}^W > s+k, K_{y_0}^M > t+k \middle| K_{x_0}^W > s, K_{y_0}^M > t\right) \\
&= \frac{P\left(K_{x_0}^W > s+k, K_{y_0}^M > t+k\right)}{P\left(K_{x_0}^M > s, K_{y_0}^W > t\right)} = \frac{S_{x_0y_0}(s+k, t+k)}{S_{x_0y_0}(s,t)} = \frac{C_0^*\left({}_{s+k}p_{x_0}^W, {}_{t+k}p_{y_0}^M\right)}{C_0^*\left({}_sp_{x_0}^W, {}_tp_{y_0}^M\right)}.
\end{aligned}
\tag{4.73}
$$

The study assessed the probability (4.73) for $m = 1$ based on the Life Table for Lower Silesia from 2011 using survival copula $C_0^*(u,v) = u + v - 1 + C(1-u, 1-v)$ introduced by the dependent structure, was estimated by using the Gumbel copula (see (4.57)) with parameter $\alpha = 1.0786$, when the reference ages equal $x_0 = y_0 = 60$, and Kendall's tau coefficient of correlation $\tau = 0.073$. The details are provided in Section 4.2.3 before Example 4.3.

Example 4.8

Assume that conditions C1 to C4 are satisfied, and compare the benefit obtained from marital reverse annuity contract in three cases: independence, dependences described by the Markov model, and the Gumbel copula. The results are presented in Table 4.8 in five situations of (x, y) for different x (wife's age) and y (husband's age).

Table 4.8. The reverse annuity contract's benefit payment for marriage (x, y) in three cases

(x, y)	**Benefit**					
	Gumbel		**Markov model**		**independent**	
	JLS (R = 0)	**LSS (R = 1)**	**JLS (R = 0)**	**LSS (R = 1)**	**JLS (R = 0)**	**LSS (R = 1)**
(65,65)	5745.4	3916.1	*5721.7*	3848.2	5836.9	3896.3
(65,70)	6386.6	4018.4	*6360.9*	3961.7	6495.3	4016.5
(70,65)	6154.0	4167.1	*6140.8*	4089.9	6280.8	4146.5
(70,70)	*6734.6*	4351.0	6740.1	4273.9	6897.8	4341.1
(80,80)	*10202.0*	5912.3	10480.0	5841.5	10811.0	5983.2

Source: own elaboration.

It can be observed that the benefit increases with the spouses' age. Its value grows faster for the older people. Under the assumption that the future lifetimes of spouses are independent, the benefit is always higher than in other in JLS cases which is the analogical result as in (Dębicka et al., 2020). In the case of dependence, similar amounts of annuity are obtained. For the Gumbel function, the JLS contract benefit payments are higher for 'younger' marriages than in the Markov model, and *vice versa* for older people. The annuity for LSS contract is always higher in the case of the Gumbel copula compared with the benefit in the Markov model.

Chapter 5

COMPREHENSIVE MARRIAGE CONTRACT WITH HEALTH PROTECTION

5.1. Concept of new contracts

In many countries, also in the European Union, the problems of the pension system and health care especially affect older people. Pensioners have very low incomes and belong to the poorest group of society. In addition to financial difficulties, pensioners also face deteriorating health, especially in late old age. In this group, the problem of the growth of demand for medical services is visible, and together with limited resources in the health care system cause the common phenomenon of the rationing of resources for treatment. There are many ways of dealing with the deficit, for example, by reducing the basic health premium or removing some products from the basket of benefits. The other possibilities are connected with introducing a cost-sharing concept or complementary private health insurance. In Poland, costs of healthcare are covered in 30% from private sources. Chapter 1 mentions that retirees rarely purchase private health insurance despite incurring out-of-pocket expenses.

Some motivation to increase private health insurance funding may come from new financial-insurance contracts operating as hybrids. A popular solution could be proposed as a combination of critical health insurance with life insurance. Considering that 80% of adult population in Poland have their own properties (see Chapter 1), another solution could be combining a reverse annuity contract with health insurance (Biernacki et al., 2021).

In the case of a life insurance policy, early benefit payment while the insured is still alive can be used to cover medical expenses or long-term care. In contrast, funds received from a reverse annuity contract can be used to pay the health premium for critical illness insurance. This chapter presents the new insurance and financial contract – as a combination of the reverse annuity contract and two dread disease insurances. The benefits of the new contract can help improve living conditions and quality of life and provide additional financial resources in case of critical illness. Additionally, combining of these products in one reduces maintenance costs. Due to the fact that most spouses are usually the owners of the property, the contract should include both of them. This agreement is referred to as the *Comprehensive Marriage*

Contract with Health Protection as it reflects both the financial aspect (equity release) and the health aspect (critical health insurance for both husband and wife) of the agreement, highlighting its integrated and comprehensive nature.

An individual model describing cash flows related to this type of contract when only one person owns the property is defined in (Dębicka et al., 2015). The first component of such a contract is a reverse annuity contract. The legal basis is a notarial act. The owner transfers ownership onto the company, in return receiving benefits and the right to live in the property until death. The value of benefits depends on the pensioner's further life expectancy and property value W. In practice, the value is calculated based on a certain percentage of the property's value. This percentage is denoted as α which takes a value from 0 to 0.5. The second element of such a contract is critical health insurance which usually covers various types of diseases. There are typically two stages of disease progression: mild and critical. The mild stage does not significantly affect life expectancy, only slightly increases the risk of death, and the benefits of this condition can be used for treatment. The critical stage has a huge impact on life expectancy. At this stage, the probability of death depends on the progression of the illness. Combining a reverse annuity contract with critical health insurance requires consideration of the extended multistate model for critical health insurance with annuities (described in Chapter 3 as Model III), which involves modelling the life expectancy of a critically ill person. Dębicka et al. (2015) described the multiple-state model for such individual contracts, considering a division of the disease state into more substates.

Modelling insurance and financial contracts always compromises accuracy and practicality because the so-called full models require detailed data that can be difficult to obtain. The authors decided to present the basic *Comprehensive Marriage Contract with Health Protection* model in this monograph. In this case, it is not the fact that simplified models are often based on more general or more easily available data, which makes them more practical to use, but rather the complexity of the model in its presentation, according to the principle that the importance of simplicity is the most important in conveying ideas and understanding concepts.

In a *Comprehensive Marriage Contract with Health Protection*, three random events like health, death, diagnosis and course of diseases should be modelled for two insured persons. Note that generally, if one distinguishes N states for an individual model, $N \times N$ states should be considered in the case of a model for the spouses. For this reason, to make the presentation of the idea of the new contract more transparent and easier to present, which will also facilitate the interpretation of numerical results, the authors adopted a simple version of the model described in Figure 3.4 (*multistate model of health and accident insurance*) for the wife and husband. Thus, for each spouse, three possible states were considered: the first state (A) meaning that the insured person is active (i.e. healthy), the second state (I) meaning that the insured person is sick, and the third state (D) meaning the death of the insured person. For that model, by the assumption of critical disease, the following transition matrix was introduced:

$$\mathbf{Q}(k) = \begin{pmatrix} 1 - (q_{x+k} - \varpi_{x+k}) - \zeta_{x+k} & \zeta_{x+k} & q_{x+k} - \varpi_{x+k} \\ 0 & 1 - d_{x+k} & d_{x+k} \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

where q_{x+k} is the probability of death in the whole population, ϖ_{x+k} is the diseases--specific mortality rate in the entire population, ζ_{x+k} is the morbidity rate, d_{x+k} is the fatality rate, which defines the probability of death in the critically ill population.

Based on the two models (3.26) of critical health insurance contracts (for wife and husband) with a probabilistic structure given by (5.1) and marital reverse annuity contract given by (4.1), the *Comprehensive Marriage Contract with Health Protection* is modelled in this chapter. In particular, Section 5.2 presents a multistate model along with its probabilistic structure for the proposed contract, considering two scenarios depending on the type of health benefit chosen by the spouses. Section 5.3 is dedicated to valuing the benefits resulting from the agreement. Section 5.4 contains numerical examples demonstrating the practical applicability of the proposed solution. The chapter concludes with a summary and a discussion of potential directions for further research and possibilities for expanding the proposed solutions (Section 5.5).

5.2. Multistate contract modelling

The idea of the *Comprehensive Marriage Contract with Health Protection* is as follows. Spouses who own real estate whose value is W, in accordance with the principles of mortgage annuity, in exchange for waiving the rights to the real estate, receive a reverse annuity paid as long as one of the spouses is alive (LSS annuity). This determines the contract period, which means that $n = max\{\omega_x^W, \omega_y^M\}$. The value of the annuity is determined based on the amount of αW (in practice $\alpha \in [0,0.5]$). Assume that reverse annuity amount b is constant throughout the contract period (for $k = 0,1,\ldots,n-1$) and paid at the beginning of the year (cash flow paid in advance). The spouses decide what part of it remains in their budget ($\beta \cdot b$) and what part is allocated to purchasing health insurance ($(1-\beta)\cdot b$), where β is the so-called *reverse annuity parameter* ($\beta \in [0,1]$). Assume also that the amount allocated for health insurance may be unevenly divided. Let $\gamma \in [0,1]$ be a *health parameter*, therefore

- the amount allocated to the wife's insurance (net annual premium for the wife's critical illness insurance):

$$p^x = \gamma(1-\beta)b, \tag{5.2}$$

- the amount allocated to the husband's insurance (net annual premium for the husband's critical illness insurance):

$$p^y = (1-\gamma)(1-\beta)b. \tag{5.3}$$

The critical illness contract provides one of two types of benefits: an annuity benefit payable after a critical illness diagnosis or a lump sum benefit payable at any time $k + 1$ if the insured diagnosis of critical illness occurred in the time interval $[k, k + 1)$ before the end of the contract. Both types of illness benefits are paid on completion. Therefore, we can consider the *Comprehensive Marriage Contract with Health Protection* (CMCHP*)* under one of four scenarios:

Scenario I – CMCHP, according to which *health annuity benefits* for both spouses are paid.

Scenario II – CMCHP, according to which *the wife's health annuity benefits and the husband's lump-sum health benefits* are paid.

Scenario III – CMCHP, according to which *the wife's lump-sum health benefits and the husband's health annuity benefits* are paid.

Scenario IV – CMCHP, according to which *lump-sum health benefits* are paid for both spouses.

Further considerations in this section are made for Scenario I and Scenario IV, as the remaining scenarios are a mixture of the two.

Start by designing the multiple-state model and its probabilistic structure for such a contract, and then we modify MSM depending on the scenario.

The combination of the marital reverse annuity contract with two dread disease insurance policies leads to the following random events: a healthy life (A), life with serious illness (I), and death (D), which may occur for one spouse or both simultaneously. Therefore, the state space of the model includes 3 × 3 = 9 possibilities. Thus, state space S = {1, 2, …, 9} consists of the following states:

1 – X^AY^A – both spouses are active, i.e. they are alive and healthy,
2 – X^AY^I – the wife is active, and the husband is ill,
3 – X^AY^D – the wife is active, and the husband is dead,
4 – X^IY^A – the wife is ill, and the husband is active,
5 – X^IY^I – both spouses are ill,
6 – X^IY^D – the wife is ill, and the husband is dead,
7 – X^DY^A – the wife is dead, the husband is active,
8 – X^DY^I – the wife is dead, and the husband is ill,
9 – X^DY^D – both spouses are dead.

The appropriate multiple-state model with all possible direct transitions is presented in Figure 5.1.

For the model presented in Figure 5.1, when $x + k \le \omega_x^W$ and $y + k \le \omega_y^M$, the transition probability matrix is given by the following formula:

$$\mathbf{Q}(k)=\begin{bmatrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) & q_{15}(k) & q_{16}(k) & q_{17}(k) & q_{18}(k) & q_{19}(k) \\ 0 & q_{22}(k) & q_{23}(k) & 0 & q_{25}(k) & q_{26}(k) & 0 & q_{28}(k) & q_{29}(k) \\ 0 & 0 & q_{33}(k) & 0 & 0 & q_{36}(k) & 0 & 0 & q_{39}(k) \\ 0 & 0 & 0 & q_{44}(k) & q_{45}(k) & q_{46}(k) & q_{47}(k) & q_{48}(k) & q_{49}(k) \\ 0 & 0 & 0 & 0 & q_{55}(k) & q_{56}(k) & 0 & q_{58}(k) & q_{59}(k) \\ 0 & 0 & 0 & 0 & 0 & q_{66}(k) & 0 & 0 & q_{69}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{77}(k) & q_{78}(k) & q_{79}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{88}(k) & q_{89}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.4)$$

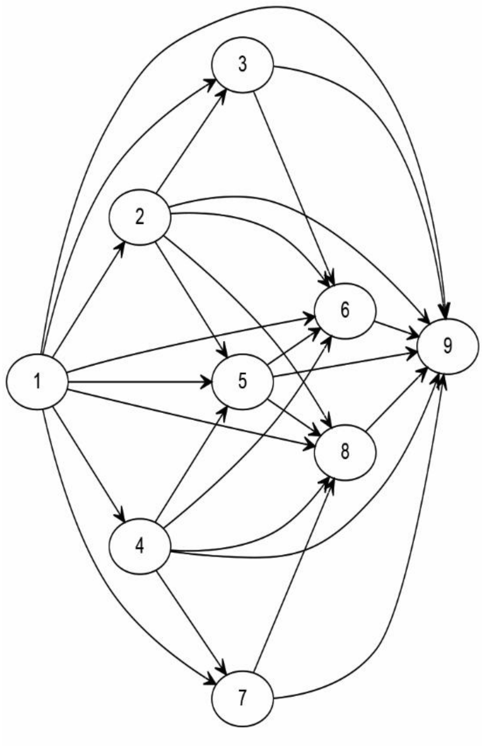


Figure 5.1. Multistate model for the Comprehensive Marriage Contract with Health Protection

Source: own elaboration. The diagram was created using the online tool (Dreampuf, 2024).

Assuming that the spouses' future lifetimes are independent, matrix $\mathbf{Q}(k)$ elements are determined based on the transition probability matrices in the marital insurance (4.3) and the health insurance for terminal illness risk given (5.1) separately for x-year--old wife and y-year-old husband. The values of these probabilities for each row of the matrix (5.4) are presented in Table 5.1.

Table 5.1. The formulas for estimators of the transition probability for Comprehensive Marriage Contract with Health Protection

The number of state (row of the matrix (5.4))	Estimators of transition probability
1	$q_{11}(k)=\left(1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k}\right)\cdot\left(1-\left(q_{y+k}-\varpi_{y+k}\right)-\zeta_{y+k}\right)$, $q_{12}(k)=\left(1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k}\right)\cdot\zeta_{y+k}$, $q_{13}(k)=\left(1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k}\right)\cdot\left(q_{y+k}-\varpi_{y+k}\right)$, $q_{14}(k)=\zeta_{x+k}\cdot\left(1-\left(q_{y+k}-\varpi_{y+k}\right)-\zeta_{x+k}\right)$, $q_{15}(k)=\zeta_{x+k}\cdot\zeta_{y+k}$, $q_{16}(k)=\zeta_{x+k}\cdot\left(q_{y+k}-\varpi_{y+k}\right)$, $q_{17}(k)=\left(q_{x+k}-\varpi_{x+k}\right)\cdot\left(1-\left(q_{y+k}-\varpi_{y+k}\right)-\zeta_{y+k}\right)$, $q_{18}(k)=\left(q_{x+k}-\varpi_{x+k}\right)\cdot\zeta_{y+k}q_{19}(k)=\left(q_{x+k}-\varpi_{x+k}\right)\cdot\left(q_{y+k}-\varpi_{y+k}\right)$
2	$q_{22}(k)=\left(1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k}\right)\cdot\left(1-d_{y+k}\right)$, $q_{23}(k)=\left(1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k}\right)\cdot d_{y+k}$, $q_{25}(k)=\zeta_{x+k}\cdot\left(1-d_{y+k}\right)$, $q_{26}(k)=\zeta_{x+k}\cdot d_{y+k}$, $q_{28}(k)=\left(q_{x+k}-\varpi_{x+k}\right)\cdot\left(1-d_{y+k}\right)$, $q_{29}(k)=\left(q_{x+k}-\varpi_{x+k}\right)\cdot d_{y+k}$
3	$q_{33}(k)=\left(1-\left(q_{x+k}-\varpi_{x+k}\right)-\zeta_{x+k}\right)$, $q_{36}(k)=\zeta_{x+k}$, $q_{39}(k)=\left(q_{x+k}-\varpi_{x+k}\right)$
4	$q_{44}(k)=\left(1-d_{x+k}\right)\cdot\left(1-\left(q_{y+k}-\varpi_{y+k}\right)-\zeta_{y+k}\right)$, $q_{45}(k)=\left(1-d_{x+k}\right)\cdot\zeta_{y+k}$, $q_{46}(k)=\left(1-d_{x+k}\right)\cdot\left(q_{y+k}-\varpi_{y+k}\right)$, $q_{47}(k)=d_{x+k}\cdot\left(1-\left(q_{y+k}-\varpi_{y+k}\right)-\zeta_{y+k}\right)$, $q_{48}(k)=d_{x+k}\cdot\zeta_{y+k}$, $q_{49}(k)=d_{x+k}\cdot\left(q_{y+k}-\varpi_{y+k}\right)$
5	$q_{55}(k)=\left(1-d_{x+k}\right)\cdot\left(1-d_{y+k}\right)$, $q_{56}(k)=\left(1-d_{x+k}\right)\cdot d_{y+k}$, $q_{58}(k)=d_{x+k}\cdot\left(1-d_{y+k}\right)$, $q_{59}(k)=d_{x+k}\cdot d_{y+k}$
6	$q_{66}(k)=\left(1-d_{x+k}\right)$, $q_{69}(k)=d_{x+k}$
7	$q_{77}(k)=1-\left(q_{y+k}-\varpi_{y+k}\right)-\zeta_{y+k}$, $q_{78}(k)=\zeta_{y+k}$, $q_{79}(k)=q_{y+k}-\varpi_{y+k}$
8	$q_{88}(k)=1-d_{y+k}$, $q_{89}(k)=d_{y+k}$
9	$q_{99}(k)=1$

Source: own elaboration.

When $x+k\le\omega_x^W$ and $y+k>\omega_y^M$ (the wife is younger than the husband), matrix $\mathbf{Q}(k)$ given by (5.4) is reduced to the following form:

$$\mathbf{Q}(k)=\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33}(k) & 0 & 0 & q_{36}(k) & 0 & 0 & q_{39}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{66}(k) & 0 & 0 & q_{69}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}. \tag{5.5}$$

When $x+k>\omega_x^W$ and $y+k\le\omega_y^M$ (the husband is younger than the wife), matrix $\mathbf{Q}(k)$ is as follows:

$$\mathbf{Q}(k)=\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{77}(k) & q_{78}(k) & q_{79}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{88}(k) & q_{89}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}. \tag{5.6}$$

Note that matrices (5.5) and (5.6) are the reduced probability matrices for the MSM model presented in Figure 5.1. The explanation is identical to (4.5) and (4.6) in Section 4.1.

To use the matrix notation for calculating benefits arising from the contract (this means the scenario of the contract), it is essential to check if conditions B1 and B2 are satisfied and, if necessary, modify the MSM presented in Figure 5.1.

Scenario I. *Comprehensive Marriage Contract with Health Protection according to which health annuity benefits for both spouses are paid*

Cash flows resulting from the contract, from the perspective of the insured spouses, are either income (reverse annuity, health benefits) or expenses (health insurance premiums) and are specified as follows:

- reverse annuity

$$b_j(k)=\begin{cases} b & \text{for} \quad j=1 \text{ and } k=0,1,\ldots,n-1 \\ & \qquad\ \ j=2,3,\ldots,8 \text{ and } k=1,2,\ldots,n-1, \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

- net health insurance premiums

$$p_j(k)=\begin{array}{lll} -(p^x+p^{y)} & \text{for} & j=1 \text{ and } k=0,1,\ldots,n-1 \\ -p^x & \text{for} & j=2,3 \text{ and } k=1,2,\ldots,n-1, \\ -p^y & \text{for} & j=4,7 \text{ and } k=1,2,\ldots,n-1\text{'} \\ 0 & & \text{otherwise} \end{array} \tag{5.8}$$

- immediate health annuity

$$b_j(k) = \begin{cases} b^x + b^y & \text{for} \quad j = 5 \text{ and } k = 2,3,\ldots,n \\ b^x & \text{for} \quad j = 4,6 \text{ and } k = 2,3,\ldots,n \\ b^y & \text{for} \quad j = 2,8 \text{ and } k = 2,3,\ldots,n \\ 0 & \text{otherwise} \end{cases}. \tag{5.9}$$

In Figure 5.2, the number of states where reverse annuity instalments are paid is marked in red. Additionally, the circles related to the states associated with the payment of the health annuity for the wife are in pink, and the circles related to the health annuity for the husband are in blue.

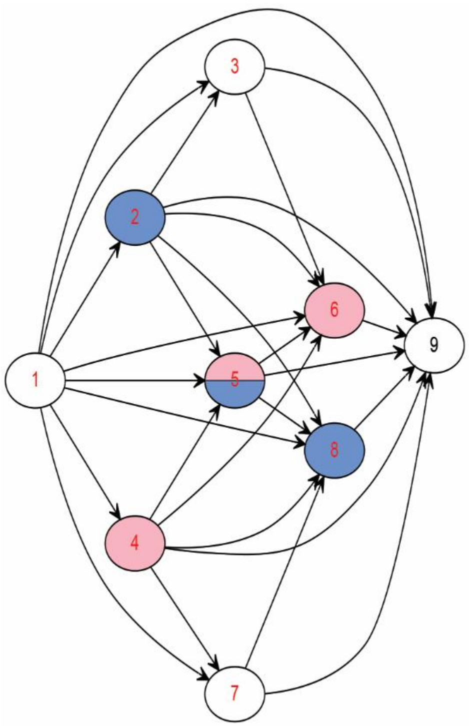


Figure 5.2. Cash flows for the Comprehensive Marriage Contract with Health Protection – Scenario I

Source: own elaboration. The diagram was created using the online tool (Dreampuf, 2024).

Note that as a result of the execution of the marriage contract under Scenario I, only cash flows of the not-*cf* type are generated, therefore $(S,T) = (S^*,T^*)$ and $\{X(k)\} \equiv \{X^*(k)\}$, so applying the matrix notation to calculate the actuarial values directly is possible. Cash flow matrix $\mathbf{C}_{in}$ resulting from the implementation of such a contract is determined as follows:

$$
\mathbf{C}_{in} = \begin{pmatrix}
b-(p^x+p^y) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b-(p^x+p^y) & b-p^x & b-p^x & b-p^y & b & b & b-p^y & b & 0 \\
b-(p^x+p^y) & b+b^y-p^x & b-p^x & b+b^x-p^y & b+b^x+b^y & b+b^x & b-p^y & b+b^y & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b-(p^x+p^y) & b+b^y-p^x & b-p^x & b+b^x-p^y & b+b^x+b^y & b+b^x & b-p^y & b+b^y & 0 \\
0 & b^y & 0 & b^x & b^x+b^y & b^x & 0 & b^y & 0
\end{pmatrix}. \quad (5.10)
$$

By substituting (5.2) and (5.3) into (5.10), we obtain $\mathbf{C}_{in} \in R^{(n+1)\times 9}$, which depends on the benefits of the contract (that is b, b^x, b^y) and the parameters (β, γ) established by the spouses.

Scenario IV. *Comprehensive Marriage Contract with Health Protection, accor-ding to which lump-sum health benefits for both spouses are paid*

According to Scenario IV, reverse annuity and health insurance premiums resulting from the marriage contract are the same as in the case of Scenario I (compare (5.7) and (5.8), respectively). Under the assumption that lump sum health benefits are constant throughout the whole insurance contract (c^x for wife, c^y for husband), then

$$
c_{ij}(k) = \begin{cases}
c^x + c^y & \text{for} \quad (i,j) = (1{,}5) \quad \text{and } k = 2{,}3, \dots, n \\
c^x & \text{for} \quad (i,j) \in \{(1{,}4), (1{,}6), (2{,}5), (2{,}6), (3{,}6)\} \text{ and } k = 2{,}3, \dots, n \\
c^y & \text{for} \quad (i,j) \in \{(1{,}2), (1{,}8), (4{,}5), (4{,}8), (7{,}8)\} \text{ and } k = 2{,}3, \dots, n \\
0 & \text{otherwise}
\end{cases}. \quad (5.11)
$$

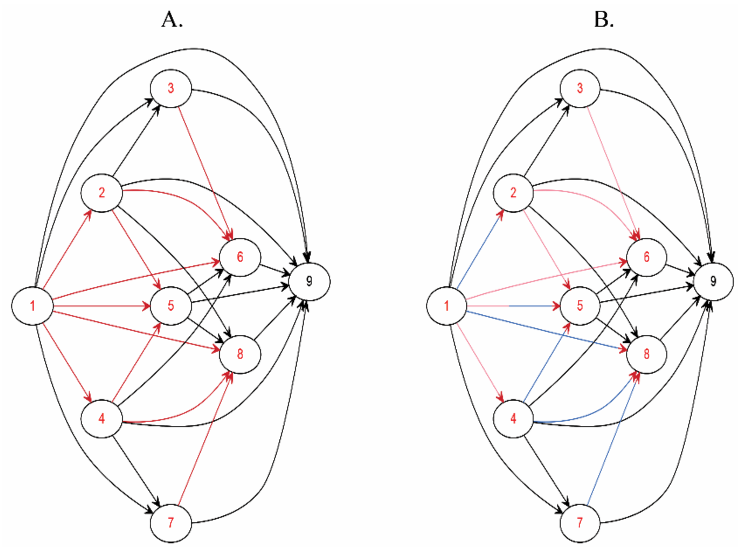


Figure 5.3. Cash flows for the Comprehensive Marriage Contract with Health Protection – Scenario IV

Source: own elaboration. The diagram was created using the online tool (Dreampuf, 2024).

Note that as a result of the execution of the marriage contract under Scenario II, cash flows of the not-*cf* type are generated (reverse annuity and net premiums), but also cash flows of *cf* type (lump sum health benefits). Compare Figure 5.3, where in scheme A, the number of states in which the not-*cf* type cash flows are marked in red, and the direct transitions associated with cash flows of the *cf* type (arrows) are also in red. Additionally, in scheme B, the tail arrows are in pink, representing the lump sum health benefit for the wife (c^x) and in blue, which means that the husband's lump sum health benefit (c^y) is paid.

The direct transitions of the *cf* type (coloured arrows) make it impossible to apply the matrix notation directly, therefore a modified multistate model should be built. To check assumptions B1 and B2,start from the control array, which is as follows:

$$W(k) = \begin{matrix} - & c^y & + & c^x & c^x + c^y & c^x & + & c^y & + \\ - & - & + & - & c^x & c^x & - & + & + \\ - & - & - & - & - & c^x & - & - & + \\ - & - & - & - & c^y & + & - & c^y & + \\ - & - & - & - & - & + & - & + & + \\ - & - & - & - & - & - & - & - & + \\ - & - & - & - & - & - & - & c^y & + \\ - & - & - & - & - & - & - & - & + \\ - & - & - & - & - & - & - & - & - \end{matrix}. \tag{5.12}$$

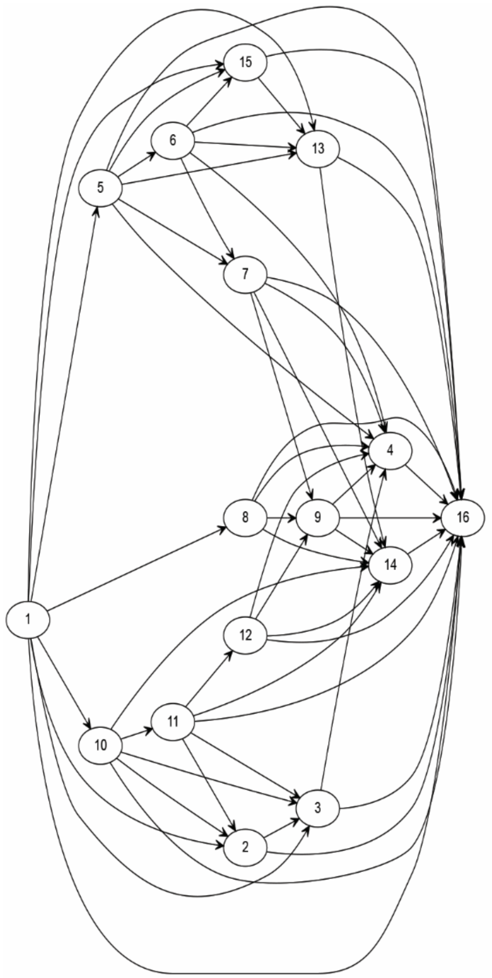
It appears that assumption B1 was not met (column 5 in (5.12)), and assumption B2 was not met either (columns 6 and 8 in (5.12)). Therefore, according to the procedure, the model presented in Figure 5.4 should be redefined, and then the modification procedure should be applied (according to the diagram presented in Figure 2.7). The result is a modified multistate model for Scenario IV (S^*, T^*) presented in Figure 5.4.

For the model presented in Figure 5.4, transition probability matrix $\mathbf{Q}^*(k)$ was described based on transition probabilities as for matrix $\mathbf{Q}(k)$ (cf. (5.4) and Table 5.1). Thus for $x + k \le \omega_x^W$ and $y + k \le \omega_y^M$

$$Q^*(k) = \begin{matrix}
q_{11}(k) & q_{13}(k) & q_{16}(k) & 0 & q_{14}(k) & 0 & 0 & q_{15}(k) & 0 & q_{12}(k) & 0 & 0 & q_{18}(k) & 0 & q_{17}(k) & q_{19}(k) \\
0 & q_{33}(k) & q_{36}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{39}(k) \\
0 & 0 & 0 & q_{66}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{69}(k) \\
0 & 0 & 0 & q_{66}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{69}(k) \\
0 & 0 & 0 & q_{46}(k) & 0 & q_{44}(k) & q_{45}(k) & 0 & 0 & 0 & 0 & 0 & q_{48}(k) & 0 & q_{47}(k) & q_{49}(k) \\
0 & 0 & 0 & q_{46}(k) & 0 & q_{44}(k) & q_{45}(k) & 0 & 0 & 0 & 0 & 0 & q_{48}(k) & 0 & q_{47}(k) & q_{49}(k) \\
0 & 0 & 0 & q_{56}(k) & 0 & 0 & 0 & 0 & q_{55}(k) & 0 & 0 & 0 & 0 & q_{58}(k) & 0 & q_{59}(k) \\
0 & 0 & 0 & q_{56}(k) & 0 & 0 & 0 & 0 & q_{55}(k) & 0 & 0 & 0 & 0 & q_{58}(k) & 0 & q_{59}(k) \\
0 & 0 & 0 & q_{56}(k) & 0 & 0 & 0 & 0 & q_{55}(k) & 0 & 0 & 0 & 0 & q_{58}(k) & 0 & q_{59}(k) \\
0 & q_{23}(k) & q_{26}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{22}(k) & q_{25}(k) & 0 & q_{28}(k) & 0 & q_{29}(k) \\
0 & q_{23}(k) & q_{26}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{22}(k) & q_{25}(k) & 0 & q_{28}(k) & 0 & q_{29}(k) \\
0 & 0 & 0 & q_{56}(k) & 0 & 0 & 0 & 0 & q_{55}(k) & 0 & 0 & 0 & 0 & q_{58}(k) & 0 & q_{59}(k) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{88}(k) & 0 & q_{89}(k) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{88}(k) & 0 & q_{89}(k) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{78}(k) & 0 & q_{77}(k) & q_{79}(k) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix} \tag{5.13}$$

Hence, when the spouses are not of the same age $x \neq y$, then for $x + k > \omega_x^W$ in (5.13), the rows from 1 to 6 contain only zeros, and similarly for $y + k > \omega_y^M$, rows 1,2,4,5,7 and 8 in matrix (5.13) contain only zeros.

In Figure 5.5, the number of states where reverse annuity instalments are paid is marked in red. Additionally, the circles related to the states associated with the payment of the lump sum health benefit for the wife are pink, and the circles related to the lump sum health benefit for the husband are blue.



State space S^*:

1 – $X^A Y^A$
2 – $X^A Y^D$
3 – $X^{I^+} Y^D$
4 – $X^I Y^D$
5 – $X^{I^+} Y^A$
6 – $X^I Y^A$
7 – $X^I Y^{I^+}$
8 – $X^{I^+} Y^{I^+}$
9 – $X^I Y^I$
10 – $X^A Y^{I^+}$
11 – $X^A Y^I$
12 – $X^{I^+} Y^I$
13 – $X^D Y^{I^+}$
14 – $X^D Y^I$
15 – $X^D Y^A$
16 – $X^D Y^D$

Legend:
X – wife
Y – husband
A – the insured is active (healthy)
I^+ – the insured person fell ill with a a critical illness
I – the insured is ill for at least one year
D – the insured is dead

Figure 5.4. Modified multistate model for the Comprehensive Marriage Contract with Health Protection – Scenario IV

Source: own elaboration. The diagram was created using the online tool (Dreampuf, 2024).

Note that lump sum benefits were transformed into benefits related to the stay of process $\{X^*(t)\}$ in specific states. These states are associated with the diagnosis of a serious disease in the wife (X^{I^+}) and the husband (Y^{I^+}), and due to their nature, $\{X^*(t)\}$ process cannot remain in this state for longer than one unit of time (i.e. states 3,5,7,8,1012,13 are the so-called reflex states).

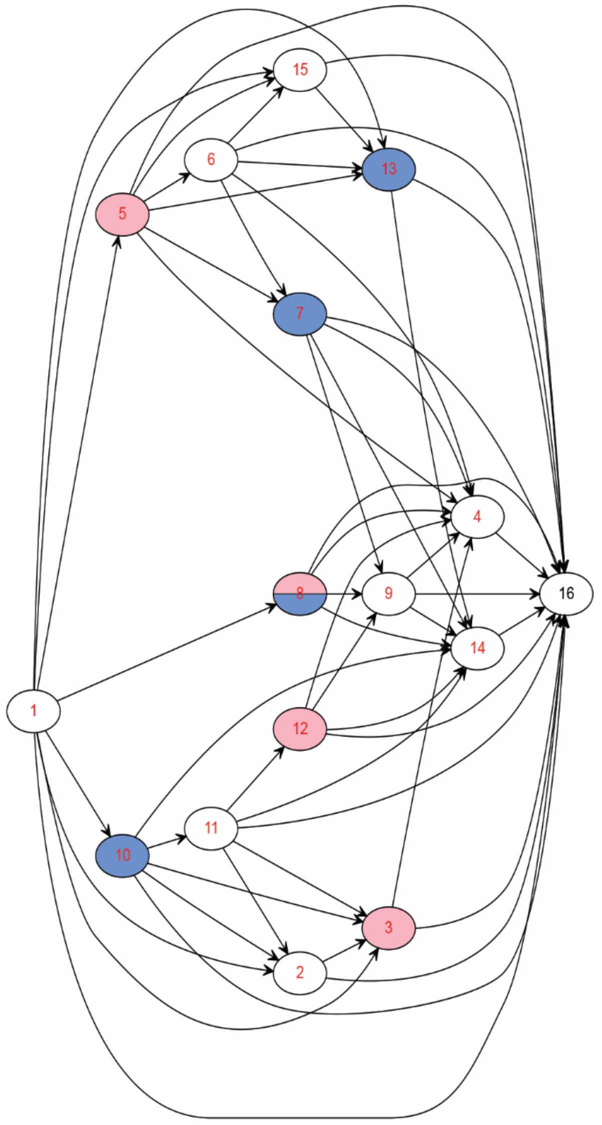


Figure 5.5. Cash flows for the modified multistate model for the Comprehensive Marriage Contract with Health Protection – Scenario IV

Source: own elaboration. The diagram was created using the online tool (Dreampuf, 2024).

Cash flow matrix $\mathbf{C}_{in}$ resulting from the implementation of the MMSM for a marriage contract under Scenario IV is determined as follows:

$$
\boldsymbol{C}_{in} = \begin{matrix}
b-(p^x+p^y) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b-(p^x+p^y) & b-p^x & b & b & b-p^y & b-p^y & b & b & b & b-p^x & b-p^x & b & b & b & b-p^y & 0 \\
b-(p^x+p^y) & b-p^x & b+c^x & b & b+c^x-p^y & b-p^y & b+c^y & b+c^x+c^y & b & b+c^y-p^x & b-p^x & b+c^x & b+c^y & b & b-p^y & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b-(p^x+p^y) & b-p^x & b+c^x & b & b+c^x+b^y & b-p^y & b+c^y & b+c^x+c^y & b & b+c^y-p^x & b-p^x & b+c^x & b+c^y & b & b-p^y & 0 \\
0 & 0 & c^x & 0 & c^x & 0 & c^y & c^x+c^y & 0 & c^y & 0 & c^x & c^y & 0 & 0 & 0
\end{matrix} \quad (5.14)
$$

Substituting (5.2) and (5.3) into (5.14) gives $\mathbf{C}_{in} \in R^{(n+1)\times 16}$, which depends on the benefits resulting from the contract (that is b, c^x, c^y) and the parameters (β, γ) established by the spouses.

5.3. Cash flow valuation

This section is dedicated to deriving matrix formulas for determining the benefits (reverse annuity and health benefits) arising from realising the *Comprehensive Marriage Contract with Health Protection* based on property value W, parameters α, and the two β, γ established by the spouses.

Note that cash flow $\wp_i(k)$, realised according to the contract modelled by (S^*, T^*), is paid when $X^*(k) = i$ and may be realised in more than one state (such as reverse annuity in CMCHW). Let $S^{\wp} \subseteq S^*$ be the subset of state space S^* such that $X^*(k) \in S^{\wp}$ for $k = 0, 1, 2, \ldots, n$ which implies that the cash flow of type $\wp_i(k)$ is realised. The period of its realisation $\wp_i(k)$ during the insurance contract depends, among other factors, on $\delta(1, i)$ the shortest path from state 1 (assuming $X^*(0) = 1$) to state i. This path can be determined in a multistate model using Dijkstra's algorithm (Dębicka & Zmyślona, 2018). In this case, benefits are associated with marriage insurance according to Scenario I $S^{\wp}$, and $\delta(1, i)$ are defined in Table 5.2 and Scenario IV in Table 5.3. Similarly, these quantities can be determined for the remaining marriage insurance scenarios.

Table 5.2. Description of the annual cash flows for Comprehensive Marriage Contract with Health Protection – Scenario I

Type of cash flow	$\wp =$	$S^{\wp} =$	$\delta(1, i) =$
Marriage reverse annuity	b	$\{1,2,\ldots,8\}$	$\begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 2,3,\ldots,8 \end{cases}$
Health period premium for wife	p_x	$\{1,2,3\}$	$\begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 2,3 \end{cases}$
Immediate health annuity for wife	b_x	$\{4,5,6\}$	1 for $i = 4,5,6$
Health period premium for husband	p_y	$\{1,4,7\}$	$\begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 4,7 \end{cases}$
Immediate health annuity for husband	b_y	$\{2,5,8\}$	1 for $i = 2,5,8$

Source: own elaboration.

Table 5.3. Description of type of benefit for Comprehensive Marriage Contract with Health Protection – Scenario IV

Type of cash flow	$\wp =$	$S^{\wp} =$	$\delta(1, i) =$
Marriage reverse annuity	b	$\{1,2,\ldots,15\}$	$\begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 2,3,\ldots,15 \end{cases}$
Health period premium for wife	p_x	$\{1,2,10,11\}$	$\begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 2,10 \\ 2 & \text{for } i = 11 \end{cases}$
Lump-sum health benefit for wife	c_x	$\{3,5,8,12\}$	$\begin{cases} 1 & \text{for } i = 3,5,8 \\ 2 & \text{for } i = 12 \end{cases}$
Health period premium for husband	p_y	$\{1,5,6,15\}$	$\begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 5,15 \\ 2 & \text{for } i = 6 \end{cases}$
Lump-sum health benefit for husband	c_y	$\{7,8,10,13\}$	$\begin{cases} 1 & \text{for } i = 8,10,13 \\ 2 & \text{for } i = 7 \end{cases}$

Source: own elaboration.

If $\wp_i(k) = \wp$ is a constant cash flow paid in advance when $X^*(k) = i$, its actuarial value can be expressed as (cf. (Dębicka & Zmyślona, 2018))

$$\wp \cdot a_{1(i)}(\delta(1,i), n) = \wp \cdot \mathbf{V}^T \left(\sum\nolimits_{k=\delta(1,i)}^{n-1} \mathbf{I}_{k+1} \mathbf{I}_{k+1}^T \right) \mathbf{D} \mathbf{J}_i, \tag{5.15}$$

where a *temporary life annuity due* $a_{1(i)}(\delta(1,i), n)$ denotes the actuarial value of the stream of unit benefits arising from a life annuity-due contract payable in period $\delta(1,i), n)$ if $X^*(k) = i$ for $k \in \delta(1,i), n)$. Actuarial value is calculated at the beginning of the insurance period (i.e. for $k = 0$). If we assume that $\wp$ is realised when $X^*(k) \in S^{\wp}$, then

$$\wp \sum\nolimits_{i \in S^{\wp}} a_{1(i)}(\delta(1,i), n) = \wp \cdot \mathbf{V}^T Diag\left(\mathbf{C}_{S^{\wp}} \mathbf{D}^T\right) \mathbf{S} = \mathbf{V}^T Diag\left(\wp \mathbf{C}_{S^{\wp}} \mathbf{D}^T\right) \mathbf{S}, \tag{5.16}$$

where $\mathbf{C}_{S^{\wp}} \in R^{(n+1) \times N^*}$ is a cash flow matrix of unit benefits paid in advance, defined as follows:

$$c_{ki} = \begin{cases} 1 & \text{for } X^*(k) = i \text{ and } i \in S^{\wp},\ k = 0,1,\ldots,n-1 \\ 0 & \text{otherwise} \end{cases}. \tag{5.17}$$

Analogously, if $\wp_i(k) = \wp$ is a constant cash flow paid from below when $X^*(k) = i$, its actuarial value can be expressed as (Dębicka & Zmyślona, 2018)

$$\wp \cdot a_{1(i)}(\delta(1,i), n) = \wp \cdot \mathbf{V}^T \left(\sum\nolimits_{k=\delta(1,i)+1}^{n} \mathbf{I}_{k+1} \mathbf{I}_{k+1}^T \right) \mathbf{D} \mathbf{J}_i, \tag{5.18}$$

where an *immediate life annuity* $a_{1(i)}(\delta(1,i), n)$ denotes the actuarial value of the stream of unit benefits arising from an immediate life annuity contract payable in period $\delta(1,i), n)$ if $X^*(k) = i$ for $k \in \delta(1,i), n)$. Moreover, we have:

$$\wp \sum_{i \in S^{\wp}} a_{1(i)}(\delta(1,i),n) = \wp \cdot \mathbf{V}^T Diag\left(\mathbf{C}_{S^{\wp}}\mathbf{D}^T\right)\mathbf{S} = \mathbf{V}^T Diag\left(\wp\mathbf{C}_{S^{\wp}}\mathbf{D}^T\right)\mathbf{S}, \quad (5.19)$$

where $\boldsymbol{C}_{S^{\wp}} \in \boldsymbol{R}^{(n+1)\times N^*}$ is a cash flow matrix of unit benefits paid from below, which is defined as follows:

$$c_{ki} = \begin{cases} 1 & \text{for } X^*(k-1) = i \text{ and } i \in S^{\wp},\ k = 1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}. \quad (5.20)$$

Note that in the case of the marriage insurance under consideration, we are dealing with the following cash flows $\wp \in \left\{b, p^x, p^y\right\}$ and $\wp \in \{b^x, c^x, b^y, c^y\}$. For further consideration, assume the following notations $\wp^x \in \{b^x, c^x\}$ and $\wp^y \in \{b^y, c^y\}$. The choice of the type of cash flow $\wp^x$ and $\wp^y$ determines the kind of scenario adopted by the spouses, namely

- $\wp^x = b^x \wedge \wp^y = b^y$ – Scenario I,
- $\wp^x = b^x \wedge \wp^y = c^y$ – Scenario II,
- $\wp^x = c^x \wedge \wp^y = b^y$ – Scenario III,
- $\wp^x = c^x \wedge \wp^y = c^y$ – Scenario IV.

Theorem 5.1

Suppose that for the Comprehensive Marriage Contract with Health Protection equivalence principle holds and assumptions A1-A2 are satisfied. Moreover, the modified multiple-state model (S^*, T^*) is defined for the particular scenario (i.e. for given $\wp^x$ and $\wp^y$). For established values of property W and parameters α, β, γ, the formulas for net benefits payable during the contract's period are

$$b = \frac{\alpha W}{\mathbf{V}^T Diag\left(\mathbf{C}_{S^b}\mathbf{D}^T\right)\mathbf{S}}, \quad (5.21)$$

$$\wp^x = \gamma(1-\beta)\alpha W \frac{\mathbf{V}^T Diag\left(\mathbf{C}_{S^{p^x}}\mathbf{D}^T\right)\mathbf{S}}{\mathbf{V}^T Diag\left(\mathbf{C}_{S^b}\mathbf{D}^T\right)\mathbf{S} \cdot \mathbf{V}^T Diag(\mathbf{C}_{S^{\wp^x}}\mathbf{D}^T)\mathbf{S}}, \quad (5.22)$$

$$\wp^y = (1-\gamma)(1-\beta)\alpha W \frac{\mathbf{V}^T Diag\left(\mathbf{C}_{S^{p^y}}\mathbf{D}^T\right)\mathbf{S}}{\mathbf{V}^T Diag\left(\mathbf{C}_{S^b}\mathbf{D}^T\right)\mathbf{S} \cdot \mathbf{V}^T Diag(\mathbf{C}_{S^{\wp^y}}\mathbf{D}^T)\mathbf{S}}. \quad (5.23)$$

Proof. The principle of equivalence is satisfied when the value of capital αW invested by the spouses is balanced by the cash flows resulting from the realisation of the insurance, i.e. from the perspective of the spouses, the inflows (reverse annuities, health benefits) and the outflows (health insurance premiums), thus:

$$\alpha W = \mathbf{V}^T Diag\left(\mathbf{C}_{in}\mathbf{D}^T\right)\mathbf{S}. \quad (5.24)$$

Note that according to (2.48), a cash flow realised at moment k, if $X^*(k) = i$ for CMCHP has the following form $cf_j^*(k) = b_j(k) - p_j^x(k) + \wp_j^x(k) - p_j^y(k) + \wp_j^y(k)$. Therefore, based on (5.16) and (5.19), matrix $\mathbf{C}_{in}$ can be decomposed as follows:

$$\mathbf{C}_{in} = b\mathbf{C}_{S^b} - p^x\mathbf{C}_{S^{p^x}} + \wp^x\mathbf{C}_{S^{\wp^x}} - p^y\mathbf{C}_{S^{p^y}} + \wp^y\mathbf{C}_{S^{\wp^y}}. \tag{5.25}$$

Applying (5.25) with (5.16) and (5.19) into (5.24), we obtain

$$\begin{aligned} \alpha W &= \mathbf{V}^T Diag\left(\left(b\mathbf{C}_{S^b} - p^x\mathbf{C}_{S^{p^x}} + \wp^x\mathbf{C}_{S^{\wp^x}} - p^y\mathbf{C}_{S^{p^y}} + \wp^y\mathbf{C}_{S^{\wp^y}}\right)\mathbf{D}^T\right)\mathbf{S} \\ &= b\mathbf{V}^T Diag\left(b\mathbf{C}_{S^b}\mathbf{D}^T\right)\mathbf{S} \\ &\quad - p^x\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{p^x}}\mathbf{D}^T\right)\mathbf{S} + \wp^x\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{\wp^x}}\mathbf{D}^T\right)\mathbf{S} \\ &\quad - p^y\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{p^y}}\mathbf{D}^T\right)\mathbf{S} + \wp^y\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{\wp^y}}\mathbf{D}^T\right)\mathbf{S}. \end{aligned} \tag{5.26}$$

Note that $p^x$and $\wp^x$ (similarly $p^y$and $\wp^y$) must satisfy the equivalence principle for the wife's (respectively, the husband's) health insurance, hence

$$p^x\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{p^x}}\mathbf{D}^T\right)\mathbf{S} = \wp^x\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{\wp^x}}\mathbf{D}^T\right)\mathbf{S} \tag{5.27}$$

and

$$p^y\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{p^y}}\mathbf{D}^T\right)\mathbf{S} = \wp^y\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{\wp^y}}\mathbf{D}^T\right)\mathbf{S}. \tag{5.28}$$

Using (5.27) and (5.28) in (5.26), we straightforwardly obtain (5.21).

Note that from (5.2), (5.27) can be rewritten as follows:

$$\gamma(1-\beta)b \cdot \mathbf{V}^T Diag\left(b\mathbf{C}_{S^{p^x}}\mathbf{D}^T\right)\mathbf{S} = \wp^x\mathbf{V}^T Diag\left(b\mathbf{C}_{S^{\wp^x}}\mathbf{D}^T\right)\mathbf{S}. \tag{5.29}$$

Applying (5.21) to (5.29), gives (5.22).

Analogously, substituting (5.3) and (5.21) into (5.28) yields (5.23), which completes the proof.

According to Theorem 5.1, the benefits resulting from the *Comprehensive Marriage Contract with Health Protectio*n can determine whether the spouses' future lifetimes are independent or dependent, provided that matrix $\mathbf{D}$ is appropriately defined. It should also be emphasised that this can be applied to the four scenarios described in Section 5.3.

5.4. Numerical examples

In this section, the authors present some numerical examples to illustrate the value of benefits arising from the marriage contract proposed in Section 5.2 and the application of Theorem 5.1 presented in Section 5.3. The calculations performed for this chapter are based on our programs written in MATLAB.

The *Comprehensive Marriage Contract with Health Protection* with Scenario IV was selected for numerical analyses. Since, according to this option, both spouses decide on lump sum health benefits, the MMSM describing this contract is the most extensive (it has the largest number of states), cf. the scheme in Figure 5.4.

The authors tried to maintain comparability with earlier examples in the monograph, therefore analogous assumptions were adopted (compare examples in Sections 4.3.2 and 4.3.3):

C1. The *value of property* $W = 100\,000$ euros.
C2. $\alpha = 50\%$
C3. Due to data availability, *lung cancer* was selected as the critical illness, hence the probability of death, morbidity and mortality rates were estimated based on life expectancy tables and epidemiological reports related to the morbidity and mortality of critical diseases (i.e. lung cancer). Details connected with the estimation of the parameters were presented in (Dębicka & Zmyślona, 2019).
C4. Fixed interest rate $r = 5.79\%$ (see Section 3.2.3).
C5. *Health parameter* $\gamma = 0.5$ means that spouses allocate the same amount for health insurance ($p^x = p^y$).
C6. In the numerical analysis, consider the *reverse annuity parameter* $\beta \in [0.9,1]$.
C7. Assume that spouses' ages at entry are $x, y \in \{65,70,75,80\}$.

First, assume that $\beta = 0.99$. Table 5.4 presents marriage reverse annuity b and dread disease insurance benefits for wife c^x and husband c^ycalculated according to Theorem 1.

Table 5.4. Benefits of the Comprehensive Marriage Contract with Health Protection – Scenario IV for different ages at entry of spouses ($\beta = 0.99$)

		x = 65			**x = 70**		
		b	c^y	c^x	***b***	c^y	c^x
y	65	4070.52	4479.96	19030.66	4373.63	4820.26	20367.78
	70	4209.26	4359.62	19678.21	4606.35	4776.54	21445.88
	75	4326.07	4675.04	20217.21	4818.45	5212.50	22422.00
	80	4414.55	5663.87	20621.26	4989.76	6407.51	23205.16
		x = 75			**x = 80**		
		b	c^y	c^x	***b***	c^y	c^x
y	65	4688.05	5175.39	22046.96	4967.62	5493.10	24281.40
	70	5051.70	5249.30	23760.96	5480.51	5708.50	26812.42
	75	5426.29	5878.52	25493.50	6069.66	6593.50	29670.72
	80	5761.06	7406.45	27041.93	6675.44	8593.96	32559.07

Source: own elaboration.

The marriage reverse annuity contract benefit increases with the age of the spouses. Again, we can see that the LSS payment is higher when the wife is older, which is reflected better in Figure 5.6 and Table 5.5.

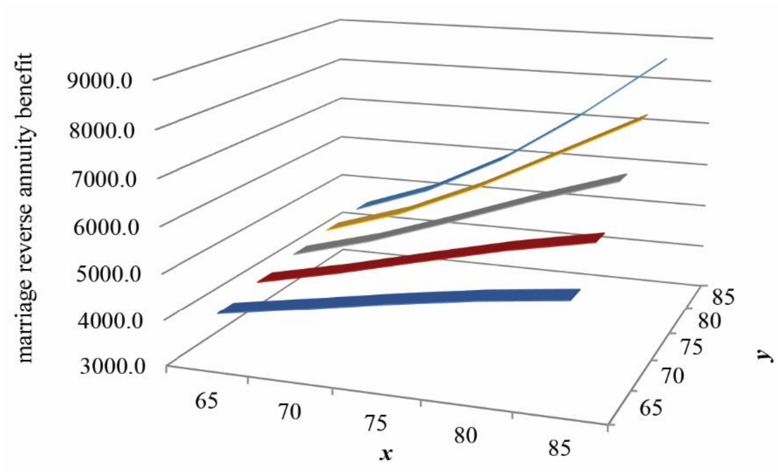


Figure 5.6. The Comprehensive Marriage Contract with Health Protection – Scenario IV. The marriage reverse annuity for different ages at entry of spouses ($\beta = 0.99$)

Source: own elaboration.

Table 5.5. The Comprehensive Marriage Contract with Health Protection – Scenario IV. The marriage reverse annuity for different ages at entry of spouses ($\beta = 0.99$)

		x				
		65	**70**	**75**	**80**	**85**
y	**65**	4070.5	4373.6	4688.0	4967.6	5185.8
	70	4209.3	4606.4	5051.7	5480.5	5833.8
	75	4326.1	4818.4	5426.3	6069.7	6652.6
	80	4414.5	4989.8	5761.1	6675.4	7595.9
	85	4479.8	5115.5	6026.0	7206.4	8562.4

Source: own elaboration.

The benefit of critical illness insurance also increases with the age of the spouses. The benefit for a wife is significantly higher than for a husband, since women suffer less often from lung cancer. The relative differences between the benefits obtained from critical illness insurance for a wife and a husband are presented in Table 5.6 and Figure 5.7.

It is clear that regardless of a woman's age, the highest difference is for 70-year--old men (even up to almost 370%), and the lowest for 85-year-old men (over 213%). The differences between payments are the highest for younger husbands, and the smallest for the elderly, and *vice versa* for wives.

Table 5.6. The Comprehensive Marriage Contract with Health Protection – Scenario IV.
The relative differences between critical illness's lump sum of wife and husband for different ages at entry of spouses ($\beta = 0.99$)

		x				
		65	**70**	**75**	**80**	**85**
y	**65**	324.8%	322.5%	326.0%	342.0%	337.5%
	70	351.4%	349.0%	352.7%	369.7%	364.9%
	75	332.4%	330.2%	333.7%	350.0%	345.4%
	80	264.1%	262.2%	265.1%	278.9%	275.0%
	85	214.9%	213.2%	215.8%	227.7%	224.3%

Source: own elaboration.

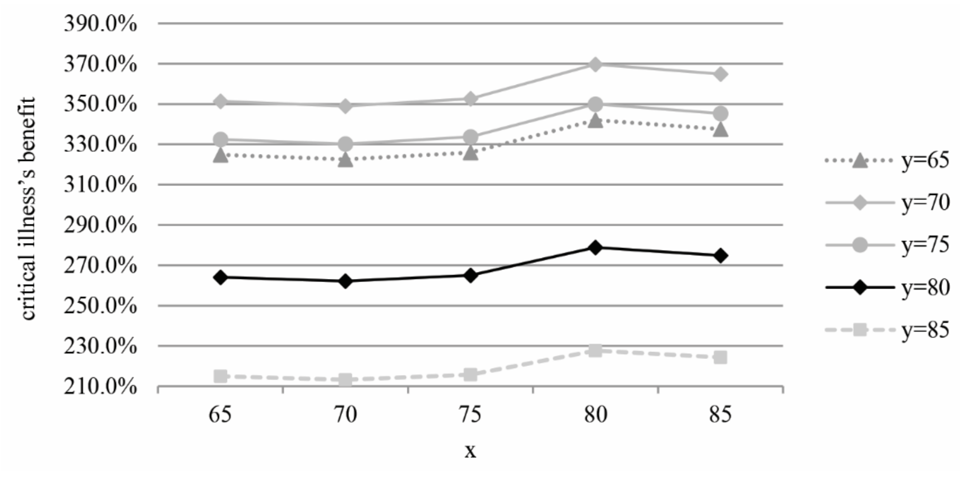


Figure 5.7. The Comprehensive Marriage Contract with Health Protection – Scenario IV.
The relative differences between critical illness's lump sum of wife and husband for different ages at entry of spouses ($\beta = 0.99$)

Source: own elaboration.

The following example shows all cash flows and the premium to be paid for critical illness insurance, depending on reverse annuity parameter β for spouses of the same age ($x = y = 70$). The remaining assumptions (C1-C5) do not change. Table 5.7 presents the results. The number of states where the particular cash flows are paid is indicated in brackets.

The spouses can decide for themselves on the level of paid health premiums by choosing the value of reverse β and health parameters γ (compare (5.2) and (5.3)). In Table 5.7, it can be easily observed that a relatively low premium for each spouse (in this example $0.5 \cdot b(1-\beta)$) does little to reduce the amount of the marital reversionary annuity obtained by the spouses when they are active βb, the health benefits c^x and c^y are very significant.

Table 5.7. The Comprehensive Marriage Contract with Health Protection – Scenario IV. The value of all types of cash flows for different reverse parameters β

Type of cash flow	Number of states (column)	β					
		1.00	0.98	0.96	0.94	0.92	0.90
b	4, 9, 14 and 3, 7, 8, 12, 13 (second row)	4606.35	4606.35	4606.35	4606.35	4606.35	4606.35
$p^x = p^y = 0.5b(1-\beta)$		0.00	46.06	92.13	138.19	184.25	230.32
c^x	3, 5, 12 (last row)	0.00	42891.76	85783.52	128675.28	171567.04	214458.80
c^y	7, 10, 13 (last row)	0.00	9553.09	19106.17	28659.26	38212.34	47765.43
$b-(p^x+p^y)=\beta b$	1	4606.35	4514.23	4422.10	4329.97	4237.85	4145.72
$b-p^x=b-p^y$ $=0.5b(1+\beta)$	2, 6, 11, 15	4606.35	4468.16	4329.97	4191.78	4053.59	3915.40
$b+c^x$	3, 12	4606.35	47498.11	90389.87	133281.63	176173.39	219065.15
$b+c^x-p^y$ $=0.5b(1+\beta)+c^x$	5	4606.35	47359.92	90113.49	132867.06	175620.63	218374.20
$b+c_y$	7, 13	4606.35	14159.44	23712.53	33265.61	42818.70	52371.78
$b+c^y-p^x$ $=0.5b(1+\beta)+c^y$	10	4606.35	14021.25	23436.14	32851.04	42265.94	51680.83
$b+c_x+c_y$	8	4606.35	57051.20	109496.05	161940.89	214385.74	266830.58
c_x+c_y	8 (last row)	0.00	52444.85	104889.69	157334.54	209779.38	262224.23

Source: own elaboration.

In conclusion, the comprehensive contract proposed in this chapter offers a new way to protect against the financial impacts of longevity in marriage. Although reverse annuity benefits are slightly lower, the health benefits are significantly higher, providing extra financial protection during serious illness. Unsurprisingly, the benefits' value depends on the spouses' age (and the contract's length, consequently). Yet, the calculations show the strong influence of sex on the amount of dread disease benefits (up to 370%). Therefore, the application that distinguishes sex in actuarial calculations seems appropriate in dread disease insurance.

5.5. Recommendations for further study and research

Intermediate variants of *the Comprehensive Marriage Contract with Health Protection* are possible when one spouse prefers a lump-sum health benefit while the other prefers a health annuity. In such cases, an intermediate model is created between Scenario I and Scenario IV models.

The simplest model of the marriage contract presented in Section 5.2 can be extended to include illness conditions in more detail. The only drawback is the multitude of states and the possible transitions between them. Although this does not pose a computational problem, it increases the complexity of the model. If the *Comprehensive Marriage Contract with Health Protection* is built based on the *Critical Health Insurances with Annuity Benefits* (Model III presented in Figure 3.8), the MSM would have $(3+g)\times(3+g)$ states. In particular, for lung cancer as a terminal illness, by constructing a marital model based on the comprehensive individual contract with health protection presented in the article (Dębicka et al., 2015), one obtains a model consisting of 64 states.

Due to the assumption of the independence of future lifetime, the problem of combining marital reverse annuity contracts with dread disease insurances can be approached differently by creating a marital insurance that is essentially a combination of three separate agreements, or in other words, a hybrid of three individual contracts (one marital and two individual for the husband and wife). In the context of multistate modelling, this type of solution does not constitute a typical marital agreement (as described in Chapters 4 and 5); however, it is a convenient method for calculating net benefits. This approach (for benefits according to Scenario IV and a detailed model of lung cancer incidence) is described in other studies, where in their numerical analysis the authors took into account factors such as the age of the wife and husband (Zmyślona & Marciniuk, 2020), the contribution of the health premium rate (Marciniuk & Zmyślona, 2022), and the effect of sex (Zmyślona & Marciniuk, 2024) on the net benefits.

Research has shown that not only a spouse's death but also the health status of one spouse can significantly impact the health status of the other due to a range of medical, psychological, sociological, and economic factors.

From a medical point of view, a shared living environment, such as level of physical activity, exposure to risk factors (e.g., smoking) and diet, can affect the health of both spouses. For example, if one spouse smokes cigarettes, the other may be exposed to secondhand smoke, which increases the risk of lung and heart diseases.

Research presented by (Castelnuovo et al., 2009) shows a high concordance between spouses regarding the mentioned coronary risk factors.

Psychological aspects, also known as the *mental health effects of spousal illness*, are also important. On the one hand, the stress associated with caring for a spouse with a critical illness can lead to an increased risk of mortality for the caregiver, known as the effect of caregiving on health (Schulz & Beach, 1999). Additionally, research presented in (Christakis & Allison, 2006) showed that the hospitalisation of one spouse is associated with an increased risk of mortality for the other spouse, whilst strong emotional support and closeness can positively impact both partners' mental and physical health. Generally, the quality of social roles, including that of the spouse, significantly impacts on psychological and physical health, as confirmed by studies (Hibbard & Pope, 1993). Conversely, lifestyle changes in one spouse, caused, for example, by a disease diagnosis, can affect the daily habits and behaviour of the other spouse. For instance, a diet adapted to the needs of the sick spouse may be adopted by both, which can have both positive and negative effects (see a systematic review of studies on this topic in (Meyler et al., 2007).

The economic impacts of bearing the costs of one spouse's treatment are also significant. They can indirectly affect the other spouse's health through financial stress and limited access to healthcare.

The advantage of considering a single contract that jointly incorporates the health status of both spouses (as demonstrated by the MSM and MMSM in Section 5.2) is the ability to capture the interdependence between the future lifetimes of the spouses due to their health status. This approach allows for a more accurate representation of the financial risks and benefits associated with joint health conditions and longevity.

One key benefit of this integrated model is that it can incorporate the health status of both spouses into a unified framework, using linking functions such as copulas to model the dependency between their health states (analogous to the probabilistic structure of the marital insurance described in Sections 4.2.3 and 4.3.3). This method enables the determination of the appropriate transition probabilities in matrix (5.2), providing a more realistic assessment of the risks compared to the hybrid models that consider the health states' of the spouses independently. Additionally, incorporating health dependencies into actuarial calculations opens up new research avenues – for instance, it allows for exploring how correlations between health conditions can impact on the combined pricing and risk management of the contracts.

Understanding these dependencies can improve the accuracy of the actuarial models by accounting for the probability of joint events, such as both spouses experiencing critical illnesses simultaneously or sequentially. This, in turn, enhances the precision of the financial projections and the adequacy of reserves required to cover potential benefits, as well as paves the way for advanced actuarial research. It facilitates a deeper understanding of health-related dependencies and their impact on the financial stability of insurance products designed for couples.

To sum up, *Comprehensive Marriage Contract with Health Protection* poses a new proposal of protection against the effects of longevity. This agreement is not offered on the market. Although the benefits obtained from the reverse annuity contract, available to the spouses, are slightly lower than when health insurance premiums are not paid, the illness benefits are considerably high. This gives the owners of real estate additional financial protection in times of a critical disease. Thanks to reducing the acquisition and maintenance costs charged to the customers, a combined contract can be less expensive for spouses than independently purchasing each single component.

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