Optimal and effective stopping times for some families of graphs

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Chapter 1

Introduction

1.1 Historical overview

The most classical optimal stopping problem is the following secretary problem (known also as the marriage problem). There are n linearly ordered candidates for a job as a secretary (or for a wife as another story tells). They come to a selector one by one in some random order. The selector knows the total number of candidates as well as the relative ranks of the girls examined so far. However, he gets no information about the candidates that are still to come. His task is to hire (marry) the presently examined candidate maximizing the probability of choosing the absolute best one. The optimal strategy is of threshold type. It tells the selector to reject asymptotically first n/e candidates and after this time to hire (marry) the first which is presently the best. The probability of choosing the absolute best candidate according to this optimal algorithm is asymptotically $1/e$. This solution was first written down by Lindley [19] in 1961 though, it seems, the problem and its solution were known before. For a comprehensive treatment of the subject consult Ferguson's history of the secretary problem [3].

This beautiful problem and its elegant solution were the inspiration for considering various generalizations. One of them was replacing the linear order of candidates with a partial order and trying to stop the search on any element which is maximal in the whole poset. Some efficient stopping rules were found by Stadje [26] and Gnedin [8]. Morayne [22] found the optimal algorithm for choosing the root in a complete binary tree of given height (its probability of success tends to 1 with the height of the tree tending to infinity). The optimal stopping rule for n pairs of "twins", i.e., for a poset consisting of n levels with two twin elements on each, was found by Garrod, Kubicki and Morayne in [5]. The optimal stopping times for other regular or simple posets were presented by Kaźmierczak and Tkocz in [13], [14], [15] and [28].

A further interesting generalization was to narrow down the selector's a priori knowledge only up to the number of candidates and try to find a universal algorithm that would be reasonably successful on any poset. In 1999 Preater showed [23], quite surprisingly, that there exists an algorithm that wins with probability at least $1/8$ on any poset. This lower bound was later improved for the same algorithm by Georgiou, Kuchta, Morayne and Niemiec in [7] to $1/4$. Kozik in [16] found an algorithm giving a better bound than $1/4$, $1/4 + \varepsilon$ for $\varepsilon > 0$. However a natural goal to achieve was to get a bound of $1/e$ (the probability of success of the optimal algorithm for a linear order) and thus to find an algorithm that would be not improvable. Such an algorithm was found by Freij and Wästlund in [4]. For families with certain restrictions better algorithms were found by Garrod and Morris in [6] and Kumar, Lattanzi, Vassilvitskii and Vattani in [18].

Realizing that partially ordered sets may be treated as very rich directed graphs led to the next generalization. Dealing with a directed graph instead of a poset a selector can see at a given moment the induced graph generated by the vertices that have already arrived. One has to stop the search maximizing the probability (or ensuring a relatively high probability) that the presently examined vertex belongs to a particular part of a graph. The first and most natural thing to do was to consider a directed path (instead of the linear order) and search for the maximal vertex, i.e., the vertex with no outgoing edges, the sink (instead of the best candidate). Kubicki and Morayne presented in [17] the optimal algorithm for such a case, which tells the selector to wait as long as possible, i.e., to stop when there is still a positive probability that currently examined vertex is the maximal one and the probability that the maximal one is still to come is equal to zero. Directed graphs are much poorer structures than partial orders therefore it seems that one cannot hope for too large probabilities of success. In the case of the directed path of length n the probability of success p_n according to the optimal rule satisfies $p_n\sqrt{n} \to \sqrt{\pi}/2$ as $n \to \infty$. The optimal stopping time for choosing one of the two top vertices from a directed path (so-called Gusein-Zade problem) was found by Przykucki and Sulkowska in [25]. In [24] Przykucki presented an optimal rule in a search for a vertex with full degree in a random graph.

Throughout this thesis we will be looking for both optimal and, simply, effective stopping times for some families of directed graphs. By effective we understand strategies which give relatively large probability of success. The expression "relatively large" will be made more precise and placed in the context of the known optimal algorithms for graphs.

Such problems have real life interpratations. For instance, one may consider on-line decision problems on structures that are useful for storing data or on structures that model computer networks. We can easily imagine the task of browsing a computer network with a known or unknown topology in a search for a server with a certain (possibly good) feature.

1.2 Results of the thesis

This thesis consists of three chapters. Two of them (3 and 4) are devoted to the universal effective algorithms for upward directed graphs. The third one generalizes the result of Kubicki and Morayne from [17] and describes the optimal algorithm for a certain family of graphs.

Chapter 3 is based on a joint work with Graham Brightwell, Paul Balister and Michał Morayne, [1]. We analyze a very simple universal algorithm for choosing a maximal vertex from an upward directed rooted graph. We show that it is effective for the whole special families of graphs, for instance, for k-ary trees, graphs we call natural pyramids or half-cubes and for structures with large minimal indegree. We also indicate the connection between the stated problem and the theory of branching processes and the percolation theory.

Chapter 4 is based on article [27]. It presents a universal strategy for choosing a maximal element from a directed acyclic graph that belongs to a fairly general family of graphs when a selector knows in advance only the number n of its vertices. The problem considered in this chapter is complementary to the problem presented in Chapter 3 in such a sense that this time we consider upward directed graphs with bounded indegrees. Precisely, the number of elements dominated directly by the maximal ones is not greater than $c_1\sqrt{n}$ for some positive constant c_1 and the indegree of remaining vertices is bounded by a common constant D. We show that the probability p_n of the right choice according to our strategy satisfies $\liminf_{n\to\infty} p_n\sqrt{n} \geq \delta > 0$, where δ is a constant depending on c_1 and D. As it can be seen from the optimal result for the directed path mentioned in the previous section one cannot hope, up to a constant, for a better result.

Chapter 5 is based on a joint work with Andrzej Grzesik and Michał Morayne, [10]. We consider the optimal algorithm for the kth power of a directed path. At first we assume that the selector knows in advance not only the underlying graph but also the distance in the underlying path between each two vertices that are joined by an edge in the induced graph. When $k = 1$ this problem reduces simply to a directed path case from [17]. We give the optimal algorithm for any kth power and for $k = 2$ its exact probability of success. We also prove that for any k the probability of success p_n (where n is the length of the path) satisfies $p_n = \Omega(n^{-1/(k+1)})$ and for k such that $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$ we have also $p_n = O(n^{-1/(k+1)})$. We show this result also for the case when the selector is not given the additional information about the distances, although without the optimal stopping time.

Chapter 2

Notation, definitions and formal model

This chapter contains notation and basic definitions that are used throughout the whole thesis. Notation and definitions that are specific for particular chapters are going to be presented later on.

2.1 Basic definitions

A *directed graph* G is a pair $G = (V, E)$, where V is a set of *vertices* and E is a set of *edges*, i.e., ordered pairs of elements from V (which means that each edge has a direction). The cardinality of V will be denoted by n.

Throughout this thesis we consider only *simple graphs*, i.e., $(v, v) \notin E$ for all $v \in V$.

A *directed cycle* in G is a subgraph $H = (W, F)$ of G such that $W = \{w_1, w_2, \dots w_k\}$ and $F = \{(w_i, w_{i+1}) : i \in \{1, 2, \dots, k\}\}$ setting $w_{k+1} = w_1$.

An *upward directed (acyclic) graph* is a simple directed graph with no directed cycles. Note that the family of upward directed graphs coincides with the family of Hasse diagrams for posets.

A *directed path* or a *chain* is a graph $P_n = (V_n, E_n)$ such that $V_n = \{v_1, v_2, \dots, v_n\}$ and $E_n = \{(v_i, v_{i-1}) : i \in \{2, 3, \dots, n\}\}$. The *length* of P_n is defined here as n.

We call $v \in V$ a *maximal vertex* if v has no outgoing edges. For a directed graph G the set of its maximal vertices will be denoted by $Max(G)$ or $Max(V)$ if E is known from the context. Whenever there is only one maximal vertex in G it is called *root* and denoted by $\mathbb{1}$. Then G is called *rooted.*

By the *depth* of $v \in V$ in G we understand the length of the shortest directed path that starts in v and ends in an element from $Max(G)$.

By the *height* of G we understand the length of the longest directed path in G.

A *leaf* in a directed graph is a vertex with no incoming edges.

For v and w from V we say that w is a *parent* of v and v is a *child* of w if $(v, w) \in E$ (note that a maximal vertex is a vertex which has no parent).

For a graph $G = (V, E)$, its maximal connected induced subgraph $G' = (W, E \cap W^2)$, $W \subseteq V$, is called a *connected component.*

Let S_n denote the family of all permutations of the set V. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$. By $G_{(m)} = G_{(m)}(\pi) = (V_{(m)}, E_{(m)})$, $m \leq n$, we denote the subgraph of G induced by $\{\pi_1, \ldots, \pi_m\}$, i.e.,

$$V_{(m)} = \{\pi_1, \pi_2, \ldots, \pi_m\},$$
$$E_{(m)} = \{(v_i, v_j) \colon \{v_i, v_j\} \subseteq \{\pi_1, \pi_2, \ldots, \pi_m\} \wedge (v_i, v_j) \in E\}.$$

Let $c(G_{(m)})$ denote the number of connected components in $G_{(m)}$.

Quite generally, let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \mathcal{F}_n \subseteq \mathcal{F}$ be a sequence of σ-algebras, i.e., a *filtration.* A random variable $\tau : \omega \to \{1, 2, \ldots, n\}$ is a *stopping time* with respect to a filtration $(\mathcal{F}_t)_{t=1}^{n}$ if $\tau^{-1}(\{t\}) \in \mathcal{F}_t$ for each $t \leq n$. If we think of $\tau(\omega)$, $\omega \in \Omega$, as of a moment when to stop observing a certain process depending on ω and $t = 1, 2, \ldots, n$, then the condition $\tau^{-1}(\{t\}) \in \mathcal{F}_t$ means that our decision to stop at t is based only on the past and present events and it does not depend on any information about the future events.

Let $(v_1, v_2, \ldots, v_m)$ be a sequence of distinct vertices of a directed graph $G = (V, E)$. Let $R \subseteq \mathbb{N}^2$. We write $(v_1, v_2, \ldots, v_m) \cong R$ if for all $i, j \leq m, i \neq j$, $(v_i, v_j) \in E$ if and only if $(i, j) \in R$.

2.2 Formal model

Defining the formal model we follow [17]. We give a general probabilistic model concerning any directed graph. Let $G = (V, E)$ be a fixed graph. We will work with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = S_n$, $\mathcal{F} = \mathcal{P}(\Omega)$ and the probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ will be defined by setting

$\mathbb{P}(\{\pi\}) \;=\; 1/n!$ for each $\pi \in S_n$. Let

$$\mathcal{F}_t = \sigma(\{\pi \in \Omega : (\pi_1, \pi_2, \ldots, \pi_t) \cong R\} : R \subseteq \mathbb{N}^2), \qquad 1 \leq t \leq n.$$

Let D be a subset of vertices of the graph G (i.e. $D \subseteq V$). When looking for a vertex from D an *optimal stopping time* is any stopping time τ^* for which

$$\mathbb{P}[\pi_{\tau^*} \in D] = \max_{\tau \in \mathcal{T}} \mathbb{P}[\pi_\tau \in D],$$

where $\mathcal{T}$ is the family of all stopping times and $[\pi_\tau \in D]$ denotes the set $\{\pi \in \Omega : \pi_{\tau(\pi)} \in D\}$. We will also consider *effective stopping times*, i.e., stopping times τ^* for which $\mathbb{P}[\pi_{\tau^*} \in D]$ is relatively large. The expression "relatively large" will be made more precise in Chapters 3 and 4 by placing it in the context of the known optimal stopping times.

2.3 Notation

Symbol	Meaning
$\mathbb{N}$	set of natural numbers $\{0, 1, 2, 3, \ldots\}$
$B(n,p)$	binomial distribution with parameters n, p
$G = (V, E)$	directed graph, V - set of vertices, E - set of edges
P_n	directed path of length n
$Max(G), Max(V)$	set of maximal vertices of $G = (V, E)$
$\mathbb{1}$	root, the only one maximal vertex of G
S_n	family of all permutations of set V
$\pi = (\pi_1, \pi_2, \ldots, \pi_n)$	random permutation of vertices from V
$G_{(m)} = G_{(m)}(\pi)$	subgraph of G induced by $\{\pi_1, \pi_2, \ldots, \pi_m\}$
$c(G_{(m)})$	number of connected components in $G_{(m)}$

Chapter 3

Analysis of a simple effective on-line algorithm for the graph-theoretic generalization of the best choice problem

3.1 Introduction

Throughout this chapter we present and analyze a simple deterministic on-line algorithm for some families of upward directed rooted graphs. We assume that the selector knows the height of the graph. His task is to choose the root with a relatively large probability.

The chapter is organized as follows. In Section 3.2 we introduce necessary definitions. In Section 3.3 we state a simple deterministic algorithm for choosing the root and discuss some connection between the best choice problem and the theory of branching processes and the percolation theory. In Sections 3.4 and 3.5 we consider two families of graphs: the complete k-ary trees and, so called, natural pyramids, analyzing the effectiveness of our algorithm for these families of graphs. In Section 3.6 we present an asymptotic analysis of the same deterministic algorithm applied to the structures with a large minimal indegree. We consider a d-dimensional half-cube as an example. In Section 3.7 we present the main theorem considering the effectiveness of our algorithm for the family of path-homogeneous graphs. The theorem is proved using the second moment method and, as in previous sections, the continuous time approach to arrival of vertices. Section 3.8 concludes the chapter with a short discussion about the choice of such an algorithm.

3.2 Definitions

The *complete k-ary trees* are upward directed rooted graphs in which each vertex except leaves has exactly k incoming edges and all the paths from the root to the leaves have the same length. (Ternary tree of height 3 is presented in Fig.3.1.)

The *pyramids* are upward directed rooted graphs. They have at least as many vertices at the level $k+1$ as at the level k. Each vertex from the $k+1$st level is connected by an edge with each vertex from the kth level. The *natural pyramid* is a pyramid with exactly k vertices at the kth level. (The natural pyramid of height 4 is presented in Fig.3.2.)

The *half-cubes* are upward directed rooted graphs with a diagram of an upper half of a d-dimensional cube. All the paths from the root to the leaves have the same length equal to $\lfloor d/2 \rfloor + 1$. (The 4-dimensional half-cube is presented in Fig.3.3.)

In this chapter we consider only induced subgraphs of upward directed rooted graphs. Let $H = (W, F)$ be such a subgraph. For $w \in W$ the *height* of w in H is the length of the longest directed path of vertices from W that ends in w.

Let $\pi \in S_n$ and $m \leq n$. Recall that $G_{(m)} = (V_{(m)}, E_{(m)})$ is a subgraph of G induced by $\{\pi_1, \dots, \pi_m\}$. Let $v \in V_{(m)}$. Let $h_m(v)$ denote the height of the vertex v in $G_{(m)}$.

Throughout this chapter G will be an upward directed rooted graph whose all leaves have depth N and the set of vertices we would like to stop on will be $D = \{\mathbb{1}\}$.

3.3 Stopping time

Let $G = (V, E)$ be an upward directed rooted graph whose all leaves have depth N. Recall that n denote the cardinality of the set of its vertices. Throughout this chapter we are going to consider the effectiveness of the following simple algorithm for choosing a root.

Strategy: Let a stopping time τ_N be equal to m if $h_m(\pi_m) = N$. If it never happens, let $\tau_N = n$.

We do not have the exact formulas for $\mathbb{P}[\pi_{\tau_N} = \mathbb{1}]$ for structures being considered. However, we are able to obtain the asymptotic effectiveness of τ_N using the second moment method and the following approach to the arrivals of vertices. Let us associate with each element v_i, $i = 1, 2, \ldots, n$, from V_N, where v_1 is the root, a random variable A_i of a value drawn uniformly from the interval $[0, 1]$, where all A_i's are independent. Let us treat A_i as the time of arrival of v_i. We have thus generated the uniform random order of arrivals of vertices from V_N. The arrival time of the root will be denoted by p ($A_1 = p$). This continuous time approach is equivalent to the discrete time one in the sense that all permutations of vertices are equiprobable.

Remark 3.1. Since A_i's are independent random variables, if the arrival time of the root is $A_1 = p$, the probability that a particular vertex appears before the root is equal to p.

By $c_{p,N}$ we denote the probability that at lest one chain of length $N - 1$ appears before the root (provided $A_1 = p$).

One can notice that such a statement of our problem generalizes and connects some issues from the theory of branching processes and percolation. For instance, if we consider only the graphs with maximum outdegree equal to 1 then $c_{p,N}$ is exactly the probability of survival of a branching process till time N, where the number of children of each parent is a random variable following the distribution depending on the structure of G (e.g. for k-ary tree a binomial distribution with parameters k, p).

On the other hand we can state this issue as follows. Reveal randomly and independently with probability p the vertices of G. What is the probability that a path of length N appears? Or, in the limit, what is the probability that an infinite component appears? This is a classical question of site percolation theory. The intensive study of percolation process grown following the work of Broadbent and Hammersley [2]. For instance, the 2-dimensional lattices were considered in [20].

As $\mathbb{P}[\pi_{\tau_N} = \mathbb{1} | A_1 = p] = c_{p,N}$ and all p's are equiprobable, the following lemma holds.

Lemma 3.2. *For an upward directed rooted graph whose all leaves have the* *same depth N we have*

$$\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] = \int_0^1 c_{p,N}\, dp.$$

□

3.4 Complete k-ary tree

In this section we prove a result for k-ary trees using the continuous time approach to arrivals of vertices (this approach was described in Section 3.3).

Since now the probability of success of our deterministic algorithm depends not only on N, but also on k we will write $\tau_{(k,N)}$ instead of τ_N. By $T_{k,N} = (V_{k,N}, E_{k,N})$ we denote the upward directed complete k-ary tree of height N. As usually, let n denote the cardinality of $V_{k,N}$; of course, $n = \frac{k^N-1}{k-1}$. ($T_{3,3}$ is presented in Fig.3.1.)

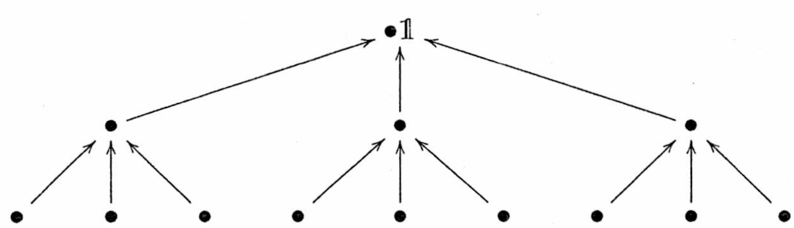


Figure 3.1: $T_{3,3}$.

In the case of $T_{k,N}$ also $c_{p,N}$ depends not only on p and N, but also on k. Nevertheless, as the value of k will always be clear from the context, we will simply write $c_{p,N}$. Note that by Remark 3.1 $c_{p,N}$ is exactly the probability of survival of the Galton-Watson branching process till time N in which the offspring distribution is binomial with parameters k, p. Let us denote by X the corresponding random variable, $X \sim B(k,p)$. We have $\mathbb{E}X = kp$. Let us define $c(p,k) = \lim_{N\to\infty} c_{p,N}$. From the theory of Galton-Watson branching processes we know the following.

Lemma 3.3. *If* $\mathbb{E}X > 1$, *i.e.,* $p > 1/k$, *then* $c(p,k)$ *is the unique root in* $(0,1)$ *of the equation* $1-\alpha = G(1-\alpha)$ *where* $G(x) = (1-p+px)^k$ *is the generating function for the offspring distribution. If* $\mathbb{E}X \leq 1$, *i.e.,* $p \leq 1/k$, *then* $c(p,k) = 0$.

Theorem 3.4. *Let π be a random permutation of vertices of $T_{k,N}$. Let $c(p,k) = \lim_{N\to\infty} c_{p,N}$. Then*

$$\lim_{N\to\infty} \mathbb{P}[\pi_{\mathcal{T}(k,N)} = \mathbb{1}] = \int_0^1 c(p,k)\,dp.$$

In particular, for the binary tree $\lim_{N\to\infty} \mathbb{P}[\pi_{\mathcal{T}(2,N)} = \mathbb{1}] = 2\ln 2 - 1$ and for the ternary tree $\lim_{N\to\infty} \mathbb{P}[\pi_{\mathcal{T}(3,N)} = \mathbb{1}] = 1.5\ln 3 - 2 + \pi/(2\sqrt{3})$.

Proof. Let us generate the uniform random order of arrivals of vertices from $V_{k,N}$ as it is done in Section 4. By Lemma 3.2 $\mathbb{P}[\pi_{\mathcal{T}(k,N)} = \mathbb{1}] = \int_0^1 c_{p,N}\,dp$. By Lebesgue's dominated convergence theorem (see [11]) and Lemma 3.3 we obtain

$$\lim_{N\to\infty} \mathbb{P}[\pi_{\mathcal{T}(k,N)} = \mathbb{1}] = \lim_{N\to\infty} \int_0^1 c_{p,N}\,dp = \int_0^1 \lim_{N\to\infty} c_{p,N}\,dp =$$
$$\int_0^1 c(p,k)\,dp = \int_{1/k}^1 c(p,k)\,dp.$$

By the same lemma we get $c(p,2) = (2p-1)/p^2$ for $k=2$ and $c(p,3) = \frac{3p^2-\sqrt{4p^3-3p^4}}{2p^3}$ for $k=3$. Finally,

$$\lim_{N\to\infty} \mathbb{P}[\pi_{\mathcal{T}(2,N)} = \mathbb{1}] = \int_{1/2}^1 (2p-1)/p^2\,dp = 2\ln 2 - 1$$

and

$$\lim_{N\to\infty} \mathbb{P}[\pi_{\mathcal{T}(3,N)} = \mathbb{1}] = \int_{1/3}^1 \frac{3p^2 - \sqrt{4p^3-3p^4}}{2p^3}\,dp = 1.5\ln 3 - 2 + \pi/(2\sqrt{3}).$$

□

3.5 Natural pyramid

In this section we show that the asymptotic effectiveness of τ_N for the natural pyramid of height N is approximately 0.516203. We use the same idea as for $T_{k,N}$ (the previous section) - generating the uniform random order of arrivals of vertices.

By $Z_N = (V_N, E_N)$ we denote the natural pyramid of height N. Let n denote the cardinality of V_N (of course $n = \frac{N(N+1)}{2}$). (Z_4 is presented in Fig.3.2.) Let $q = 1 - p$ and $\tilde{c}_{q,N} = c_{p,N}$.

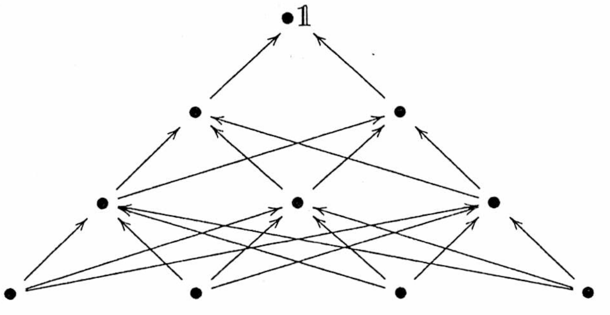


Figure 3.2: Z_4.

Lemma 3.5. *The probability* $\tilde{c}_{q,N}$ *satisfies*

$$\lim_{N\to\infty} \tilde{c}_{q,N} = \exp\left\{-\sum_{i=1}^{\infty} \frac{q^i}{i(1-q^i)}\right\} \frac{1}{1-q}.$$

Proof. We have

$$\tilde{c}_{q,N} = (1-q^2)(1-q^3)(1-q^4)\dots(1-q^{N-1})(1-q^N).$$

Thus

$$\ln \tilde{c}_{q,N} = \ln(1-q^2) + \ln(1-q^3) + \ldots + \ln(1-q^{N-1}) + \ln(1-q^N) =$$
$$-q^2 - \frac{q^4}{2} - \frac{q^6}{3} - \frac{q^8}{4} - \ldots - q^3 - \frac{q^6}{2} - \frac{q^9}{3} - \frac{q^{12}}{4} - \ldots$$
$$-q^{N-1} - \frac{q^{2(N-1)}}{2} - \frac{q^{3(N-1)}}{3} - \frac{q^{4(N-1)}}{4} - \ldots$$
$$-q^N - \frac{q^{2N}}{2} - \frac{q^{3N}}{3} - \ldots = -\sum_{j=2}^{N} q^j - \sum_{j=2}^{N} \frac{q^{2j}}{2} - \sum_{j=2}^{N} \frac{q^{3j}}{3} - \ldots =$$
$$-\sum_{i=1}^{\infty} \frac{q^{2i}(1-q^{i(N-1)})}{i(1-q^i)} \xrightarrow{N\to\infty} -\sum_{i=1}^{\infty} \frac{q^{2i}}{i(1-q^i)}.$$

We obtain

$$\lim_{N\to\infty} \ln \tilde{c}_{q,N} = -\sum_{i=1}^{\infty} \frac{q^{2i}}{i(1-q^i)} = -\sum_{i=1}^{\infty} \frac{q^i}{i(1-q^i)} + \sum_{i=1}^{\infty} \frac{q^i}{i} =$$
$$-\sum_{i=1}^{\infty} \frac{q^i}{i(1-q^i)} - \ln(1-q)$$

hence

$$\lim_{N\to\infty} \tilde{c}_{q,N} = \exp\left\{-\sum_{i=1}^{\infty} \frac{q^i}{i(1-q^i)}\right\} \frac{1}{1-q}.$$

□

Theorem 3.6. *Let π be a random permutation of vertices of Z_N. We have* $\lim_{N\to\infty} \mathbb{P}[\pi_{\tau_N} = \mathbb{1}] \approx 0.516203$.

Proof. By Lemma 3.2 and Lebesgue's dominated convergence theorem (see [11]) we get

$$\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] = \int_0^1 \tilde{c}_{q,N}\, dq \xrightarrow{N\to\infty}$$
$$\int_0^1 \exp\left\{-\sum_{i=1}^{\infty} \frac{q^i}{i(1-q^i)}\right\} \frac{1}{1-q}\, dq \approx 0.516203.$$

(Wolfram Mathematica 8.)

□

3.6 Structures with a large minimal indegree

In this section we prove that the asymptotic effectiveness of τ_N applied to structures with respectively large minimal indegrees equals 1. We give the structure of half-cube as an example. Let again $q = 1 - p$ and $\tilde{c}_{q,N} = c_{p,N}$.

Theorem 3.7. *Let G be the upward directed rooted graph whose all leaves have the same depth N and let the minimal indegree of its vertices excluding leaves be $\delta = \delta(N)$. Let $\frac{\delta(N)}{\log N} \xrightarrow{N\to\infty} \infty$. Then $\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] \xrightarrow{N\to\infty} 1$.*

Proof. We have $\tilde{c}_{q,N} \geq (1 - q^\delta)^{N-1}$. Thus $\liminf_{N\to\infty} \tilde{c}_{q,N} \geq \lim_{N\to\infty} e^{-q^\delta N}$. Since $\frac{\delta(N)}{\log N} \xrightarrow{N\to\infty} \infty$ we obtain

$$\lim_{N\to\infty} \log\left(q^\delta N\right) = \lim_{N\to\infty} \left(\delta \log q + \log N\right) = -\infty.$$

Thus $q^\delta N \xrightarrow{N\to\infty} 0$ which gives $\lim_{N\to\infty} \tilde{c}_{q,N} = 1$. By lemma 3.2 and Lebesgue's dominated convergence theorem (see [11]) we get

$$\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] = \int_0^1 \tilde{c}_{q,N}\, dq \xrightarrow{N\to\infty} \int_0^1 1\, dq = 1.$$

□

Example 3.8. Half-cube case. We denote the d-dimensional half-cube by $Q_d = (V_d, E_d)$. We have $N = \lfloor d/2 \rfloor + 1$. (4-dimensional half-cube is presented in Fig.3.3.) As N does not identify the Q_d in this case we will write $\tau_{(d)}$ instead of τ_N.

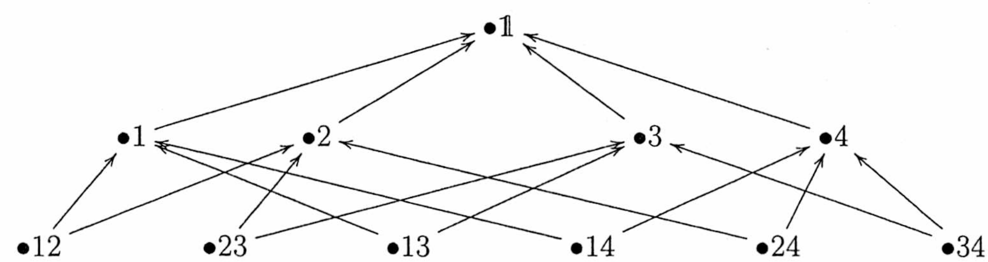


Figure 3.3: Q_4.

Note that the minimal indegree of Q_d is $\delta(d) = d - N + 2 = \lceil \frac{d}{2} \rceil + 1$. We have $\frac{\delta(d)}{\log N} = \frac{\lceil \frac{d}{2} \rceil + 1}{\log(\lfloor d/2 \rfloor + 1)} \xrightarrow{d \to \infty} \infty$. Thus applying Theorem 3.7 we get $\mathbb{P}[\pi_{\tau(d)} = \mathbb{1}] \xrightarrow{d \to \infty} 1$.

3.7 Path-homogeneous structures

In this section we show that our deterministic algorithm τ_N gives good results (the probability of its success tends to 1 with N tending to infinity) also when applied to a broader family of path-homogeneous structures. We prove it using the second moment method and the idea of generating the uniform random order of arrivals of vertices. We give the structures of k-ary trees and half-cubes as examples.

The second moment method is based on Chebyshev's inequality.

Lemma 3.9 (Chebyshev's Inequality, [21]). *Let X be a random variable. Then for all $\varepsilon > 0$*

$$\mathbb{P}[|X - \mathbb{E}X| \geq \varepsilon] \leq \frac{\mathbb{V}ar[X]}{\varepsilon^2}.$$

In particular, for $\varepsilon = \mathbb{E}X$ we have

$$\mathbb{P}[X = 0] \leq \mathbb{P}[|X - \mathbb{E}X| \geq \mathbb{E}X] \leq \frac{\mathbb{V}ar[X]}{(\mathbb{E}X)^2}.$$

□

We call $G = (V, E)$ *path-homogeneous* if for any two paths of the same length r, $P_1 = (\mathbb{1}, v_1, v_2, \ldots, v_{r-1}), P_2 = (\mathbb{1}, w_1, w_2, \ldots, w_{r-1})$ there exists an isomorphism of G, $\rho : V \to V$ such that $\rho(v_i) = w_i$ for each $i = 1, 2, \ldots, r-1$.

Let M denote the total number of paths of length N in a given structure. Let M_s denote the total number of paths of length $N-1$ which do not contain the root and have exactly s vertices in common with a given path of length N. Note that M_s is well defined because our structure is path-homogeneous.

Theorem 3.10. *Let $G = (V, E)$ be path-homogeneous with all leaves of depth N. If $\sup_s (\frac{M_s}{M})^{1/s} \xrightarrow{N \to \infty} 0$ for $s \in \{1, 2, \ldots, N-2\}$ and $\lim_{N \to \infty} M^{-1/N} = 0$ then*

$$\lim_{N \to \infty} \mathbb{P}[\pi_{\tau_N} = \mathbb{1}] = 1.$$

Remark 3.11. Note that in most cases the condition $\sup_s \left(\frac{M_s}{M}\right)^{1/s} \xrightarrow{N\to\infty} 0$ implies $M^{-1/N} \xrightarrow{N\to\infty} 0$. It is enough that we deal with a structure for which there exists $\alpha > 0$ such that $M_{\alpha N} \geq 1$.

Proof. Let us enumerate in G all the paths of length $N-1$ which do not contain the root by $1, 2, 3, \ldots, M$ and define a sequence of random variables for $i = 1, 2, 3, \ldots, M$:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th path appears before the root } \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let also $X = \sum_{i=1}^{M} X_i$. Now X is the total number of paths of length $N-1$ that appear before the root $\mathbb{1}$. Note that $c_{p,N} = \mathbb{P}[X > 0]$. We have

$$\mathbb{E}X = \sum_{i=1}^{M} \mathbb{E}X_i = Mp^{N-1},$$

$$\begin{aligned}\mathbb{E}[X^2] =& \mathbb{E}\left[\sum_{i=1}^{M} X_i\right]^2 = \mathbb{E}\left[\sum_{i=1}^{M} X_i^2\right] + 2 \sum_{1\leq i<j\leq M} \mathbb{E}[X_i X_j] = \\ & Mp^{N-1} + 2 \sum_{1\leq i<j\leq M} \mathbb{E}[X_i X_j].\end{aligned}$$

For $1 \leq i < j \leq M$

$$X_i X_j = \begin{cases} 1 & \text{if the } i\text{th and } j\text{th paths appear before the root } \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases}$$

If the ith and jth paths have s vertices in common then $\mathbb{E}[X_i X_j] = p^{2(N-1)-s}$. Hence $p^{2(N-1)} \leq \mathbb{E}[X_i X_j] \leq p^N$. Since G is path-homogeneous there are $\frac{MM_s}{2}$ pairs of paths of length $N-1$ which do not contain the root that have s vertices in common. Therefore

$$\sum_{1\leq i<j\leq M} \mathbb{E}[X_i X_j] = \sum_{s=0}^{N-2} \frac{MM_s}{2} p^{2(N-1)-s}.$$

Using Chebyshev's inequality (3.9) we get

$$\mathbb{P}[X=0] \leq \frac{\mathbb{V}ar[X]}{(\mathbb{E}X)^2} = \frac{\mathbb{E}X^2-(\mathbb{E}X)^2}{(\mathbb{E}X)^2} =$$
$$\frac{\mathbb{E}X + 2\sum_{1\leq i<j\leq M}\mathbb{E}X_iX_j}{(\mathbb{E}X)^2} - 1 =$$
$$\frac{Mp^{N-1}+\sum_{s=0}^{N-2}MM_sp^{2(N-1)-s}}{M^2p^{2(N-1)}} - 1 = \frac{1+\sum_{s=0}^{N-2}M_sp^{N-1-s}}{Mp^{N-1}} - 1 =$$
$$\frac{M_0}{M}+\frac{M_1}{Mp}+\frac{M_2}{Mp^2}+\frac{M_3}{Mp^3}+\ldots+\frac{M_{N-2}}{Mp^{N-2}}+\frac{1}{Mp^{N-1}}-1.$$

Let $L_N = \sup_s (M_s/M)^{1/s}$ for $s \in \{1,2,\ldots,N-2\}$. Since $L_N \xrightarrow{N\to\infty} 0$

$$\sum_{s=1}^{N-2}\frac{M_s}{Mp^s} = \sum_{s=1}^{N-2}\left(\left(\frac{M_s}{M}\right)^{1/s}\frac{1}{p}\right)^s \leq$$
$$\sum_{s=1}^{N-2}\left(\frac{L_N}{p}\right)^s = \frac{(L_N/p)(1-(L_N/p)^{N-2})}{1-L_N/p} \xrightarrow{N\to\infty} 0.$$

Thus also $\sum_{s=1}^{N-2}\frac{M_s}{M} \xrightarrow{N\to\infty} 0$ and

$$\frac{M_0}{M} = \frac{M-(1+M_1+M_2+\ldots+M_{N-2})}{M} =$$
$$\frac{M-1}{M} - \sum_{s=1}^{N-2}\frac{M_s}{M} \xrightarrow{N\to\infty} 1.$$

Since $M^{-1/N} \xrightarrow{N\to\infty} 0$ we also get $1/(Mp^{N-1}) \xrightarrow{N\to\infty} 0$. Hence we obtain $\mathbb{P}[X=0] \xrightarrow{N\to\infty} 0$ which gives

$$c_{p,N} = 1-\mathbb{P}[X=0] \xrightarrow{N\to\infty} 1.$$

By Lemma 3.2 and Lebesgue's dominated convergence theorem (see [11]) we finally get

$$\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] = \int_0^1 c_{p,N}\,dp \xrightarrow{N\to\infty} 1.$$

□

Example 3.12. Complete k-ary tree $T_{k(N),N}$. Let us recall that by $T_{k(N),N}$ we denote the upward directed complete $k(N)$-ary tree of height N. Let $k(N) \to \infty$ with $N \to \infty$.

For $T_{k(N),N}$ we have $M = k(N)^{N-1}$ and $M_s = (k(N)-1)k(N)^{N-(s+2)}$ for $s = 0, 1, \ldots, N-2$. Thus for $s = 1, 2, \ldots, N-2$ we get $\sup_s \left(\frac{M_s}{M}\right)^{1/s} \xrightarrow{N\to\infty} 0$ which in this case implies also $\lim_{N\to\infty} M^{-1/N} = 0$. Since $T_{k(N),N}$ is a path-homogeneous structure we may apply theorem 3.10 and therefore get $\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] \xrightarrow{N\to\infty} 1$.

Example 3.13. Half-cube case. Let us recall that by $Q_d = (V_d, E_d)$ we denote the d-dimensional half-cube. In this case we will write again $\tau_{(d)}$ instead of τ_N. As usually, n denote the cardinality of V_d. We get the following relations:

$$N = \lfloor d/2 \rfloor + 1,$$

$$n = \begin{cases} 2^{d-1}, & \text{n - odd} \\ 2^{d-1} + \frac{1}{2}\binom{d}{d/2}, & \text{n - even} \end{cases}$$

$$M = d(d-1)(d-2)\ldots(d-(N-2)) = (N-1)!\binom{d}{N-1} = \frac{d!}{\lfloor d/2 \rfloor!}.$$

Lemma 3.14. *We have for Q_d and $s \in \{1, 2, \ldots, \lfloor d/2 \rfloor - 1\}$:*

$$\sup_s \left(\frac{M_s}{M}\right)^{1/s} \xrightarrow{N\to\infty} 0.$$

Proof. Note that $M_s \le \binom{N-1}{s}(N-1-s)!$ where $\binom{N-1}{s}$ is the number of possibilities of choosing s points on a given path of length $N-1$ and $(N-1-s)!$ is the upper estimation of the number of paths that go through those s points. Recalling that $N = \lfloor d/2 \rfloor + 1$ and $M = \frac{d!}{\lfloor d/2 \rfloor!}$ we obtain

$$\left(\frac{M_s}{M}\right)^{1/s} \le \left(s!\binom{d}{\lfloor d/2 \rfloor}\right)^{-1/s}.$$

We are going to show that the function $f(s) = \left(s!\binom{d}{\lfloor d/2 \rfloor}\right)^{1/s}$ is decreasing and that $f(\lfloor d/2 \rfloor - 1) \xrightarrow{d\to\infty} \infty$. We have

$$\left(\frac{f(s+1)}{f(s)}\right)^{s(s+1)} = \frac{(s+1)^s}{s!\binom{d}{\lfloor d/2 \rfloor}}.$$

Since $\frac{(s+1)^s}{s!}$ is increasing, in order to obtain $\frac{f(s+1)}{f(s)} < 1$ it is enough to show $\frac{(s+1)^s}{s!} < \binom{d}{\lfloor d/2 \rfloor}$ for $s = \lfloor d/2 \rfloor - 1$. We have

$$\frac{\lfloor d/2 \rfloor^{(\lfloor d/2 \rfloor - 1)}}{(\lfloor d/2 \rfloor - 1)!} = \frac{\lfloor d/2 \rfloor^{\lfloor d/2 \rfloor}}{\lfloor d/2 \rfloor !} < \frac{(\lceil d/2 \rceil + 1)(\lceil d/2 \rceil + 2)\dots d}{\lfloor d/2 \rfloor !} = \binom{d}{\lfloor d/2 \rfloor}.$$

We have shown that $f(s)$ is decreasing. We have also

$$f(\lfloor d/2 \rfloor - 1) = \left((\lfloor d/2 \rfloor - 1)! \frac{d!}{\lfloor d/2 \rfloor ! \lceil d/2 \rceil !} \right)^{\frac{1}{\lfloor d/2 \rfloor - 1}} =$$

$$\left((\lfloor d/2 \rfloor - 1)! \frac{(\lceil d/2 \rceil + 1)(\lceil d/2 \rceil + 2)\dots d}{\lfloor d/2 \rfloor !} \right)^{\frac{1}{\lfloor d/2 \rfloor - 1}} >$$

$$\left((\lfloor d/2 \rfloor - 1)! \frac{\lfloor d/2 \rfloor (\lfloor d/2 \rfloor)^{(\lfloor d/2 \rfloor - 1)}}{\lfloor d/2 \rfloor !} \right)^{\frac{1}{\lfloor d/2 \rfloor - 1}} = \lfloor d/2 \rfloor \xrightarrow{d \to \infty} \infty.$$

□

Q_d is path-homogeneous and for Q_d we have $\lim_{N \to \infty} M^{-1/N} = 0$. Thus, by Lemma 3.14 and Theorem 3.10, we obtain $\mathbb{P}[\pi_{\tau_{(d)}} = \mathbb{1}] \xrightarrow{d \to \infty} 1$.

This result for half-cubes was already obtained in the previous section by simpler methods. However, there are families of upward directed graphs for which we can apply Theorem 3.10 while Theorem 3.7 does not work.

Example 3.15. Complete $\lfloor \log \log N \rfloor$-ary tree case. Let us consider $T_{k(N),N}$ with $k(N) = \lfloor \log \log N \rfloor$. For this structure the minimal indegree $\delta(N) = \lfloor \log \log N \rfloor$ hence $\frac{\delta(N)}{\log N} \xrightarrow{N \to \infty} 0$ and Theorem 3.7 can not be applied. However Example 3.12 shows that for our $\lfloor \log \log N \rfloor$-ary tree by Theorem 3.10 we get $\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] \xrightarrow{N \to \infty} 1$.

3.8 Short discussion on the choice of the algorithm τ_N

We have proved throughout this chapter that our deterministic algorithm τ_N for choosing the root, despite being very simple, works very well for many families of directed rooted graphs. One can however wonder why should not we stop earlier. After all τ_N tells the selector to wait till the very end. Let us show that stopping at the top of a shorter path does not give a better result when playing on the structures presented in this chapter.

Let us consider the following algorithm. Let a stopping time $\tilde{\tau}_N$ be equal to the first m such that $h_m(\pi_m) = N-1$. If it never happens, let $\tilde{\tau}_N = n$. Note that the algorithms τ_N and $\tilde{\tau}_N$ are disjoint, i.e., whenever $\tilde{\tau}_N$ stops, τ_N plays further and whenever τ_N stops, $\tilde{\tau}_N$ does not stop. Thus whenever one of them gives the probability of success greater than 0.5 the other one can not perform better. The probability of success of τ_N for the structures presented throughout the chapter either tends asymptotically to 1 with N tending to infinity or is equal to a value greater than 0.5 (except the complete binary tree case which is discussed below). This shows the advantage of τ_N over $\tilde{\tau}_N$ for the presented families of structures.

In the binary tree case we have $\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] \xrightarrow{N\to\infty} 2\ln 2 - 1 \approx 0.386$. For $a, b, c \in V_N$ let $E_{a,b,c}$ denote the event that the vertices a, b, c appear in a random permutation in exactly this oder. Let $\mathbf{2}, \mathbf{3}$ be the children of the root $\mathbb{1}$. Note that $\mathbb{P}[\pi_{\tau_N} = \mathbb{1} | E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}}] = \mathbb{P}[\pi_{\tilde{\tau}_N} = \mathbb{1} | E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}}] = 0$. Thus $\mathbb{P}[\pi_{\tau_N} = \mathbb{1}] = \mathbb{P}[\pi_{\tau_N} = \mathbb{1} | (E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C]\mathbb{P}[(E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C]$ and analogously for $\tilde{\tau}_N$. Since $\mathbb{P}[(E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C] = 2/3$ we get

$$\mathbb{P}[\pi_{\tau_N} = \mathbb{1} | (E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C] \xrightarrow{N\to\infty} 3\ln 2 - 3/2.$$

Recall that τ_N and $\tilde{\tau}_N$ are never equal, whence

$$\mathbb{P}[\pi_{\tilde{\tau}_N} = \mathbb{1} | (E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C] \leq \\ 1 - \mathbb{P}[\pi_{\tau_N} = \mathbb{1} | (E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C] \xrightarrow{N\to\infty} 5/2 - 3\ln 2.$$

Finally,

$$\lim_{N\to\infty} \mathbb{P}[\pi_{\tilde{\tau}_N} = \mathbb{1}] = \\ \lim_{N\to\infty} \mathbb{P}[\pi_{\tilde{\tau}_N} = \mathbb{1} | (E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C]\mathbb{P}[(E_{\mathbb{1},\mathbf{2},\mathbf{3}} \cup E_{\mathbb{1},\mathbf{3},\mathbf{2}})^C] \leq \\ (5/2 - 3\ln 2)2/3 = 5/3 - 2\ln 2 \approx 0.28.$$

Thus also in the binary tree case τ_N performs better than $\tilde{\tau}_N$.

Chapter 4

The best choice problem for upward directed graphs

4.1 Introduction

In [23] Preater presented a universal algorithm for partial orders in the case of restricted information, precisely, when a selector knows in advance only the cardinality of a poset. Surprisingly, this algorithm wins with the probability at least 1/4 (original Preater's bound of 1/8 was improved in [7] to 1/4) on any partial order. (For further development of the subject see [16], [4], [6], [18]; compare also the Historical overview in Chapter 1.) A natural question is if such an efficient universal stopping time exists for upward directed graphs.

In this chapter we assume that the selector is told in advance only the total number n of vertices of an upward directed graph (actually there will be also some restriction on the structure of the graph). We describe a universal strategy for choosing a maximal element. We prove that, as long as the number of elements dominated directly by the maximal vertices is not greater than $c_1\sqrt{n}$ for some positive constant c_1 and the indegree of remaining vertices is bounded by a constant D, the probability of success p_n satisfies $\liminf_{n\to\infty} p_n\sqrt{n} \geq \delta > 0$, where δ is some constant depending on c_1 and D. As it can be seen from the optimal result for the directed path from [17] one cannot hope, up to a constant, for a better result. The similar results have been obtained independently by Goddard, Kubicka and Kubicki in [9].

The chapter is organized as follows. In Section 4.2 we introduce necessary definitions. Section 4.3 presents our strategy and contains an analysis of its effectiveness. Section 4.4 contains a comparison of our universal algorithm with the optimal strategy in the directed path case and a short discussion about the universal algorithm for the structures with large minimal indegree that was introduced in Chapter 3.

4.2 Definitions

Let $G = (V, E)$ be a directed graph. By *in(v)* we denote the *indegree* of v in G which is the number of edges incoming to v.

The depth of $v \in V$ in G will be denoted by $d(v)$. The set of the elements of depth 2 will be denoted by $Sec(G)$. (See Fig.4.1.)

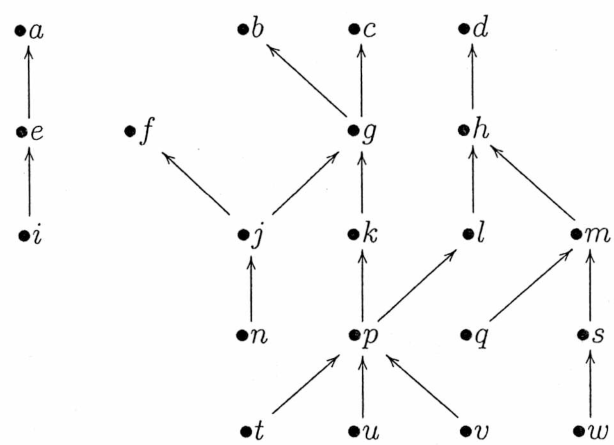


Figure 4.1: An upward directed graph G with $Max(G) = \{a, b, c, d, f\}$ and $Sec(G) = \{e, g, h, j\}$.

Throughout this chapter G will be any upward directed graph and the set of vertices we would like to stop on will be $D = Max(G)$.

4.3 Universal algorithm

In this section we present a stopping time τ_n for choosing a maximal element from an upward directed graph $G_n = (V_n, E_n)$, $|V_n| = n$. Our strategy uses randomization that was first introduced in [23] to construct a universal best choice algorithm for posets. We show that as long as $|Sec(G_n)| \leq c_1\sqrt{n}$ for some positive constant c_1 and $in(v)$ is bounded by a constant for all $v \in V_n \setminus (Max(G_n) \cup Sec(G_n))$ the probability of success of our strategy satisfies $\liminf_{n\to\infty} \mathbb{P}[\pi_{\tau_n} \in Max(G_n)]\sqrt{n} \geq \delta > 0$, for some constant δ independent of n and the considered sequence of graphs.

Strategy: Let us define a stopping time τ_n as follows. Flip an asymmetric coin, having some probability p of coming down tails, n times. If it comes down tails M times reject the first M elements. After this time pick the first element which is maximal in the induced graph. In other words, τ_n is equal to the first $j > M$ such that $\pi_j \in Max(G_{(j)})$. If no such j is found let $\tau_n = n$.

Before we move on to analyzing the effectiveness of our strategy we are going to prove the following lemma.

Lemma 4.1. *Let $\pi \in S_n$ be a random permutation of vertices in V. Suppose that we have a coin that comes down tails with probability p. Let M denote the number of occurrences of tails in n tosses. Then all vertices from V appear in $\{\pi_1, \pi_2, \dots, \pi_M\}$ with probability p independently.*

Proof. Let $V_M = \{\pi_1, \pi_2, \dots, \pi_M\}$. Let $v \in V$. We start with proving that $\mathbb{P}[v \in V_M] = p$. Since $M \sim B(n,p)$ we have

$$\mathbb{P}[v \in V_M] = \sum_{i=1}^{n} \mathbb{P}[v \in V_M | M = i]\mathbb{P}[M = i] = \sum_{i=1}^{n} \frac{i}{n}\binom{n}{i}p^i(1-p)^{n-i} =$$
$$\sum_{i=1}^{n}\binom{n-1}{i-1}p^i(1-p)^{n-i} = \sum_{j=0}^{n-1}\binom{n-1}{j}p^{j+1}(1-p)^{n-j-1} =$$
$$p\sum_{j=0}^{n-1}\binom{n-1}{j}p^j(1-p)^{(n-1)-j} = p \cdot 1 = p.$$

Now we are going to prove that all vertices from V appear in V_M independently. We need to show that for each $1 \leq r \leq |V|$:

$$\mathbb{P}[v_1 \in V_M, v_2 \in V_M, \dots, v_r \in V_M] = \mathbb{P}[v_1 \in V_M]\mathbb{P}[v_2 \in V_M] \dots \mathbb{P}[v_r \in V_M].$$

We already know that $\mathbb{P}[v_1 \in V_M]\mathbb{P}[v_2 \in V_M] \dots \mathbb{P}[v_r \in V_M] = p^r$. Thus we need to show $\mathbb{P}[v_1 \in V_M, v_2 \in V_M, \ldots, v_r \in V_M] = p^r$. We have

$$\mathbb{P}[v_1 \in V_M, \ldots, v_r \in V_M] =$$

$$\sum_{i=1}^{n} \mathbb{P}[v_1 \in V_M, \ldots, v_r \in V_M | M = i]\mathbb{P}[M = i] =$$

$$\sum_{i=r}^{n} \frac{\binom{i}{r}}{\binom{n}{r}} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=r}^{n} \binom{n-r}{i-r} p^i (1-p)^{n-i} =$$

$$\sum_{j=0}^{n-r} \binom{n-r}{j} p^{j+r} (1-p)^{n-j-r} =$$

$$p^r \sum_{j=0}^{n-r} \binom{n-r}{j} p^j (1-p)^{(n-r)-j} = p^r \cdot 1 = p^r.$$

□

Theorem 4.2. *Let $G_n = (V_n, E_n)$ be an upward directed graph, $|V_n| = n$. Let $\pi \in S_n$ be a random permutation of vertices in V_n. Let also $|Sec(G_n)| \le c_1\sqrt{n}$ for some positive constant c_1, $in(v) \le D$ for $v \in V_n \setminus (Max(G_n) \cup Sec(G_n))$ and let p (from the definition of τ_n) be equal to $1 - c/\sqrt{n}$ where c is some positive constant. Then*

$$\liminf_{n\to\infty} \mathbb{P}[\pi_{\tau_n} \in Max(G_n)]\sqrt{n} \ge \delta = \delta(c, c_1, D) > 0.$$

Before stating the proof of Theorem 4.2 let us briefly discuss why the suggested stopping time τ_n with $p = 1 - c/\sqrt{n}$ would work. We are aiming at the universal algorithm that gives the probability of success of the order $\Theta(1/\sqrt{n})$. If we then let, on average, all but $c\sqrt{n}$ vertices pass, the probability that at least one maximal vertex is still to come is of the order $\Omega(1/\sqrt{n})$. Since the number of vertices that are still to come is "small" the probability that our algorithm will encounter among remaining $c\sqrt{n}$ vertices a "misleading" one which is maximal in the induced graph but not in G_n is also small whenever we deal with the graphs considered in this chapter.

Proof. Let M be the value mentioned in the definition of the stopping time τ_n and let $V_M = \{\pi_1, \pi_2, \ldots, \pi_M\}$. Let us partition the set $V \setminus (Max(G_n) \cup Sec(G_n))$ into two sets: $V_{odd} = \{v \in V \setminus (Max(G_n) \cup Sec(G_n)) : d(v) \text{ is odd}\}$

and $V_{even} = \{v \in V \setminus (Max(G_n) \cup Sec(G_n)) : d(v) \text{ is even}\}$. Let $|V_{even}| = L_n$. Let us also assign to every vertex $v \in V \setminus (Max(G_n) \cup Sec(G_n))$ one of its parents whose depth is smaller than the depth of v and denote it by $r(v)$. Let $s(v) = \{w : (w, v) \in E\}$. Let

$$R = \{r(v) : v \in V_{even} \setminus V_M\},$$

$$S = \left(\bigcup_{v \in V_{even} \setminus V_M} s(v) \right) \cap V_{odd}.$$

Let us give an example of the sets R and S based on Figure 4.1. Consider $\pi = (l, e, v, f, u, m, g, b, t, h, c, j, s, d, q, w, k, n, a, p, i)$ and $M = 14$. Then $V_M = \{l, e, v, f, u, m, g, b, t, h, c, j, s, d\}$ and $V \setminus V_M = \{q, w, k, n, a, p, i\}$. We have $V_{even} \setminus V_M = \{p, q\}$ thus $S = \{t, u, v\}$. Choosing $r(p) = l$ we also have $R = \{l, m\}$.

We have

$$\mathbb{P}[\pi_{\tau_n} \in Max(G_n)] =$$
$$\sum_{k=0}^{L_n} \mathbb{P}[\pi_{\tau_n} \in Max(G_n) || V_M \cap V_{even}| = k] \mathbb{P}[|V_M \cap V_{even}| = k].$$

By Lemma 4.1 we have $\mathbb{P}[|V_M \cap V_{even}| = k] = \binom{L_n}{k} p^k (1-p)^{L_n - k}$. Let A be the event that after time M some maximal vertex is still to come, i.e., $A = [Max(G_n) \not\subset V_M]$. Note that if the event $A \cap [(Sec(G_n) \cup R \cup S) \subseteq V_M]$ occurs then the vertex $\pi_j \in V \setminus V_M$ is maximal in the induced graph $G_{(j)}$ if and only if it belongs to $Max(G_n)$ (each vertex from $V \setminus V_M$ with even depth in G_n has already one of its parents in $R \subseteq V_M$; also each vertex from $V \setminus (V_M \cup Max(G_n))$ with odd depth in G_n has at least one ("even") parent in V_M since all the "odd" children of vertices from $V_{even} \setminus V_M$ are already in $S \subseteq V_M$). Therefore $\pi_{\tau_n} \in Max(G_n)$ whenever the event $A \cap [(Sec(G_n) \cup R \cup S) \subseteq V_M]$ occurs. Thus we have

$$\mathbb{P}[\pi_{\tau_n} \in Max(G_n) || V_M \cap V_{even}| = k] \geq$$
$$\mathbb{P}[A \cap ((Sec(G_n) \cup R \cup S) \subseteq V_M) || V_M \cap V_{even}| = k].$$

If $|V_M \cap V_{even}| = k$ we have $|R| \leq L_n - k$ and $|S| \leq D(L_n - k)$. The vertices from $Max(G_n) \cup Sec(G_n) \cup R \cup S$ do not belong to V_{even}, hence by Lemma 4.1 we get

$$\mathbb{P}[\pi_{\tau_n} \in Max(G_n) || V_M \cap V_{even}| = k] \geq (1-p) p^{|Sec(G_n)|} p^{L_n - k} p^{D(L_n - k)}.$$

Thus

$$\mathbb{P}[\pi_{\tau_n} \in Max(G_n)] \geq$$
$$\sum_{k=0}^{L_n} (1-p)p^{|Sec(G_n)|} p^{L_n-k} p^{D(L_n-k)} \binom{L_n}{k} p^k (1-p)^{L_n-k} =$$
$$(1-p)p^{|Sec(G_n)|} p^{L_n(D+1)} \sum_{k=0}^{L_n} \binom{L_n}{k} (p^{-D})^k (1-p)^{L_n-k} =$$
$$(1-p)p^{|Sec(G_n)|} p^{L_n(D+1)} (1-p+p^{-D})^{L_n}.$$

Since $|Sec(G_n)| \leq c_1\sqrt{n}$ we have

$$\liminf_{n\to\infty} \sqrt{n}(1-p)p^{|Sec(G_n)|} \geq \lim_{n\to\infty} \sqrt{n}\frac{c}{\sqrt{n}}\left(1-\frac{c}{\sqrt{n}}\right)^{c_1\sqrt{n}} = ce^{-cc_1}. \quad (4.1)$$

Hence we will be done if we show that $\liminf_{n\to\infty} p^{L_n(D+1)}(1-p+p^{-D})^{L_n}$ is some positive constant. We have

$$p^{L_n(D+1)}(1-p+p^{-D})^{L_n} = (p(1+p^D(1-p)))^{L_n} =$$
$$\left(\left(1-\frac{c}{\sqrt{n}}\right)\left(1+\left(1-\frac{c}{\sqrt{n}}\right)^D \frac{c}{\sqrt{n}}\right)\right)^{L_n} =$$
$$\left(1-\frac{c}{\sqrt{n}}\left(1-\left(1-\frac{c}{\sqrt{n}}\right)^D\right)-\frac{c^2}{n}\left(1-\frac{c}{\sqrt{n}}\right)^D\right)^{L_n} =$$
$$\left(1-\frac{c\sqrt{n}(1-(1-c/\sqrt{n})^D)+c^2(1-c/\sqrt{n})^D}{n}\right)^{L_n}.$$

We have $c\sqrt{n}(1-(1-c/\sqrt{n})^D) \xrightarrow{n\to\infty} c^2 D$ and $c^2(1-c/\sqrt{n})^D \xrightarrow{n\to\infty} c^2$. Quite formally, since $L_n = |V_{even}| \leq |V_n|$, there exists a constant $c_2 \leq 1$ such that (we introduce c_2 only because we will refer to it in Section 4.4)

$$\liminf_{n\to\infty} \left(p^{D+1}(1-p+p^{-D})\right)^{L_n} \geq$$
$$\lim_{n\to\infty}\left(1-\frac{c\sqrt{n}(1-(1-c/\sqrt{n})^D)+c^2(1-c/\sqrt{n})^D}{n}\right)^{c_2 n} = e^{-c_2c^2(D+1)}.$$

Since $c_2 \leq 1$ we finally get

$$\liminf_{n\to\infty} \mathbb{P}[\pi_{\tau_n} \in Max(G_n)]\sqrt{n} \geq ce^{-c(c_1+c_2c(D+1))} \geq$$
$$ce^{-c(c_1+c(D+1))} = \delta(c, c_1, D) > 0.$$

Note that the constant c is the initial parameter of the algorithm, the constants c_1 and D refer to the assumptions of Theorem 4.2 hence δ does not depend on a particular sequence of graphs satisfying our hypothesis. $\square$

4.4 Comments and remarks

Let us analyze how well our strategy works for the directed path. From [17] we know that the optimal algorithm τ for choosing the top vertex $\mathbb{1}$ of a directed path satisfies $\mathbb{P}[\pi_\tau = \mathbb{1}]\sqrt{n} \xrightarrow{n\to\infty} \sqrt{\pi}/2 \approx 0.89$. For G_n being the directed path of length n we have $|Sec(G_n)| = 1, D = 1$ and $c_2 = 1/2$, where D and c_2 are constants from the proof of Theorem 4.2. Then we obtain (setting $|Sec(G_n)| = 1$ already in (4.1)) $\liminf_{n\to\infty} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}]\sqrt{n} \geq ce^{-c^2}$. The value $c = 1/\sqrt{2}$ maximizes this lower bound which is then equal to $1/\sqrt{2e} \approx 0.43$.

In this chapter we have presented a universal stopping time τ_n for structures with bounded indegree. A simple universal algorithm τ_N for structures with large minimal indegree was presented in Chapter 3. It was shown that if the selector knows in advance the height N of the structure the probability of success of τ_N tends to 1 with N tending to infinity if the minimal indegree is $\omega(\log N)$ (Section 3.6). This justifies considering here a universal algorithm only for graphs with bounded indegrees.

Chapter 5

The best choice problem for the kth power of a directed path

5.1 Introduction

In this chapter we generalize the optimal algorithm for a directed path from [17] to any kth power of a directed path. However, we additionally assume that the selector knows the distance in the underlying path between each two vertices that are joined by an edge in the induced graph. We give the exact probability of success for $k = 2$. We show that the probability of success p_n (where n is the length of the path) according to the optimal algorithm for the kth power of the directed path satisfies $p_n = \Omega(n^{-1/(k+1)})$ and also for k such that $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$, $p_n = O(n^{-1/(k+1)})$, no matter whether the selector gets the additional information about distances or not. Quite surprisingly, one of the cases considered here turns out to be a case of the classical secretary problem (with the linear order) with extra information.

This chapter is organized as follows. In Section 5.2 a few definitions are introduced. In Section 5.3 we present the stopping time for choosing the root from the kth power of a directed path when the selector is given the additional information about distances. We prove its optimality. In Section 5.4 we give the exact probability of success of our algorithm for $k = 2$. In Section 5.5 we prove two theorems estimating the probability of success of the optimal algorithm. We consider the probability of success of the optimal algorithm also for the case when the selector is not given the additional information about distances. Section 5.6 discusses separately the case $k = n - 1$ where the graph problem turns out to be the classical secretary problem where the selector knows the differences between the ranks of examined candidates.

5.2 Definitions

A *kth power* of a graph $G = (V, E)$ is a graph with the set of vertices V and an edge between two vertices if and only if there is a path of length at most $k + 1$ between them in G.

Let us define a function $d_G : E \to \mathbb{N}$ by $d_G((v, w)) = l_G((v, w)) - 2$ where $l_G((v, w))$ is the length of the longest directed path in G joining the vertices v and w.

Throughout this chapter G will be a power of a directed path. It has only one maximal element $\mathbb{1}$ on which we would like to stop ($D = \{\mathbb{1}\}$). We are also going to assume that the selector will be given some additional information, namely the value d_G of each edge that appears in the induced graph.

5.3 Optimal stopping time

Let $P_n^k = (V_n^k, E_n^k)$ be the kth power of the directed path P_n $(1 \leq k < n)$. The example of the second power of the directed path P_9 may be found in Fig.5.1. (Whenever the context is clear we omit the indices n and k for clarity of notation and write P instead of P_n^k.) In this section we present a stopping time τ_n for choosing the root from P_n^k search and show that it is optimal.

Let $\pi \in S_n$ be a random permutation of vertices from V_n^k and $P_{(t)}$ be the graph induced by $\{\pi_1, \pi_2, \ldots, \pi_t\}$. Suppose that $H = (W, F)$ is a connected component in $P_{(t)}$ and that w and z are two extreme vertices of H. Since one knows the value $d_P(e)$ for each $e \in F$, one can also tell how many of the remaining vertices are going to be placed between w and z on the full path P_n. Let us sum the number of those remaining "inner" vertices over all connected components in $P_{(t)}$ and denote the result by b_t. (Compare Fig.5.1.)

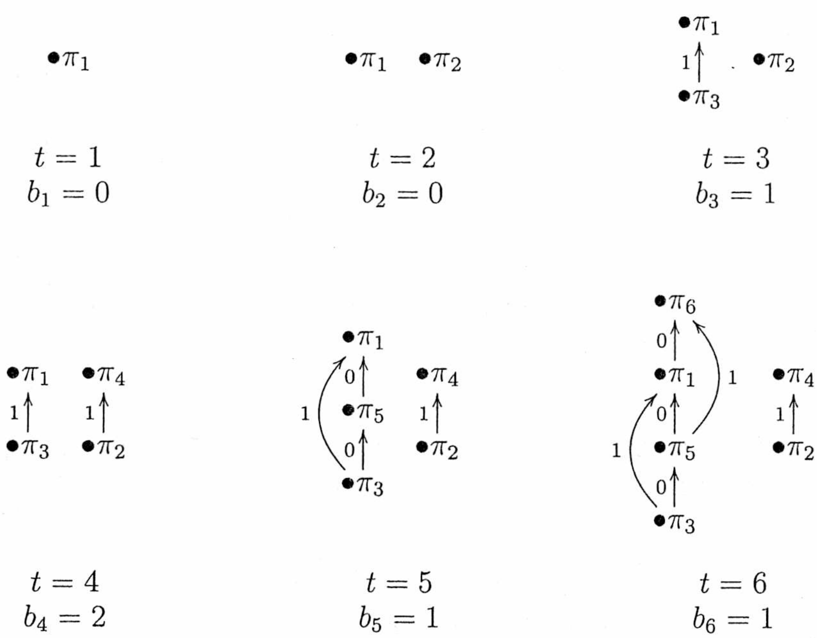
Strategy: Let us define a stopping time τ_n as follows.

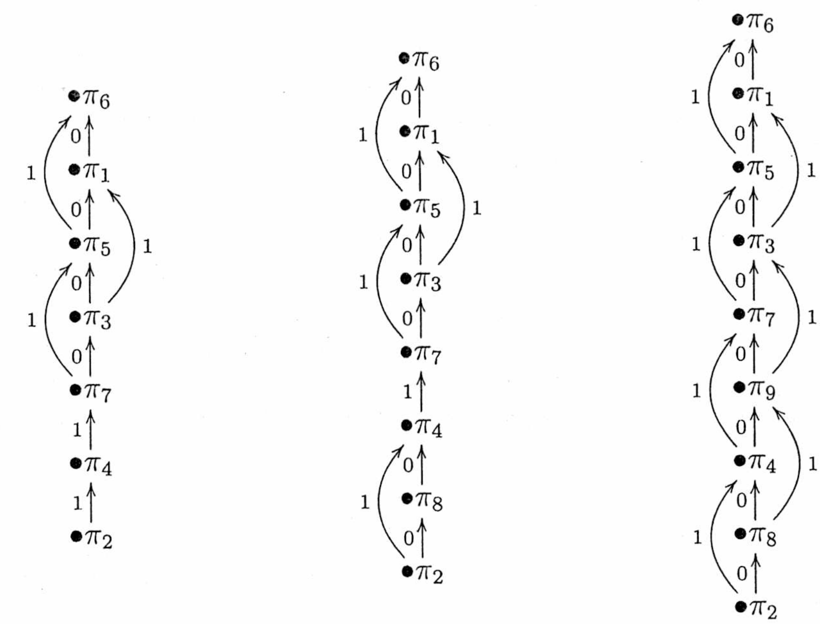
$$\tau_n(\pi) = \min\{t \leq n : n - t = k(c(P_{(t)}) - 1) + b_t,\ \pi_t \in Max\{\pi_1, \pi_2, \ldots \pi_t\}\},$$

using the convention $\min \emptyset = n$.

Note that τ_n tells the selector not to stop as long as there is still a chance to win in the future. (For instance, we have $\tau_9 = 6$ in Fig.5.1.) The condition $n - t = k(c(P_{(t)}) - 1) + b_t$ means that the probability that $\mathbb{1}$ is still to come is equal to zero because among $n - t$ remaining vertices we need at least $k(c(P_{(t)}) - 1)$ to connect the components that we have at the time t and b_t is exactly the number of vertices that will join already existing components falling somewhere between their vertices. Thus the strategy τ_n can be stated exactly as the analogue of the optimal strategy for a directed path that was given by Kubicki and Morayne in [17].

Stop when there is a positive conditional (given history) probability that the presently examined candidate is the maximal one and the probability that the maximal one can be among the future candidates is equal to zero.





$t = 7$ $b_7 = 2$ $t = 8$ $b_8 = 1$ $t = 9$ $b_9 = 0$

Figure 5.1: Induced graphs of P_9^2 at time t for the permutation $\pi = (v_2, v_9, v_4, v_7, v_3, v_1, v_5, v_8, v_6)$.

Theorem 5.1. *Let π be a random permutation of vertices of P_n^k. For P_n^k, $1 \leq k < n$, the stopping time τ_n is optimal, i.e.,*

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \max_{\tau \in \mathcal{T}} \mathbb{P}[\pi_\tau = \mathbb{1}],$$

where $\mathcal{T}$ is the set of all stopping times.

Proof. 1 At first, let us observe that it is reasonable to stop at time m only if $\pi_m \in Max(P_{(m)})$. Of course, we should definitely stop whenever $\mathbb{P}[\mathbb{1} \in \{\pi_{m+1}, \dots, \pi_n\} | \pi_m \in Max(P_{(m)})] = 0$. From [17] we know that if we play on a directed path and $\mathbb{P}[\mathbb{1} \in \{\pi_{m+1}, \dots, \pi_n\} | \pi_m \in Max(P_{(m)})] > 0$ then it always pays off to play further, for instance simple waiting for the next maximal element in the induced graph is profitable. Now we are going to explain the full analogy of the situation at time m between the game on a directed path and the game on its kth power. Since we assume that playing on the kth power one has the additional information about the values d_P, one knows that at least b_m of the remaining vertices are "dummy". They do not play any more any role in our game since we know that they are not going to appear as the maximal ones in the induced graph. We have $k(c(P_{(m)})-1)$ more "dummy" vertices that will appear immediately under the components seen a time $t=m$ (they are also not going to appear as the maximal ones in the induced graph). Note that it is exactly the directed path case at time $\tilde{m}=m+b_m+(k-1)c(P_{(m)}-1)$ when $\pi_{\tilde{m}}$ is maximal in the induced graph, the number of components of the induced graph is $c(P_{(m)})$ and we know about $c(P_{(m)})-1$ "dummy" vertices (supporting the existing at $t=\tilde{m}$ components). Recall that probability that $\mathbb{1}$ is still to come is positive thus we know that in a directed path case ($k=1$) we should play further. Thus we should also play further in the kth power case since throughout the game we are going to obtain at least as many information as playing for $k=1$.

As sometimes the intuitive type argument may contain a hidden bug, to be on the safe side, we also present the fully formal proof of optimality of τ_n below. □

Proof. 2. This proof is analogous to the one that shows the optimality of τ_n for $k=1$ presented in [17]. At first, let us make the observation that it is reasonable to stop only if the currently examined vertex is maximal in the induced graph. Now, aiming for a contradiction, let us assume that there exists a stopping time τ such that $\mathbb{P}[\pi_\tau=\mathbb{1}]>\mathbb{P}[\pi_{\tau_n}=\mathbb{1}]$ which is optimal and that there is no optimal stopping time $\tilde{\tau}>\tau$. By our observation we may also assume that $\tau(\pi)=t$ if and only if $\pi_t\in Max(P_{(t)})$ or $t=n$.

Whenever $\pi_n=\mathbb{1}$ we have $\tau_n(\pi)=\mathbb{1}$ thus

$$\mathbb{P}[\pi_\tau=\mathbb{1}|\tau=n]\leq\mathbb{P}[\pi_{\tau_n}=\mathbb{1}|\tau=n].$$

Hence now let us consider the event $[\tau<n]$.

We have $\tau(\pi)=m<n$ and $\pi_m\in Max(P_{(m)})$. Let $a_m=k(c(P_{(m)})-1)$. Let us calculate the probability that τ wins counting simply all the possible settings of the remaining vertices. We need at least a_m out of the remaining vertices to connect the components of $P_{(m)}$ (which refers to the term $\binom{n-m}{a_m}a_m!$ in (5.1)). Moreover, we need b_m more vertices out of the remaining ones that will fall between the extreme vertices of the components in $P_{(m)}$ (which refers to the term $\binom{n-m-a_m}{b_m}b_m!$). Finally, all the $n-m-a_m-b_m$ remaining vertices may be arbitrarily permuted together with $c(P_{(m)})$ components (which refers to the term $(n-m-a_m-b_m+c(P_{(m)}))!$). If we wish to have the component

containing π_m at the top of the whole graph, then we can arbitrarily permute the $n-m-a_m-b_m$ remaining vertices with $c(P_{(m)})-1$ components (which refers to the term $(n-m-a_m-b_m+c(P_{(m)})-1)!$). Hence we get

$$\mathbb{P}[\pi_m = \mathbb{1} | \pi_m \in Max(P_{(m)})] = \frac{\binom{n-m}{a_m} a_m! \binom{n-m-a_m}{b_m} b_m! (n-m-a_m-b_m+c(P_{(m)})-1)!}{\binom{n-m}{a_m} a_m! \binom{n-m-a_m}{b_m} b_m! (n-m-a_m-b_m+c(P_{(m)}))!} = \frac{1}{n-m-a_m-b_m+c(P_{(m)})}. \tag{5.1}$$

Since all the components of $P_{(m)}$ have equiprobable chance to be placed at the top of the whole underlying graph, we obtain

$$\mathbb{P}[\mathbb{1} \in P_{(m)} | \pi_m \in Max(P_{(m)})] = \frac{c(P_{(m)})}{n-m-a_m-b_m+c(P_{(m)})}$$

which implies

$$\mathbb{P}[\mathbb{1} \notin P_{(m)} | \pi_m \in Max(P_{(m)})] = \frac{n-m-a_m-b_m}{n-m-a_m-b_m+c(P_{(m)})}.$$

Let us consider the following stopping time

$$\overline{\tau}(\pi) = \begin{cases} \min\{t > m : \pi_t \in Max(P_{(t)})\} & \text{if } \tau(\pi) = m < n, \\ n & \text{in the remaining cases,} \end{cases}$$

using the convention $\min \emptyset = n$. Because $\tau \neq \tau_n$ there exists m such that $\{t > m : \pi_t \in Max(P_{(t)})\} \neq \emptyset$. We will show that

$$\mathbb{P}[\pi_{\overline{\tau}(\pi)} = \mathbb{1} | \pi_m \in Max(P_{(m)})] \geq \mathbb{P}[\pi_m = \mathbb{1} | \pi_m \in Max(P_{(m)})].$$

Note that among $n-m$ vertices that are still to come there are at most $n-m-a_m-b_m$ which may arrive as the maximal ones in the induced graph. Therefore if $\mathbb{1}$ is among the remaining vertices then with probability at least $1/(n-m-a_m-b_m)$ it will appear as the first maximal vertex in the induced graph after time m (note that whenever $\mathbb{1}$ is among the remaining vertices, $n-m-a_m-b_m > 0$). Therefore

$$\begin{aligned} &\mathbb{P}[\pi_{\overline{\tau}(\pi)} = \mathbb{1} | \pi_m \in Max(P_{(m)})] = \\ &\quad \mathbb{P}[\overline{\tau}(\pi) = \mathbb{1} | \mathbb{1} \notin P_{(m)}, \pi_m \in Max(P_{(m)})] \mathbb{P}[\mathbb{1} \notin P_{(m)} | \pi_m \in Max(P_{(m)})] \geq \\ &\frac{1}{(n-m-a_m-b_m)} \frac{(n-m-a_m-b_m)}{(n-m-a_m-b_m+c(P_{(m)}))} = \\ &\quad \mathbb{P}[\pi_m = \mathbb{1} | \pi_m \in Max(P_{(m)})]. \end{aligned}$$

Thus we have found the stopping time $\overline{\tau}$ which is at least equally effective as τ and stops later than τ which contradicts with the assumption that there is no optimal stopping time $\tilde{\tau} > \tau$. This proves the optimality of τ_n. $\square$

5.4 Square of a directed path

In this section we give the exact probability of success of the optimal algorithm τ_n for the square of a directed path, i.e., for P_n^2. Let

$$\begin{aligned} B_m &= [\pi_m \in Max\{\pi_1, \pi_2, \ldots, \pi_m\}], \\ C_m &= [n - m = 2(c(P_{(m)}) - 1) + b_m], \\ A_m &= B_m \cap C_m. \end{aligned}$$

Since $C_m = \emptyset$ for $m < (n+2)/3$, we have

$$\begin{aligned} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \sum_{m=\lceil \frac{n+2}{3} \rceil}^{n} \mathbb{P}[\pi_{\tau_n} = \mathbb{1} | A_m] \mathbb{P}[A_m] = \\ \sum_{m=\lceil \frac{n+2}{3} \rceil}^{n} \mathbb{P}[\pi_{\tau_n} = \mathbb{1} | A_m] \mathbb{P}[B_m | C_m] \mathbb{P}[C_m]. \end{aligned}$$

Note that C_m means that at the time m all the remaining vertices are going to fall between the vertices of $P_{(m)}$ (none of the remaining vertices can be the extreme vertex of P_n). Moreover, since we deal with the square of a directed path, not more than two vertices of those that are still to come can be finally placed on P_n next to each other. Therefore we have

$$\mathbb{P}[C_m] = \frac{1}{\binom{n}{m}} \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \binom{m-1}{n-m-k} \binom{n-m-k}{k}.$$

In this formula the kth term corresponds to $P_{(m)}$ having $k+1$ components. The term $\binom{m-1}{n-m-k}$ refers to forming a sequence of m 1s and then out of all but the last element choosing $n-m-k$ 1s that will be followed by 0. That is the way we choose $n-m-k$ spaces between m vertices of $P_{(m)}$ for the elements that are still to come. We are going to have k spaces for the pairs of vertices (because we have $k+1$ components) and $n-m-2k$ spaces for single vertices that are still to arrive (the term $\binom{n-m-k}{k}$ refers to choosing which spaces are going to be "single" and which "double"). From $n-m$ remaining vertices we may form at most $\lfloor (n-m)/2 \rfloor$ pairs which explains the upper limit of the summation. Let $W_m = \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \binom{m-1}{n-m-k}\binom{n-m-k}{k}$. We have

$$\mathbb{P}[B_m|C_m] = \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \mathbb{P}[B_m|[c(P_{(m)}) = k+1] \cap C_m]\mathbb{P}[c(P_{(m)}) = k+1|C_m].$$

Obviously,

$$\mathbb{P}[B_m|[c(P_{(m)}) = k+1] \cap C_m] = \mathbb{P}[B_m|c(P_{(m)}) = k+1] = \frac{|Max(P_{(m)})|}{m} = \frac{k+1}{m}. \tag{5.2}$$

We also have

$$\mathbb{P}[c(P_{(m)}) = k+1|C_m] = \frac{\binom{m-1}{n-m-k}\binom{n-m-k}{k}}{W_m} \tag{5.3}$$

thus

$$\mathbb{P}[B_m|C_m] = \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \frac{k+1}{m}\frac{\binom{m-1}{n-m-k}\binom{n-m-k}{k}}{W_m} \\ \frac{1}{mW_m}\sum_{k=0}^{\lfloor (n-m)/2 \rfloor}(k+1)\binom{m-1}{n-m-k}\binom{n-m-k}{k}. \tag{5.4}$$

Let $T_m = \sum_{k=0}^{\lfloor (n-m)/2 \rfloor}(k+1)\binom{m-1}{n-m-k}\binom{n-m-k}{k}$. We have

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}|A_m] = \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}|[c(P_{(m)}) = k+1] \cap A_m]\mathbb{P}[c(P_{(m)}) = k+1|A_m].$$

Furthermore,

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}|[c(P_{(m)}) = k+1] \cap A_m] = \frac{1}{k+1}$$

and, by (5.2), (5.3) and (5.4) for the second equality below,

$$\mathbb{P}[c(P_{(m)}) = k+1|A_m] = \\ \frac{\mathbb{P}[B_m|[c(P_{(m)}) = k+1] \cap C_m]\mathbb{P}[c(P_{(m)}) = k+1|C_m]}{\mathbb{P}[B_m|C_m]} = \\ \frac{\frac{k+1}{m}\frac{\binom{m-1}{n-m-k}\binom{n-m-k}{k}}{W_m}}{\frac{T_m}{mW_m}} = \frac{(k+1)\binom{m-1}{n-m-k}\binom{n-m-k}{k}}{T_m}.$$

Hence

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}|A_m] = \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \frac{1}{k+1} \frac{(k+1)\binom{m-1}{n-m-k}\binom{n-m-k}{k}}{T_m} = \frac{W_m}{T_m}.$$

Thus

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}|A_m]\mathbb{P}[B_m|C_m]\mathbb{P}[C_m] = \frac{W_m}{T_m}\frac{T_m}{mW_m}\frac{W_m}{\binom{n}{m}} = \frac{W_m}{m\binom{n}{m}}$$

which finally gives

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \sum_{m=\lceil (n+2)/3 \rceil}^{n} \frac{1}{m\binom{n}{m}} \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} \binom{m-1}{n-m-k}\binom{n-m-k}{k}.$$

In the next section we are going to prove that $\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \Omega(n^{-1/3})$.

5.5 Probability of success

In this section we show that the probability of success p_n according to our optimal strategy τ_n for P_n^k (the kth power of a directed path P_n) satisfies $p_n = \Omega(n^{-1/(k+1)})$ for $1 \leq k < n$. (Again for clarity of notation we write P instead of P_n^k.) For $k < n$ such that $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$ we also prove $p_n = O(n^{-1/(k+1)})$. We show this result also for the case when the selector is not given the additional information about the values of d_P in the induced graph. In order to prove this result we use the continuous time approach to arrivals of vertices and a probabilistic methods, more precisely: Markov's inequality and Janson's inequality.

Recall that $V_n = \{v_1, v_2, \dots, v_n\}$ and $E_n = \{(v_i, v_{i-1}), i = 2, 3, \dots, n\}$ are the sets of vertices and edges of P_n respectively; thus $v_1 = \mathbb{1}$ is the root. Note that if $\mathbb{1} = \pi_t$ and v_n is still to appear at the time t, then at the time t the condition $n - t = k(c(P_{(t)}) - 1) + b_t$ is not satisfied (we have then $n - t > k(c(P_{(t)}) - 1) + b_t$). Thus the condition that v_n precedes $\mathbb{1}$ in π is necessary for the event $[\pi_{\tau_n} = \mathbb{1}]$. Note also that in the two easy cases, when $k = n - 2$ or $k = n - 1$ this condition is also sufficient. Then we get $\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = 1/2$. Throughout the rest of this section we always assume that $1 \leq k < n - 2$.

Now, let us recall what we understand by the continuous time approach to arrivals of vertices (we have already explained it in Section 3.3). We associate with each v_i, $i = 1, 2, \dots, n$, a random variable A_i of a value drawn uniformly from the interval $[0, 1]$, where all A_i's are independent. We treat A_i as the time of arrival of v_i. Thereby we have generated the uniform random order of arrivals of vertices from P_n^k. The arrival time of the root will be again denoted by p ($A_1 = p$). We have already said that this continuous time variant is equivalent to the discrete time one in a sense that all the permutations of vertices are equiprobable. Let us recall also the following remark.

Remark 5.2. Since all A_i's are independent and the arrival time of the root is $A_1 = p$, the probability that a particular vertex appears before the root is equal to p.

Let us define the following sequence of the indicator random variables

$$X_i^{(p)} = \begin{cases} 1 & \text{if } A_{i+1} > p \wedge A_{i+2} > p \wedge \dots \wedge A_{i+k+1} > p, \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

for $1 \leq i \leq n - k - 2$. Let also $X^{(p)} = \sum_{i=1}^{n-k-2} X_i^{(p)}$. The event $[X^{(p)} = 0]$ means that in the induced graph at the time p there are no two components such that they are neighbors (no other element from the induced graph is between them) and the distance between them in P_n is greater than $k + 2$ (by the distance between two components we understand the length of the shortest path in P_n that joins vertices from the different components). Hence the event $[X^{(p)} = 0, A_n < p]$ ensures that at the time p when the root comes (suppose $\mathbb{1} = \pi_t$) the condition $n - t = k(c(P_{(t)}) - 1) + b_t$ is satisfied. It works also the other way round, i.e., whenever $A_n > p$ or $X^{(p)} > 0$ we have $n - t > k(c(P_{(t)}) - 1) + b_t$. Thus $\pi_{\tau_n} = \mathbb{1}$ if and only if $X^{(p)} = 0$ and $A_n < p$. Thus, since $\mathbb{P}[\pi_{\tau_n} = \mathbb{1} | A_1 = p] = \mathbb{P}[X^{(p)} = 0, A_n < p | A_1 = p]$ and all p's are equiprobable, we have the following lemma (which is analogous to Lemma 3.2).

Lemma 5.3. *For P_n^k $(1 \leq k < n-2)$ we have*

$$P[\pi_{\tau_n} = \mathbb{1}] = \int_0^1 \mathbb{P}[X^{(p)} = 0, A_n < p | A_1 = p]\, dp.$$

□

In this section we are going to use the first moment method which is based on Markov's inequality.

Lemma 5.4 (Markov's Inequality, [21]). *Let X be a nonnegative random variable. Then for all $\varepsilon > 0$*

$$\mathbb{P}[X \geq \varepsilon] \leq \frac{\mathbb{E}X}{\varepsilon}.$$

In particular, for $\varepsilon = 1$ we have

$$\mathbb{P}[X \geq 1] \leq \mathbb{E}X.$$

We will also need the exponentially small bound on a lower tail of sums of not independent random variables. Whenever the dependence is relatively weak, the bound is given by Janson's inequality.

Lemma 5.5 (Janson's Inequality, [12]). *Let F be a finite set. A subset R is drawn randomly from F such that the inclusions of individual elements from F are independent. Let $\mathcal{A}$ be a family of subsets of F. For each $A \in \mathcal{A}$ we define*

$$X_A = \begin{cases} 1 & \text{if } A \subseteq R \\ 0 & \text{otherwise} \end{cases}$$

and $X = \sum_{A \in \mathcal{A}} X_A$. For the ordered pairs (A, B), $A, B \in \mathcal{F}$, write $A \sim B$ if $A \neq B$ and $A \cap B \neq \emptyset$. Then, for every $0 < \varepsilon \leq 1$,

$$\mathbb{P}[X \leq (1-\varepsilon)\mathbb{E}X] \leq \exp\left\{-\frac{1}{2}\frac{(\varepsilon\mathbb{E}X)^2}{\mathbb{E}X + \sum_{A\sim B}\mathbb{E}[X_A X_B]}\right\}.$$

In particular, for $\varepsilon = 1$ we have

$$\mathbb{P}[X = 0] \leq \exp\left\{-\frac{1}{2}\frac{(\mathbb{E}X)^2}{\mathbb{E}X + \sum_{A\sim B}\mathbb{E}[X_A X_B]}\right\}.$$

Before we move on to our main theorems, let us prove the following lemma.

Lemma 5.6. *For P_n^k $(1 \leq k < n-2)$ let $X_i^{(p)}$'s be defined by (5.5) and, as earlier, $X^{(p)} = \sum_{i=1}^{n-k-2} X_i^{(p)}$. Let $\varepsilon \in (0,1)$, $a_{n,\varepsilon} = 1-(1+\varepsilon)n^{-1/(k+1)}$ and $b_{n,\varepsilon} = 1-(1-\varepsilon)n^{-1/(k+1)}$. Then*

$$\begin{cases} \mathbb{P}[X^{(p)} = 0] \xrightarrow{n\to\infty} 1, & \text{if } p \geq b_{n,\varepsilon} \text{ and } k(n) \xrightarrow{n\to\infty} \infty, \\ \mathbb{P}[X^{(p)} = 0] \geq C > 0, & \text{if } p \geq b_{n,\varepsilon} \text{ and } k \text{ is a constant,} \end{cases}$$

where C is some constant. When k is such that $\liminf_{n\to\infty} n^{1/(k+1)} > 1$ and ε is chosen in such a way that $a_{n,\varepsilon} > 0$

$$\begin{cases} \limsup_{n\to\infty} \mathbb{P}[X^{(p)} = 0] \leq c < 1, & \text{if } p \leq a_{n,\varepsilon} \text{ and } k \text{ is a constant,} \\ \mathbb{P}[X^{(p)} = 0] \xrightarrow{n\to\infty} 0, & \text{if } p \leq a_{n,\varepsilon} \text{ and } k(n) \xrightarrow{n\to\infty} \infty, \end{cases}$$

where c is some constant. More precisely

$$1 - n(1-p)^{k+1} \leq \mathbb{P}[X^{(p)} = 0] \leq \exp\left\{-\frac{(n-k-2)^2(1-p)^{k+1}}{2n(2k(1-p)+1)}\right\}.$$

Remark 5.7. Note that k may be a constant independent of n or a function $k = k(n)$ such that $k(n) \xrightarrow{n\to\infty} \infty$. As it is always known from the context, we simply write k.

Proof. We begin with proving the first two statements using the first moment method. We have $\mathbb{P}[X_i^{(p)} = 1] = (1-p)^{k+1}$, therefore

$$\mathbb{E}X^{(p)} = (n-k-2)(1-p)^{k+1} \leq n(1-p)^{k+1}.$$

By Markov's inequality (Lemma 5.4) $\mathbb{P}[X^{(p)} \geq 1] \leq \mathbb{E}X^{(p)}$, thus we obtain

$$\mathbb{P}[X^{(p)} = 0] = 1 - \mathbb{P}[X^{(p)} \geq 1] \geq 1 - \mathbb{E}X^{(p)} \geq 1 - n(1-p)^{k+1}.$$

Hence, for $p \geq 1-(1-\varepsilon)n^{-1/(k+1)}$

$$\mathbb{P}[X^{(p)} = 0] \geq 1 - n(1-p)^{k+1} \geq \\ 1 - n((1-\varepsilon)n^{-1/(k+1)})^{k+1} = 1-(1-\varepsilon)^{k+1}.$$

Thus, for $p \geq 1-(1-\varepsilon)n^{-1/(k+1)}$ and $k = k(n) \xrightarrow{n\to\infty} \infty$ $(1 \leq k < n-2)$, we get

$$\mathbb{P}[X^{(p)} = 0] \geq 1-(1-\varepsilon)^{k+1} \xrightarrow{n\to\infty} 1,$$

and for $p \geq 1-(1-\varepsilon)n^{-1/(k+1)}$ and k being a constant

$$\mathbb{P}[X^{(p)}=0] \geq 1-(1-\varepsilon)^{k+1} = C > 0.$$

Now we are going to justify the last two statements using Janson's inequality. Let us write $i \sim j$, $i,j \in \{1,2,\ldots,n-k-2\}$, when $i \neq j$ and $X_i^{(p)}, X_j^{(p)}$ are not independent. Obviously, $(\mathbb{E}X^{(p)})^2 = (n-k-2)^2(1-p)^{2(k+1)}$. Let us calculate then $\Delta^{(p)} = \sum_{i\sim j} \mathbb{E}[X_i^{(p)}X_j^{(p)}]$. If $i \sim j$ then the total number of vertices s on which $X_i^{(p)}$ and $X_j^{(p)}$ depend satisfies $k+2 \leq s \leq 2k+1$ and we have $\mathbb{E}[X_i^{(p)}X_j^{(p)}] = (1-p)^s$. Moreover, there are $2(n-s-1)$ pairs $i \sim j$ such that $X_i^{(p)}$ and $X_j^{(p)}$ depend on s vertices altogether. Therefore

$$\Delta^{(p)} = \sum_{s=k+2}^{2k+1} 2(n-s-1)(1-p)^s \leq 2kn(1-p)^{k+2}.$$

Thus we get

$$\frac{(\mathbb{E}X^{(p)})^2}{\mathbb{E}X^{(p)}+\Delta^{(p)}} \geq \frac{(n-k-2)^2(1-p)^{2(k+1)}}{(n-k-2)(1-p)^{k+1}+2kn(1-p)^{k+2}} \geq \frac{(n-k-2)^2(1-p)^{k+1}}{n(2k(1-p)+1)}.$$

The function $f(p) = \frac{(n-k-2)^2(1-p)^{k+1}}{n(2k(1-p)+1)}$ is decreasing in p thus for $p \leq 1-(1+\varepsilon)n^{-1/(k+1)}$ and by Lemma 5.5

$$\mathbb{P}[X^{(p)}=0] \leq \exp\left\{-\frac{1}{2}\frac{(\mathbb{E}X^{(p)})^2}{\mathbb{E}X^{(p)}+\Delta^{(p)}}\right\} \leq \exp\left\{-\frac{(n-k-2)^2(1-p)^{k+1}}{2n(2k(1-p)+1)}\right\} \leq \exp\left\{-\frac{(n-k-2)^2(1+\varepsilon)^{k+1}}{2n^2(2k(1+\varepsilon)n^{-1/(k+1)}+1)}\right\}.$$

Note that whenever $k=k(n) \xrightarrow{n\to\infty} \infty$, $\frac{(n-k-2)^2(1+\varepsilon)^{k+1}}{2n^2(2k(1+\varepsilon)n^{-1/(k+1)}+1)} \xrightarrow{n\to\infty} \infty$. Thus for $p \leq 1-(1+\varepsilon)n^{-1/(k+1)}$ and $k=k(n) \xrightarrow{n\to\infty} \infty$ we obtain

$$\lim_{n\to\infty} \mathbb{P}[X^{(p)}=0] = 0.$$

Now assume that k is a constant. We have

$$\frac{(n-k-2)^2(1+\varepsilon)^{k+1}}{2n^2(2k(1+\varepsilon)n^{-1/(k+1)}+1)} \xrightarrow{n\to\infty} \frac{(1+\varepsilon)^{k+1}}{2},$$

and thus for $p \leq 1-(1+\varepsilon)n^{-1/(k+1)}$ setting $c = \exp\left\{-\frac{(1+\varepsilon)^{k+1}}{2}\right\}$ we obtain

$$\limsup_{n\to\infty} \mathbb{P}[X^{(p)} = 0] \leq c.$$

□

Since τ_n is optimal there is no other strategy that performs better. In order to bound the effectiveness of τ_n from below we shall analyze the effectiveness of another stopping time τ_p^* that was already introduced in Chapter 4.

Strategy: Flip an asymmetric coin, having some probability p of coming down tails, n times. If it comes down tails M times reject the first M elements. After this time pick the first element which is maximal in the induced graph. In other words, τ_p^* is equal to the first $j > M$ such that $\pi_j \in Max(P_{(j)})$. If no such j is found let $\tau_p^* = n$.

The strategy τ_p^* uses randomization that was introduced by Preater in [23] to construct a universal best choice algorithm for posets. In Chapter 4 the following lemma was proved.

Lemma 5.8. *Let $\pi \in S_n$ be a random permutation of vertices in V. Suppose that we have a coin that comes down tails with probability p. Let M denote the number of occurrences of tails in n tosses. Then all vertices from V appear in $\{\pi_1, \pi_2, \ldots, \pi_M\}$ with probability p independently.*

□

Theorem 5.9. *Let P_n^k be the kth power of a directed path, $1 \leq k < n$. Let π be a random permutation of its vertices. There exists a constant $c > 0$ such that*

$$\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \geq c.$$

Proof. We have already discussed the two easy cases for $k = n-2$ or $k = n-1$. Then $\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = 1/2$ and thus we get $\lim_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = 1/2$. Throughout the rest of the proof we always assume that $1 \leq k < n-2$.

Let us consider the stopping time τ_p^* with $p = 1 - (1-\varepsilon)n^{-1/(k+1)}$ for some $\varepsilon \in (0,1)$. Let V_p^* be the set $\{\pi_1, \pi_2, \ldots, \pi_M\}$ from Lemma 5.8. Since τ_n is optimal, we have $\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \geq \mathbb{P}[\pi_{\tau_p^*} = \mathbb{1}]$. Hence we are going to show that the probability of success of τ_p^* satisfies the statement of the theorem.

Let us define the following sequence of the indicator random variables

$$X_i^{(M)} = \begin{cases} 1 & \text{if } \{v_{i+1}, v_{i+2}, \ldots, v_{i+k+1}\} \subseteq V_n \setminus V_p^*, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n-k-2$. Let also $X^{(M)} = \sum_{i=1}^{n-k-2} X_i^{(M)}$. Note that if $[X^{(M)} = 0, v_n \in V_p^*, \mathbb{1} \in V_n \setminus V_p^*]$ then $\mathbb{1}$ is the only element which comes as the maximal one in the induced graph after time M. Thus we have

$$\mathbb{P}[\pi_{\tau_p^*} = \mathbb{1}] \geq \mathbb{P}[X^{(M)} = 0, v_n \in V_p^*, \mathbb{1} \in V_n \setminus V_p^*].$$

Since $p = 1 - (1-\varepsilon)n^{-1/(k+1)}$, by Lemma 5.8 and by the lower bound from Lemma 5.6, we obtain

$$\begin{aligned}\mathbb{P}[\pi_{\tau_p^*} = \mathbb{1}] \geq & \mathbb{P}[X^{(M)} = 0]\mathbb{P}[v_1 \in V_p^*]\mathbb{P}[\mathbb{1} \in V_n \setminus V_p^*] \geq \\ & (1 - n(1-p)^{k+1})p(1-p) = \\ & (1-(1-\varepsilon)^{k+1})(1-(1-\varepsilon)n^{-1/(k+1)})(1-\varepsilon)n^{-1/(k+1)}.\end{aligned}$$

Thus for $1 \leq k < n-2$

$$\begin{aligned}n^{1/(k+1)}\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \geq & n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbb{1}] \geq \\ & (1-(1-\varepsilon)^{k+1})(1-(1-\varepsilon)n^{-1/(k+1)})(1-\varepsilon).\end{aligned}$$

We have

$$\liminf_{n\to\infty}(1-(1-\varepsilon)^{k+1})(1-(1-\varepsilon)n^{-1/(k+1)})(1-\varepsilon) = \tilde{c} > 0,$$

thus setting $c = \min\{\tilde{c}, 1/2\}$ we obtain for $1 \leq k < n$

$$\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \geq c.$$

□

Theorem 5.10. *Let P_n^k be the kth power of a directed path and let π be a random permutation of its vertices. There exists $\varepsilon > 0$ such that for k satisfying $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$, $k < n$, we have*

$$\limsup_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \leq 1 + \varepsilon.$$

Proof. First, let us assume $\liminf_{n\to\infty} n^{1/(k+1)} > 1$. Let again $X_i^{(p)}$'s be defined by (5.5) and $X^{(p)} = \sum_{i=1}^{n-k-2} X_i^{(p)}$. By Lemma 5.3, the independence of A_i's and the fact that $\mathbb{P}[A_n < p | A_1 = p] = \mathbb{P}[A_n < p] = p$ we have

$$\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \int_0^1 p\mathbb{P}[X^{(p)} = 0]\, dp.$$

We need to show that there exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} n^{1/(k+1)} \int_0^1 p\mathbb{P}[X^{(p)} = 0]\, dp \leq 1 + \varepsilon.$$

Let $a_{n,\varepsilon} = 1 - (1+\varepsilon)n^{-\frac{1}{k+1}}$ and let $\varepsilon > 0$ be such that $a_{n,\varepsilon} > 0$. We have

$$\int_0^1 p\mathbb{P}[X^{(p)} = 0]\, dp = \int_0^{a_{n,\varepsilon}} p\mathbb{P}[X^{(p)} = 0]\, dp + \int_{a_{n,\varepsilon}}^1 p\mathbb{P}[X^{(p)} = 0]\, dp.$$

We are going to estimate the following expressions

$$n^{1/(k+1)} \int_0^{a_{n,\varepsilon}} p\mathbb{P}[X^{(p)} = 0]\, dp, \tag{5.6}$$

$$n^{1/(k+1)} \int_{a_{n,\varepsilon}}^1 p\mathbb{P}[X^{(p)} = 0]\, dp. \tag{5.7}$$

Let us start with (5.7). We have

$$\begin{aligned} n^{1/(k+1)} \int_{a_{n,\varepsilon}}^1 p\mathbb{P}[X^{(p)} = 0]\, dp \leq \\ n^{1/(k+1)} \int_{a_{n,\varepsilon}}^1 1\, dp = n^{1/(k+1)}(1 - a_{n,\varepsilon}) = 1 + \varepsilon. \end{aligned} \tag{5.8}$$

Now let us estimate (5.6). By the upper bound from Lemma 5.6 and the fact that $f(p) = \frac{(n-k-2)^2(1-p)^{k+1}}{2n(2k(1-p)+1)}$ is decreasing in p we have

$$
\begin{aligned}
&n^{1/(k+1)} \int_0^{a_{n,\varepsilon}} p\mathbb{P}[X^{(p)} = 0]\, dp \leq \\
&n^{1/(k+1)} \int_0^{a_{n,\varepsilon}} \exp\left\{-\frac{(n-k-2)^2(1-p)^{k+1}}{2n(2k(1-p)+1)}\right\} dp \leq \\
&n^{1/(k+1)} \int_0^{a_{n,\varepsilon}} \exp\left\{-\frac{(n-k-2)^2(1-a_{n,\varepsilon})^{k+1}}{2n(2k(1-a_{n,\varepsilon})+1)}\right\} dp = \\
&n^{1/(k+1)} a_{n,\varepsilon} \exp\left\{-\frac{(n-k-2)^2(1-a_{n,\varepsilon})^{k+1}}{2n(2k(1-a_{n,\varepsilon})+1)}\right\} = \\
&n^{1/(k+1)}(1-(1+\varepsilon)n^{-1/(k+1)}) \cdot \\
&\exp\left\{-\frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\} = \\
&n^{1/(k+1)} \exp\left\{-\frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\} - \\
&(1+\varepsilon) \exp\left\{-\frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\}.
\end{aligned}
\tag{5.9}
$$

From the calculations in the proof of Lemma 5.6 we know that the second term converges to 0 when $k = k(n) \xrightarrow{n\to\infty} \infty$. Now we will show that the first term $n^{1/(k+1)} \exp\left\{-\frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\}$ also tends to 0 with n tending to infinity. We have

$$
\begin{aligned}
&n^{1/(k+1)} \exp\left\{-\frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\} = \\
&\exp\left\{\frac{\ln n}{k+1} - \frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\}.
\end{aligned}
$$

It is easy to notice that whenever $\frac{\ln n}{k+1}$ is bounded by some positive constant, we are done. Let us assume now that $\frac{\ln n}{k+1} \xrightarrow{n\to\infty} \infty$. We have

$$\exp\left\{\frac{\ln n}{k+1} - \frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\} =$$

$$\exp\left\{\frac{\ln n}{k+1}\left(1 - \frac{(n-k-2)^2}{2n^2}\frac{(k+1)(1+\varepsilon)^{k+1}}{(2k(1+\varepsilon)n^{-1/(k+1)}+1)\ln n}\right)\right\}.$$

Since now $\frac{(n-k-2)^2}{2n^2} \xrightarrow{n\to\infty} \frac{1}{2}$ it is enough to show

$$\liminf_{n\to\infty} \frac{(k+1)(1+\varepsilon)^{k+1}}{(2k(1+\varepsilon)n^{-1/(k+1)}+1)\ln n} > 2.$$

We have

$$\frac{(2k(1+\varepsilon)n^{-1/(k+1)}+1)\ln n}{(k+1)(1+\varepsilon)^{k+1}} = \frac{\left(2k(1+\varepsilon)+\exp\left\{\frac{\ln n}{k+1}\right\}\right)\ln n}{\exp\left\{\frac{\ln n}{k+1}\right\}(k+1)(1+\varepsilon)^{k+1}} =$$

$$\frac{2k\ln n}{(k+1)\exp\left\{\frac{\ln n}{k+1}\right\}(1+\varepsilon)^k} + \frac{\ln n}{(k+1)(1+\varepsilon)^{k+1}} \le$$

$$\frac{2\ln n}{\exp\left\{\frac{\ln n}{k+1}\right\}(1+\varepsilon)^k} + \frac{\ln n}{k(1+\varepsilon)^k} = \frac{\ln n}{(1+\varepsilon)^k}\left(\frac{2}{\exp\left\{\frac{\ln n}{k+1}\right\}} + \frac{1}{k}\right).$$

From the assumptions $\frac{\ln n}{k+1} \xrightarrow{n\to\infty} \infty$ and $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$ we obtain

$$\limsup_{n\to\infty} \frac{\ln n}{(1+\varepsilon)^k}\left(\frac{2}{\exp\left\{\frac{\ln n}{k+1}\right\}} + \frac{1}{k}\right) < 1/2.$$

Thus indeed

$$\liminf_{n\to\infty} \frac{(k+1)(1+\varepsilon)^{k+1}}{(2k(1+\varepsilon)n^{-1/(k+1)}+1)\ln n} > 2$$

which yields

$$\lim_{n\to\infty} n^{1/(k+1)} \exp\left\{-\frac{(n-k-2)^2}{2n^2}\frac{(1+\varepsilon)^{k+1}}{2k(1+\varepsilon)n^{-1/(k+1)}+1}\right\} = 0.$$

From (5.9) we obtain

$$\lim_{n\to\infty} n^{1/(k+1)} \int_0^{a_{n,\varepsilon}} p\mathbb{P}[X^{(p)} = 0]\, dp = 0. \tag{5.10}$$

From (5.8) and (5.10) for k satisfying both, $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$ and $\liminf_{n\to\infty} n^{1/(k+1)} > 1$ we obtain

$$\limsup_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \le 1 + \varepsilon.$$

Thus for k such that $\limsup_{n\to\infty} \frac{\ln n}{k(1+\varepsilon)^k} < 1/2$ and $k < n$ we obtain

$$\limsup_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}] \le 1 + \varepsilon.$$

Thus we have proved the conclusion under one of the assumptions

1. $\frac{\ln n}{k+1}$ is bounded by a constant or

2. $\frac{\ln n}{k+1} \xrightarrow{n\to\infty} \infty$.

Assume now that in general the conclusion does not hold, i.e.,

$$\limsup_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \infty.$$

Passing to a subsequence we can write

$$\lim_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = \infty.$$

Thus we know that $\frac{\ln n}{k+1}$ is unbounded. Again passing to a subsequence we can get $\frac{\ln n}{k+1} \xrightarrow{n\to\infty} \infty$. But this contradicts the second case. Thus we have proved the conclusion under the assumption $\liminf_{n\to\infty} n^{1/(k+1)} > 1$.

Note that if $\lim_{n\to\infty} n^{1/(k+1)} = 1$ the conclusion holds trivially. Using again the argument of passing to subsequences we obtain the conclusion in the general situation. □

Note that although we do not formulate the optimal algorithm for the case when the selector does not know the values of d_P in the induced graph, we are able to tell the order of the probability of its success.

Corollary 5.11. *For P_n^k being the kth power of a directed path let $\tilde{\tau}_n$ be the optimal stopping time for choosing the root when the selector does not know the values d_P of each edge that appears in the induced graph. Then*

$$\begin{aligned} &\mathbb{P}[\pi_{\tilde{\tau}_n} = \mathbb{1}] = \Omega(n^{-1/(k+1)}) \quad \textit{for } 1 \le k < n, \\ &\mathbb{P}[\pi_{\tilde{\tau}_n} = \mathbb{1}] = O(n^{-1/(k+1)}) \quad \textit{when } k < n \textit{ and } \limsup_{n\to\infty} \tfrac{\ln n}{k(1+\varepsilon)^k} < 1/2. \end{aligned}$$

Proof. We have $\mathbb{P}[\pi_{\tilde{\tau}_n} = \mathbb{1}] \leq \mathbb{P}[\pi_{\tau_n} = \mathbb{1}]$ because when the values of d_P are known in the induced graph one can take at least as efficient decision as when they are not known. On the other hand note that our lower estimation of $\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_n} = \mathbb{1}]$ in Theorem 5.9 does not use the information about the values d_P at all. Thus the estimation is also true for $\tilde{\tau}_n$. □

Remark 5.12. For now we do not know if for $\mathbb{P}[\pi_{\tilde{\tau}_n} = \mathbb{1}] = O(n^{-1/(k+1)})$ the additional assumption about k may be dropped.

5.6 Comments and remarks

Note that for $k = 1$ our problem reduces to the directed path case from [17]. On the other hand when $k = n - 1$ we deal with a graph which gives the linear order of n elements. Recall that then, for π being the random permutation of vertices of P_n^{n-1}, $\mathbb{P}[\pi_{\tau_n} = \mathbb{1}] = 1/2$ because $\pi_{\tau_n} = \mathbb{1}$ if and only if v_n precedes $\mathbb{1}$ in π. Note that for $k = n-1$ when we do not assume that the selector knows the values of $d_P(e)$ for each edge e that appears in the induced graph, we, in fact, talk about the classical linear order secretary problem. From [19] we know that the probability of success of the optimal algorithm for the linear order is asymptotically $1/e$. It is quite surprising that revealing this additional information about distances increases the probability of success of the optimal algorithm only to $1/2$.

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