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Analityczne i numeryczne podejście do modelowania ryzyka operacyjnego firmy

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Contents

Introduction.

For the last several years, the nature of operational risk has become a field of intensive studies due to its growing importance as against market and credit risk. Appropriate defining and quantifying operational risk is a hard task, thus suitable regulatory is still in development. Although The New Basel Accord gives the methodology for managing banking operational risk, the corporate risk seems not to be recognized enough. Here in the thesis we make an attempt to put some insight into operational risk measurement in the non-financial corporation. The objective is to apply suitable results from insurance ruin theory to build a framework for measuring corporate operational risk and finding required capital charge.

In the second Chapter, a brief description of banking operational risk regulatory and methodological proposals for risk measurement is presented. The detailed discussion on the topic can be found in consultative documents by Basel Committee (i.e. [**2**], [**3**], [**4**]) and review papers by Pezier ([**36**], [**37**]). Refined analysis with exploiting Extreme Value Theory techniques are presented in papers by Embrechts with others 2004–2006 ([**17**], [**18**], [**11**], [**32**]). We focus on the differences between a bank and commodity-branch corporation in the field of operational risk management. We introduce also a motivation to apply ruin theory methods to operational risk measurement.

The next Chapter is devoted to some aspects of commodity market risk measurement with emphasizing such differences from banking methodology as time horizon for decisions, managers activity and different risk measures. The correlation-based, analytical approach to Revenues at Risk (RaR) and Earnings at Risk (EaR) measures calculation is proposed instead of Monte Carlo simulation methods.

Chapters 3, 4 and 5 consist of wide studies of ruin probability estimates with the distinction for finite and infinite time horizon methods. Basic aspects of insurance risk theory are presented and several ruin probability approximations are described. They are numerically compared with each other and illustrated as a functions of capital and (in the finite horizon case) time horizon. Moreover, we propose two promising, new approximations. Chapters 3–5 are based on earlier papers by Burnecki, Miśta, Weron ([**9**], [**7**]) and Chapter 15 in [**12**].

Next, in the Chapter 6 we consider more complicated model of risk process, allowing for diffusion component. Numerically tractable formula for ruin probability is found in the case of losses distribution with Laplace transform being rational function. Moreover, an analytic result is given when the claim size is described by mixture of exponentials distribution, being of high importance when modelling operational risk (cf. Chapter 1). The general ideas and parts of the Chapter are heavily borrowed from papers by Jacobsen ([**25**], [**26**], [**27**]).

Finally, in the last Chapter, based on Otto & Miśta [**30**], we deal with setting appropriate level of capital charge for operational risk with possible risk transfer through insurance. By inverting various approximations of ruin probability we arrive at suitable capital charge with predetermined level of such probability. In the case of operational risk modelling in non-financial corporation , this approach seems to be a proper alternative to high confidence level Value at Risk (VaR) measure.

Banking operational risk established, corporate risk measurement as a new challenge.

1.1. The New Accord (Basel II)

The short brief of the history of risk management regulatory takes us back to the 1988, when the Basel Accord (Basel I) was established. The document was concerning the minimal capital requirements against credit risk with one standardized approach, namely Cooke ratio. Next, in 1996, in amendment to Basel I, the necessary regulatory for managing market risk appeared, i.e. internal models, netting and finally Value at Risk measure. Consequently, for the next years we came to the term *operational risk*. In 1999 Basel Committee on Banking Supervision published the first consultative paper on the New Accord (Basel II), introducing definition of operational risk and submitting some proposals of measuring methods and suitable regulatory. Until now several consultative papers on the New Basel Capital Accord appeared but the full implementation of Basel II is not expected before 2007.

First of all, the New Accord brings more flexibility and risk sensitivity in the structure of three-pillar framework: minimal capital requirements (Pillar 1), supervisory review of capital adequacy (Pillar 2) and public disclosure (Pillar 3). Pillar 1 sets out the minimum capital requirements (Cook Ratio, McDonough Ratio) to 8%:

$$(1.1) \qquad \frac{\text{total amount of capital}}{\text{risk-weighted assets}} \geq 8\%,$$

resulting in the definition of *minimum regulatory capital* (MRC):

$$(1.2) \qquad MRC \stackrel{\text{def}}{=} 8\% \text{ of risk-weighted assets.}$$

Due to New Accord, the most accurately describing definition of operational risk can be formulated in the following way.

DEFINITION 1.1.1. *Operational risk is the risk of (direct or indirect) losses resulting from inadequate or failed internal processes and procedures, people and systems, or external events.*

It should be remarked that this definition includes legal risk, but excludes strategic and reputational risk.

Let us denote by C_{op} the capital charge for operational risk. It was initially feared that the Basel II proposals would reduce the capital requirements for credit and market risks by 20% on average. However, still growing risks such as fraud, terrorism, technology failures and trade settlements errors, may leave the banking industry more exposed to operational risk than ever before. This led the Basel Committee to propose a new tranche of capital charges for operational risk equal to 20% of purely credit and market risk

minimum capital requirements. After further studies on how much of the economic capital the banking industry allocates to operational risk, and in response to other industry concerns, the Basel Committee proposed to reduce the operational risk minimum regulatory capital figure from 20% down to 12% of MRC. Finally we should notice that it is not uncommon to find that $C_{op} > C_{mr}$ (C_{mr} being market risk capital charge).

1.2. Risk measurement methods for operational risk

In the New Accord documents, the Committee proposes three different approaches to operational risk measurement:

- Basic Indicator Approach (BIA),
- Standardized Approach (SA),
- Advanced Measurement Approach (AMA).

The first and the most crude, but simple approach relies just on taking capital charge on operational risk (say C_{op}^{bia}) as some percentage of average annual gross income:

$$(1.3) \qquad C_{op}^{bia} = \alpha GI,$$

where Gross Income (GI) means average annual gross income over the previous three years and $\alpha = 15\%$ is an indicator set by the Committee based on Collective Investment Schemes (CIS).

The Standardized Approach, similar to BIA and also simple, takes into account indicators on the level of each business line:

$$(1.4) \qquad C_{op}^{sa} = \sum_{i=1}^{8} \beta_i GI_i,$$

with $\beta_i \in [12\%, 18\%]$, 3-year averaging and 8 business lines specified by Committee:

TABLE 1.1. Business lines and corresponding indicator's levels.

Corporate finance (18%)
Payment & Settlement (18%)
Trading & sales (18%)
Agency Services (15%)
Retail banking (12%)
Asset management (12%)
Commercial banking(15%)
Retail brokerage (12%)

From our point of view, the most interesting and refined is the latter approach, considering more sophisticated methods of risk measurement - Advanced Measurement Approach (AMA). On the one hand, there has been a lot of critique put on the naive, linear approaches (BIA and SA), but on the other, the AMA approach is with no doubt more demanding (for detailed critique see i.e. Pezier [36],[37]). Involving

advanced methods we encounter the serious problem of estimating very rare events (having probability less than 0.1%).

Generally, with the Advanced Measurement Approach, the Basel Committee allows banks to use their internally generated risk estimates after meeting qualitative and quantitative standards. Also the risk mitigation via insurance is possible (20% of C_{op}) and incorporation of risk diversification benefits is allowed. Although AMA leads directly to *Loss Distribution Approach* (LDA), the Committee "is not specifying the approach or distributional assumptions used to generate the operational risk measures for regulatory capital purposes." Except LDA, the Committee also considers the Scorecard Approaches and Internal Measurement Approaches (similar to Basel II model for Credit Risk):

$$(1.5) \qquad C_{op}^{ima} = \sum_{i=1}^{8} \sum_{k=1}^{7} \gamma_{ik} ES^{ik},$$

with ES^{ik} being expected aggregated loss for business line i and risk type k, γ_{ik} scaling factors and the specified 7 loss types:

TABLE 1.2. Classified 7 risk/loss types.

Internal fraud
External fraud
Employment practices and workplace safety
Clients, products & business practices
Damage to physical assets
Business disruption and system failures
Execution, delivery & process management

We omit the detailed discussion on all approaches presented above as it is not a part of further analysis here, but we focus on the Loss Distribution Approach as the most refined and challenging method. We refer interested Reader to regulatory and consultative papers by Basel Committee ([2], [3], [4]), moreover the constructive review of Basel proposals can be found in papers by Pezier ([36], [37]).

1.3. Loss Distribution Approach

The Loss Distribution Approach tends to identify the one year loss distribution in each business line / loss type cell (i, k) and then to find the appropriate risk measure on the basis of total one year loss distribution including all cells. This leads to modelling each cell variable as a compound variable:

$$(1.6) \qquad S_t^{i,k} = \sum_{l=1}^{N_t^{i,k}} X_l^{i,k},$$

with $N_t^{i,k}$ being some counting process (measuring frequency of losses) in time t for cell (i, k) and $X_l^{i,k}$ corresponding variables describing severity of losses in cell (i, k). We should notice here for the first time the similarity to the actuarial theory and collective risk models in insurance. The severity variables X can

also be modelled with the most popular (nonnegative) distributions in insurance (exponential, gamma, lognormal, Pareto) and N_t can be counting process like Poisson, binomial, negative binomial or one of more complex point processes (see i.e. Burnecki & Weron Chapter 14 in [12]).

All in all, the total one year loss in year t is then given by

$$(1.7) \qquad S_t = \sum_{i=1}^{8} \sum_{k=1}^{7} S_t^{i,k} = \sum_{i=1}^{8} \sum_{k=1}^{7} \sum_{l=1}^{N_t^{i,k}} X_l^{i,k}.$$

Now what has to be done is to choose and calibrate the distribution of $S_t^{i,k}$ for each cell, finding possible correlations between cells and specifying risk measure g_α at confidence level α close to 1. The total operational loss capital charge should be found on the basis of $g_\alpha(S_t^{i,k})$ calculated for each cell.

Going further into details of Basel II proposals, we find the risk measure to be popular Value at Risk at a very high confidence level $\alpha = 99.9\%$ or even higher ($99.95\% - 99.97\%$). The distribution chosen should be based on internal data and models, external data and expert opinions and period taken into consideration equal to one year. Finally, the total capital charge C_{op} should be found as a sum of $VaRs$ with possible reduction due to correlation effects:

$$(1.8) \qquad C_{op} = \sum_{i,k} VaR_\alpha(L^{i,k}).$$

Loss Distribution Approach in such a form, although being refined enough, still encounters very serious difficulties in implementation. For the first, such high confidence level causes distribution estimation very difficult if possible, due to obvious lack of data. In solving these problems, Extreme Value Theory (EVT) enters and such methods as Peaks Over Threshold (POT). The POT method focuses on that realizations of variables X, that exceed some threshold u: $Y = \max(X - u, 0)$. Distribution of Y, called *conditional excess distribution*, is formulated in the following way:

$$(1.9) \qquad F_u(y) = P(X - u \le y | X > u), \qquad 0 \le y \le x_F - u,$$

where $x_F \le \infty$ is the right endpoint of the original distribution F of variable X. To handle the difficulties of estimation of F_u (due to lack of data in that region), the EVT comes with strong, limiting result given in the theorem of Picands (1975), Belkam and de Haan (1974).

THEOREM 1.3.1. *For every $\xi \in \mathbb{R}$ distribution F belongs to Maximal Domain of Attraction of Generalized Extreme Value (GEV) distribution if and only if*

$$(1.10) \qquad \lim_{u \nearrow x_F} \sup_{0 < x < x_F - u} |F_u(x) - G_{\xi,\sigma}(x)| = 0,$$

with $G_{\xi,\sigma}$ being Generalized Pareto Distribution (GPD) given by df

$$G_{\xi,\sigma}(y) = \begin{cases} 1 - \left(1 + \frac{\xi}{\sigma} y\right)^{-1/\xi}, & \xi \ne 0 \\ 1 - \exp(-y/\sigma), & \xi = 0, \end{cases}$$

with $y \in [0, x_F - u]$ for $\xi \ge 0$, $y \in \left[0, -\frac{\sigma}{\xi}\right]$ for $\xi < 0$ and some positive σ, depending on the value of threshold level u.

The above result gives us the analytical form of conditional excess distribution and makes the modelling and estimation of total loss distribution much easier and clearer in a variety of possible single-loss distributions. The GEV family of distributions contains three standard classes, namely Fréchet, Gumbel and Weibull, and thus it includes almost all popular insurance and financial distributions like Pareto, Loggamma, Exponential-like, Weibull-like, Beta, Gamma, Normal and Lognormal. The theorem 1.3.1 in more extended form likewise the proof of it can be found in Embrechts, Klüppelberg, Mikosch [**16**]. For more details on the EVT and GEV, GPD distributions with application to operational risk measurement we refer Reader to papers by Embrechts, Kaufmann, Samorodnitsky (2004) ([**17**], [**18**]) and Chavez-Demoulin, Embrechts, Neslehova (2006) ([**17**], [**18**]).

However, some properties of data like non-stationarity, dependence and inhomogeneity still remain the serious problem and make the use of multivariate extreme value theory and copulas necessary. Finally, choosing VaR as a risk measure may lead to wrong conclusions because of lack of subadditivity in the presence of dependent variables, whereas other risk measures require finite mean. All observed difficulties cause the need for more refined analysis to be done and more complicated models to be applied. We refer reader to [**17**] and [**18**] for more details on applying EVT and POT methodology, and lately the infinite mean models, in measuring banking operational risk.

1.4. Motivation to model operational risk with ruin theory

Returning to main objective of the thesis – the operational risk modelling in non-financial corporation, we have to emphasize, that Basel II proposals refer predominantly to operational risk in banking industry. The whole classification with business lines and risk types and alike suggestion to consider one-year risk, relates to banking specification and does not capture individual features of non-financial corporations. In corporation and especially in commodity-branch corporation, the horizon of planning and the horizon of making decisions is much longer than one year. Due to natural, several year cycles in commodity markets and often the expensive, long-term investments, also the risk management decisions should encompass at least a few-year horizon. Thus the operational risk policy should be adapted to such a situation, likewise the market risk management and hedging in corporations differs from the banking VaR-based, short term risk management.

As we said before, the Loss Distribution Approach to operational risk modelling seems to be dual to actuarial models in insurance, widely exploiting the sums of random variables, what results in studying compound distributions. The most popular in insurance is the classical risk model with the variety of extensions and generalizations. The definition and many properties of it will be widely described in following chapters, now let us just focus on the similarities to LDA. The classical insurance risk model for the reserve of company is given by equation

$$(1.11) \qquad\qquad R_t = u + ct - \sum_{i=1}^{N_t} X_i,$$

where u is assumed to be an initial capital, c – premium paid to insurer in time unit and X_i – losses that happen in random moments modelled by jumps of counting process N_t. In LDA we focus only on

the total loss S_t, modelled by random sum of the same type and finding the appropriate quantile to apply the VaR method. We have to remark here, that according to theorem of combining compound Poisson risks (see Panjer & Willmot [**34**]), the sum of the form (1.7) can be reformulated again as a $\sum_{i=1}^{\tilde{N}_t} \tilde{X}_i$ with other Poisson counting process $\tilde{N}_t$ and the distribution of $\tilde{X}_i$ being a mixture of original distributions.

The aim of LDA is to find on the basis of the model, the capital charge C_{op} for the next year. Our proposal is to exploit the insurance risk model in a longer time horizon (at least several years) and use the probability of ruin as a risk measure instead of VaR. In such a representation, c would model every year capital charge C_{op} (we will call it operational reserves in next chapters) and u would be arbitrary amount of capital that should never be exceeded under threat of bankruptcy (ruin). It could be for instance the economic capital reduced by C_{op} or yearly net profit assumed in budget for next few years. Quite well developed actuarial methods of ruin probability estimation and finding the suitable level of premium results in promising proposals of finding the appropriate level of capital charge on operational risk in non-financial corporation. Furthermore, the variety of generalizations of insurance risk model allow to make the modelling closer to real life. These could be for instance taking $c(t)$ instead of ct, randomizing ct or making N_t much more complex. There are also models allowing to add the diffusion component.

The proposed approach, although having the same basis as LDA with VaR, is substantially different in the way the modelling with random variable is different from modelling with stochastic process. Instead of finding a quantile of estimated variable, the problem extends to finding first the probability of ruin (first passage of the stochastic process problem) and then to invert it in order to find the required capital charge.

CHAPTER 2

Market risk management in corporation. Hedging as a key tool.

2.1. Setting up the problem

On the contrary to new regulations and freshly developing methodology in the field of operational risk measurement, the market risk seems to be quite well recognized. There is a variety of publications considering market risk measurement, modelling risk factors and methods of reducing the danger of market risk. For last several years, hedging and financial engineering has become even a separate part of science and papers on new interesting financial instruments are still published at very high intensity. The field of market risk measurement is so wide that we do not intent here to make any brief or summary but just to show some aspects, that could be interesting for risk manager in his practice. First, we would like to emphasize again the difference between banking market risk measurement and non-financial, corporate methodology. In the previous chapter we noticed the difference in the time horizon to be considered, the other issue is the market activity.

The exposure of bank to market risk relates always to a set of portfolios. So every asset in each portfolio will be a risk factor for a bank. The portfolio theory comes then with methodologies of managing such risks, and many types of financial instruments and derivatives become necessary. The important issue is that portfolio manager can always decide to increase or reduce his exposure to risk by taking any (long or short) position on the market. Taking now the producer of metal, oil, gas or any commodity as an example of non-financial corporation, the situation looks different. The firm is also exposed to high risks resulting from changing prices of commodities and currencies, however it has to be viewed as already having natural long position. The scope of risk manager in such a firm is to hedge some part of commodity that is to be sold, to minimize the volatility of cash flows and potential future earnings, likewise to possibly ensure reasonable, minimal price of production sale covering company's costs. That obliges the manager to take only short positions on the market, otherwise his activity would be a speculation, not hedging. The corporation being consumer not producer of commodities exposed to market risk has similar but opposite situation. All of these differences from banking industry cause not only the activity of risk managers to be specified in other way, but also the applied risk measures to be different. When the most popular risk measure exploited in banks is VaR (Value at Risk - value of portfolio exposed to loss), the measures used in non-financial corporations are RaR (Revenues at Risk), EaR (Earnings at Risk) and $CFaR$ (Cash Flow at Risk). When VaR in bank should be calculated once a day, week or month with such a short time horizon, RaR, EaR and $CFaR$ in corporation should be prepared for quarter, year, three-year horizon or even longer.

As it was stated before, we focus on non-financial corporation acting in the field of commodities as an example. The first thing to be done to work out the system of risk measurement, is to set the main risk factors and establish the map of exposure to classify which field of activity is exposed to risk that corporation is willing to measure and manage. Then, the most adequate stochastic model should be chosen to each risk factor with determining all possible correlations. Next, according to the most popular methodology there is a need for building the stochastic system that calibrates the models and applies Monte Carlo simulations of each risk factor, to finally compute all the risk measures on that basis. This standard approach seems to be very accurate and correct from methodological point of view, however in practice risk managers encounter computational problems. As simulations have to be done in quite a long time horizon, in order to obtain reasonable effects, it does require a great number of Monte Carlo simulations of possibly several, dependent stochastic processes describing risk factors. Next, a lot of calculations has to be done to obtain the revenues and earnings at each time point in the future and at the desired level of probability α. Especially if the portfolio of hedging positions is complicated and consists of several thousands of instruments.

Thus, making some necessary assumptions and simplifications, we would like to propose an analytical approach to calculate RaR or EaR measures referring to simple example, that can be however extended to more general situations. The approach exploits some aspects of portfolio theory with correlation-based calculations of total risk exposure (see RiskMetrics methodology), that can be illustrated by following simple formula for VaR of the sum of two dependent portfolios X_1 and X_2:

$$(2.1) \qquad VaR(X_1 + X_2) = \sqrt{VaR(X_1)^2 + VaR(X_2)^2 + 2VaR(X_1)VaR(X_2)\rho_{X_1,X_2}}.$$

Let us consider the producer of some commodity X_t, that is denominated and quoted in foreign currency. Its plans of production are established for next several years. Assume only two main sources of risk: the commodity price X_t and foreign currency rate of exchange Y_t. In fact, other market risks can be often neglected due to its little importance comparing to X_t and Y_t. The corporation has hedged its production and possesses portfolio of commodity "sell forward" contracts on X_t and portfolio of currency "sell forwards". All contracts are monthly settled, as that is exactly the basis of production sell. In fact, hedging portfolio can contain much more complicated derivative instruments (including options), if only there is a possibility of obtaining linear form of portfolio payout around its desired level of quantile in each settlement month in the future.

Once the models of risk factors are chosen and calibration is done properly, the problem of analytical determining of RaR and EaR measures leads to finding the distribution (in fact the quantiles and mean) of revenues from sale of commodity in time t, reduced by settlement results from hedging portfolios in time t, all valued in domestic currency. To find the distribution of earnings we need only to include costs, that we assume do not depend on risk factors.

For convenience purpose, let USD be the foreign currency with PLN being domestic one. Then X_t is USD price of commodity and $X_t Y_t$ is its PLN value. We propose the following steps in reaching the final target.

RaR algorithm

For every t starting from 0 to the end of time horizon for our analysis:

(1) First, compute the desired statistics (mean and quantiles of level α) for revenues from sell of commodity X_t valued in PLN.

(2) Second, compute the statistics for PLN value of settlement result from commodity hedging portfolio.

(3) Then, obtain the summary statistics in PLN (revenues from X_t + hedging of X_t + premiums paid and received for commodity options) as a sum of results from point (1) and (2), corrected by including obvious negative (close to -1) correlation between revenues X_t valued in PLN and hedging X_t valued in PLN.

(4) Next, compute the statistics for settlement resulting from currency hedging portfolio.

(5) Now calculate the main outcome – total statistics for revenues from X_t reduced by result of currency and commodity hedging portfolios. It will be found by summing values obtained in points (3) and (4) with necessary correction by correlation between (3) (revenues from X_t + hedging of X_t) and (4) (hedging of Y_t).

(6) Finally, as a supplement we can compute the statistics for the whole hedging portfolio as a sum of values calculated in points (2) and (4), corrected by correlation between commodity portfolio and currency portfolio.

Proceeding this way allows to skip the calculation of redundant correlations, searching only for necessary ones. Moreover, beyond the final result, we receive the future distribution of separate portfolios and revenues, what can be very useful in examining the structure of risk exposure.

2.2. Modelling risk factors

2.2.1. Schwartz commodity model. According to the best practices in commodity market, we decide to model the commodity price by geometric Ornstein-Uhlenbeck process (Schwartz mean-reverting commodity model):

$$(2.2) \qquad dX_t = \eta \left(\frac{\sigma^2}{2\eta} + \log(k) - \log(X_t) \right) X_t dt + \sigma X_t dW_t,$$

with k being mean reversion level, η - speed of mean reversion, σ -volatility and W_t standard Wiener process.

Thus the price of commodity has the feature of returning to some long-term mean level and the variance stabilizing with time flow. The model seems to be widely exploited in the commodity market, moreover its expected value in time often indicates similarity to commodity forward curve.

Straightforward from the model we find the distribution of X_t to be lognormal

$$\log(X_t) \overset{\mathrm{d}}{=} N(\mu_{t,X}, \sigma^2_{t,X})$$

with parameters

$$\mu_{t,X} = \log(X_{t_0}) \exp(-\eta(t - t_0)) + \log(k)(1 - \exp(-\eta(t - t_0))), \qquad \sigma^2_{t,X} = \frac{\sigma^2 \exp(-\eta(t - t_0))}{2\eta}.$$

Calculating the mean, variance and α-quantile of X_t comes directly as

$$
\begin{aligned}
\mathbb{E}(X_t) &= \exp\left(\mu_{t,X} + \frac{\sigma_{t,X}^2}{2}\right), \\
\mathbb{V}ar(X_t) &= \exp(2\mu_{t,X} + \sigma_{t,X}^2)(\exp(\sigma_{t,X}^2) - 1), \\
\mathbb{Q}_\alpha(X_t) &= \exp\left(\mu_{t,X} + \sigma_{t,X}\Phi^{-1}(\alpha)\right),
\end{aligned}
$$

(2.3)

with $\Phi^{-1}(\alpha)$ denoting the inverse of standard normal distribution function.

2.2.2. Calibration of Schwartz model. To calibrate the Schwartz model, we need to transform Geometric Ornstein-Uhlenbeck process to Arithmetic one by working with logarithms of price X_t (say $\tilde{X}_t = \log(X_t)$). Then the equation takes a form:

$$
d\tilde{X}_t = \eta\left(\log(k) - \tilde{X}_t\right)dt + \sigma dW_t. \tag{2.4}
$$

There are two basic methods for calibrating the Arithmetic Ornstein-Uhlenbeck process parameters: the method of moments and maximum likelihood method. The first is quite crude, however it gives reasonable outcomes. The second, more refined, requires three-dimensional optimization of likelihood function, possibly driving to serious numerical problems, but it seems to effect with more reliable results. Moreover, according to paper by José Carlos Garcia Franco ([10]), we can reduce the dimension of optimization needed. Given $n + 1$ observations $\mathbf{x} = (x_{t_0}, \ldots, x_{t_n})$ of logarithms of prices in time points t_i, the log-likelihood function corresponding to conditional density of x_{t_i} (with constant terms omitted) can be found as:

(2.5)
$$
\mathcal{L}(\mathbf{x}) = -n\log\left(\frac{\sigma^2}{2\eta}\right) - \sum_{i=1}^n \log\left(1 - e^{-2\eta(t_i - t_{i-1})}\right) - \frac{2\eta}{\sigma^2}\sum_{i=1}^n \frac{(x_{t_i} - \log(k) - (x_{t_{i-1}} - \log(k))e^{-\eta(t_i - t_{i-1})})^2}{1 - e^{-2\eta(t_i - t_{i-1})}}.
$$

In his paper, exploiting some elementary analytical manipulations of the first order conditions of MLE, author finds convenient relations between MLE estimators $(\hat{\log(k)}, \hat{\eta}$ and $\hat{\sigma})$:

(2.6)
$$
\hat{\log(k)}(\hat{\eta}) = \sum_{i=1}^n \frac{x_{t_i} - x_{t_{i-1}}e^{-\hat{\eta}(t_i - t_{i-1})}}{1 + e^{-\hat{\eta}(t_i - t_{i-1})}}\left(\sum_{i=1}^n \frac{1 - e^{-\hat{\eta}(t_i - t_{i-1})}}{1 + e^{-\hat{\eta}(t_i - t_{i-1})}}\right)^{-1},
$$

(2.7)
$$
\hat{\sigma}(\hat{\eta}, \hat{\log(k)}) = \sqrt{\frac{2\hat{\eta}}{n}\sum_{i=1}^n \frac{(x_{t_i} - \hat{\log(k)} - (x_{t_{i-1}} - \hat{\log(k)})e^{-\hat{\eta}(t_i - t_{i-1})})^2}{1 - e^{-2\hat{\eta}(t_i - t_{i-1})}}}.
$$

Now, substituting $\hat{\log(k)}(\hat{\eta})$ and $\hat{\sigma}(\hat{\eta}, \hat{\log(k)})$ directly into the likelihood function and maximizing with respect to $\hat{\eta}$ yields the desired, numerically tractable one-dimensional optimization problem.

Finally, it is worth to notice, that correction of the historical data of commodity price by adjusting it with the PPI inflation, yields more reliable estimates from economic point of view and definitely helps to avoid numerical exceptional difficulties in the case of unexpected, large price movements.

2.2.3. Geometric Brownian Motion for currency price. For modelling currency price we simply put the Geometric Brownian Motion, commonly identified with Black-Scholes model for option price valuation.

$$(2.8) \qquad dY_t = \mu Y_t dt + \sigma Y_t dW_t,$$

with drift parameter μ and volatility σ.

Analogously, distribution of Y_t is lognormal

$$\log Y_t \overset{\mathrm{d}}{=} N(\mu_{t,Y}, \sigma_{t,Y}^2).$$

with parameters

$$\mu_{t,Y} = \log(Y_{t_0}) + (\mu - \frac{1}{2}\sigma^2)(t - t_0), \qquad \sigma_{t,Y}^2 = \sigma^2(t - t_0).$$

We leave calibration of currency model to subjective decision of any interested reader, the methods are quite standard and require calculating historical trend and standard deviation of logarithmic returns from currency price data. To obtain volatility closer to current market, one may apply exponentially weighted standard deviation. Drift can be also fitted to current market currency forward curve.

Formulas for mean, variance and α-quantile of Y_t are same as in (2.3).

2.3. How to calculate the *RaR* measure

based on analytical correlations approach Having chosen the risk factor models and estimated its parameters, we can formulate the main theorem of this chapter, introducing the analytical correlations approach to *RaR* measure calculation.

THEOREM 2.3.1. *The RaR algorithm proposed in section (2.1) can be realized by an analytical correlations approach exploiting formula (2.1).*

PROOF. Proceeding with the first step of *RaR* calculations we search for statistics of revenues from sale valued in PLN. Denote by β_t the amount of commodity planned for sale in moment t. Thus, the revenue from sale in time t is given by $\beta_t X_t Y_t$. Define the value of commodity in PLN by $U_t = X_t Y_t$. Assuming independence X_t of Y_t, the expected value of revenues in time t results in

$$(2.9) \qquad \beta_t \mathbb{E}(U_t) = \beta_t \mathbb{E}(X_t)\mathbb{E}(Y_t) = \beta_t \exp\left(\mu_{t,X} + \mu_{t,Y} + \frac{\sigma_{t,X}^2 + \sigma_{t,Y}^2}{2}\right).$$

For further calculations purpose it is important to observe, that U_t has again lognormal distribution of the form

$$(2.10) \qquad \log U_t \overset{\mathrm{d}}{=} N(\mu_{t,U}, \sigma_{t,U}^2) = N(\mu_{t,X} + \mu_{t,Y}, \sigma_{t,X}^2 + \sigma_{t,Y}^2).$$

Observe by (2.10) that the desired quantile of level α for the revenues reads $\beta_t \mathbb{Q}_\alpha(U_t)$ and

$$(2.11) \qquad \beta_t \mathbb{Q}_\alpha(U_t) = \beta_t \exp\left(\mu_{t,X} + \mu_{t,Y} + \sqrt{\sigma_{t,X}^2 + \sigma_{t,Y}^2}\,\Phi^{-1}(\alpha)\right).$$

In this way, the first step of *RaR* algorithm is finished.

Next, statistics for the hedging portfolio need to be found. For this purpose, define the commodity hedging portfolio settlement result in moment t as a variable

$$(2.12) \qquad H_t^X \overset{\text{def}}{=} \gamma_t Y_t (K_t - X_t),$$

with γ_t denoting suitable quantity and K_t – the average level of price hedged (strike).

Finding quantile $\mathbb{Q}_\alpha(H_t^X)$ appears not to be so straightforward, since H_t^X can be viewed as a difference between two dependent, lognormal variables $\gamma_t K_t Y_t$ and $\gamma_t X_t Y_t$, and as such, it does not have a closed form analytical distribution. However we can make an attempt to find it via numerical integration. First, omitting constant factor γ_t, we realize the joint distribution of the pair $(U_t, V_t) \overset{\text{def}}{=} (X_t Y_t, K_t Y_t)$ to be 2-dimensional lognormal distribution:

$$(2.13) \qquad (\log(U_t), \log(V_t)) \overset{\text{d}}{=} N(\mu_{t,U}, \mu_{t,V}, \sigma_{t,U}^2, \sigma_{t,V}^2, \rho_{t,UV}),$$

with $\rho_{t,UV}$ meaning correlation coefficient between normal variables $(\log(V_t), \log(U_t))$, $(\mu_{t,U}, \sigma_{t,U}^2)$ given by (2.10), and $(\mu_{t,V} = \mu_{t,Y} + \log(K_t), \sigma_{t,V}^2 = \sigma_{t,Y}^2)$. Then, corresponding 2-dimensional lognormal density reads

(2.14)

$g(u, v) =$

$$\frac{1}{2\pi\sigma_{t,U}\sigma_{t,V} uv \sqrt{1-\rho_{t,UV}^2}} \exp\left[\frac{-1}{2(1-\rho_{t,UV}^2)}\left(\left(\frac{\log u - \mu_{t,U}}{\sigma_{t,U}}\right)^2 + \left(\frac{\log v - \mu_{t,V}}{\sigma_{t,V}}\right)^2 - \frac{2\rho_{t,UV}(\log u - \mu_{t,U})(\log v - \mu_{t,V})}{\sigma_{t,U}\sigma_{t,V}}\right)\right].$$

Therefore, putting $q = \frac{\mathbb{Q}_\alpha(H_t^X)}{\gamma_t}$ and omitting t-indexation for convenience purpose, yields

(2.15)

$$P\left(H_t^X < \mathbb{Q}_\alpha(H_t^X)\right) = P\left(V_t - U_t < \frac{\mathbb{Q}_\alpha(H_t^X)}{\gamma_t}\right) = P(V_t < U_t + q) = \int_{\max\{0,-q\}}^{\infty} \int_0^{q+u} g(u,v) dv du$$

$$= \int_{\max\{0,-q\}}^{\infty} \int_0^{q+u} \frac{1}{2\pi\sigma_U\sigma_V uv \sqrt{1-\rho_{UV}^2}} \exp\left[\frac{-1}{2(1-\rho_{UV}^2)}\left((\frac{\log u - \mu_U}{\sigma_U})^2 + (\frac{\log v - \mu_V}{\sigma_V})^2 - \frac{2\rho_{UV}(\log u - \mu_U)(\log v - \mu_V)}{\sigma_U\sigma_V}\right)\right] dv du$$

$$= \int_{\max\{0,-q\}}^{\infty} \int_{-\infty}^{\frac{\log(q+u)-\mu_V}{\sigma_V}} \frac{1}{2\pi\sigma_U u \sqrt{1-\rho_{UV}^2}} \exp\left[\frac{-1}{2(1-\rho_{UV}^2)}\left((\frac{\log u - \mu_U}{\sigma_U})^2 + z^2 - \frac{2\rho_{UV}(\log u - \mu_U)z}{\sigma_U}\right)\right] dz du$$

$$= \int_{\frac{\log(\max\{0,-q\})-\mu_U}{\sigma_U}}^{\infty} \int_{-\infty}^{\frac{\log(q+\exp(\mu_U+\sigma_U y))-\mu_V}{\sigma_V}} \frac{1}{2\pi\sqrt{1-\rho_{UV}^2}} \exp\left[\frac{-1}{2(1-\rho_{UV}^2)}\left(y^2 + z^2 - 2\rho_{UV}yz\right)\right] dz dy$$

$$= \int_{\frac{\log(\max\{0,-q\})-\mu_U}{\sigma_U}}^{\infty} \int_{-\infty}^{\frac{\log(q+\exp(\mu_U+\sigma_U y))-\mu_V}{\sigma_V}} \frac{1}{2\pi\sqrt{1-\rho_{UV}^2}} \exp\left(-\frac{y^2}{2} - \frac{(z-\rho_{UV}y)^2}{2(1-\rho_{UV}^2)}\right) dz dy$$

$$= \int_{\frac{\log(\max\{0,-q\})-\mu_U}{\sigma_U}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \Phi\left(\frac{\log(q+\exp(\mu_U+\sigma_U y))-\mu_V-\rho_{UV}\sigma_V y}{\sigma_V\sqrt{1-\rho_{UV}^2}}\right) dz dy.$$

The last term does not have analytical solution, however it can be computed via numerical integration with little care needed when cutting bounds of infinite integration (lower bound tends to $-\infty$ when $q \nearrow 0$). The only unknown parameter in (2.15) left to find, is correlation coefficient ρ_{UV}. In order to figure it out, we have to search for correlation between U_t and V_t and that is essentially correlation $\rho_{XY,Y}$ between

X_tY_t and Y_t. Independence X_t of Y_t and elementary algebra yields

$$\mathbb{C}ov(X_tY_t, Y_t) = \mathbb{E}(X_t)\mathbb{V}ar(Y_t)$$

and

$$\mathbb{V}ar(X_tY_t) = \mathbb{V}ar(X_t)(\mathbb{E}(Y_t))^2 + \mathbb{V}ar(Y_t)(\mathbb{E}(X_t))^2 + \mathbb{V}ar(X_t)\mathbb{V}ar(Y_t).$$

Thus, involving (2.3) obtains

(2.16)

$$\rho_{XY,Y} = \frac{\mathbb{E}(X_t)}{\sqrt{\frac{\mathbb{V}ar(X_t)}{\mathbb{V}ar(Y_t)}(\mathbb{E}(Y_t))^2 + (\mathbb{E}(X_t))^2 + \mathbb{V}ar(X_t)}} = \frac{\exp\left(\mu_{t,X} + \frac{\sigma_{t,X}^2}{2}\right)\sqrt{\exp(\sigma_{t,Y}^2) - 1}}{\sqrt{\exp(2\mu_{t,X} + \sigma_{t,X}^2)(\exp(\sigma_{t,X}^2 + \sigma_{t,Y}^2) - 1)}}.$$

The desired correlation coefficient ρ_{UV} arises after suitable transformation of $\rho_{XY,Y}$, complying the relation between correlation of original variables and correlation between log-variables. Thus

(2.17)
$$\rho_{UV} = \frac{\log\left(1 + \rho_{XY,Y}\sqrt{(\exp(\sigma_{t,U}^2) - 1)(\exp(\sigma_{t,V}^2) - 1)}\right)}{\sigma_{t,U}\sigma_{t,V}}.$$

Finally, finding the desired quantile $\mathbb{Q}_\alpha(H_t^X)$ requires inverting (2.15) equal to α. It can be done for instance with any optimization method without special difficulties. The end of step (2) of RaR algorithm, comes with obvious formula for $\mathbb{E}(H_t^X)$

(2.18)
$$\mathbb{E}(H_t^X) = \gamma_t\mathbb{E}(Y_t)(K_t - \mathbb{E}(X_t)).$$

Next step (3) of algorithm reads to find $\mathbb{E}(\beta_tU_t + H_t^X)$ and $\mathbb{Q}_\alpha(\beta_tU_t + H_t^X)$. The first is trivial and approximation of the second follows from applying suitable version of (2.1):

(2.19)
$$\mathbb{Q}_\alpha(\beta_tU_t + H_t^X) \approx \mathbb{E}(\beta_tU_t + H_t^X) + p(H_t^X)$$

$$\pm\sqrt{(\mathbb{Q}_\alpha(\beta_tU_t) - \mathbb{E}(\beta_tU_t))^2 + (\mathbb{Q}_\alpha(H_t^X) - \mathbb{E}(H_t^X))^2 + 2(\mathbb{Q}_\alpha(\beta_tU_t) - \mathbb{E}(\beta_tU_t))(\mathbb{Q}_\alpha(H_t^X) - \mathbb{E}(H_t^X))\rho_{UH}},$$

with $p(H_t^X)$ denoting PLN value of premiums paid for options settling in moment t, the sign $\pm$ before square root depending on level α of the quantile (upper/lower) and ρ_{UH} being correlation coefficient between variables β_tU_t and H_t^X. Introducing auxiliary variable $\phi = \sqrt{\frac{\mathbb{V}ar(Y_t)}{\mathbb{V}ar(X_tY_t)}}$ and again omitting t-indexation, it can be found in the following way:

(2.20)
$$\rho_{UH} = corr(\beta_tU_t, H_t^X) = corr(X_tY_t, (K_t - X_t)Y_t) = \frac{\mathbb{C}ov(XY,(K-X)Y)}{\sqrt{\mathbb{V}ar(XY)\mathbb{V}ar((K-X)Y)}} = \frac{K\mathbb{C}ov(Y,XY) - \mathbb{V}ar(XY)}{\sqrt{\mathbb{V}ar(XY)\mathbb{V}ar(KY-XY)}}$$

$$= \frac{K\mathbb{C}ov(Y,XY) - \mathbb{V}ar(XY)}{\sqrt{K^2\mathbb{V}ar(Y) + \mathbb{V}ar(XY) - 2K\mathbb{C}ov(Y,XY)}\sqrt{\mathbb{V}ar(XY)}\frac{\sqrt{\mathbb{V}ar(Y)}}{\sqrt{\mathbb{V}ar(Y)}}} = \frac{K\rho_{XY,Y} - \sqrt{\mathbb{V}ar(XY)/\mathbb{V}ar(Y)}}{\sqrt{K^2 + \frac{\mathbb{V}ar(XY)}{\mathbb{V}ar(Y)} - 2K\frac{\rho_{XY,Y}\sqrt{\mathbb{V}ar(XY)\mathbb{V}ar(Y)}}{\mathbb{V}ar(Y)}}}$$

$$= \frac{K\rho_{XY,Y} - \frac{1}{\phi}}{\sqrt{K^2 + \frac{1}{\phi^2} - 2K\rho_{XY,Y}\frac{1}{\phi}}} = \frac{\phi K\rho_{XY,Y} - 1}{\sqrt{1 - \rho_{XY,Y}^2 + (K\phi - \rho_{XY,Y})^2}}.$$

Putting now (2.20) into (2.19) yields the finish of point (3).

Obtaining step (4) – the mean and quantiles for currency hedging $H_t^Y \stackrel{\text{def}}{=} \delta_t(L_t - Y_t)$ is obvious and reads:

$$\mathbb{E}(H_t^Y) = \delta_t(L_t - \mathbb{E}(Y_t)),$$

(2.21)
$$\mathbb{Q}_\alpha(H_t^Y) = \delta_t(L_t - \mathbb{Q}_{1-\alpha}(Y)).$$

To approach the main target – calculation of statistics for revenues reduced by hedging, we apply the same methodology and results obtained so far. Thus

(2.22) $$\mathbb{E}(\beta_t U_t + H_t^X + H_t^Y) = \beta_t \mathbb{E}(X_t)\mathbb{E}(Y_t) + \gamma_t(K_t - \mathbb{E}(X_t))\mathbb{E}(Y_t) + \delta_t(L_t - \mathbb{E}(Y_t)),$$

and
(2.23)
$$\mathbb{Q}_\alpha(\beta_t U_t + H_t^X + H_t^Y) \approx \mathbb{E}(\beta_t U_t + H_t^X + H_t^Y) + p(H_t^X) + p(H_t^Y)$$

$$\pm \sqrt{(\mathbb{Q}_\alpha(\beta_t U_t + H_t^X) - \mathbb{E}(\beta_t U_t + H_t^X))^2 + (\mathbb{Q}_\alpha(H_t^Y) - \mathbb{E}(H_t^Y))^2 + 2(\mathbb{Q}_\alpha(\beta_t U_t + H_t^X) - \mathbb{E}(\beta_t U_t + H_t^X))(\mathbb{Q}_\alpha(H_t^Y) - \mathbb{E}(H_t^Y))\rho_{UHH}}.$$

Same as before, the only unknown parameter to be found is ρ_{UHH}, being the correlation coefficient between $\beta_t U_t + H_t^X$ and H_t^Y. And that is essentially the same as $-\rho_{UHY}$ with $\rho_{UHY} \stackrel{\text{def}}{=} corr(\beta_t U_t + H_t^X, Y)$. To calculate ρ_{UHY}, first we need to introduce auxiliary corelation coefficient $\rho_{HY} \stackrel{\text{def}}{=} corr(H_t^X, Y)$. Calculation of ρ_{HY} follows by:

$$\rho_{HY} = \frac{K\mathbb{V}ar(Y) - \mathbb{C}ov(XY,Y)}{\sqrt{\mathbb{V}ar(KY - XY)\mathbb{V}ar(Y)}} = \frac{K\mathbb{V}ar(Y) - \mathbb{C}ov(XY,Y)}{\sqrt{K^2\mathbb{V}ar(Y) + \mathbb{V}ar(XY) - 2K\mathbb{C}ov(Y,XY)}\sqrt{\mathbb{V}ar(Y)}\frac{\sqrt{\mathbb{V}ar(XY)}}{\sqrt{\mathbb{V}ar(XY)}}}$$

(2.24)
$$= \frac{\frac{K\mathbb{V}ar(Y)}{\sqrt{\mathbb{V}ar(XY)\mathbb{V}ar(Y)}} - \rho_{XY,Y}}{\sqrt{K^2\frac{\mathbb{V}ar(Y)}{\mathbb{V}ar(XY)} + 1 - 2K\rho_{XY,Y}\frac{\sqrt{\mathbb{V}ar(Y)}}{\sqrt{\mathbb{V}ar(XY)}} + \rho_{XY,Y}^2 - \rho_{XY,Y}^2}} = \frac{K\phi - \rho_{XY,Y}}{\sqrt{1 - \rho_{XY,Y}^2 + (K\phi - \rho_{XY,Y})^2}}.$$

Finally, correlation coefficient ρ_{UHY} can be directly found as:

(2.25)
$$\rho_{UHY} = \frac{\mathbb{C}ov(\beta XY,Y) + \mathbb{C}ov(H_t^X,Y)}{\sqrt{\mathbb{V}ar(\beta XY + \gamma(KY - XY))\mathbb{V}ar(Y)}} = \frac{\mathbb{C}ov(\beta XY,Y) + \mathbb{C}ov(\gamma KY - \gamma XY + \frac{K\gamma^2}{\beta-\gamma}Y - \frac{K\gamma^2}{\beta-\gamma}Y, Y)}{\frac{\beta-\gamma}{\gamma}\sqrt{\mathbb{V}ar(\gamma XY + \frac{K\gamma^2}{\beta-\gamma}Y)\mathbb{V}ar(Y)}}$$

$$= \frac{\mathbb{C}ov(\beta XY,Y) + \mathbb{C}ov(H_t^X(K'=-\frac{K\gamma}{\beta-\gamma}),Y) + \mathbb{C}ov((\gamma K + \frac{K\gamma^2}{\beta-\gamma})Y,Y)}{\frac{\beta-\gamma}{\gamma}\sqrt{\mathbb{V}ar(H_t^X(K'=-\frac{K\gamma}{\beta-\gamma}))\mathbb{V}ar(Y)}} = \frac{\gamma}{\beta-\gamma}\rho_{HY}(K') + \frac{\mathbb{C}ov(\beta XY,Y) + \frac{\beta\gamma K}{\beta-\gamma}\mathbb{V}ar(Y)}{\frac{\beta-\gamma}{\gamma}\sqrt{\mathbb{V}ar(H_t^X(K'))\mathbb{V}ar(Y)}}$$

$$= \frac{\gamma}{\beta-\gamma}\rho_{HY}(K') + \frac{\beta\gamma(\beta-\gamma)\rho_{XY,Y}\sqrt{\mathbb{V}ar(XY)} + \beta\gamma^2 K\sqrt{\mathbb{V}ar(Y)}}{(\beta-\gamma)^2\sqrt{\mathbb{V}ar(H_t^X(K'))}} = \frac{\gamma}{\beta-\gamma}\rho_{HY}(K')$$

$$+ \frac{\beta\gamma(\beta-\gamma)\rho_{XY,Y}\sqrt{\mathbb{V}ar(XY)} + \beta\gamma^2 K\sqrt{\mathbb{V}ar(Y)}}{(\beta-\gamma)^2\gamma\sqrt{\mathbb{V}ar(XY)}\sqrt{1 - \rho_{XY,Y}^2 + (K'\phi - \rho_{XY,Y})^2}} = \frac{\gamma}{\beta-\gamma}\frac{K'\phi - \rho_{XY,Y}}{\sqrt{1 - \rho_{XY,Y}^2 + (K'\phi - \rho_{XY,Y})^2}} + \frac{\beta(\beta-\gamma)\rho_{XY,Y} + \beta\gamma K\phi}{(\beta-\gamma)^2\sqrt{1 - \rho_{XY,Y}^2 + (K'\phi - \rho_{XY,Y})^2}}$$

$$= \frac{\gamma(\beta-\gamma)(K'\phi - \rho_{XY,Y}) + \beta(\beta-\gamma)\rho_{XY,Y} + \beta\gamma K\phi}{(\beta-\gamma)^2\sqrt{1 - \rho_{XY,Y}^2 + (K'\phi - \rho_{XY,Y})^2}} = \frac{(\beta-\gamma)^2\rho_{XY,Y} + \gamma\phi((\beta-\gamma)K' + \beta K)}{(\beta-\gamma)^2\sqrt{1 - \rho_{XY,Y}^2 + (K'\phi - \rho_{XY,Y})^2}} = \frac{(\beta-\gamma)^2\rho_{XY,Y} + \gamma\phi(\beta K - \gamma K)}{(\beta-\gamma)^2\sqrt{1 - \rho_{XY,Y}^2 + (\frac{K\gamma\phi}{\beta-\gamma} + \rho_{XY,Y})^2}}$$

$$= \frac{(\beta-\gamma)\rho_{XY,Y} + \gamma\phi K}{\sqrt{(\beta-\gamma)^2(1 - \rho_{XY,Y}^2) + (K\gamma\phi + \rho_{XY,Y}(\beta-\gamma))^2}}.$$

The final result of (2.23) comes with applying $-\rho_{UHY}$ instead of ρ_{UHH}. Remark, that point (5) of *RaR* algorithm can be calculated in the same way, if only applying ρ_{HY} found in (2.24). That yields to quantifying the separate risk exposure of hedging portfolio alone.

Especially important and useful with taking hedging decisions is lower quantile $\mathbb{Q}_\alpha(\beta_t U_t + H_t^X + H_t^Y)$ with $\alpha = 0.05$ or $\alpha = 0.01$, as leading directly to *RaR* measure:

$$(2.26) \qquad RaR_t^\alpha \stackrel{\text{def}}{=} \mathbb{E}(\beta_t U_t + H_t^X + H_t^Y) - \mathbb{Q}_\alpha(\beta_t U_t + H_t^X + H_t^Y)$$

It tells the managers how big is the exposure to market risk, i.e. how much the corporation is risking by not hedging totally its revenues. On the *RaR* basis, the corporation is able to decide whether to hedge some additional part of revenues or not. Either some system of strategic limits could be introduced exploiting *RaR* measure. $\qquad\qquad\square$

Finally, it is worth to mention, that *RaR* can be also defined as the value of revenues exposed to risk with respect to revenues predetermined in the budget:

$$(2.27) \qquad RaR_t^{B,\alpha} \stackrel{\text{def}}{=} \beta_t X_t^B Y_t^B + H_t^{X^B} + H_t^{Y^B} - \mathbb{Q}_\alpha(\beta_t U_t + H_t^X + H_t^Y)$$

Involving $RaR_t^{B,\alpha}$ in taking hedging decisions could be however not effective due to possible mismatch of budget prices to the current market situation.

CHAPTER 3

Some basic aspects of actuarial risk theory.

3.1. Collective risk model and ruin probabilities

In examining the nature of the risk associated with a portfolio of business, it is often of interest to assess how the portfolio of the corporation may be expected to perform over an extended period of time. One approach concerns the use of ruin theory (Panjer & Willmot [**34**]). Similarly as in the case of operational risk management in banking, the compound Poisson distributions can be used to model corporate operational risk and in particular, ruin theory should be applied. Ruin theory is concerned with the excess of the income (with respect to a portfolio of business) over the outgo, or claims paid. This quantity, referred to as insurer's surplus, varies in time. Specifically, ruin is said to occur if the insurer's surplus reaches a specified lower bound, e.g. minus the initial capital. One measure of risk is the probability of such an event, clearly reflecting the volatility inherent in the business. In addition, it can serve as a useful tool in long range planning for the use of insurer's funds.

We start with a definition of a classical risk model (see e.g. Grandell [**21**], and Rolski et al. [**38**]).

DEFINITION 3.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying Poisson process $\{N_t\}_{t \geq 0}$ with intensity λ, and sequence $\{X_k\}_{k=1}^{\infty}$ of independent, positive, identically distributed random variables, with mean μ and variance σ^2. Furthermore, we assume that $\{X_k\}$ and $\{N_t\}$ are independent. The classical risk process $\{R_t\}_{t \geq 0}$ is given by*

$$(3.1) \qquad R_t = u + ct - \sum_{i=1}^{N_t} X_i,$$

where c is some positive constant and u is nonnegative.

This is the standard mathematical model for insurance risk. The initial capital is u, the Poisson process N_t describes the number of claims in $(0, t]$ interval and claim severities are random, given by sequence $\{X_k\}_{k=1}^{\infty}$ with mean value μ and variance σ^2, independent of N_t. To cover its liability, the insurance company receives premium at a constant rate c, per unit time, where $c = (1 + \theta)\lambda\mu$ and $\theta > 0$ is often called the relative safety loading. The loading has to be positive, otherwise c would be less than $\lambda\mu$ and thus with probability 1 the risk business would become negative in infinite time.

For mathematical purposes, it is sometimes more convenient to work with a claim surplus process process process $\{S_t\}_{t \geq 0}$ (see e.g. Asmussen [**1**]), namely

$$S_t = u - R_t = \sum_{i=1}^{N_t} X_i - ct.$$

To introduce the term *ruin probability*, i.e. the probability that the risk process drops below zero, first define the time to ruin as

$$\tau(u) = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : S_t > u\}. \tag{3.2}$$

Let

$$L = \sup_{0 \leq t < \infty} \{S_t\} \quad \text{and} \quad L_T = \sup_{0 \leq t \leq T} \{S_t\}.$$

DEFINITION 3.1.2. *The ruin probability in finite time T is given by*

$$\psi(u, T) = \mathbb{P}(\tau(u) \leq T) = \mathbb{P}(L_T > u)$$

and ruin probability in infinite time is defined as

$$\psi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(L > u). \tag{3.3}$$

We also note that obviously $\psi(u, T) < \psi(u)$. However, the infinite time ruin probability may be sometimes also relevant for the finite time case.

In the sequel we assume $c = 1$, but it is not a restrictive assumption. Following Asmussen [1], let $c \neq 1$ and define $\tilde{R}_t = R_{\frac{t}{c}}$. Then relations between ruin probabilities $\psi(u)$, $\psi(u, T)$ for the process R_t and $\tilde{\psi}(u)$, $\tilde{\psi}(u, T)$ for the process $\tilde{R}_t$ are given by equations:

$$\psi(u) = \tilde{\psi}(u), \qquad \psi(u, T) = \tilde{\psi}(u, Tc).$$

3.2. Adjustment coefficient

In financial and actuarial mathematics there's a distinction between light- and heavy-tailed distributions (see, e.g. Embrechts et al. [16]). Distribution $F_X(x)$ is said to be light-tailed, if there exist constants $a > 0$, $b > 0$ such that $\overline{F_X}(x) = 1 - F_X(x) \leq ae^{-bx}$ or, equivalently, if there exist $z > 0$, such that $M_X(z) < \infty$, where $M_X(z)$ is the moment generating function. Distribution $F_X(x)$ is said to be heavy-tailed, if for all $a > 0$, $b > 0$ $\overline{F_X}(x) > ae^{-bx}$, or, equivalently, if $\forall z > 0$ $M_X(z) = \infty$.

The main claim size distributions to be studied are presented in Table 1.

Adjustment coefficient (called also the Lundberg exponent) plays a key role in calculating the ruin probability in the case of light-tailed claims.

DEFINITION 3.2.1. *Let $\gamma = \sup_z M_X(z) < \infty$ and let R be a positive solution of the equation*

$$1 + (1 + \theta)\mu R = M_X(R), \qquad R < \gamma. \tag{3.4}$$

If there exists a non-zero solution to the above equation, we call such R an adjustment coefficient.

Analytical solution to equation (3.4) exists only for few claim distributions. However, it is quite easy to obtain a numerical solution. The coefficient R satisfies the inequality

$$R < \frac{2\theta\mu}{\mu^{(2)}}, \tag{3.5}$$

TABLE 3.1. Claim size distributions

Light-tailed distributions		
name	parameters	pdf
exponential	$\beta > 0$	$f_X(x) = \beta e^{-\beta x},$ $\qquad x \geq 0$
gamma	$\alpha > 0,\ \beta > 0$	$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$ $\qquad x \geq 0$
Weibull	$c > 0,\ \tau \geq 1$	$f_X(x) = c\tau x^{\tau-1} e^{-cx^\tau},$ $\qquad x \geq 0$
mixed exp's	$\beta_i > 0,\ \sum\limits_{i=1}^{n} a_i = 1$	$f_X(x) = \sum\limits_{i=1}^{n} (a_i \beta_i e^{-\beta_i x}),$ $\qquad x \geq 0$

Heavy-tailed distributions		
name	parameters	pdf
Weibull	$c > 0,\ 0 < \tau < 1$	$f_X(x) = c\tau x^{\tau-1} e^{-cx^\tau},$ $\qquad x \geq 0$
lognormal	$\mu \in \mathbb{R}\ \ \sigma > 0$	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}},$ $\qquad x \geq 0$
loggamma	$\alpha > 0,\ \beta > 0$	$f_X(x) = \frac{\beta^\alpha (\log(x))^{\alpha-1}}{x^{\beta+1}\Gamma(\alpha)},$ $\qquad x \geq 1$
Pareto	$\alpha > 0,\ \nu > 0$	$f_X(x) = \frac{\alpha}{\nu+x}\left(\frac{\nu}{\nu+x}\right)^\alpha,$ $\qquad x \geq 0$
Burr	$\alpha > 0,\ \nu > 0,\ \tau > 0$	$f_X(x) = \frac{\alpha\tau\nu^\alpha x^{\tau-1}}{(\nu+x^\tau)^{\alpha+1}},$ $\qquad x \geq 0$

where $\mu^{(2)} = \mathbb{E}X_i^2$ (cf. [1]). Let $D(z) = 1 + (1+\theta)\mu z - M_X(z)$. Thus, the adjustment coefficient $R > 0$ satisfies the equation $D(R) = 0$. In order to get the solution one may use the Newton-Raphson formula

$$(3.6) \qquad R_{j+1} = R_j - \frac{D(R_j)}{D'(R_j)},$$

with the initial condition $R_0 = \frac{2\theta\mu}{\mu^{(2)}}$, where $D'(z) = (1+\theta)\mu - M_X(z)'$.

Moreover, if it is possible to calculate the third raw moment $\mu^{(3)}$, we can obtain a sharper bound than (3.5) (see [34]):

$$R < \frac{12\mu\theta}{3\mu^{(2)} + \sqrt{9(\mu^{(2)})^2 + 24\mu\mu^{(3)}\theta}},$$

and use it as a initial condition in (3.6).

We note that most of the methods of estimating the ruin probability discussed in the next chapters require only the existence of first two or three moments of the claim size distribution, and some of them also the existence of the moment generating function.

Now, let us we consider the aggregate loss process S_t with $c = 1$. Put $\xi(u) = S_{\tau(u)} - u$, where $\tau(u)$ is the time to ruin defined by (3.2). The following statement presents a general formula for the ruin probability in infinite time (see e.g. Asmussen [1]).

PROPOSITION 3.2.1. *Let us assume that for some $R > 0$ the process $\left\{e^{RS_t}\right\}_{t \geq 0}$ is a martingale and $S_t \to -\infty$ a.s. on $\{\tau(u) = \infty\}$. Then*

$$(3.7) \qquad \psi(u) = \frac{e^{-Ru}}{\mathbb{E}(e^{R\xi(u)}\,|\,\tau(u) < \infty)}.$$

For the classical risk model the foregoing assumptions hold and R is the adjustment coefficient.

CHAPTER 4

Ruin probability in finite time horizon.

From a practical point of view, $\psi(u, T)$, where T is related to the planning horizon of the company, may perhaps sometimes be regarded as more interesting than $\psi(u)$. Most insurance managers will closely follow the development of the risk business and increase the premium or the if the risk business behaves badly. The planning horizon may be thought of as the sum of the following: the time until the risk business is found to behave "badly", the time until the management reacts and the time until a decision of a premium increase takes effect. Therefore, in non-life insurance, it may be natural to regard T equal to four or five years as reasonable (Grandell [21]). Analogously, applying actuarial methodology to operational risk management in the corporation within five year planning horizon, $\psi(u, T)$ would be a very interesting tool providing useful risk measurement, essential for taking accurate management decisions and establishing required level of operational capital charge.

We also note that the situation in infinite time is markedly different from the finite horizon case as the ruin probability in finite time can always be computed directly using Monte Carlo simulations. It is worth to remark that generalizations of the classical risk process which are studied in Čižek, Härdle and Weron [12], Chapter 14, where the occurrence of the claims is described by point processes other than the Poisson process (i.e., non-homogeneous, mixed Poisson and Cox processes) do not alter the ruin probability in infinite time. This stems from the following fact ([12], Chapter 14).

FACT 4.0.1. *Consider a risk process $\tilde{R}_t$ driven by a Cox process $\tilde{N}_t$ with the intensity process $\tilde{\lambda}(t)$, namely*

$$\tilde{R}_t = u + (1 + \theta)\mu \int_0^t \tilde{\lambda}(s)ds - \sum_{i=1}^{\tilde{N}_t} X_i.$$

Define now $\Lambda_t = \int_0^t \lambda(s)ds$ and $R_t = \tilde{R}(\Lambda_t^{-1})$. Then the point process $N_t = \tilde{N}(\Lambda_t^{-1})$ is a standard Poisson process, and therefore,

$$\tilde{\psi}(u) = P(\inf_{t \geq 0}\{\tilde{R}_t\} < 0) = P(\inf_{t \geq 0}\{R_t\} < 0) = \psi(u).$$

The time scale defined by Λ_t^{-1} is called the operational time scale. It naturally affects the time to ruin, hence the finite time ruin probability, but not the ultimate ruin probability.

The ruin probabilities in infinite and finite time can only be calculated for a few special cases of the claim amount distribution. Thus, finding a reliable approximation, especially in the ultimate case, when the Monte Carlo method can not be utilized, is really important from a practical point of view.

In this section, first the exact ruin probabilities in finite time are discussed, then the most important approximations of the finite time ruin probability are presented and illustrated. One new approximation, namely Finite De Vylder approximation, is proposed.

To illustrate and compare approximations in this and the next sections, we use the PCS (Property Claim Services) catastrophe data (for details, introduction and the estimations see [**12**], Chapter 13). The data describes losses resulting from natural catastrophic events in USA that occurred between 1990 and 1999. This data set was used to obtain the parameters of the discussed distributions.

4.1. Exact ruin probabilities in finite time

We are now interested in the probability that the company's capital as defined by (3.1) remains non-negative for a finite period T rather than permanently. We assume that the number of losses process N_t is a Poisson process with rate λ, and consequently, the total claims (aggregate loss) process is a compound Poisson process. Premiums are payable at rate c per unit time. We recall that the intensity of the process N_t is irrelevant in the infinite time case provided that it is compensated by the premium.

In contrast to the infinite time case there is no general formula for the ruin probability like the Pollaczek–Khinchin one given in the next chapter by (5.11). In the literature one can only find a partial integro-differential equation which satisfies the probability of non-ruin, see [**34**]. An explicit result is merely known for the exponential losses, and even is this case a numerical integration is needed [**1**].

4.1.1. Exponential loss amounts. First, in order to simplify the formulae, let us assume that losses have the exponential distribution with $\beta = 1$ and the amount of premium is $c = 1$. Then

$$(4.1) \qquad \psi(u, T) = \lambda \exp\left\{-(1 - \lambda)u\right\} - \frac{1}{\pi} \int_0^\pi \frac{f_1(x) f_2(x)}{f_3(x)} dx,$$

where

$$f_1(x) = \lambda \exp\left\{2\sqrt{\lambda}T \cos x - (1 + \lambda)T + u\left(\sqrt{\lambda}\cos x - 1\right)\right\},$$

$$f_2(x) = \cos\left(u\sqrt{\lambda}\sin x\right) - \cos\left(u\sqrt{\lambda}\sin x + 2x\right), \qquad \text{and} \qquad f_3(x) = 1 + \lambda - 2\sqrt{\lambda}\cos x.$$

Now, notice that the case $\beta \neq 1$ is easily reduced to $\beta = 1$, using the formula:

$$(4.2) \qquad \psi_{\lambda,\beta}(u, T) = \psi_{\frac{\lambda}{\beta},1}(\beta u, \beta T).$$

Moreover, the assumption $c = 1$ is not restrictive since we have

$$(4.3) \qquad \psi_{\lambda,c}(u, T) = \psi_{\lambda/c,1}(u, cT).$$

As an example, Table 4.1 shows the exact values of the ruin probability for exponential claims with $\beta = 6.3789 \cdot 10^{-9}$ (see Chapter 13 in [**12**]) with respect to the initial capital u and the time horizon T. The relative safety loading θ equals 30%. Compare to Table 15.2 in [**12**], Chapter 15, to see that the values converge to those calculated in infinite case as T is getting larger. The speed of convergence decreases as the initial capital u grows.

TABLE 4.1. The ruin probability for exponential claims with $\beta = 6.3789 \cdot 10^{-9}$ and $\theta = 0.3$ (u in USD billion).

u	0	1	2	3	4	5
$\psi(u, 1)$	0.757164	0.147954	0.025005	0.003605	0.000443	0.000047
$\psi(u, 2)$	0.766264	0.168728	0.035478	0.007012	0.001286	0.000218
$\psi(u, 5)$	0.769098	0.176127	0.040220	0.009138	0.002060	0.000459
$\psi(u, 10)$	0.769229	0.176497	0.040495	0.009290	0.002131	0.000489
$\psi(u, 20)$	0.769231	0.176503	0.040499	0.009293	0.002132	0.000489

TABLE 4.2. Monte Carlo results (50 x 10000 simulations) for mixture of two exponentials claims with $\beta_1 = 3.5900 \cdot 10^{-10}$, $\beta_2 = 7.5088 \cdot 10^{-9}$, $a = 0.0584$ and $\theta = 0.3$ (u in USD billion).

u	0	1	5	10	20	50
$\psi(u, 1)$	0.672550	0.428150	0.188930	0.063938	0.006164	0.000002
$\psi(u, 2)$	0.718254	0.501066	0.256266	0.105022	0.015388	0.000030
$\psi(u, 5)$	0.753696	0.560426	0.323848	0.159034	0.035828	0.000230
$\psi(u, 10)$	0.765412	0.580786	0.350084	0.184438	0.049828	0.000726
$\psi(u, 20)$	0.769364	0.587826	0.359778	0.194262	0.056466	0.001244

4.2. Approximations of the ruin probability in finite time

In this section, we present 5 different approximations. We illustrate them on a common claim size distribution example, namely the mixture of two exponentials claims with $\beta_1 = 3.5900 \cdot 10^{-10}$, $\beta_2 = 7.5088 \cdot 10^{-9}$ and $a = 0.0584$ (see Chapter 13 in [12]). Numerical comparison of the approximations is given in Section 4.3.

4.2.1. Monte Carlo method. The ruin probability in finite time can always be approximated by means of Monte Carlo simulations. Table 4.2 shows the output for mixture of two exponentials claims with β_1, β_2, a, with respect to the initial capital u and the time horizon T. The relative safety loading θ is set to 30%. For the Monte Carlo method purposes we generated 50 x 10000 simulations. We see that the values approach those calculated in infinite case as T increases, cf. Table 15.2 in [12]. We note that the Monte Carlo method will be used as a reference method when comparing different finite time approximations in Section 4.3.

4.2.2. Segerdahl normal approximation. The following result due to Segerdahl [39] is said to be a time-dependent version of the Cramér–Lundberg approximation given by (5.5).

$$(4.4) \qquad \psi_S(u, T) = C \exp(-Ru)\Phi\left(\frac{T - u m_L}{\omega_L \sqrt{u}}\right),$$

where $C = \theta\mu / \{M_X'(R) - \mu(1 + \theta)\}$, $m_L = C\{\lambda M_X'(R) - 1\}^{-1}$ and $\omega_L^2 = \lambda M_X''(R) m_L^3$.

This method requires existence of the adjustment coefficient. This implies that only light-tailed distributions can be used. Numerical evidence shows that the Segerdahl approximation gives the best results for huge values of the initial capital u, see [1].

4.2.3. Diffusion approximation. The idea of the diffusion approximation is first to approximate the claim surplus process S_t by a Brownian motion with drift (arithmetic Brownian motion) by matching the first two moments, and next, to note that such an approximation implies that the first passage probabilities are close. The first passage probability serves as the ruin probability.

The diffusion approximation is given by:

$$(4.5) \qquad \psi_D(u, T) = IG\left(\frac{T\mu_c^2}{\sigma_c^2}; -1; \frac{u|\mu_c|}{\sigma_c^2}\right),$$

where $\mu_c = \lambda\theta\mu$, $\sigma_c = \lambda\mu^{(2)}$, and $IG(\cdot; \zeta; u)$ denotes the distribution function of the passage time of the Brownian motion with unit variance and drift ζ from the level 0 to the level $u > 0$ (often referred to as the inverse Gaussian distribution function), namely $IG(x; \zeta; u) = 1 - \Phi\left(u/\sqrt{x} - \zeta\sqrt{x}\right) + \exp\left(2\zeta u\right)$ $\cdot \Phi\left(-u/\sqrt{x} - \zeta\sqrt{x}\right)$, see [1]. We also note that in order to apply this approximation we need the existence of the second moment of the claim size distribution.

4.2.4. Corrected diffusion approximation. The idea presented above of the diffusion approximation ignores the presence of jumps in the risk process (the Brownian motion with drift is skip-free) and the value $S_{\tau(u)} - u$ in the moment of ruin. The corrected diffusion approximation takes this and other deficits into consideration [1]. Under the assumption that $c = 1$, cf. relation (4.3), we have

$$(4.6) \qquad \psi_{CD}(u, t) = IG\left(\frac{T\delta_1}{u^2} + \frac{\delta_2}{u}; -\frac{Ru}{2}; 1 + \frac{\delta_2}{u}\right),$$

where R is the adjustment coefficient, $\delta_1 = \lambda M_X''(\gamma_0)$, $\delta_2 = M_X'''(\gamma_0)/\{3M_X''(\gamma_0)\}$, and γ_0 satisfies the equation: $\kappa'(\gamma_0) = 0$, where $\kappa(s) = \lambda\{M_X(s) - 1\} - s$.

Similarly as in the Segerdahl approximation, the method requires existence of the moment generating function, so we can use it only for light-tailed distributions.

4.2.5. Finite time De Vylder approximation. Let us recall the idea of the De Vylder approximation in infinite time: we replace the claim surplus process with the one with $\theta = \bar{\theta}$, $\lambda = \bar{\lambda}$ and exponential claims with parameter $\bar{\beta}$, fitting first three moments, see the next chapter. Here, the idea is the same. First, we compute

$$\bar{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}, \qquad \bar{\lambda} = \frac{9\lambda\mu^{(2)^3}}{2\mu^{(3)^2}}, \qquad \text{and} \qquad \bar{\theta} = \frac{2\mu\mu^{(3)}}{3\mu^{(2)^2}}\theta.$$

Next, we employ relations (4.2) and (4.3) and finally use the exact, exponential case formula presented in Section 4.1.1. Obviously, the method gives the exact result in the exponential case. For other claim distributions, the first three moments have to exist in order to apply the approximation.

Summarizing methods presented above, Table 4.3 shows which approximation can be used for each claim size distribution. Moreover, the necessary assumptions on the distribution parameters are presented.

TABLE 4.3. Survey of approximations with an indication when they can be applied

Distribution	Exp.	Gamma	Weibull	Mix.Exp.	Lognormal	Pareto	Burr
Method							
Monte Carlo	+	+	+	+	+	+	+
Segerdahl	+	+	−	+	−	−	−
Diffusion	+	+	+	+	+	$\alpha > 2$	$\alpha\tau > 2$
Corr. diff.	+	+	−	+	−	−	−
Fin. De Vylder	+	+	+	+	+	$\alpha > 3$	$\alpha\tau > 3$

4.3. Numerical comparison of the finite time approximations

Now, we illustrate all 5 approximations presented in Section 4.2. We consider three claim amount distributions which were best fitted to the catastrophe data in Chapter 13 in[**12**], namely the mixture of two exponentials, log-normal and Pareto distributions. The parameters of the distributions are: $\beta_1 = 3.5900 \cdot 10^{-10}$, $\beta_2 = 7.5088 \cdot 10^{-9}$, $a = 0.0584$ (mixture), $\mu = 18.3806$, $\sigma = 1.1052$ (log-normal), and $\alpha = 3.4081$, $\lambda = 4.4767 \cdot 10^8$ (Pareto). The ruin probability will be depicted as a function of u, ranging from USD 0 to 30 billion, with fixed $T = 10$ or with fixed value of $u = 20$ billion USD and varying T from 0 to 20 years. The relative safety loading is set to 30%. Figures have the same form of output. In the left panel, the exact ruin probability values obtained via Monte Carlo simulations are presented. The right panel describes the relative error with respect to the exact values. We also note that for the Monte Carlo method purposes we generated 50 x 10000 simulations.

FIGURE 4.1. The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). The Segerdahl (short-dashed blue line), diffusion (dotted red line), corrected diffusion (solid black line) and finite time De Vylder (long-dashed green line) approximations. The mixture of two exponentials case with T fixed and u varying.

First, we consider the mixture of two exponentials case. As we can see in Figures 4.1 and 4.2 the diffusion approximation almost for all values of u and T gives highly incorrect results. Segerdahl and corrected diffusion approximations yield similar error, which visibly decreases when the time horizon gets bigger. The finite time De Vylder method is a unanimous winner and always gives the error below 10%.
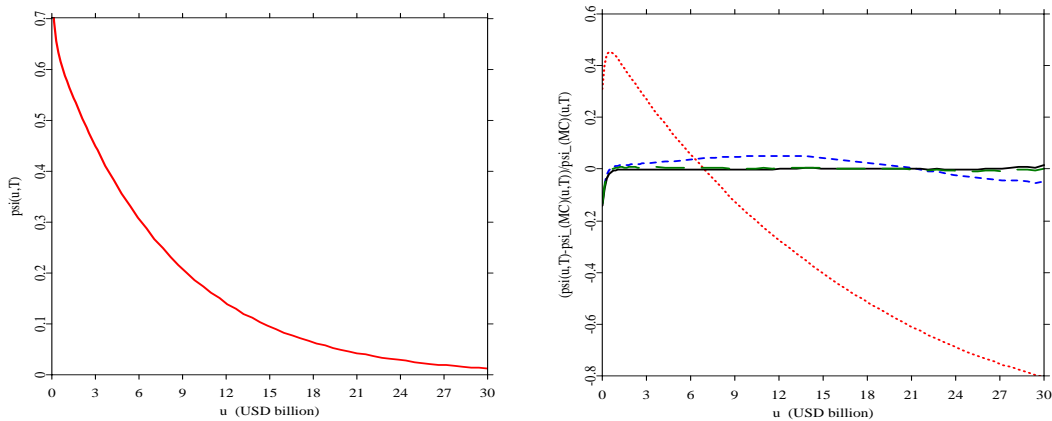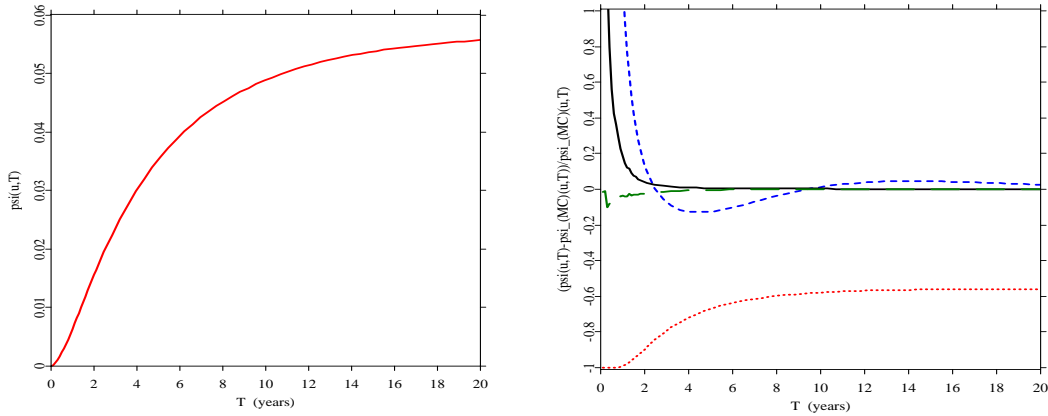

FIGURE 4.2. The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). The Segerdahl (short-dashed blue line), diffusion (dotted red line), corrected diffusion (solid black line) and finite time De Vylder (long-dashed green line) approximations. The mixture of two exponentials case with u fixed and T varying.

In the case of log-normally distributed claims, we can only apply two approximations: diffusion and finite time De Vylder ones, cf. Table 4.3. Figures 4.3 and 4.4 depict the exact ruin probability values obtained via Monte Carlo simulations and the relative error with respect to the exact values. Again, the finite time De Vylder approximation works much better than the diffusion one.

FIGURE 4.3. The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). Diffusion (dotted red line) and finite time De Vylder (long-dashed green line) approximations. The log-normal case with T fixed and u varying.

FIGURE 4.4. The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). Diffusion (dotted red line) and finite time De Vylder (long-dashed green line) approximations. The log-normal case with u fixed and T varying.

Finally, we take into consideration the Pareto claim size distribution. Figures 4.5 and 4.6 depict the exact ruin probability values and the relative error with respect to the exact values for the diffusion and finite time De Vylder approximations. We see that now we cannot claim which approximation is better. The error strongly depends on the values of u and T. We may only suspect that a combination of the two methods could give interesting results.
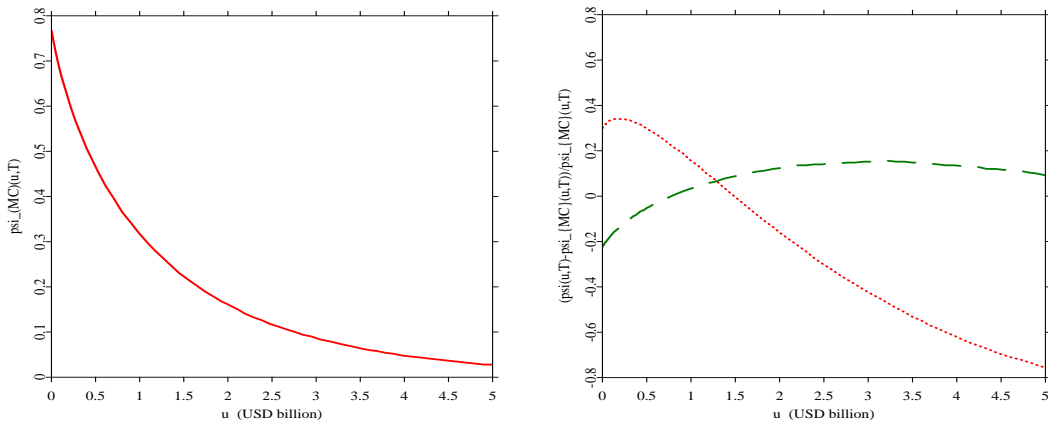

FIGURE 4.5. The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). Diffusion (dotted red line) and finite time De Vylder (long-dashed green line) approximations. The Pareto case with T fixed and u varying.

For more detailed analysis on the ruin approximations in finite time, see Chapter 15 in [12].
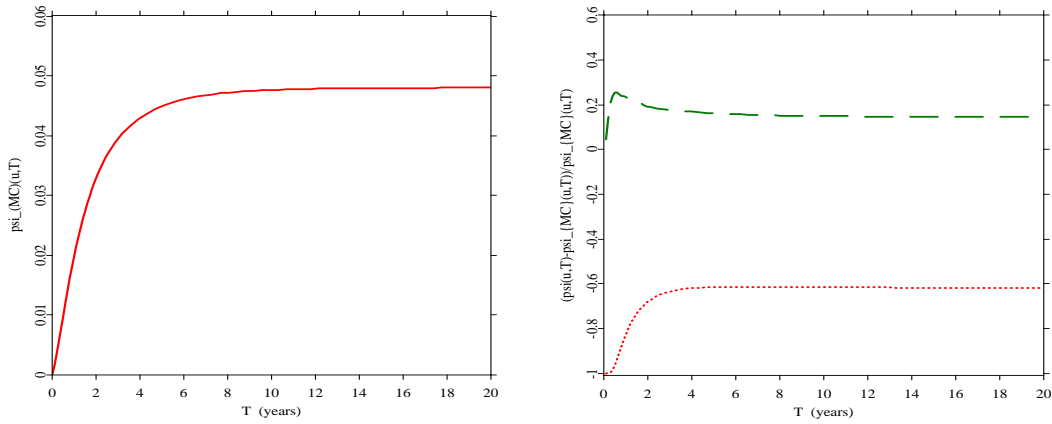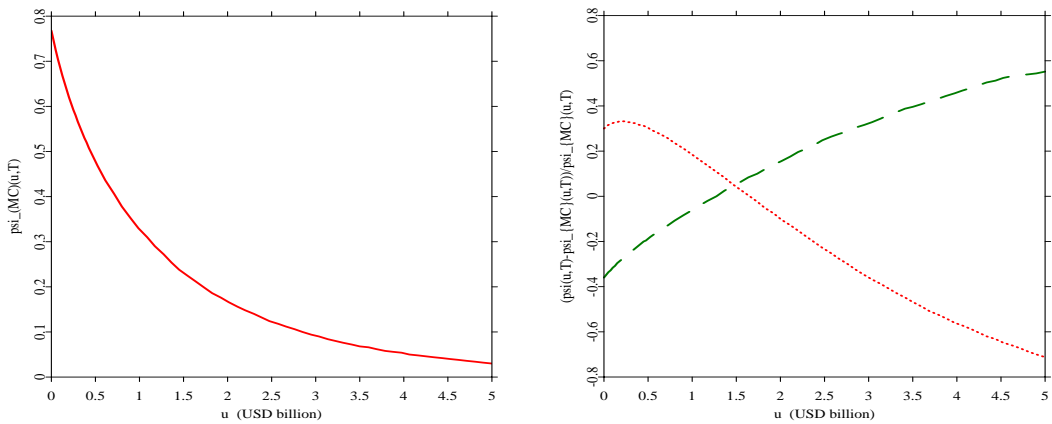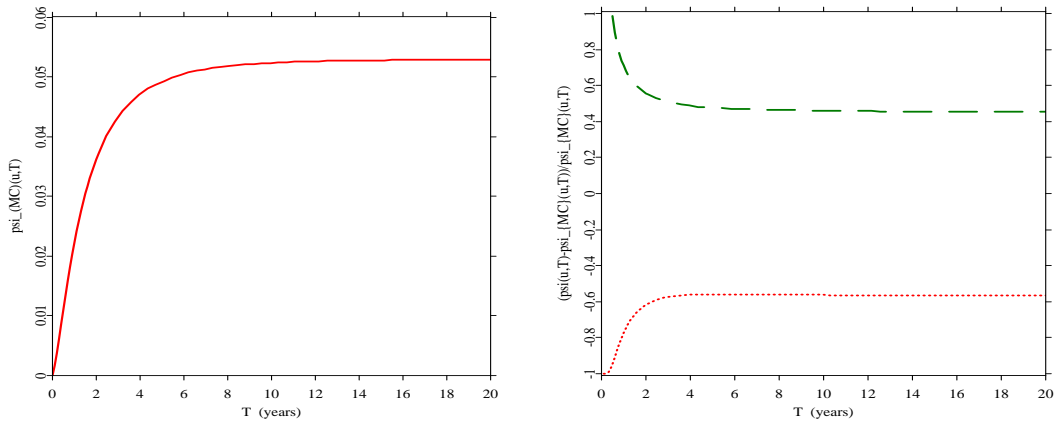
FIGURE 4.6. The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). Diffusion (dotted red line) and finite time De Vylder (long-dashed green line) approximations. The Pareto case with u fixed and T varying.

CHAPTER 5

Infinite horizon.

Let us now switch to the infinite time horizon management. Studying infinite horizon approximations we start from conclusions of Grandell [**22**]. In his paper, Grandell demonstrates that between possible simple approximations of ruin probabilities in infinite time the most successful is the De Vylder approximation, which is based on the idea to replace the risk process with the one with exponentially distributed claims and ensuring that the first three moments coincide.

First, we introduce in Section 5.2 a modification to the De Vylder approximation by changing the exponential to the gamma distribution and fitting first three moments. This modification is promising and works in many cases even better than the original method. Second, in contrast to the above paper, we drop here the assumption 'simple' and show in Section 5.4 that approximation based on the Pollaczek-Khinchin formula gives the best results. Moreover, it works for all possible distributions of claims and can be chosen as the reference method, see Section 5.5.

When the claim size distribution is exponential (or closely related), simple analytic results for the ruin probability in infinite time may be possible, see Section 5.1. For more general claim amount distributions, e.g. heavy-tailed, the Laplace transform technique does not work and one may need some estimates. In this section we will present 12 different well-known and not so well-known approximations. Numerical comparison of the approximations is given in Section 5.4. We also note that new approximations have been recently proposed in the literature, see e.g. Lima et al. [**29**] and Usábel [**42**], but as they work for specific classes of distributions and are far from computational simplicity, we will not consider them.

5.1. Exact ruin probabilities

Now, we are going to present a collection of basic exact results on the ruin probability in infinite time.

5.1.1. No initial capital. When $u = 0$ it is easy to obtain the exact formula

$$\psi(u) = \frac{1}{1 + \theta}.$$

For more details see e.g. Grandell [**21**]. Notice that the formula depends only on θ, regardless of the claim size distribution.

Exponential claims. The explicit, easy to calculate formula exists for exponential claims, namely

(5.1) $$\psi(u) = \frac{1}{1 + \theta} e^{\frac{-\theta \beta u}{1 + \theta}}.$$

Gamma claims. It was shown by Grandell and Segerdahl [**20**] that for the gamma claim distribution with mean 1 and $\alpha \leq 1$

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(5.2) $$\psi(u) = \frac{\theta(1 - \frac{R}{\alpha})e^{-Ru}}{1 + (1+\theta)R - (1+\theta)(1 - \frac{R}{\alpha})} + \frac{\alpha\theta sin(\alpha\pi)}{\pi} \cdot I,$$

where

$$I = \int_0^\infty \frac{x^\alpha e^{-(x+1)\alpha u}\,dx}{\left[x^\alpha\left(1 + \alpha(1+\theta)(x+1)\right) - cos(\alpha\pi)\right]^2 + sin^2(\alpha\pi)}.$$

The integral I has to be calculated numerically, but with some care near 0 it can be done precisely. We notice that the assumption on the mean is no restriction since for claims X with arbitrary mean μ we have that $\psi_X(u) = \psi_{\frac{X}{\mu}}(\frac{u}{\mu})$. As the gamma distribution is closed under scale changes we obtain that $\psi_{G(\alpha,\beta)}(u) = \psi_{G(\alpha,\alpha)}(\frac{\beta u}{\alpha})$ and we can now calculate the exact ruin probability via equation (5.2).

Mixture of n exponentials claims. For the claim size distribution being a mixture of n exponentials with the parameters $\beta_1 < \cdots < \beta_n$ and weights $a_1,\ldots,a_n$, using the Laplace transform inversion, one may obtain an exact formula of the form (Dufresne and Gerber, [15]):

(5.3) $$\psi(u) = \sum_{k=1}^n C_k e^{-r_k u},$$

with $r_1, r_2, \ldots, r_n$ being the n positive solutions to the equation

$$(1+\theta)\mu = \sum_{j=1}^n \frac{a_j}{\beta_j - r},$$

with $0 < r_1 = R < \beta_1 < r_2 < \beta_2 < \cdots < r_n < \beta_n$.

The coefficients C_k are given by the formula

$$C_k = \frac{1}{r_k} \frac{\sum_{j=1}^n \frac{a_j}{\beta_j - r_k} - \mu}{\sum_{j=1}^n \frac{a_j}{(\beta_j - r_k)^2}}.$$

In the case of mixture of two exponentials claim amounts $(n = 2)$ a simple analytic result is given (Panjer & Willmot [34]):

(5.4) $$\psi(u) = \frac{1}{(1+\theta)(r_2 - r_1)} \left\{(\rho - r_1)\exp(-r_1 u) + (r_2 - \rho)\exp(-r_2 u)\right\},$$

where

$$r_1 = \frac{\rho + \theta(\beta_1 + \beta_2) - \left[\{\rho + \theta(\beta_1 + \beta_2)\}^2 - 4\beta_1\beta_2\theta(1+\theta)\right]^{1/2}}{2(1+\theta)},$$

$$r_2 = \frac{\rho + \theta(\beta_1 + \beta_2) + \left[\{\rho + \theta(\beta_1 + \beta_2)\}^2 - 4\beta_1\beta_2\theta(1+\theta)\right]^{1/2}}{2(1+\theta)}$$

and

$$\rho = \beta_1(1-p) + \beta_2 p, \qquad p = \frac{a_1\beta_1^{-1}}{a_1\beta_1^{-1} + a_2\beta_2^{-1}}.$$

5.2. A survey of approximations

5.2.1. Cramér–Lundberg approximation. The following approximation holds.

$$(5.5) \qquad \psi_{CL}(u) = Ce^{-Ru},$$

where $C = \frac{\theta\mu}{M'_X(R) - \mu(1+\theta)}$.

For the proof we refer to Grandell [21]. The classical Cramér–Lundberg approximation yields quite accurate results, however we must remember that in order to use it the adjustment coefficient has to exist, therefore merely the light-tailed distributions can be taken into consideration.

For exponential claims formula (5.5) yields the exact result.

5.2.2. Exponential approximation. This approximation was proposed and derived by De Vylder [14].

$$\Psi_E(u) = exp\left(-1 - \frac{2\mu\theta u - \mu^{(2)}}{\sqrt{(\mu^{(2)})^2 + (4/3)\theta\mu\mu^{(3)}}}\right).$$

5.2.3. Lundberg approximation. The following formula, called a Lundberg approximation, comes from Grandell [22].

$$\Psi_L(u) = \left[1 + \left(\theta u - \frac{\mu^{(2)}}{2\mu}\right)\frac{4\theta\mu^2\mu^{(3)}}{3(\mu^{(2)})^3}\right]e^{\frac{-2\mu\theta u}{\mu^{(2)}}}.$$

5.2.4. Beekman–Bowers approximation. The Beekman–Bowers approximation uses the following representation of the ruin probability.

$$\psi(u) = \mathbb{P}(M > u) = \mathbb{P}(M > 0)\mathbb{P}(M > u|M > 0).$$

The idea of the approximation is to replace the conditional probability $1 - \mathbb{P}(M > u|M > 0)$ with a gamma distribution function $G(u)$ by fitting first two moments (see Grandell, [22]). This leads to

$$(5.6) \qquad \Psi_{BB}(u) = \frac{1}{1+\theta}(1 - G(u)),$$

where the parameters α, β of G are given by

$$\alpha = \frac{(1 + (\frac{4\mu\mu^{(3)}}{3(\mu^{(2)})^2} - 1)\theta)}{1 + \theta}, \qquad \beta = \frac{2\mu\theta}{\mu^{(2)} + \left(\frac{4\mu\mu^{(3)}}{3\mu^{(2)}} - \mu^{(2)}\right)\theta}.$$

The Beekman–Bowers approximation gives rather accurate results, in the exponential case it becomes the exact formula. It can be used for distributions with finite first three moments which is always true for exponential, gamma, lognormal, truncated normal and Weibull distributions. For loggamma distribution we have to set $\beta > 3$, for Pareto $\alpha > 3$, and for Burr $\alpha\tau > 3$.

5.2.5. Renyi approximation. The Renyi approximation is based on a classical result about p-thinning, Renyi's theorem (see [22]). It may be also derived from (5.6) when we replace the gamma distribution function G with an exponential one, matching only the first moment. It could be regarded as a simplified version of the Beekman–Bowers approximation.

$$\Psi_R(u) = \frac{1}{1+\theta}e^{-\frac{2\mu\theta u}{\mu^{(2)}(1+\theta)}}.$$

5.2.6. De Vylder approximation. The idea of this approximation is to replace the risk process with the one with $\theta = \bar{\theta}$, $\lambda = \bar{\lambda}$ and exponential claims with parameter $\bar{\beta}$, fitting first three moments (De Vylder [14]).

Let

$$\bar{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}, \qquad \bar{\lambda} = \frac{9\lambda\mu^{(2)^3}}{2\mu^{(3)^2}}, \qquad and \qquad \bar{\theta} = \frac{2\mu\mu^{(3)}}{3\mu^{(2)^2}}\theta.$$

Then the De Vylder's approximation is given by

$$\Psi_{DV}(u) = \frac{1}{1+\theta}e^{-\frac{\bar{\theta}\bar{\beta}u}{1+\bar{\theta}}}.$$

Obviously, in the exponential case the method gives the exact result. For other claim distributions in order to apply the approximation, similarly as in the Beekman–Bowers approximation, the first three moments have to exist.

5.2.7. 4-moment gamma De Vylder approximation. Following [7] we now introduce 4-moment gamma approximation based on the De Vylder's idea to replace the risk process with another one for which the expression for $\psi(u)$ is explicit. We fit the four moments in order to calculate the parameters of the new process with gamma distributed claims and apply the exact formula for the ruin probability in this case which is given in [20]. The risk process with gamma claims is determined by the four parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$. Since

$$
\begin{aligned}
\mathrm{E}(S_t) &= -\theta\lambda\mu t, \\
\mathrm{E}(S_t^2) &= \lambda\mu^{(2)}t + (\theta\lambda\mu t)^2, \\
\mathrm{E}(S_t^3) &= \lambda\mu^{(3)}t - 3(\lambda\mu^{(2)}t)(\theta\lambda\mu t) - (\theta\lambda\mu t)^2, \\
\mathrm{E}(S_t^4) &= \lambda\mu^{(4)}t - 4(\lambda\mu^{(3)}t)(\theta\lambda\mu t) + 3(\lambda\mu^{(2)}t)^2 + 6(\lambda\mu^{(2)}t)(\theta\lambda\mu t)^2 + (\theta\lambda\mu t)^4
\end{aligned}
$$

and for the gamma distribution

$$\bar{\mu}^{(3)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}}(2\bar{\mu}^{(2)} - \bar{\mu}^2), \quad \bar{\mu}^{(4)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2),$$

the parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$ must satisfy the equations

$$
\begin{aligned}
\theta\lambda\mu &= \bar{\theta}\bar{\lambda}\bar{\mu}, \\
\lambda\mu^{(2)} &= \bar{\lambda}\bar{\mu}^{(2)}, \\
\lambda\mu^{(3)} &= \bar{\lambda}\frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2), \\
\lambda\mu^{(4)} &= \bar{\lambda}\frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\bar{\lambda} &= \frac{\lambda(\mu^{(3)})^2(\mu^{(2)})^3}{(\mu^{(2)}\mu^{(4)} - 2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)} - 3(\mu^{(3)})^2)}, \\
\bar{\theta} &= \frac{\theta\mu(2(\mu^{(3)})^2 - \mu^{(2)}\mu^{(4)})}{(\mu^{(2)})^2\mu^{(3)}}, \\
\bar{\mu} &= \frac{3(\mu^{(3)})^2 - 2\mu^{(2)}\mu^{(4)}}{\mu^{(2)}\mu^{(3)}}, \\
\bar{\mu}^{(2)} &= \frac{(\mu^{(2)}\mu^{(4)} - 2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)} - 3(\mu^{(3)})^2)}{(\mu^{(2)}\mu^{(3)})^2}.
\end{aligned}
$$

We also need to assume that $\mu^{(2)}\mu^{(4)} < \frac{3}{2}(\mu^{(3)})^2$ to ensure that $\bar{\mu}, \bar{\mu}^{(2)} > 0$ and $\bar{\mu}^{(2)} > \bar{\mu}^2$. In case this assumption can not be fulfilled, we simply set $\bar{\mu} = \mu$ and do not calculate the fourth moment. This case leads to

$$
(5.7) \qquad \bar{\lambda} = \frac{2\lambda(\mu^{(2)})^2}{\mu(\mu^{(3)} + \mu^{(2)}\mu)}, \quad \bar{\theta} = \frac{\theta\mu(\mu^{(3)} + \mu^{(2)}\mu)}{2(\mu^{(2)})^2}, \quad \bar{\mu} = \mu, \quad \bar{\mu}^{(2)} = \frac{\mu(\mu^{(3)} + \mu^{(2)}\mu)}{2\mu^{(2)}}.
$$

All in all, we get the approximation

$$
(5.8) \qquad \psi_{4MG}(u) = \frac{\bar{\theta}(1 - \frac{R}{\bar{\alpha}})e^{-\frac{\bar{\beta}R}{\bar{\alpha}}u}}{1 + (1 + \bar{\theta})R - (1 + \bar{\theta})(1 - \frac{R}{\bar{\alpha}})} + \frac{\bar{\alpha}\bar{\theta}\sin(\bar{\alpha}\pi)}{\pi} \cdot I,
$$

where

$$
I = \int_0^\infty \frac{x^{\bar{\alpha}}e^{-(x+1)\bar{\beta}u}\,dx}{\left[x^{\bar{\alpha}}\left(1 + \bar{\alpha}(1 + \bar{\theta})(x + 1)\right) - \cos(\bar{\alpha}\pi)\right]^2 + sin^2(\bar{\alpha}\pi)},
$$

R is the adjustment coefficient for the gamma distribution and $(\bar{\alpha}, \bar{\beta})$ are given by $\bar{\alpha} = \frac{\bar{\mu}^2}{\bar{\mu}^{(2)} - \bar{\mu}^2}$, $\bar{\beta} = \frac{\bar{\mu}}{\bar{\mu}^{(2)} - \bar{\mu}^2}$.

In the exponential and gamma case this method gives the exact results. For other claim distributions in order to apply the approximation, the first four (or three) moments have to exist. In Section 5.4 will show that it gives a slight correction to the De Vylder approximation, which is said in Grandell [22] to be the best among simple approximations.

5.2.8. Heavy traffic approximation. The term 'heavy traffic' comes from queuing theory. In risk theory it means that on the average the premiums exceed only slightly the expected claims. It implies that safety loading θ is positive and small. Asmussen [1] suggests the following approximation.

$$
\psi_{HT}(u) = \exp\left(-\frac{2\theta\mu u}{\mu^{(2)}}\right).
$$

This method requires the existence of the first two moments of the claim size distribution, so we assume: $\beta > 2$ for the loggamma case, $\alpha > 2$ for the Pareto case, and $\alpha\tau > 2$ for the Burr case.

5.2.9. Light traffic approximation. As for heavy traffic, the term 'light traffic' comes from queuing theory, but has an obvious interpretation also in risk theory, namely, on the average, the premiums are much larger than the expected claims. It implies that the safety loading θ is positive and large. We may obtain the following asymptotic formula.

$$\psi_{LT}(u) = \lambda \int_u^\infty \bar{F}_X(x)dx.$$

In risk theory heavy traffic is most often argued to be the typical case rather than light traffic. However, light traffic is of some interest as a complement to heavy traffic, as well as it is needed for the interpolation approximation to be studied in the next point.

5.2.10. Heavy-light traffic approximation. The idea is to combine heavy and light approximations:

$$\psi_{HLT}(u) = \frac{\theta}{1+\theta}\psi_{LT}\left(\frac{\theta u}{1+\theta}\right) + \frac{1}{(1+\theta)^2}\psi_{HT}(u),$$

see [**1**]. The particular features of this approximation is that it is exact for the exponential distribution and asymptotically correct both in light and heavy traffic.

5.2.11. Heavy-tailed claims approximation. First, let us introduce the class of subexponential distributions $\mathcal{S}$ (see e.g. Embrechts et al., [**16**]), namely

$$\mathcal{S} = \left\{F : \lim_{x\to\infty}\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2\right\} \equiv \left\{F : \lim_{x\to\infty}\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n; \quad n \geq 2\right\}.$$

The class contains lognormal and Weibull (for $\tau < 1$) distributions. Moreover, all distributions with a regularly varying tail (e.g. loggamma, Pareto and Burr distributions) are subexponential. For subexponential distributions we can formulate the following approximation of the ruin probability. If $F \in \mathcal{S}$, then the asymptotic formula for large u is given by

(5.9)
$$\psi_{HTC}(u) = \frac{1}{\theta\mu}\left(\mu - \int_0^u \overline{F}(x)dx\right),$$

see [**1**]. This method can be used for Weibull, lognormal, loggamma, Pareto and Burr distributions.

5.2.12. Computer approximation via the Pollaczek–Khinchin formula. This time we use the representation (3.3) of the ruin probability and the decomposition of the maximum M as a sum of ladder heights. Let L_1 be the value that process $\{S_t\}$ reaches for the first time above the zero level. Next, let L_2 be the value which is obtained for the first time above the level L_1; $L_3, L_4, \ldots$ are defined in the same way. The values L_k are called ladder heights. Since the process $\{S_t\}$ has stationary and independent increments, $\{L_k\}_{k=1}^\infty$ is the sequence of independent and identically distributed variables. One may show that the number of ladder heights K to the moment of ruin is given by a geometric distribution with parameters $p = \frac{1}{1+\theta}$ and $q = \frac{\theta}{1+\theta}$. Thus, random variable M may be expressed by

$$M = \sum_{i=1}^K L_i,$$

This implies that random variable M has a compound geometric distribution given by the distribution function

$$F_M(x) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} G^{*n}(x),$$

where G is the defective density

$$g(x) = \frac{1}{\mu(1+\theta)} \overline{F_X}(x) = \frac{1}{1+\theta} b_0(x)$$

and the density

(5.10)
$$b_0(x) = \frac{\overline{F_X}(x)}{\mu}.$$

The above fact together with the representation (3.3) leads to the Pollaczek–Khinchin formula for the ruin probability:

(5.11)
$$\psi(u) = \mathbb{P}(M > u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n \overline{B_0^{*n}}(u),$$

where $\overline{B_0}$ is the tail of the distribution function corresponding to the density b_0 and $\overline{B_0^{*0}}(u) \equiv I_{\{u \geq 0\}}$.

One can use the formula to derive explicit solutions for a number of claim amount distributions, see e.g. [1] or [34]. If it is not possible, this formula can be applied directly to calculate the ruin probability. It incorporates an infinite sum, hence we use the Monte Carlo method. From (5.11) the ruin probability $\psi(u) = EZ$, where $Z = 1(M > u)$, may be generated as follows.

SIMULATION ALGORITHM

(1) Generate a random variable K from the geometric distribution with the parameters $p = \frac{1}{1+\theta}$ and $q = \frac{\theta}{1+\theta}$.
(2) Generate random variables $X_1, X_2, \cdots, X_K$ from the density $b_0(x)$.
(3) Calculate $M = X_1 + X_2 + \cdots + X_K$.
(4) If $M > u$, let $Z = 1$, otherwise let $Z = 0$.

The main problem seems to be simulating random variables from the density $b_0(x)$.

PROPOSITION 5.2.1. *The density $b_0(x)$ has a closed form only for four of the considered distributions, namely*

(i) *for exponential claims, $b_0(x)$ is the density of the same exponential distribution,*
(ii) *for mixture of exponentials claims, $b_0(x)$ is the density of the mixture of exponential distribution with the weights $\left(\frac{\frac{a_1}{\beta_1}}{\sum_{i=1}^{n}(\frac{a_i}{\beta_i})}, \cdots, \frac{\frac{a_n}{\beta_n}}{\sum_{i=1}^{n}(\frac{a_i}{\beta_i})}\right),$*
(iii) *for Pareto claims, $b_0(x)$ is the density of the Pareto distribution with the parameters $\alpha - 1$ and ν,*
(iv) *for Burr claims, $b_0(x)$ is the density of the transformed beta distribution.*

PROOF. (i) For exponential claims

$$\overline{F_X}(x) = e^{-\beta x}, \qquad \mu = \frac{1}{\beta},$$

thus

$$b_0(x) = \beta\overline{F_X}(x) = \beta e^{-\beta x},$$

which yields again the exponential distribution, with parameter β.

(ii) For mixture of exponentials claims

$$\overline{F_X}(x) = \sum_{i=1}^{n}(a_i e^{-\beta_i x}), \qquad \mu = \frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n},$$

hence

$$
\begin{aligned}
b_0(x) &= \frac{1}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}\overline{F_X}(x) = \frac{a_1}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}\overline{F_{X_1}}(x) + \cdots + \frac{a_n}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}\overline{F_{X_n}}(x) \\
&= \frac{\frac{a_1}{\beta_1}}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}\beta_1\overline{F_{X_1}}(x) + \cdots + \frac{\frac{a_n}{\beta_n}}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}\beta_n\overline{F_{X_n}}(x) \\
&= \frac{\frac{a_1}{\beta_1}}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}f_{X_1}(x) + \cdots + \frac{\frac{a_n}{\beta_n}}{\frac{a_1}{\beta_1} + \cdots + \frac{a_n}{\beta_n}}f_{X_n}(x),
\end{aligned}
$$

which is again a mixture of exponential distributions, with the weights $\left(\frac{\frac{a_1}{\beta_1}}{\sum_{i=1}^{n}(\frac{a_i}{\beta_i})}, \cdots, \frac{\frac{a_n}{\beta_n}}{\sum_{i=1}^{n}(\frac{a_i}{\beta_i})}\right)$.

(iii) For Pareto claims

$$\overline{F_X}(x) = \left(\frac{\nu}{\nu + x}\right)^{\alpha}, \qquad \mu = \frac{\nu}{\alpha - 1}, \qquad \alpha > 1,$$

so

$$b_0(x) = \frac{\alpha - 1}{\nu}\overline{F_X}(x) = \frac{\alpha - 1}{\nu}\left(\frac{\nu}{\nu + x}\right)^{\alpha} = \frac{\alpha - 1}{\nu + x}\left(\frac{\nu}{\nu + x}\right)^{\alpha - 1},$$

which again gives the Pareto distribution with parameters $(\alpha - 1, \nu)$.

(iv) For Burr claims

$$\overline{F_X}(x) = \left(\frac{\nu}{\nu + x^{\tau}}\right)^{\alpha}, \qquad \mu = \nu^{\frac{1}{\tau}}\frac{\Gamma\left(\alpha - \frac{1}{\tau}\right)\Gamma\left(1 + \frac{1}{\tau}\right)}{\Gamma(\alpha)}, \qquad \alpha\tau > 1,$$

therefore

$$b_0(x) = \frac{\Gamma(\alpha)}{\nu^{\frac{1}{\tau}}\Gamma\left(\alpha - \frac{1}{\tau}\right)\Gamma\left(1 + \frac{1}{\tau}\right)}\left(\frac{\nu}{\nu + x^{\tau}}\right)^{\alpha},$$

Let us put

$$a = \alpha - \frac{1}{\tau}, \qquad b = \frac{1}{\tau}, \qquad c = \tau, \qquad d = \nu.$$

Then

$$
\begin{aligned}
b_0(x) &= \frac{\Gamma(a + b)}{d^b\Gamma(a)\Gamma(1 + b)}\left(\frac{d}{d + x^c}\right)^{a+b} = \frac{\Gamma(a + b)d^a}{\Gamma(a)b\Gamma(b)(d + x^c)^{a+b}} \\
&= \frac{\Gamma(a + b)cd^a x^{cb-1}}{\Gamma(a)\Gamma(b)(d + x^c)^{a+b}}.
\end{aligned}
$$

The foregoing formula represents the density from the transformed beta distribution with parameters a, b, c and d. This distribution comes as a quotient of two variables from generalized gamma distribution with corresponding parameters (for details see [**34**]). □

For other distributions treated in this paper in order to generate random variables X_k we use formula (5.10) and controlled, numerical integration. The described above computer approximation via the Pollaczek–Khinchin formula will be called in short the Pollaczek–Khinchin approximation. We note that the approximation works for all considered distributions of claims.

5.3. Summary of the approximations

Table 5.1 shows which approximation can be used for a particular choice of a claim size distribution. Moreover, the necessary assumptions on the distribution parameters are included.

TABLE 5.1. Survey of approximations with an indication when they can be applied

distribution → method ↓	Exp.	Gamma	Weibull	Mix.Exp.	Lognormal	Loggamma	Pareto	Burr
Cramér-Lundberg	+	+	−	+	−	−	−	−
Exponential	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
Lundberg	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
Beekman–Bowers	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
Renyi	+	+	+	+	+	$\beta > 2$	$\alpha > 2$	$\alpha\tau > 2$
De Vylder	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
4-m. gamma De Vylder	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
Heavy Traffic	+	+	+	+	+	$\beta > 2$	$\alpha > 2$	$\alpha\tau > 2$
Light Traffic	+	+	+	+	+	+	+	+
Heavy-Light Traffic	+	+	+	+	+	$\beta > 2$	$\alpha > 2$	$\alpha\tau > 2$
Heavy-tailed	−	−	$0 < \tau < 1$	−	−	+	+	+
Pollaczek–Khinchin	+	+	+	+	+	+	+	+

5.4. Numerical comparison of the methods

We now aim to compare all 12 approximations presented in the preceding section in few cases of loss amount distribution. To this end we consider the ruin probability as a function of the initial capital u. In order to show the relative errors of the methods we compare results of the approximations with the exact values, which can be done in the exponential, gamma and mixture of exponentials case, partially in the lognormal case, or the results obtained via the Pollaczek–Khinchin formula, which we feel, and justify it numerically, can be the reference method. For the Monte Carlo method purposes we generate 100 blocks of 100000 simulations.

For the exponential case Cramér-Lundberg, Renyi, Beekman–Bowers, De Vylder and 4-moment gamma De Vylder approximations yield the exact result given by formula (5.1). We will usually assume that the mean of the claim distribution is equal to 1 and $\theta = 0.1$.

In the gamma case we can obtain exact values via formula (5.2) and use them in order to compare all methods except heavy-tailed and 4-moment gamma De Vylder approximations, which yield the exact result. When $\alpha = 0.01$ and $\beta = 0.01$, see Figure 5.1, all approximations except the heavy, light (disastrous results) and heavy-light traffic, give the relative error less than 3%.
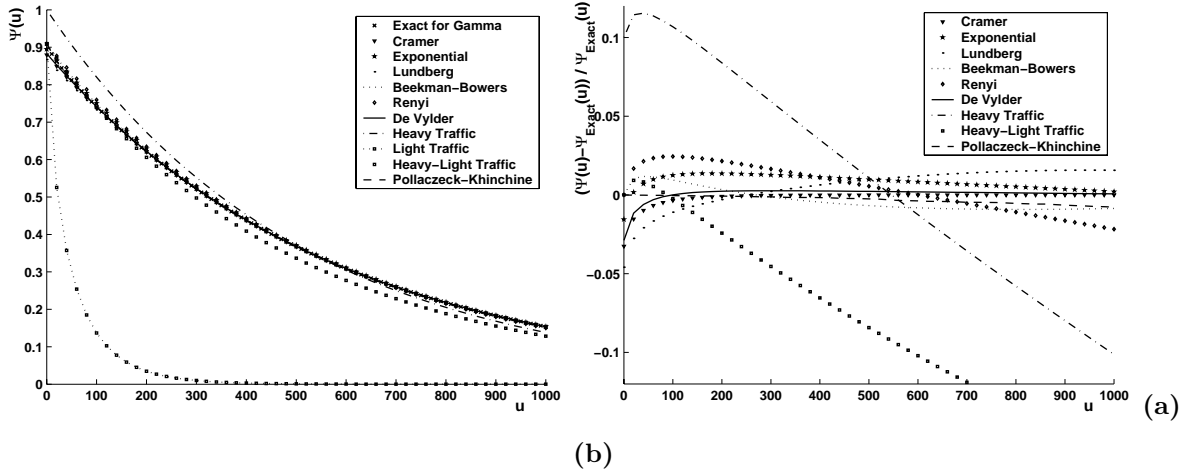
FIGURE 5.1. Illustration of the ruin probability (a) and the relative error (b) of the approximations. The gamma case with $\alpha = 0.01$, $\beta = 0.01$, $\theta = 0.1$ and $u \leq 1000$.

When the claim distribution is a mixture of three exponentials, see Figure 5.2, Cramér–Lundberg, De Vylder 4-moment gamma De Vylder and exponential approximations give quite accurate results, Beekman–Bowers and Lundberg approximations are just acceptable.
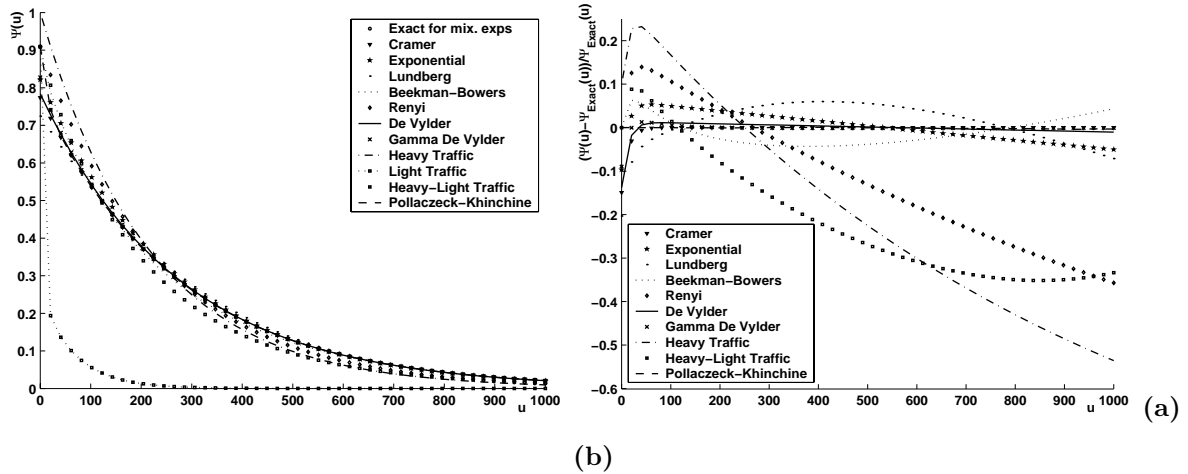
FIGURE 5.2. Illustration of the ruin probability (a) and the relative error (b) of the approximations. The mixture of three exponentials case with $\beta_1 = 0.014631$, $\beta_2 = 0.190206$, $\beta_3 = 5.514588$, weight $a_1 = 0.0039793$, $a_2 = 0.1078392$, $a_3 = 0.8881815$, $\theta = 0.1$ and $u \leq 1000$.

Since there are no exact methods for other considered distributions, we are going to calculate the relative errors with respect to the most accurate method. From Figure 5.1-5.2 the possible candidates are Cramér–Lundberg, De Vylder, 4-moment gamma De Vylder and Pollaczek–Khinchin approximations. However, we must notice that the Cramér–Lundberg approximation works only for light-tailed distributions, hence we have to decide between both De Vylder and Pollaczek–Khinchin approximations.

To this end we take into consideration a mixture of three exponential distributions and a lognormal distribution. For simple analytic results in the former case see Section 5.2. In the latter case, with

TABLE 5.2. Comparison of De Vylder, 4-moment gamma De Vylder and Pollaczek–Khinchin approximations under mixture of three exponentials and lognormal distributions. Relative errors in (%).

			mixture of three exponentials					lognormal		
θ	u	$\Psi(u)$	E_{PK}	E_{DV}	E_{3MGDV}	u	$\Psi(u)$	E_{PK}	E_{DV}	E_{3MGDV}
0.05	10	0.8897	0.0151	-3.2089	-1.6062	100	0.55074	-0.0182	-20.6159	-19.2468
0.10	10	0.7993	-0.0281	-5.4247	-2.6773	100	0.34395	**0.0087**	**-19.4825**	**-18.4852**
0.15	10	0.7242	-0.0320	-6.9981	-3.4102	100	0.23573	-0.02545	-14.2281	-13.6088
0.20	10	0.6611	0.0075	-8.1485	-3.8859	100	0.17309	0.0983	-8.0883	-7.8052
0.25	10	0.6073	**-0.00280**	**-8.9791**	**-4.1874**	100	0.13384	0.0448	-2.0547	-2.0622
0.30	10	0.5610	-0.0366	-9.5811	-4.3583	100	0.10765	0.21367	3.5300	3.2606
1.00	10	0.2634	-0.0566	-10.6644	-2.9803	100	0.02535	0.0789	41.7278	39.6726
0.05	100	0.7144	**-0,0018**	**0.3737**	**0.2967**	1000	0.04199	2.6530	55.0917	53.1769
0.10	100	0.5393	-0.0170	1.1125	0.6082	1000	0.01099	4.0218	85.5323	87.9436
0.15	100	0.4247	-0.0590	1.9143	0.8429	1000	0.00574	3.92334	79.6690	87.7003
0.20	100	0.3455	0.0179	2.7120	1.0304	1000	0.00384	**2.7917**	**68.6979**	**81.0365**
0.25	100	0.2886	0.0237	3.3784	1.1019	1000	0.00288	3.2431	59.1632	74.5937
0.30	100	0.2461	0.0264	3.9862	1.1418	1000	0.00230	3.3304	51.8174	69.5435
1.00	100	0.0724	-0.2033	7.2086	-0.0773	1000	0.00060	6.3000	19.8233	46.7283
0.05	1000	0.1149	-0.0159	0.0087	-0.0087	10000	0.00008	**2.5000**	**-99.9996**	**-99.999**
0.10	1000	0.0210	-0.2071	-0.9429	0.0190	10000	0.00004	-9.5000	-100.0000	-100.0000
0.15	1000	0.0054	-0.7574	-3.7000	-0.2889	10000	0.00002	27.0000	-100.0000	-100.0000
0.20	1000	0.0018	-3.5056	-8.4000	-1.3167	10000	0.00002	-15.0000	-100.0000	-100.0000
0.25	1000	0.0007	**0.2714**	**-10.5714**	**1.4343**	10000	0.00001	50.0000	-100.0000	-100.0000
0.30	1000	0.0003	7.7333	-8.8700	9.7900	10000	0.00001	22.0000	-100.0000	-100.0000
1.00	1000	0.0000	–	–	–	10000	0.00000	–	–	–

a choice of specific parameters, exact values of the ruin probability can be computed using numerical inversion of Laplace transform technics, see Wikstad [43], Thorin & Wikstad [41]. Let us now compare the three approximations. In Table 5.2 the exact and approximated values, and relative errors are shown with respect to u and θ with the distribution parameters like in [20], cf. Figure 5.2 and 5.3.

It is easy to notice that De Vylder and 4-moment gamma De Vylder are no match for the Pollaczek–Khinchin approximation, see the boldfaced results. This and Figure 5.1-5.2 justify the thesis that the Pollaczek–Khinchin approximation can be chosen as the reference method. Moreover, it is worth noting that in the light-tailed cases the 4-moment gamma De Vylder approximation gives much more accurate results than the original method. Henceforth we will compare the methods with respect to the values obtained via the Pollaczek–Khinchin formula.

5.5. Pollaczek–Khinchin approximation as the reference method

In the lognormal case, see Figure 5.3, the situation is very interesting. All methods give the error greater then 50%. The lognormal case is quite important as often loss data appear to have the lognormal

distribution. Thus we may say that using the Pollaczek–Khinchin approximation is essential when dealing with real-life data.
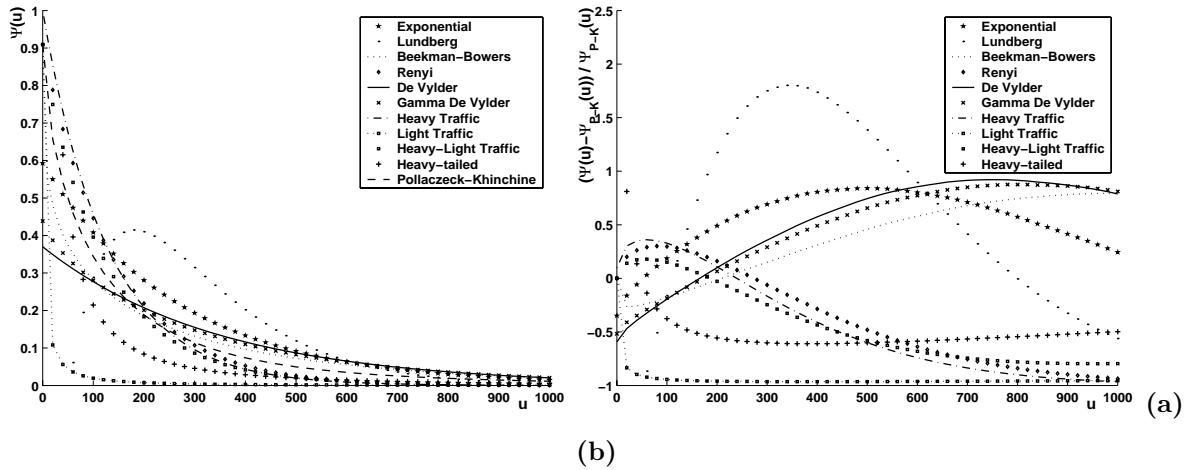

FIGURE 5.3. Illustration of the ruin probability (a) and the relative error (b) of the approximations (with respect to the Pollaczek–Khinchin approximation). The lognormal case with $\mu = -1.62$ i $\sigma = 1.8$, $\theta = 0.1$ and $u \leq 1000$.
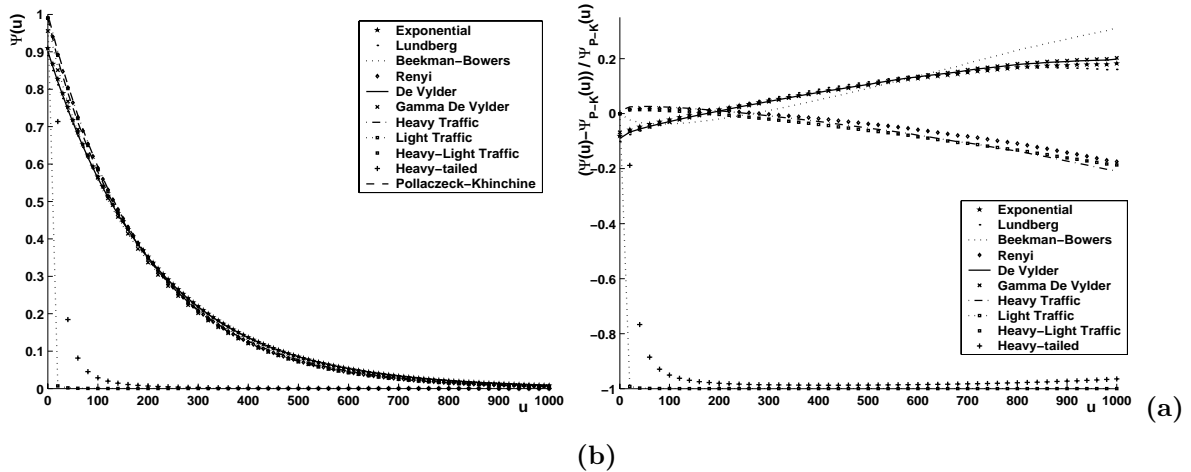

FIGURE 5.4. Illustration of the ruin probability (a) and the relative error (b) of the approximations (with respect to the Pollaczek–Khinchin approximation). The Pareto case with $\alpha = 3.1$, $\nu = 2.1$, $\theta = 0.01$ and $u \leq 1000$.

For the Pareto distributed claims, see Figure 5.4, all methods produce the error up to about 20%, light traffic and heavy-tailed approximations show a total lack of accuracy. The parameters of the Pareto distribution imply that the first three moments still exist. In the cases when it is not true, only a few approximations remain useful.

Finally, we claim, that the approximation via the Pollaczek–Khinchin formula is the best method for calculating the ruin probability in infinite time:

- Only two of 12 considered approximations work for all distributions, namely Pollaczek–Khinchin and light traffic. From Figure 5.1-5.2 it is clear that the former is much better.

- Figure 5.1-5.2 demonstrate that among all presented approximations which work for the light- and heavy-tailed distributions only De Vylder, 4-moment gamma De Vylder and Pollaczek–Khinchin behave well.

- In Table 5.2 the exact and approximated values of the three approximations, and relative errors with respect to u and θ are shown. It is easy to notice that both De Vylder approximations are no match for the Pollaczek–Khinchin approximation, see the boldfaced results. This and Figure 5.1-5.2 justify the thesis that the Pollaczek–Khinchin approximation can be chosen as the reference method

- The Pollaczek–Khinchin approximation gives the most accurate results, even for the class of heavy-tailed distributions like lognormal. We also note that in each case for the Monte Carlo method purposes we generated 100 blocks of 100000 simulations and the variance within the results derived from the blocks was always relatively small.

More detailed analysis and a wider variety of loss distributions studied can be found in [12], Chapter 15, and in papers by Burnecki, Miśta and Weron ([9], [7]).

CHAPTER 6

Diffusion model with losses given by mixture of exponentials.

This chapter is devoted to an extension of classical risk process to more general form, i.e. risk process that, between jumps, follows Brownion motion B_t with drift. An easy to compute formula for ruin probability when the Laplace transform of the claim size distribution is a rational function, is given. Moreover, for the mixture of exponentials an analytic formula is found in details. The general ideas and some parts of the proof are heavily borrowed from the papers by Jacobsen ([**25**], [**26**], [**27**]).

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P}_x)$ be a probability space with filtration $\mathcal{F}_t$ and with Markov state space $\mathbb{E}$. The set of measurable and bounded functions $f : \mathbb{E} \to \mathbb{R}$ equipped with a supremum norm is a Banach space. For all $t \in \mathbb{R}_+$ define the contracting semigroup of operators

$$P_t f(x) = \mathrm{E}_x f(x_t).$$

The first derivative of P_t in $t = 0$ is called the infinitesimal generator

$$\mathcal{A}f = \lim_{t \searrow 0} \frac{P_t f - f}{t}.$$

Dynkin formula gives

$$P_t f - f = \int_0^t \mathcal{A} P_s f \ ds = \int_0^t P_s \mathcal{A}f \ ds.$$

For the risk process of the form:

$$(6.1) \qquad R_t = R_0 + \beta t + \sigma B_t - \sum_{n=1}^{N_t} X_n,$$

the infinitesimal generator is given by

$$(6.2) \qquad \mathcal{A}f(x) = \beta f'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda \int_0^\infty F_X(dy)(f(x-y) - f(x)).$$

LEMMA 6.0.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with $f \in C^2 B^2(\mathbb{R}_+)$ – twice continuously differentiable with up to second derivative bounded on $\mathbb{R}_+$. Then, by Itô's formula and the martingale representation, we get the following formula*

$$(6.3) \qquad f(R_{\tau \wedge t}) = f(R_0) + \int_0^{\tau \wedge t} \mathcal{A}f(R_s) \ ds + M_t,$$

for M being $\mathcal{F}_t$-martingale starting from zero ($M_0 \equiv 0$).

For the proof involving martingale representations and suitable version of Itô's formula, we refer reader to [**27**], or to any book devoted to martingale theory.

6.1. Laplace transform of claims being rational function

Let us distinguish now between the ruin caused by a jump and ruin by continuity. Denote the corresponding sets of events by:

$$A_j = \{R_\tau < 0,\ \tau < \infty\},\ \text{ and }\ A_c = \{R_\tau = 0,\ \tau < \infty\}.$$

Then the ruin probability of the risk process given by (6.1) starting from R_0 has the form

$$(6.4) \qquad\qquad \mathrm{P}_{R_0}(\tau < \infty) = \mathrm{P}_{R_0}(A_j) + \mathrm{P}_{R_0}(A_c).$$

We will need also the following notation: for $z \in \mathbb{C}$ let $q(z) = \phi(z) - \lambda = \beta z + \frac{1}{2}\sigma^2 z^2 - \lambda$, where $\phi(z) = \beta z + \frac{1}{2}\sigma^2 z^2$ is a polynomial of degree ≤ 2 associated with the scaled Brownian motion with drift that risk process follows between jumps caused by claims occurrences.

The further important assumption we require is that the Laplace transform L_X of the distribution of claims is a rational function, i.e.

$$(6.5) \qquad\qquad L_X(\nu) = \mathrm{E}\exp(-\nu X) = \frac{P_X(\nu)}{Q_X(\nu)}$$

with $\nu \geq 0$ and P_X, Q_X being the polynomials with no common complex roots and the leading coefficient for Q_X equal to 1. It follows that if the degree of Q_X is m, P has to be of degree $\leq m$. However, we need $P_X(z)$ and $Q_X(z)$ for all $z \in \mathbb{C}$, the extension of L_X to $\bar{L}_X(z) = \frac{P_X(z)}{Q_X(z)}$ leads to $\mathrm{E}\exp(-zX)$, that is only guaranteed for z with $\mathcal{R}e(z) > -\epsilon$ and ϵ small enough. This implies that all the m roots of $Q_X(z)$ satisfy $\mathcal{R}e(z) < 0$.

In order to state the proposition, we need the following two versions of the Cramèr-Lundberg equation:

$$(6.6) \qquad\qquad Q_X(\gamma) = -P_X(\gamma)\frac{\lambda}{\beta\gamma + \frac{1}{2}\sigma^2\gamma^2 - \lambda}$$

and the modified version

$$(6.7) \qquad\qquad Q_X(\gamma)(\beta\gamma + \frac{1}{2}\sigma^2\gamma^2 - \lambda) = -\lambda P_X(\gamma).$$

PROPOSITION 6.1.1. *For $R_0 > 0$, ruin probability $P_{R_0}(\tau < \infty) = 1$ if and only if*

$$(6.8) \qquad\qquad \beta \leq \lambda\, EX.$$

PROOF. Since now, to simplify the notation, we denote the ruin probability $\mathrm{P}_{R_0}(\tau < \infty)$ shortly by P_{ruin}. By (6.1),

$$(6.9) \qquad\qquad \frac{R_t - R_0}{t} = \beta + \frac{\sigma B_t}{t} - \frac{1}{t}\sum_{n=1}^{N_t} X_n.$$

It is clear, that the second term on the right side converges to 0 a.s., and the term with the sum to $\lambda \mathrm{E}X$. Thus $R_t \to -\infty$ P_{R_0} −a.s. if $\beta < \lambda \mathrm{E}X$ and $R_t \to +\infty$ P_{R_0} −a.s. if $\beta > \lambda \mathrm{E}X$. So the sharp inequality in (6.8) implies ruin probability P_{ruin} equal to 1, and also if (6.8) does not hold, there's nonzero probability, that drift of R_t goes to ∞ before any claim has arrived, so $\mathrm{P}_{ruin} < 1$. For equality in (6.8) we may consider risk process with $\beta - \epsilon$ instead of β and the result comes from taking $\epsilon \searrow 0$. □

Note, that $\gamma = 0$ is always the solution to the Cramèr-Lundberg equations (6.6), (6.7). We will show, that $\mathrm{P}_{ruin} < 1$, implies that equation (6.7) has exactly $m+1$ solutions with $\mathcal{R}e(\gamma) < 0$. The next result is a slight modification of Theorem 1 in [**27**].

PROPOSITION 6.1.2. *Let us consider the risk process R_t in the form of (6.1) with Laplace transform of claim size distribution being a rational function*

$$L_X(\nu) = \frac{P_X(\nu)}{Q_X(\nu)}, \quad degree(Q) = m.$$

(i) *If $P_{ruin} < 1$, then the Cramèr-Lundberg equation (6.7) has precisely $m+1$ solutions $(\gamma_l)_{1 \le l \le m+1}$ with $\mathcal{R}e(\gamma_l) < 0$,*

(ii) *$\gamma_l : \mathcal{R}e(\gamma_l) < 0$ is a solution to (6.6) if and only if γ_l is a solution to the modified Cramèr-Lundberg equation (6.7) with $q(\gamma_l) = \beta \gamma_l + \frac{1}{2}\sigma^2 \gamma_l^2 - \lambda \ne 0$,*

(iii) *If $(\tilde{\gamma}_k)_{1 \le k \le m}$ are any m of the $(m+1)$ solutions to (6.6) with $\mathcal{R}e(\tilde{\gamma}_k) < 0$ and these solutions are distinct with $q(\tilde{\gamma}_k) = \beta \tilde{\gamma}_k + \frac{1}{2}\sigma^2 \tilde{\gamma}_k^2 - \lambda \ne 0$, it holds for all $R_0 > 0$ that*

(6.10)
$$\sum_{k=1}^{m} r_k \frac{\lambda}{q(\tilde{\gamma}_k)} P_{R_0}(A_c) - \left(\sum_{k=1}^{m} r_k \right) P_{R_0}(A_j) = \sum_{k=1}^{m} r_k \frac{\lambda \exp(\tilde{\gamma}_k R_0)}{q(\tilde{\gamma}_k)},$$

with r_k given by

(6.11)
$$r_k = -\frac{P_X(\tilde{\gamma}_k)}{\tilde{\gamma}_k \prod_{k' \ne k}(\tilde{\gamma}_k - \tilde{\gamma}_{k'})},$$

(iv) *If $P_{ruin} < 1$ and all the solutions $(\gamma_l)_{1 \le l \le m+1}$ to (6.6) with $\mathcal{R}e(\gamma_l) < 0$ are distinct and $q(\gamma_l) \ne 0$, using (6.10) twice with, say, $(\tilde{\gamma}_k)_{1 \le k \le m} = (\gamma_1, \ldots, \gamma_{m-1}, \gamma_{m+s})$, $s = \{0, 1\}$, we obtain a system of 2 linear equations with unknowns $P_{R_0}(A_j)$ and $P_{R_0}(A_c)$, that can be solved uniquely provided matrix of coefficients is non-singular.*

PROOF. We proceed to discuss (iii), as (ii) and (iv) seems to be obvious. Let $(\gamma_1, \ldots, \gamma_m)$ be m distinct roots of equation (6.6) and consider $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \begin{cases} \sum_{k=1}^{m} c_k \exp(\gamma_k x) & x \ge 0 \\ K & x < 0, \end{cases} \qquad (6.12)$$

where

(6.13)
$$c_k = \frac{\lambda r_k}{q(\gamma_k)}, \quad K = -\sum_{k=1}^{m} r_k,$$

and r_k as in (6.11) above.

Next we aim to show

(6.14) $$\mathcal{A}f(x) = 0, \qquad x \geq 0.$$

Then consequently by (6.3), for any $t \geq 0$ holds

(6.15) $$\mathrm{E}_{R_0}\left[f(R_\tau); \tau \leq t\right] + \mathrm{E}_{R_0}\left[f(R_t); \tau > t\right] = f(R_0).$$

The limit

(6.16) $$\lim_{t \to \infty} \mathrm{E}_{R_0}\left[f(R_t); \tau > t\right] = 0,$$

for $\mathrm{P}_{ruin} = 1$ is obvious since f is bounded. If $\mathrm{P}_{ruin} < 1$ by Proposition (6.1.1) $\lim_{t \to \infty} R_t = \infty$ a.s., which together with $\mathcal{R}e(\gamma_k) < 0$ and definition of f implies that $\lim_{t \to \infty} f(R_t) = 0$. Now (6.16) follows by dominated convergence. Clearly letting $t \to \infty$ in (6.3) yields

$$\mathrm{E}_{R_0}\left[f(R_\tau); \tau < \infty\right] = f(R_0),$$

that is the same as

$$\mathrm{E}_{R_0}\left[\sum_{k=1}^{m} c_k; R_\tau = 0\right] + \mathrm{E}_{R_0}\left[K; R_\tau < 0\right] = \sum_{k=1}^{m} c_k \exp(\gamma_k R_0),$$

and at this part (iii) will be proven if only we can show (6.14) with f, c_k, K as in (6.12), (6.13). Writing out (6.14):

$$
\begin{aligned}
\mathcal{A}f(x) &= \sum_{k=1}^{m} c_k (\beta\gamma_k + \sigma^2\gamma_k^2)e^{\gamma_k x} + \lambda \int_0^x F_X(dy) \sum_{k=1}^{m} c_k e^{\gamma_k(x-y)} + \lambda K \int_x^\infty F_X(dy) \\
&\quad - \lambda \int_0^x \sum_{k=1}^{m} c_k e^{\gamma_k x} F_X(dy) \\
&= \sum_{k=1}^{m} c_k q(\gamma_k) e^{\gamma_k x} + \lambda \left[\int_0^x F_X(dy) \sum_{k=1}^{m} c_k e^{\gamma_k(x-y)} + K \int_x^\infty F_X(dy)\right] \\
&= \lambda \left[\sum_{k=1}^{m} r_k e^{\gamma_k x} + \int_0^x F_X(dy) \sum_{k=1}^{m} c_k e^{\gamma_k(x-y)} + K \int_x^\infty F_X(dy)\right] = 0.
\end{aligned}
$$

With $x = 0$, the above takes the form $\sum_{k=1}^{m} r_k + K$ and is equal to 0 by definition of K.

Let $x > 0$, we have to show

$$\sum_{k=1}^{m} \left[r_k \exp(\gamma_k x) + \int_0^x F_X(dy) c_k \exp(\gamma_k(x-y))\right] + K(1 - F_X(x)) = 0,$$

and because all $\mathcal{R}e(\gamma_k) < 0$, it holds if and only if it holds for the Laplace transform. Multiplying by $e^{-\nu x}$ for $\nu \geq 0$ and integrating from 0 to ∞ we get

$$\sum_{k=1}^{m} \left[r_k \int_0^\infty e^{(\gamma_k-\nu)x}dx + \int_0^\infty e^{-(\nu-\gamma_k)x}dx \int_0^x c_k e^{-\gamma_k y} F_X(dy) - r_k \int_0^\infty e^{-\nu x}dx \int_x^\infty F_X(dy)\right] = 0.$$

Using some elementary calculations for Laplace transforms, we arrive at

$$\sum_{k=1}^{m}\left[r_k\left(\frac{1}{\nu-\gamma_k}-\frac{1}{\nu}(1-L_X(\nu))\right)+c_k\frac{L_X(\nu)}{\nu-\gamma_k}\right]=0,$$

and further

$$L_X(\nu) = \frac{\sum_{k=1}^{m}r_k\left(\frac{1}{\nu}-\frac{1}{\nu-\gamma_k}\right)}{\sum_{k=1}^{m}\left(c_k\frac{1}{\nu-\gamma_k}+r_k\frac{1}{\nu}\right)} = \frac{-\prod_{k=1}^{m}(\nu-\gamma_k)\sum_{k=1}^{m}\frac{r_k\gamma_k}{\nu(\nu-\gamma_k)}}{\prod_{k=1}^{m}(\nu-\gamma_k)\sum_{k=1}^{m}\frac{c_k\nu+r_k(\nu-\gamma_k)}{\nu(\nu-\gamma_k)}}$$

(6.17)
$$= \frac{-\sum_{k=1}^{m}r_k\gamma_k\pi_k(\nu)}{\sum_{k=1}^{m}(c_k\nu+r_k(\nu-\gamma_k))\pi_k(\nu)}, \qquad \text{where} \quad \pi_k(\nu)=\prod_{k'\neq k}(\nu-\gamma_{k'}).$$

LEMMA 6.1.1. *Suppose $\mathcal{S}$ is a polynomial of degree $\leq m-1$ and $\gamma_1,\ldots,\gamma_m$ are complex, distinct numbers. Then for $z\in\mathbb{C}$*

(6.18)
$$\mathcal{S}(z)=\sum_{k=1}^{m}\frac{\mathcal{S}(\gamma_k)}{\pi_k(\gamma_k)}\pi_k(z).$$

Continuing with (6.17), inserting $r_k=-\frac{P_X(\gamma_k)}{\gamma_k\pi_k(\gamma_k)}$ and using lemma 6.1.1 we identify the numerator as $P_X(\nu)$. Therefore

$$L_X(\nu)=\frac{\sum_{k=1}^{m}\frac{P_X(\gamma_k)}{\pi_k(\gamma_k)}\pi_k(\nu)}{\sum_{k=1}^{m}\left(c_k\nu-\frac{P_X(\gamma_k)(\nu-\gamma_k)}{\gamma_k\pi_k(\gamma_k)}\right)\pi_k(\nu)}=\frac{P_X(\nu)}{\nu\sum_{k=1}^{m}\left[\left(\frac{\lambda}{q(\gamma_k)}+1\right)\frac{-P_X(\gamma_k)}{\gamma_k\pi_k(\gamma_k)}\pi_k(\nu)\right]+\sum_{k=1}^{m}\frac{P_X(\gamma_k)}{\pi_k(\gamma_k)}\pi_k(\nu)},$$

and by Cramèr-Lundberg equation (6.7), $\frac{\lambda}{q(\gamma_k)}+1=\frac{P_X(\gamma_k)-Q_X(\gamma_k)}{P_X(\gamma_k)}$, so we finally arrive at

$$L_X(\nu)=\frac{P_X(\nu)}{-\nu\sum_{k=1}^{m}(P_X(\gamma_k)-Q_X(\gamma_k))\frac{\pi_k(\nu)}{\gamma_k\pi_k(\gamma_k)}+P_X(\nu)}=\frac{P_X(\nu)}{\frac{Q_X(\nu)-P_X(\nu)}{\nu}+P_X(\nu)}=\frac{P_X(\nu)}{Q_X(\nu)}.$$

The latter comes with lemma (6.1.1) applied with polynomial $\mathcal{S}$ of the form $\mathcal{S}(z)=\frac{Q_X(z)-P_X(z)}{z}$ (degree of $\mathcal{S}\leq m-1$ since $z=0$ is always the root to $Q-P$). That completes the proof of (iii).

We proceed with the proof of (i). Remind both sides of Cramèr-Lundberg equation (6.7) are polynomials and the left side of (6.7), let us denote it by $\mathcal{S}_l$, is a polynomial of degree $m+2$. Similarly $\mathcal{S}_r$ is of degree $\leq m-1$. As two roots to $q(\gamma)$ are $\gamma_{\pm}=\frac{1}{\sigma^2}(-\beta\pm\sqrt{\beta^2+2\lambda\sigma^2})$, we conclude $\mathcal{S}_l$ has exactly $m+1$ roots with $\mathcal{R}e(\gamma)<0$.

Let us recall now Rouché theorem from complex function theory (see e.g. [**24**]).

FACT 6.1.1. **(Rouché Theorem)** *Consider functions f and g on compact set $U\subseteq\mathbb{C}$, let ∂U be a curve bounding U. Assume $f,g:U\to\mathbb{C}$ are analytical on U and following inequality holds*

(6.19)
$$|f(z)|>|g(z)|, \qquad \text{for } z\in\partial U.$$

Then f and $f+g$ has the same number of roots on U.

Returning to proof of (i), let $\rho > 0$ be given and $0 < \epsilon < \rho$ be small enough to ensure $q(z) \neq 0$ for $|z| \leq \epsilon$ and $\bar{L}_X(z)$ to exist. Consider the open set $U_{\rho,\epsilon}$ given by the interior of the curve (boundary):

$$\partial U = \{z : |z| = \rho, \mathcal{Re}(z) < 0\} \cup \{z : z = iy, \, y \in \mathbb{R}, \, \epsilon \leq |y| \leq \rho, \} \cup \{z : |z| = \epsilon, \mathcal{Re}(z) < 0\}.$$

Take $f \equiv \mathcal{S}_l$, $g \equiv -\mathcal{S}_r$. To imply the claim in (i) by Rouché theorem, it is sufficient to show $|-\mathcal{S}_r| < |\mathcal{S}_l|$ for $z \in \partial U$, ρ large enough and ϵ small enough.

Now we have to show

(6.20)
$$|-\lambda P_X(z)| < \left| Q_X(z)(\beta z + \frac{1}{2}\sigma^2 z^2 - \lambda) \right|.$$

- For $|z| = \rho$ with ρ sufficiently large (6.20) holds as $degree(\mathcal{S}_l) > degree(\mathcal{S}_r)$,
- for $z = iy$ with $|y| \geq \epsilon$, since $q(z) \neq 0$ (6.20) is equivalent to

$$\left| \bar{L}_X(iy) \right| \left| \frac{\lambda}{\beta iy - \frac{1}{2}\sigma^2 y^2 - \lambda} \right| < 1.$$

The first term on the left side is ≤ 1 and $\lambda < |\beta iy - \frac{1}{2}\sigma^2 y^2 - \lambda| = \sqrt{\frac{1}{4}\sigma^4 y^4 + \beta^2 y^2 + \lambda\sigma^2 y^2 + \lambda^2}$ holds since $y \neq 0$.

- We are left with the case $|z| = \epsilon$, $\mathcal{Re}(z) < 0$. Since for ϵ sufficiently small $P_X(z) \neq 0$, inequality (6.20) is equivalent to

(6.21)
$$\left| \frac{\lambda}{\beta z + \frac{1}{2}\sigma^2 z^2 - \lambda} \right| < \frac{1}{|\bar{L}_X(z)|}.$$

Consider following functions, being sides of above inequality, $g_l(z) = \frac{\lambda}{\beta z + \frac{1}{2}\sigma^2 z^2 - \lambda}$ and $g_r(z) = \frac{1}{|\bar{L}_X(z)|}$. Both are analytical near 0 with $g_l(0) = 1$, $g_r(0) = 1$. For $z = x + iy$ close to 0, using Taylor expansion, we arrive at

$$|g(z)|^2 = g^2(0) + 2xg(0)g'(0) + y^2(g'(0)^2 - g(0)g''(0)) + o(x) + o(y^2).$$

Now, inequality (6.21) holds for $x, y \neq 0$ small enough, if following two inequalities hold

$$xg_l'(0) < xg_r'(0) \qquad \text{and} \qquad y^2(g_l'^2(0) - g_l''(0)) < y^2(g_r'^2(0) - g_r''(0)),$$

and equivalently

$$g_l'(0) > g_r'(0) \qquad \text{and} \qquad g_l'^2(0) - g_l''(0) < g_r'^2(0) - g_r''(0).$$

Since $g_r'(0) = \mathrm{E}\,X$, $g_r''(0) = -(\mathrm{E}\,X^2 - 2(\mathrm{E}\,X)^2)$, then

$$g_r'^2(0) - g_r''(0) = \mathrm{Var}\,X > 0.$$

Differentiating g_l, we come to $g_l'(0) = \frac{\beta}{\lambda}$ and $g_l''(0) = \frac{\lambda\sigma^2 + 2\beta^2}{\lambda^2}$, so

$$g_l'^2(0) - g_l''(0) = -\frac{\lambda\sigma^2 + \beta^2}{\lambda^2} < 0.$$

Finally

$$g_l'^2(0) - g_l''(0) < g_r'^2(0) - g_r''(0) \qquad \text{and} \qquad g_l'(0) > g_r'(0),$$

since $\beta > \lambda \mathrm{E}\,X$ comes with proposition (6.1.1). The latter ends the proof of (i).

$\square$

6.2. Mixture of exponentials claims

Next we aim to focus on mixture of exponential distributions as a key distribution in further considerations. Mixture of m exponential distribution, say given by $F_X(\nu) = \sum_{i=1}^{m} a_i(1 - \exp(-\delta_i \nu))$, surely belongs to family of distributions with Laplace transform being a rational function, as

$$L_X(\nu) = \sum_{i=1}^{m} a_i \frac{\delta_i}{\delta_i + \nu} = \frac{\sum_{i=1}^{m} a_i \delta_i \prod_{j \neq i}(\delta_j + \nu)}{\prod_{j=1}^{m}(\delta_j + \nu)} = \frac{P_X(\nu)}{Q_X(\nu)}$$

is well defined for $\nu > -\min_{i=1,\ldots,m}\{\delta_i\}$ and P_X, Q_X are polynomials respectively of degree m and $\leq m - 1$. In this case the modified Cramèr-Lundberg equation (6.7) takes a form:

$$(6.22) \qquad \prod_{j=1}^{m}(\delta_j + \gamma)(\beta\gamma + \frac{1}{2}\sigma^2\gamma^2 - \lambda) = -\lambda \sum_{i=1}^{m} a_i \delta_i \prod_{j \neq i}(\delta_j + \gamma).$$

Solving above equation amounts to finding the zeros of a polynomial of degree $m + 1$ with $\gamma = 0$ always a root. It can always be computed with no special difficulties. At this moment we are ready to state the main theorem that gives exact formulas for ruin probability in discussed case.

THEOREM 6.2.1. *Let us consider the risk process R_t in the form of (6.1) with claim sizes following mixture of m exponential distributions F_X with parameters $(a_1, \ldots, a_m)$ and $(\delta_1, \ldots, \delta_m)$. To ensure $P_{ruin} < 1$ assume drift coefficient $\beta < \lambda \sum_{i=1}^{m} \frac{a_i}{\delta_i}$. Then*

(i) *the Cramèr-Lundberg equation (6.22) has precisely $m+1$ solutions $(\gamma_k)_{1 \leq k \leq m+1}$ with $\mathcal{R}e(\gamma_k) < 0$,*

(ii) *If all the solutions $(\gamma_k)_{1 \leq k \leq m+1}$ to (6.22) with $\mathcal{R}e(\gamma_k) < 0$ are distinct and $q(\gamma_k) = \beta\gamma_k + \frac{1}{2}\sigma^2\gamma_k^2 - \lambda \neq 0$, then for all $R_0 > 0$*

$$(6.23) \qquad P_{ruin} = \frac{\Sigma_m^{(1)}\Sigma_{m+1}^{(3)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(3)} - \Sigma_m^{(3)}\Sigma_{m+1}^{(2)} + \Sigma_{m+1}^{(3)}\Sigma_m^{(2)}}{\Sigma_m^{(1)}\Sigma_{m+1}^{(2)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(2)}},$$

with $\Sigma_l^{(1)}, \Sigma_l^{(2)}, \Sigma_l^{(3)}$ given by:

$$(6.24) \qquad \Sigma_l^{(1)} = \sum_{k \neq l} r_k, \qquad \Sigma_l^{(2)} = \sum_{k \neq l} \frac{\lambda r_k}{q(\gamma_k)}, \qquad \Sigma_l^{(3)} = \sum_{k \neq l} \frac{\lambda r_k \exp(\gamma_k R_0)}{q(\gamma_k)}.$$

and

$$(6.25) \qquad r_k = -\frac{\sum_{i=1}^{m} a_i \delta_i \prod_{j \neq i}(\delta_j + \gamma_k)}{\gamma_k \prod_{k' \neq k}(\gamma_k - \gamma_{k'})}.$$

PROOF. The first part (i) of Theorem follows directly from (i) of Proposition 6.1.2 and Proposition 6.1.1. To arrive at (ii) one needs to use (iii) and (iv) of Proposition 6.1.2, let's say, for $(\gamma_1, \ldots, \gamma_m)$ and $(\gamma_1, \ldots, \gamma_{m-1}, \gamma_{m+1})$ being appropriate solutions to equation (6.22) to obtain a system of 2 equations:

$$\sum_{k \neq m} \frac{\lambda r_k}{q(\gamma_k)} P_{R_0}(A_c) - \sum_{k \neq m} r_k P_{R_0}(A_j) = \sum_{k \neq m} \frac{\lambda r_k \exp(\gamma_k R_0)}{q(\gamma_k)}$$

$$\sum_{k \neq m+1} \frac{\lambda r_k}{q(\gamma_k)} P_{R_0}(A_c) - \sum_{k \neq m+1} r_k P_{R_0}(A_j) = \sum_{k \neq m+1} \frac{\lambda r_k \exp(\gamma_k R_0)}{q(\gamma_k)}$$

with r_k as in (6.25). Setting notation (6.24), after some elementary algebra we arrive at

$$\mathrm{P}_{R_0}(A_c) = \frac{\Sigma_m^{(1)}\Sigma_{m+1}^{(3)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(3)}}{\Sigma_m^{(1)}\Sigma_{m+1}^{(2)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(2)}}$$

$$\mathrm{P}_{R_0}(A_j) = \frac{-\Sigma_m^{(3)}\Sigma_{m+1}^{(2)} + \Sigma_{m+1}^{(3)}\Sigma_m^{(2)}}{\Sigma_m^{(1)}\Sigma_{m+1}^{(2)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(2)}}$$

and finally

$$\mathrm{P}_{ruin} = \mathrm{P}_{R_0}(A_c) + \mathrm{P}_{R_0}(A_j) = \frac{\Sigma_m^{(1)}\Sigma_{m+1}^{(3)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(3)} - \Sigma_m^{(3)}\Sigma_{m+1}^{(2)} + \Sigma_{m+1}^{(3)}\Sigma_m^{(2)}}{\Sigma_m^{(1)}\Sigma_{m+1}^{(2)} - \Sigma_{m+1}^{(1)}\Sigma_m^{(2)}}.$$

That completes the proof of Theorem 6.2.1. □

Finally, let us emphasize the results obtained here in this chapter are important for operational risk modelling. First, modelling operational risk process by classical model with diffusion component allows to fit the model closer to real life situations. For the second, as it was noticed in Chapter 1, mixtures of distributions are naturally exploited distributions in operational risk modelling. Hence any analytical solution in this field is of significant importance.

CHAPTER 7

Building operational reserves.

7.1. Introduction

In this chapter, setting the appropriate level of capital charge c for operational risk is considered in a broader context of business decisions, concerning also risk transfer through insurance. The ideas are taken and reformulated from insurance risk theory. For detailed discussion on setting the level of insurance premium in the environment of insurance risk process with reinsurance and rate of return on capital, see Otto and Miśta [**30**].

At first, the simple version of the problem is solved to illustrate the idea of the chapter. Let us consider our model of an operational risk process based on (3.1) describing capital assets or profit of a company exposed to operational losses, in the basic form:

$$R_t = u + ct - S_t, \qquad t \geq 0,$$

where R_t denotes the current capital at time t, $u = R_0$ stands for critical level of capital, that should never be exceeded under threat of bankruptcy, c is the amount of operational capital charge to cover one year losses, and S_t is the aggregate loss process – amount of loss outlays over the period $(0, t]$. Let us assume that increments of the aggregate loss process $S_{t+h} - S_t$ are for any $t, h > 0$ normally distributed $N(\mu h, \sigma^2 h)$ and mutually independent.

In this case the probability of ruin is an exponential function of the initial capital:

$$\psi(u) = \exp(-Ru), \qquad u \geq 0,$$

where the adjustment coefficient R exists for $c > \mu$, and equals then $2(c - \mu)\sigma^{-2}$. The above formula can be easily inverted to render the operational reserves c for a given critical capital or profit u and predetermined level ψ of ruin probability:

$$c = \mu + \frac{-\log(\psi)}{2u}\sigma^2.$$

Given the safety standard ψ, the larger the expected budget gain (or critical level of capital) u of the company is, the lower reserves c it does need to cover the operational risk.

Throughout the rest of chapter, the above simplistic assumptions on the risk process can be dropped. It is shown there how to invert various approximate formulas for the ruin probability in order to calculate necessary capital charge for the whole business as well as to decompose it into individual business risks lines. Finally, an extension of the decision problem by allowing for additional insurance is considered.

We assume that we typically have at our disposal incomplete information on the distribution of the aggregate loss, and this incomplete information set consists of cumulants of order 1, 2, 3, and possibly

4. The rationale is that sensible empirical investigation of frequency and severity distributions could be done only separately for sub-portfolios (business lines) of homogeneous risks. Cumulants for the whole business are then obtained just by summing up figures over the collection of sub-portfolios, provided that sub-portfolios are mutually independent. Both the quantile of the current year loss and the probability of ruin in the long run will be approximated by formulas based on cumulants of the one-year aggregate loss W.

7.2. Ruin probability criterion

Presuming long-run horizon for operational reserves calculation we widely exploit the ruin theory. Our aim is now to obtain such a level of operational capital charge c needed to cover each year the aggregate loss W, which results from a profit u presumed in budget (or other critical level of capital, that cannot be lost) and accepted level of ruin probability ψ. This is done by inverting various approximate formulae for the probability of ruin. Information requirements of different methods are emphasized. For details on the variety of approximations look for in previous chapters.

7.2.1. Approximation based on Lundberg inequality. This is a simplest (and crude) approximation method, simply assuming replacement of the true function $\psi(u)$ by:

$$\psi_{Li}(u) = e^{-Ru}.$$

At first we obtain the approximation $R_{(Li)}$ of the desired level of the adjustment coefficient R:

$$R_{(Li)} = \frac{-\ln \psi}{u}.$$

In the next step we make use of the definition of the adjustment coefficient for the portfolio:

$$\mathrm{E}\left(e^{RW}\right) = e^{Rc(W)},$$

to obtain directly the reserve amount formula:

$$c(W) = R^{-1} \ln \left\{ \mathrm{E}\left(e^{RW}\right) \right\} = R^{-1} C_W(R),$$

where C_W denotes the cumulant generating function and $c(W)$ the yearly capital charge enough to cover each year loss W. The result is well known as the exponential premium formula in insurance. It possesses several desirable properties – not only that it is derivable from ruin theory. First of all, by the virtue of properties of the cumulant generating function, it is additive for independent risks. So there is no need to distinguish between marginal and basic reserves for individual risk lines. The formula can be practically applied once we replace the adjustment coefficient R by its approximation $R_{(Li)}$.

Under certain conditions we could rely on truncating higher order terms in the expansion of the cumulant generating function:

(7.1) $$c(W) = \frac{1}{R} C_W(R) = \mu_W + \frac{1}{2!} R\sigma_W^2 + \frac{1}{3!} R^2 \mu_{3,W} + \frac{1}{4!} R^3 c_{4,W} + ...,$$

and use for the purpose of individual risk line pricing the formula (where higher order terms are truncated as well):

$$(7.2) \qquad c\left(X\right) = \frac{1}{R}C_X\left(R\right) = \mu_X + \frac{1}{2!}R\sigma_X^2 + \frac{1}{3!}R^2\mu_{3,X} + \frac{1}{4!}R^3 c_{4,X} + ...$$

Some insight into the nature of the long-run criteria for capital charge calculation could be gained by re-arrangement of the formula (7.1). At first we could express the u level in units of standard deviation of the aggregate loss: $U = u\sigma_W^{-1}$. Now the adjustment coefficient could be expressed as:

$$R = \frac{-\ln\psi}{U\sigma_W},$$

and reserve formula (7.1) as:

$$(7.3) \qquad c\left(W\right) = \mu_W + \sigma_W\left\{\frac{1}{2!}\frac{-\ln\psi}{U} + \frac{1}{3!}\left(\frac{-\ln\psi}{U}\right)^2\gamma_W + \frac{1}{4!}\left(\frac{-\ln\psi}{U}\right)^3\gamma_{2,W} + ...\right\}$$

where in the brackets appear only unit-less figures, that form together the pricing formula for the standardized risk $\left(W - \mu_W\right)\sigma_W^{-1}$. Let us notice that the contribution of higher order terms in the expansion is neglectible when u is large enough. The above phenomenon could be interpreted as a result of risk diversification in time (as opposed to cross-sectional risk diversification). Provided the profit capital is large, the ruin (if it happens at all) will rather appear as a result of aggregation of poor results over many periods of time. However, given the skewness and kurtosis of one-year increment of the risk process, the sum of increments over n periods has skewness of order $n^{-\frac{1}{2}}$, kurtosis of order n^{-1} etc. Hence the larger the critical capital, the smaller importance of the difference between the distribution of the yearly increment and the normal distribution. In a way this is how the diversification of risk in time works (as opposed to cross-sectional diversification). In the case of a cross-sectional diversification the assumption of mutual independency of risks plays the crucial role. Analogously, diversification of risk in time works effectively when subsequent increments of the risk process are independent.

7.2.2. "Zero" approximation. The "zero" approximation is a kind of naive approximation, assuming replacement of the function $\psi(u)$ by:

$$\psi_0\left(u\right) = \left(1 + \theta\right)^{-1}\exp\left(-Ru\right),$$

where θ denotes the relative security loading, which means that $\left(1 + \theta\right) = \frac{c(W)}{E(W)}$. The "zero" approximation is applicable to the case of Poisson loss arrivals (as opposed to Lundberg inequality, which is applicable under more general assumptions). Relying on "zero" approximation leads to the system of two equations:

$$c\left(W\right) = R^{-1}C_W\left(R\right)$$

$$R = \frac{1}{u}\ln\frac{E(W)}{\psi c(W)}.$$

The system could be solved by assuming at first:

$$R^{(0)} = \frac{-\ln\psi}{u},$$

and next by executing iterations:

$$c^{(n)}(W) = \frac{1}{R^{(n-1)}} C_W \left(R^{(n-1)} \right)$$

$$R^{(n)} = \frac{1}{u} \ln \frac{\mathrm{E}(W)}{\psi c^{(n)}(W)},$$

that under reasonable circumstances converge quite quickly to the solution $R_{(0)} = \lim_{n \to \infty} R^{(n)}$, which allows applying formula (7.1) for the whole portfolio and formula (7.2) for individual risks, provided the coefficient R is replaced by its approximation $R_{(0)}$.

7.2.3. Cramér–Lundberg approximation. Operational reserve calculation could also be based on the Cramér-Lundberg approximation. In this case the problem can be reduced also to the system of equations (three this time):

$$
\begin{aligned}
c(W) &= R^{-1} C_W (R) \\
R &= \frac{1}{u} \left\{ -\ln \psi + \ln \frac{\mu_Y \theta}{M_Y'(R) - \mu_Y (1 + \theta)} \right\} \\
(1 + \theta) &= \frac{c(W)}{\mathrm{E}(W)}.
\end{aligned}
$$

where $M_Y'(\cdot)$ and μ_Y denote respectively the first order derivative of the moment generating function and the expectation of the severity distribution. Solution of the system in respect of unknowns $c(W)$, θ and R requires now a bit more complex calculations. Obtained result $R_{(CL)}$ could be used then to replace R in formulas (7.1) and (7.2). The method is applicable to the case of Poisson loss arrivals. Moreover, severity distribution has to be known in this case. It can be expected that the method will produce accurate results for large u.

7.2.4. Beekman–Bowers approximation. This method is often recommended as the one which produces relatively accurate approximations, especially for moderate amounts of u. The problem consists in solving the system of three equations:

$$
\begin{aligned}
\psi &= (1 + \theta)^{-1} \{1 - G_{\alpha, \beta}(u)\} \\
\frac{\alpha}{\beta} &= (1 + \theta) \frac{m_{2,Y}}{2\theta m_{1,Y}} \\
\frac{\alpha (\alpha + 1)}{\beta^2} &= (1 + \theta) \left\{ \frac{m_{3,Y}}{3\theta m_{1,Y}} + 2 \left(\frac{m_{2,Y}}{2\theta m_{1,Y}} \right)^2 \right\},
\end{aligned}
$$

where $G_{\alpha, \beta}$ denotes the cdf of the gamma distribution with parameters (α, β), and $m_{k,Y}$ denotes the raw moment of order k of the severity distribution. Last two equations arise from equating moments of the gamma distribution to conditional moments of the maximal loss distribution (provided the maximal loss is positive). Solving this system of equation is a bit cumbersome, as it involves multiple numerical evaluations of the cdf of the gamma distribution. The admissible solution exists provided $m_{3,Y} m_{1,Y} > m_{2,Y}^2$, that is always satisfied for arbitrary severity distribution with support on the positive part of the axis. Denoting the solution for the unknown θ by θ_{BB}, we can write the latter as a function:

$$\theta_{BB} = \theta_{BB}(u, \psi, m_{1,Y}, m_{2,Y}, m_{2,Y}),$$

and obtain the reserve $c(W)$ from the equation:

$$c_{BB}(W) = (1 + \theta_{BB}) \, E(W).$$

Formally, application of the method requires only moments of first three orders of the severity distribution to be finite. However, the problem arises when we wish to price individual risks (or business line risks). Then we have to know the moment generating function of the severity distribution, and it should obey conditions for adjustment coefficient to exist. If this is a case, we can replace the coefficient θ of the equation:

$$M_Y(R) = 1 + (1 + \theta) \, m_{1,Y} R$$

by its approximation θ_{BB}, and thus obtain the approximation $R_{(BB)}$ of the adjustment coefficient R. It allows calculating capital charge according to formulas (7.1) and (7.2). It is easy to verify that there is no danger of contradiction, as both formulas for $c_{BB}(W)$ produce the same result $(1 + \theta_{BB}) \, E(W) = R_{(BB)}^{-1} C_W(R_{(BB)})$.

7.2.5. Diffusion approximation. This approximation method requires the scarcest information. It suffices to know the first two moments of the increment of the risk process, to invert the formula:

$$\psi_D(u) = \exp\left(-R_{(D)} u\right),$$

where:

$$R_{(D)} = 2 \left\{ c(W) - \mu_W \right\} \sigma_W^{-2},$$

in order to obtain the reserve:

$$c_D(W) = \mu_W + \frac{\sigma_W^2}{2} \frac{-\log \psi}{u},$$

that again is easily decomposable for individual risks. The formula is equivalent to the exponential formula (7.1), where all terms except the first two are omitted.

7.2.6. De Vylder approximation. The method requires information on moments of the first three orders of the increment of the risk process. According to the method, ruin probability could be expressed as:

$$\psi_{dV}(u) = \frac{1}{1 + R_{(D)}\rho} \exp\left(-\frac{R_{(D)} u}{1 + R_{(D)}\rho}\right),$$

where for simplicity the abbreviated notation $\rho \stackrel{\text{def}}{=} \frac{1}{3}\sigma_W \gamma_W$ is used. Setting $\psi_{dV}(u)$ equal to ψ and rearranging the equation we obtain another form of it:

$$\left\{ -\log \psi - \log\left(1 + R_{(D)}\rho\right) \right\} \left(1 + R_{(D)}\rho\right) = R_{(D)} u$$

that can be solved numerically in respect of $R_{(D)}$, to yield as a result formula:

$$c_{dV}(W) = \mu_W + \frac{\sigma_W^2}{2} R_{(D)},$$

which again is directly decomposable.

When the analytic solution is needed, we can make some further simplifications. Namely, the equation entangling the unknown coefficient $R_{(D)}$ could be transformed to a simplified form on the basis of the following approximation:

$$\left(1 + R_{(D)}\rho\right) \log\left(1 + R_{(D)}\rho\right) =$$

$$= \left(1 + R_{(D)}\rho\right)\left\{R_{(D)}\rho - \frac{1}{2}\left(R_{(D)}\rho\right)^2 + \frac{1}{3}\left(R_{(D)}\rho\right)^3 - \ldots\right\} \approx R_{(D)}\rho.$$

Provided the error of omission of higher order terms is small, we obtain the approximation:

$$R_{(D)} \approx \frac{-\log\psi}{u + \rho(\log\psi + 1)}.$$

The error of the above solution is small, provided u is several times greater than the product $\rho\left|\log\psi + 1\right|$. Under this condition we obtain the explicit (approximated) formula:

$$c_{dV^*}(W) = \mu_W + \frac{\sigma_W^2}{2}\left\{\frac{-\log\psi}{u + \rho(\log\psi + 1)}\right\},$$

where the star symbolizes the simplification made. Applying now the method of linear approximation of marginal cost $c_{dV^*}(W + X) - c_{dV^*}(W)$ (see Otto [33] for details) yields the result:

$$c_{dV^*}(X) = \mu_X + \frac{-\log\psi\left\{u + 2\rho(\log\psi + 1)\right\}}{2\left\{u + \rho\left(\log\psi + 1\right)\right\}^2}\sigma_X^2 + \frac{\log\psi(\log\psi + 1)}{6\left\{u + \rho\left(\log\psi + 1\right)\right\}^2}\mu_{3,X}.$$

The reader can verify that the formula $c_{dV^*}(\cdot)$ is additive for independent risks, and so it can serve for marginal as well as for basic valuation.

7.2.7. Subexponential approximation. This method applies to the classical model (Poisson loss arrivals) with thick-tailed severity distribution. More precisely, when the severity cdf F_Y possesses the finite expectation μ_Y, then the integrated tail distribution cdf F_{L_1} (interpreted as the cdf of the variable L_1, being the "ladder height" of the claim surplus process) is defined as follows:

$$1 - F_{L_1}(x) = \frac{1}{\mu_Y}\int_x^\infty \{1 - F_Y(y)\}dy.$$

Assuming now that the latter distribution is subexponential (see Section 5.2), we could obtain (applying the Pollaczek-Khinchin formula) the approximation, which should work for large values of critical capital:

$$c_S(W) = \mu_W\left[1 + \frac{1}{\psi}\{1 - F_{L_1}(u)\}\right].$$

The extended study of consequences of thick-tailed severity distributions can be found in Embrechts et al. [16].

All approximation methods presented in this section are more or less standard, and more detailed information on them can be found in any actuarial textbook, as for example in Bowers et al. [5]. More advanced analysis can be found in a book by Asmussen [1] and numerical comparison of this and other approximations are given in Čižek, Härdle and R.Weron, Chapter 15 by Burnecki, Miśta and A.Weron.

7.3. Ruin probability criterion and the insurance

In this section operational reserves calculation is considered under predetermined ruin probability, with cession of some part of risk to insurance company included. At first an example involving the so called insurance with self retention limit is presented.

Example 1 We assume, that the aggregate loss W has a compound Poisson distribution with expected number of claims $\lambda_W = 1000$, and with severity distribution being truncated-Pareto distribution with with cdf given for $y \geqslant 0$ by the formula:

$$F_Y(y) = \begin{cases} 1 - \left(1 + \frac{y}{\nu}\right)^{-\alpha} & \text{when} \quad y < M_0 \\ 1 & \text{when} \quad y \geqslant M_0 \end{cases}$$

and parameters $(\alpha, \nu, M_0) = (2.5,\ 1.5,\ 500)$.

Variable W is subdivided into retained part $\underline{W}_M$ and ceded part $\overline{W}_M$, that given the subdivision parameter $M \in (0, M_0]$ have a form:

$$\underline{W}_M = \underline{Y}_{M,1} + ... + \underline{Y}_{M,N},$$

$$\overline{W}_M = \overline{Y}_{M,1} + ... + \overline{Y}_{M,N}.$$

We assume also that the excess of each loss over the limit $M \in (0, M_0]$ is ceded to the insurer using premium pricing reflected by the formula:

$$c^{(I)}\left(\overline{W}_M\right) = (1 + \theta_0)\,\mathrm{E}\left(\overline{W}_M\right) + \theta_1 Var\left(\overline{W}_M\right).$$

The problem lies in choosing such a value of the retention limit M and critical capital u, which minimize the total own operational reserves and premium paid to insurer, under predetermined values of parameters $(\psi, \theta_0, \theta_1)$. The problem could be solved with application of the De Vylder and Beekman–Bowers approximation methods, however allowing for insurance leads to numerical solutions.

Solution.

Now, the discrete-time version of the risk process is assumed:

$$R_n = u + \left\{c - c^{(I)}\left(\overline{W}_M\right)\right\} t - \sum_{i=1}^{n} \underline{W}_{M,i}.$$

where all events are assumed to be observed once a year, and notations are obviously adapted.

The problem takes a form of minimization of the capital charge c under restrictions, which in the case of De Vylder method take a form:

$$\psi \quad = \quad \left(1 + R_{(D)}\rho\right)^{-1} \exp\left\{-R_{(D)}u\left(1 + R_{(D)}\rho\right)^{-1}\right\},$$

$$R_{(D)} \quad = \quad 2\left(c - c^{(I)}\left(\overline{W}_M\right)\right)\sigma^{-2}\left(\underline{W}_M\right),$$

$$\rho \quad = \quad \tfrac{1}{3}\mu_3\left(\underline{W}_M\right)\sigma^{-2}\left(\underline{W}_M\right),$$

and in the version based on the Beekman–Bowers approximation method take a form:

$$
\begin{aligned}
c - c^{(I)}\left(\overline{W}_M\right) &= (1+\theta)\,\mathrm{E}\left(\underline{W}_M\right), \\
\psi &= (1+\theta)^{-1}\left(1 - G_{\alpha,\beta}\left(u\right)\right), \\
\alpha\beta^{-1} &= (1+\theta)\,\mathrm{E}\left(\underline{Y}_M^2\right)\left\{2\theta\,\mathrm{E}\left(\underline{Y}_M\right)\right\}^{-1}, \\
\alpha\left(\alpha+1\right)\beta^{-2} &= (1+\theta)\left\{\frac{\mathrm{E}\left(\underline{Y}_M^3\right)}{3\theta\,\mathrm{E}\left(\underline{Y}_M\right)} + 2\left(\frac{\mathrm{E}\left(\underline{Y}_M^2\right)}{2\theta\,\mathrm{E}\left(\underline{Y}_M\right)}\right)^2\right\}.
\end{aligned}
$$

The first step to solve it is to express moments of first three orders of variables $\underline{Y}_M$ and $\overline{Y}_M$ as functions of parameters (α, ν, M_0) and the real variable M. Expected value of the truncated-Pareto variable with parameters (α, ν, M) equals by definition:

$$
\int_0^M y\frac{\alpha\nu^\alpha}{(\nu+y)^{\alpha+1}}\,dy + M\left\{1 - F\left(M\right)\right\} = \int_1^{1+\frac{M}{\nu}} (x-1)\frac{\alpha\nu}{x^{\alpha+1}}\,dx + M\left(1 + \frac{M}{\nu}\right)^{-\alpha} =
$$

$$
= \alpha\nu\int_1^{1+\frac{M}{\nu}} \left(x^{-\alpha} - x^{-\alpha-1}\right)dx + M\left(1 + \frac{M}{\nu}\right)^{-\alpha}
$$

that, after integration and reordering of components produces the following formula:

$$
m_1 = \frac{\nu}{\alpha-1}\left\{1 - \left(1 + \frac{M}{\nu}\right)^{1-\alpha}\right\}.
$$

Similar calculations made for moments of higher order yield the recursive equation:

$$
m_{k,\alpha} = \frac{\nu}{\alpha-1}\left\{\alpha m_{k-1,\alpha-1} - (\alpha-1)m_{k-1,\alpha} - M^{k-1}\left(1 + \frac{M}{\nu}\right)^{1-\alpha}\right\},
$$

$$
k = 2, 3, \dots
$$

where the symbol $m_{K,A}$ means for $A > 0$ just the moment of order K of the truncated-Pareto variable with parameters (A, ν, M). No matter whether A is positive or not, in order to start the recursion we take:

$$
m_{1,A} = \begin{cases} \frac{\nu}{A-1}\left\{1 - \left(1 + \frac{M}{\nu}\right)^{1-A}\right\} & \text{when} \quad A \neq 1 \\ \nu\ln\left(1 + \frac{M}{\nu}\right) & \text{when} \quad A = 1 \end{cases}
$$

The above formulas could serve to calculate raw moments as well of the variable $\underline{Y}_M$ as the variable Y, provided we replace M by M_0. Having calculated moments for both variables $\underline{Y}_M$ and Y already, we make use of the relation:

$$
(7.4) \qquad \mathrm{E}(Y^k) = \sum_{j=0}^k \binom{k}{j}\mathrm{E}\left(\underline{Y}_M^{k-j}\overline{Y}_M^j\right),
$$

to calculate moments of the variable $\overline{Y}_M$. In the above formula we read $\underline{Y}_M^0$ and $\overline{Y}_M^0$ as equal one with probability one. Mixed moments appearing on the RHS of formula (7.4) can be calculated easily as positive values of the variable $\overline{Y}_M$ happen only when $\underline{Y}_M = M$. So mixed moments equal simply:

$$
\mathrm{E}\left(\underline{Y}_M^m\overline{Y}_M^n\right) = M^m E\left(\overline{Y}_M^n\right)
$$

for arbitrary $m, n > 0$.

TABLE 7.1. Minimization of operational reserves c with respect to choice of capital u and retention limit M. Basic characteristics of the variable W: $\mu_W = 999.8$, $\sigma_W = 74.2$, $\gamma_W = 0.779$, $\gamma_{2,W} = 2.654$

Variants of minimization problems	Method of approx. of the ruin probability	Retention limit M	Critical capita level u	Loading $\frac{c - \mu_W}{\mu_W}$
V.1: (basic)	BB	184.2	416.6	4.17%
	dV	185.2	416.3	4.16%
V.2: $\psi = 2.5\%$	BB	150.1	463.3	4.65%
	dV	156.3	461.7	4.63%
V.3: $\theta_0 = 50\%$	BB	126.1	406.2	4.13%
	dV	127.1	406.0	4.13%
V.4: $\theta_1 = 0.25\%$	BB	139.7	409.0	4.13%
	dV	140.5	408.8	4.13%
V.5: (no insurance)	BB	500.0	442.9	4.25%
	dV	500.0	442.7	4.25%

The second step is to express cumulants of both variables $\underline{W}_M$ and $\overline{W}_M$ as a product of the parameter λ_W and respective raw moments of variables $\underline{Y}_M$ and $\overline{Y}_M$. All these characteristics are functions of parameters (α, ν, λ_W) and the decision variable M.

Interpretation of solutions obtained in Example 1

Results of numerical optimization are reported in Table 7.1. In the basic variant of the problem, parameters has been set on the level $(\psi, \theta_0, \theta_1) = (\, 5\%,\ 100\%,\ 0.5\%)$. Variants 2, 3 and 4 differ from the basic variant by the value of one of parameters $(\psi, \theta_0, \theta_1)$. Results could be summarized as follows:

(i) Insurance results in operational reserve reduction (compare variant 5 with variant 1), the need for high critical capital level is also reduced.

(ii) Comparison of variants 2 and 1 shows that increasing safety (reduction of parameter ψ from 5% to 2.5%) results in significant growth of capital charge needed. This effect is caused as well by increase of critical level of capital, as by increase of costs of insurance, because of reduced retention limit. It is also worthwhile to notice that predetermining $\psi = 2.5\%$ results in significant diversification of results obtained by two methods of approximation. In the case when $\psi = 5\%$ the difference is neglectible.

(iii) Results obtained invariants 3 and 4 show that the optimal level of insurance is quite sensitive to changes of parameters reflecting costs of insurance policy.

In a general case, i.e. other then truncated-Pareto severity distribution, the only difference in solving the problem, is the calculation of the moments of this distribution as well as of the truncated distribution.

7.4. Final remarks

It should be noted that all presented models, including risk participation of insurers, lead only to a modification of the distribution of the increment of the risk process. Still the mutual independence of subsequent increments and their identical distribution is preserved. There are also models where decisions concerning capital charge and insurance depend on current size of the company's capital. In general, models of this type need the stochastic control technique to be applied. Nevertheless, models presented in this chapter preserve simplicity, and allow just to have insight on long-run consequences of some decision rules, provided they remain unchanged for a long time. This insight is worthwhile despite the fact that in reality decisions are undertaken on the basis of the current situation, and no fixed strategy remains unchanged under changing conditions of the environment. On the other hand, it is always a good idea to have some reference point, when consequences of decisions motivated by current circumstances have to be evaluated.

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