M A T H E M A T I C A L   E C O N O M I C S

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ON SOME EXTREMAL PROBLEM IN DISCRETE GEOMETRY

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Abstract. Let p, q, r be any three lines in the plane passing through a common point and suppose that O, P, Q, R are any four collinear points such that $P \in p$, $Q \in q$, $R \in r$, P and R are harmonic conjugates with respect to O and Q (that is, $|OP|/|PQ| = |OR|/|QR|$). For every $k \geq 2$, we construct a set X_n of $n = 4k$ points, which is distributed on the lines p, q, r, but each element of $X_n \cup \{O\}$ is incident to at most $n/2$ lines spanned by $X_n \cup \{O\}$.

Keywords: Dirac-Motzkin conjecture, Desargues' theorem.

JEL Classification: D8, D2.

1. Introduction

Dirac (1951) and Motzkin (1951) conjectured that any set X of n non--collinear points in the plane has an element incident to at least $n/2$ lines *spanned* by X, i.e. the lines passing through at least two points of X. Some counter-examples were shown for small values of n by Grünbaum (1972, p. 25) (see also Grünbaum, 2010), and an infinite family of counter-examples was constructed by Felsner (after Brass, Moser, Pach (2005, p. 313)), and Akiyama et al. (2011).

Given collinear points O, P, Q, R, the points P and R are *harmonic conjugates* with respect to O and Q if

$$\frac{|OP|}{|PQ|} = \frac{|OR|}{|QR|}.$$

Let p, q, r be any three lines in the plane passing through a common point. Suppose that O, P, Q, R are any four collinear points such that $P \in p$, $Q \in q$, $R \in r$, P and R are harmonic conjugates with respect to O and Q. For

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every $k \geq 2$, we construct a set X_n of $n = 4k$ points, which is distributed on the lines p, q, r, but each element of $X_n \cup \{O\}$ is incident to at most $n/2$ lines spanned by $X_n \cup \{O\}$ (see Theorem 2.2).

The "weak Dirac conjecture" proved by Beck (1983) and independently by Szemerédi, Trotter (1983) states that there is a constant $c > 0$ such that in every non-collinear set X of n points in the plane some element is incident to at least cn lines spanned by X. Brass, Moser, Pach (2005, p. 313) proposed the following "strong Dirac conjecture": there is a constant $c > 0$ such that any set X of n points in the plane, not all on a line, has an element which lies on at least $(n/2) - c$ lines spanned by X.

2. Main result

Let p, q, r be any three lines in the plane passing through a common point A. Suppose that O, P, Q, R are any four collinear points such that $P \in p$, $Q \in q$, $R \in r$, P and R are harmonic conjugates with respect to O and Q. For two points $x \neq y$ in the plane we denote by xy the straight line through x and y. Let x_1 be a point of an open segment (P, A) and $y_1 = Ox_1 \cap r$. We define the following four sequences (see Figure 1): $x_0 = P$, $y_0 = R$ and

$$
\begin{aligned}
w_n &:= x_n y_n \cap q && \text{for} \quad n \geq 0,\\
x_{n+1} &:= y_{n-1} w_n \cap p && \text{for} \quad n \geq 1,\\
y_{n+1} &:= x_{n-1} w_n \cap r && \text{for} \quad n \geq 0,\\
z_n &:= x_n y_{n+1} \cap x_{n+1} y_n && \text{for} \quad n \geq 0.
\end{aligned}
$$

Notice that

$$(*) \qquad w_n = x_{n-1}y_{n+1} \cap x_n y_n \cap x_{n+1}y_{n-1}, \ \text{for} \ n \geq 1.$$

Since P and R are harmonic conjugates with respect to O and Q, we have $Q = PR \cap Az_0$ (see Coxeter, 1961). Hence,

$$(**) \qquad q = Az_0.$$

Let us also denote

$$v_n^i := x_{n+i}\, y_n \cap x_{n+1+i}\, y_{n+1} \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad n \geq 0.$$

Note that $v_0^0 = O$.

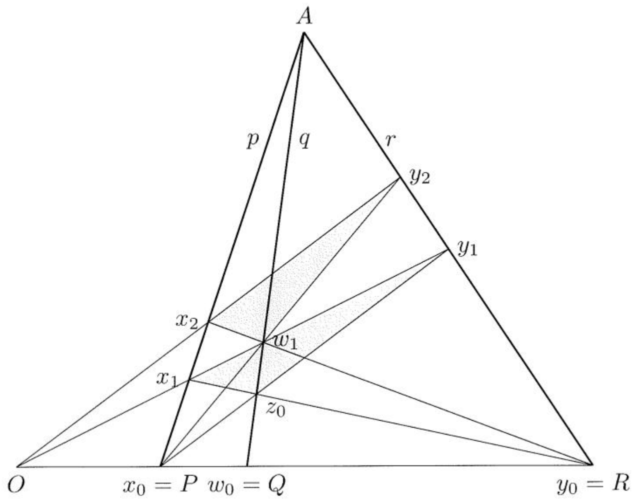

Fig. 1. The triangles $\Delta x_1 z_0 y_1$, $\Delta x_2 w_1 y_2$ are in perspective centrally and are in perspective axially

Source: author's own study.

In the proof of Lemma 2.1 below we use the following Desargues' theorem from projective geometry (Coxeter, 1961): two triangles are in perspective *centrally* if and only if they are in perspective *axially*. In a more explicit form, denote one triangle by $\Delta x_1 z_0 y_1$ and the other by $\Delta x_2 w_1 y_2$ (see Figure 1). The condition of central perspectivity is satisfied if and only if the three lines x_1x_2, z_0w_1 and y_1y_2 are concurrent at a point called "the center of perspectivity" (the point A). The condition of axial perspectivity is satisfied if and only if the points of intersection of x_1z_0 with x_2w_1, z_0y_1 with w_1y_2,and x_1y_1 with x_2y_2 are collinear on a line called "the axis of perspectivity" (the line PR).

Lemma 2.1. *We have:*

(a) $z_n \in q$; *for* $n \geq 0$;

(b) $v_n^0 = O,$ *and* $v_n^1 = v_0^1,$ *for* $n \geq 1$.

Proof. By (**), $z_0 \in$ q. Hence, it is sufficient to prove the following:

(i) If $z_{n-1} \in q$, then $v_n^0 = v_{n-1}^0$ and $z_n \in q$, for $n \geq 1$,

(ii) $v_n^1 = v_0^1,$ for $n \geq 1$.

(i). If $z_{n-1} \in q$, then the lines $x_n x_n + 1$, $z_{n-1} w_n$ and $y_n y_{n+1}$ are concurrent at the point A. Therefore, the triangles

$$\Delta\, x_n z_{n-1} y_n,\ \Delta\, x_{n-1} w_n y_{n+1}$$

are in perspective centrally, whence these triangles are in perspective axially. So by (*) the points $y_{n-1} = x_n z_{n-1} \cap x_{n+1} w_n$, $x_{n-1} = z_{n-1} y_n \cap w_n y_{n+1}$ and $v_n^0 = x_n y_n \cap x_{n+1}\, y_{n+1}$ are collinear. Thus, $v_{n-1}^0 = v_n^0 \in x_{n+1}\, y_{n+1}$. Hence, the points $v_{n-1}^0 = x_{n-1} y_{n-1} \cap x_n y_n$, $x_{n+1} = y_{n-1} w_n \cap y_n z_n$ and $y_{n+1} = x_{n-1} w_n \cap x_n z_n$ are collinear. Therefore, the triangles

$$\Delta\, x_{n-1} y_{n-1} w_n,\ \Delta\, x_n y_n z_n$$

are in perspective axially, whence these triangles are in perspective centrally. So the lines $x_{n-1} x_n$, $y_{n-1} y_n$ and $w_n z_n$ are concurrent at the point A, and finally $z_n \in q$.

(ii). By (*) the lines $x_n w_{n+1}$, $y_n y_{n+1}$ and $x_{n+1} z_{n+1}$ are concurrent at the pointy_{n+2}. Therefore, the triangles

$$\Delta\, x_n y_n x_{n+1},\ \Delta\, w_{n+1} y_{n+1} z_{n+1}$$

are in perspective centrally, whence these triangles are in perspective axially. Thus, by (i), $O = v_n^0$ and $w_{n+1} z_{n+1} = q$. Hence, the points $O = x_n y_n \cap w_{n+1} y_{n+1}$, $v_n^1 = y_n x_{n+1} \cap y_{n+1} z_{n+1}$ and $A = x_n x_{n+1} \cap w_{n+1} z_{n+1}$ are collinear. Since $v_n^1 \in OA$ for $n \geq 0$, we have $v_n^1 = v_{n-1}^1$ for $n \geq 1$.

□

Theorem 2.2. *Let X_n be the following set of $n = 4k$, $k \geq 2$, points distributed on the lines p, q, r:*

$$X_n := \{A\} \cup \{x_i : 0 \leq i < k\} \cup \{y_i : 0 \leq i < k\}$$
$$\cup \{w_i : 0 \leq i < k\} \cup \{z_i : 0 \leq i < k-1\}.$$

Any point of $X_n \cup \{O\}$ belongs to at most n/2 lines spanned by $X_n \cup \{O\}$.

Proof. Let us observe that $z_n, w_n \in q$, by Lemma 2.1(a). Moreover, the points O, x_n, w_n, y_n are collinear, by Lemma 2.1(b). Thus, we only need to show the following:

(i) If $m, n \geq 0$, then:

$$x_m y_n \cap q = \begin{cases} w_{\frac{m+n}{2}}, & \text{for } m+n \text{ even} \\ z_{\frac{m+n-1}{2}}, & \text{for } m+n \text{ odd.} \end{cases}$$

Let us denote

$$a^i_{(m,n)} := x_{m+i}y_n \cap x_{n+i}y_m \quad \text{for} \quad i = 0,1 \quad \text{and} \quad 0 \le m < n.$$

Fix $i = 0$; 1 and $m \ge 0$. We first prove the following implication:

(ii) If $a^i_{(m,n)} \in q$, then $a^i_{(m,n+1)} \in q$, for $n > m$.

By Lemma 2.1(b), $v^i_n = v^i_m \in x_{m+i}y_m$. Hence, $y_m = x_{n+i}a^i_{(m,n)} \cap x_{n+1+i}a^i_{(m,n+1)}$, $x_{m+i} = a^i_{(m,n)}y_n \cap a^i_{(m,n+1)}y_{n+1}$ and $v^i_n = x_{n+i}y_n \cap x_{n+1+i}y_{n+1}$ are collinear points. Therefore, the triangles

$$\Delta x_{n+i}a^i_{(m,n)}y_n,\ \Delta x_{n+1+i}a^i_{(m,n+1)}y_{n+1}$$

are in perspective axially, whence these triangles are in perspective centrally. So the lines $x_{n+i}x_{n+1+i}$, $a^i_{(m,n)}a^i_{(m,n+1)}$ and y_ny_{n+1} are concurrent at point A. Thus, if $a^i_{(m,n)} \in q$, then $a^i_{(m,n+1)} \in q$.

By Lemma 2.1(a) and (*), $a^0_{(m,m+1)} =|z_m \in q$ and $a^1_{(m,m+1)} \in q = w_{m+1} \in q$. From (ii) it follows that $a^i_{(m,n)} \in q$ for $i = 0$; 1 and $0 \le m < n$, which gives

$$x_my_n \cap x_ny_m = a^0_{(m,n)} = a^1_{(m,n-1)} = a^0_{(m+1,n-1)} = \ldots =$$

$$= \begin{cases} a^1_{(\frac{m+n}{2}-1,\frac{m+n}{2})} & \text{for} \quad m+n \quad \text{even} \\ a^0_{(\frac{m+n-1}{2},\frac{m+n+1}{2})} & \text{for} \quad m+n \quad \text{odd} \end{cases}$$

$$= z \begin{cases} w_{\frac{m+n}{2}} & \text{for} \quad m+n \quad \text{even} \\ z_{\frac{m+n-1}{2}} & \text{for} \quad m+n \quad \text{odd.} \end{cases}$$

Hence (i) holds.

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