M A T H E M A T I C A L   E C O N O M I C S

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AGGREGATE DEPENDENT RISKS – RISK MEASURE CALCULATION[1]

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Abstract. We investigate the sum of dependent random variables. The dependent structure is modeled by copulas. The risk measures, VaR and ES of such sums, are calculated. We present the lower and upper border of VaR. The examples when the marginals have exponential and Pareto distribution are investigated. The influence of the degree of dependence on the value of VaR of the sum of dependent random variables is analysed.

Keywords: dependent random variables, copula, Frechet bounds, Value-at-Risk, expected shortfall.

JEL Classification: G10.

1. Introduction

The paper is devoted to the calculation of the risk measures, mainly Value-at-Risk (VaR), of the sum of dependent random variables. The dependent structure between such random variables will be described by the copulas. We will investigate the extreme cases: the perfect positive (comonotonic) and negative (countermonotonic) dependence, the independence and the intermediate degrees of dependence modeled by the Clayton copula.

The Value-at-Risk is the most widely used risk measure in financial institutions. It was recommended by G-30 (an influential international body consisting of senior representatives of the private and public sectors and academia) and the Basel Accord on Banking Supervision (Basel II) (McNeil, Frey, Embrechts, 2005). But it is not a “good” risk measure. The VaR is not a superadditive, coherent risk measure. The superadditivity is the

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base of the risk diversification. So we also investigate the expected shortfall (ES), the coherent risk measure.

Embrechts, Hoing, Puccetti (2005) presented the fallacies connected with dependent random variables. Fallacy 3 stated that "the worst case VaR (quantile) for a linear portfolio $X + Y$ occurs when $\rho(X, Y)$ (coefficient of correlation) is maximal, i.e. X and Y are comonotonic". We thoroughly investigate such a fallacy in our paper. We study cases when the random variables have the exponential and Pareto distribution.

The sum of dependent random variables is investigated in Section 2. The lower and upper bounds on the cumulative distribution functions of such sums are presented. The risk measures VaR and coherent measure ES are studied in Section 3. Finally, in Section 4 we study the examples based on the different distributions of marginal random variables, the exponential and Pareto distribution.

2. Sums of random variables

Now we will study the sums of the dependent random variables $X_1, ..., X_n$, with the cumulative distribution functions (cdf.) $F_i(x) = P(X_i \le x)$, where $i = 1, ..., n$. The joint distribution of such random variables may be described by the copula C in the following way:

$$F(x_1, ... , x_n) = P(X_1 \le x_1, ... , X_n \le x_n) = C(F_1(x_1), ... , F_n(x_n)).$$

A copula is any function C: $[0, 1]^n \to [0, 1]$ non-decreasing in each argument, n-increasing, i.e. for all $a = (a_1, ... , a_n)$, $b = (b_1, ... , b_n) \in [0, 1]^n$ with $a_i \le b_i$, $i = 1, ... , n$, we have

$$\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\ldots+j_n} C\left(u_{1j_1}, \ldots, u_{nj_n}\right) \ge 0,$$

where $u_{i1} = a_i$, $u_{i2} = b_i$ for all $i = 1, ... , n$ and satisfying condition:

$$C(1, ... , 1, u_i, 1, ... , 1) = u_i$$

for all $u_i \in [0, 1]$. The copula can be treated as the cdf. of the n-dimension random variable restricted to $[0, 1]^n$ having the uniform marginals (Nelsen, 1999; Heilpern, 2007).

Any copula C can be bounded by the Frechet lower and upper bounds:

$$W(u_1, ... , u_n) \le C(u_1, ... , u_n) \le M(u_1, ... , u_n),$$

where $W(u_1, ..., u_n) = \max(u_1 + ... + u_n - n + 1, 0)$ and $M(u_1, ..., u_n) = \min(u_1, ..., u_n)$. The upper bound M is a copula, but the lower W is not a copula for $n > 2$. But, for any $n \geq 3$ and any $u \in [0, 1]^n$, there is a copula C such that $C(u_1, \dots, u_n) = W(u_1, \dots, u_n)$.

The Frechet upper bound M is also called the comonotonicity copula. It reflects the dependent structure of the perfectly positively dependent random variables $X_1, \dots, X_n$. For $n = 2$ the lower bound W is a countermonotonicity copula. The random variables X_1, X_2 are perfectly negatively dependent in this case. The independent copula takes the form:

$$C_I(u_1, \dots, u_n) = u_1 u_2 \dots u_n.$$

Let $S = X_1 + \dots + X_n$ be a sum of random variables with the dependent structure described by copula C. We can calculate the cumulative distribution function of such a sum using the following n-dimension integrals:

$$F_C(s) = P(S \leq s) = \int_{K(s)} dC\left(F_1(x_1), \dots, F_n(x_n)\right) = \int_{A(s)} dC(u_1, \dots, u_n), \quad (1)$$

where

$$K(s) = \{(x_1, \dots, x_n): x_1 + \dots + x_n \leq s\}$$

and

$$A(s) = \{(u_1, \dots, u_n): F_1^{-1}(u_1) + \dots + F_n^{-1}(u_n) \leq s\}.$$

First, we want to study the problem concerning the construction of the lower and upper bounds of such sums. This problem, attributed to A.N. Kolmogorov, was solved by Makarov (1981) for the sum of two random variables. We give the solution of the n-dimension version of such a problem.

Theorem 1 (Denuit, Genest, Marceau, 1999). *The cumulative distribution function $F_C(s)$ of the sum of random variables $S = X_1 + \dots + X_n$ satisfies the following inequalities*:

$$F_{\min}(s) \leq F_C(s) \leq F_{\max}(s),$$

where

$$F_{\min}(s) = \sup_{\mathbf{x} \in \Sigma(s)} \max\left\{\sum_{i=1}^{n} F_i^{-}(x_i) - n + 1, 0\right\}, \; F_{\max}(s) = \inf_{\mathbf{x} \in \Sigma(s)} \min\left\{\sum_{i=1}^{n} F_i(x_i), 1\right\},$$

$$\Sigma(s) = \{(x_1, \dots, x_n): x_1 + \dots + x_n = s\} \text{ and } F_i^{-}(x) = P(X_i < x).$$

For fixed marginal cdf. $F_i(x)$ the above inequalities cannot be improved, the bounds $F_{\min}(s)$, and $F_{\max}(s)$ are point-wise the best possible. For any s, there exist copulas C^- and C^+ such that

$$F_{\min}^{-}(s)=F_{C^-}^{-}(s) \quad \text{and} \quad F_{\min}(s)=F_{C^+}(s).$$

We see that the above bounds are not connected with the upper Frechet bound in the comonotonic, strictly positive dependent case. They are rather weakly related with the lower Frechet bound.

The above bounds take the following simpler form for the sum of two random variables X_1, X_2:

$$F_{\min}(s)=\sup_{x\in R}\max\left\{F_1^-(x)+F_2^-(s-x)-1,0\right\},$$

$$F_{\max}(s)=\inf_{x\in R}\min\left\{F_1(x)+F_2(s-x),1\right\}.$$

If the joint distribution of random variables X_1 and X_2 is described by copula:

$$C_\beta(u,v)=\begin{cases}\max\left\{\beta,W(u,v)\right\} & (u,v)\in[\beta,1]^2\\ M(u,v) & \text{otherwise}\end{cases},$$

then the value of cumulative distribution function of the sum of such variables is equal to lower bound (Embrechts, McNeil, Straumann, 2002), i.e.

$$F_{C_\beta}(s)=F_{\min}(s)=\beta.$$

The copula C_β,, treated as the cdf. on $[0, 1]^2$, has the density uniformly focused on the line $v = u$ for $0 \le u \le \beta$ and on $v = 1 + \beta - u$ for $\beta < u \le 1$.

Now, we present the examples of distributions of marginals for which the bounds $F_{\min}$ and $F_{\max}$ can be computed explicitly (Denuit, Genest, Marceau, 1999; Nelsen, 1999).

a) **Shifted exponential distribution:** $F_i = \text{Ex}(\lambda_i, \theta_i)$, $i = 1, 2$, where $F_i(x) = 1 - \exp(-(x - \theta_i)/\lambda_i)$ for $x \ge \theta_i$ and $F_i(x) = 0$ otherwise. Then

$$F_{\min} = \text{Ex}(\lambda_1 + \lambda_2, \theta_1 + \theta_2 + (\lambda_1 + \lambda_2)\ln(\lambda_1 + \lambda_2) - \lambda_1\ln\lambda_1 - \lambda_2\ln\lambda_2)$$

and

$$F_{\max} = \text{Ex}(\max\{\lambda_1, \lambda_2\}, \theta_1 + \theta_2).$$

b) **Pareto distribution:** $F_i = \text{Pa}(\alpha, \lambda_i, \theta_i)$, where $F(x)=1-\left(\dfrac{\lambda_i}{\lambda_i+x-\theta_i}\right)^\alpha$ for $x \ge \theta_i$ and $F_i(x) = 0$ otherwise. Then

$$F_{\min} = \mathrm{Pa}(\alpha, \lambda_0, \theta_1 + \theta_2 + \lambda_0 - \lambda_1 - \lambda_2),$$

where $\lambda_0 = \left(\lambda_1^\beta + \lambda_2^\beta\right)^{1/\beta}$, $\beta = \dfrac{\alpha}{\alpha+1}$ and

$$F_{\max} = \mathrm{Pa}(\alpha, \max\{\lambda_1, \lambda_2\}, \theta_1 + \theta_2).$$

c) **Uniform distribution:** $X_i = \mathrm{U}(a_i, b_i)$, where $b_1 - a_1 \le b_2 - a_2$. Then

$$F_{\min} = \mathrm{U}(b_1 + a_2, b_1 + b_2)$$

and

$$F_{\max} = \mathrm{U}(a_1 + a_2, \max\{a_1 + b_2, b_1 + a_2\}).$$

d) **Normal distribution:** $F_i = N(\mu_i, \sigma)$. Then

$$F_{\min}(s) = \begin{cases} 0 & s < \mu_1 + \mu_2 \\ 2\Phi\left(\dfrac{s - \mu_1 - \mu_2}{2\sigma}\right) - 1 & s \ge \mu_1 + \mu_2 \end{cases}$$

and

$$F_{\max}(s) = \begin{cases} 2\Phi\left(\dfrac{s - \mu_1 - \mu_2}{2\sigma}\right) & s < \mu_1 + \mu_2 \\ 1 & s \ge \mu_1 + \mu_2 \end{cases}$$

When $F_i = N(\mu_i, \sigma_i)$, where $\sigma_1 \ne \sigma_2$, then we have

$$F_{\min}(s) = \Phi\left(\frac{-\sigma_1\theta - \sigma_2\varphi}{\sigma_2^2 - \sigma_1^2}\right) + \Phi\left(\frac{\sigma_2\theta - \sigma_1\varphi}{\sigma_2^2 - \sigma_1^2}\right) - 1$$

and

$$F_{\min}(s) = \Phi\left(\frac{-\sigma_1\theta + \sigma_2\varphi}{\sigma_2^2 - \sigma_1^2}\right) + \Phi\left(\frac{\sigma_2\theta + \sigma_1\varphi}{\sigma_2^2 - \sigma_1^2}\right),$$

where

$$\theta = s - \mu_1 - \mu_2 \quad \text{and} \quad \varphi = \sqrt{\theta^2 + 2(\sigma_2^2 - \sigma_1^2)\ln(\sigma_2 / \sigma_1)}.$$

There exist formulas for the Cauchy and Weibull (upper bound only) distribution and multidimensional version of such formulas (Denuit, Genest, Marceau, 1999; Alsina, 1981).

3. Risk measures

The Value-at-Risk (VaR) is the most widely used risk measure in financial institutions. It is a very simple risk measure based on quantile:

$$\mathrm{VaR}_\alpha(X) = \inf\{x\colon F_X(x) \leq \alpha\},$$

where F_X is cdf. of risk X and $0 < \alpha < 1$. It is the smallest number such that the probability of the loss connected with the risk X is no larger than $1 - \alpha$. In practice, the confidence level α takes the value near one.

The risk measure VaR has been criticized on the grounds that it has poor aggregation properties. Mainly, VaR violates the subadditivity, the base of the risk diversification. Artzner et al. (1999) gave the list of axioms that a "good" so-called coherent risk measure should satisfy. We present the version of axioms done by McNeil, Frey, Embrechts (2005).

We can treat the risk measure ρ as a function defined on the set of risks, almost surely finite random variables (McNeil, Frey, Embrechts, 2005). The coherent risk measure should satisfy the following properties:

a) $\rho(X + a) = \rho(X) + a$, for any $\alpha \in R$ translation invariance,
b) $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for any risks X, Y subadditivity,
c) $\rho(aX) = a\rho(X)$ for any $a > 0$ positive homogenity,
d) If $X \leq Y$ almost surely, then $\rho(X) \leq \rho(Y)$ monotonicity.

The VaR_α is not a coherent risk measure. It does not satisfy the subadditivity. We will study this risk measure for the sum of random variables $S = X_1 + ... + X_n$, where the dependent structure is described by the copula C and we will denote it by symbol $\mathrm{VaR}_\alpha(S; C)$. It is a known fact that the VaR is additive for comonotonic risks (McNeil, Frey, Embrechts, 2005), i.e. we have

$$\mathrm{VaR}_\alpha(X_1 + ... + X_n; M) = \mathrm{VaR}_\alpha(X_1) + ... + \mathrm{VaR}_\alpha(X_n).$$

So there exist random variables X_i and copula C such that VaR is not subadditive and it satisfies inequality:

$$\mathrm{VaR}_\alpha(S; M) = \mathrm{VaR}_\alpha(X_1) + ... + \mathrm{VaR}_\alpha(X_n) < \mathrm{VaR}_\alpha(S; C).$$

So we obtain the greater value of VaR for such copula than for comonotonic case. The VaR for sum of the random variables is bounded by the quantile of the lower bound $F_{\min}$, i.e.

$$\mathrm{VaR}_\alpha(S; C) \leq \inf\{x\colon F_{\min}(x) \leq \alpha\}.$$

If the random vector (X_1, ..., X_n) has the elliptical distribution, e.g. the multivariate normal or Student distribution, then the VaR is a subadditive risk measure (McNeil, Frey, Embrechts, 2005). But we must remember that elliptical marginals X_i do not imply that the joint distribution is elliptical too. The copula which determines the dependent structure must be elliptical. For instance, there exists a copula such that the joint distribution of the normal marginals is focused on the circle (Nelsen, 1999; Heilpern, 2007).

The expected shortfall defined by formula:

$$\mathrm{ES}_{\alpha}\left(X\right)=\frac{1}{1-\alpha}\int_{\alpha}^{1}\mathrm{VaR}_{u}\left(X\right)\mathrm{d}u$$

is a coherent risk measure (McNeil, Frey, Embrechts, 2005). For continuous random variable X, we obtain the following formulas:

$$\mathrm{ES}_{\alpha}(X) = E(X \mid X > \mathrm{VaR}_{\alpha}(X)) = \mathrm{VaR}_{\alpha}\left(X\right)+\frac{1}{1-\alpha}\int_{\mathrm{VaR}_{\alpha}(X)}^{\infty}\left(1-F_{X}\left(x\right)\right)\mathrm{d}x.$$

This risk measure can be interpreted as the expected loss that is incurred in the event that VaR is exceeded (McNeil, Frey, Embrechts, 2005). The expected shortfall has better properties than the VaR. It is subadditive risk measure. But it is the conditional expectation and it is equal infinity for the heavy-tailed random variables, e.g. Pareto distribution. For such random variables, often used in the catastrophic insurance, we must use another risk measure, e.g. VaR.

The expected shortfall is additive for comonotonic random variables too, so we obtain the following inequality:

$$\mathrm{ES}_{\alpha}(S; C) \le \mathrm{ES}_{\alpha}(X_1) + ... + \mathrm{ES}_{\alpha}(X_n) = \mathrm{ES}_{\alpha}(S; M).$$

We see that the expected shortfall takes the greatest value for the comonotonic, perfect positive dependent random variables.

4. Examples

Now we will investigate the values of risk measures, mainly Value-at-Risk, for the sum of two random variables X_1 and X_2. We will study two cases. The random variables have exponential distribution in the first case and Pareto, heavy-tailed distribution in the second.

a) **Exponential distribution**

Let us assume that the random variables X_1, X_2 have the standard exponential distribution with expected value equal one, i.e. $F_i(x) = 1 - e^{-x}$ for $x \geq 0$, $i = 1, 2$. When this random variables are independent, then the sum $S = X_1 + X_2$ has the gamma distribution Ga(2, 1). For comonotonic random variables we have $S = 2X_1$, and the sum has the exponential distribution with the expectation equal two, i.e. $F_M(s) = 1 - e^{-s/2}$.

The lower and upper bound of the cdf. of sum S are done by formulas:

$$F_{\min}(s) = 1 - \exp(-(s - \ln 4)/2),$$

$$F_{\max}(s) = 1 - \exp(-s).$$

We see that the lower bound is the cdf. of the shifted exponential distributed random variables Ex(2, ln 4) and the upper is the standard exponential with expectation equals one. So the VaR of the sum S have the following bounds:

$$\mathrm{VaR}_\alpha^{\min} = -\ln(1-\alpha) \leq \mathrm{VaR}_\alpha(S;C) \leq -2\ln((1-\alpha)/2) = \mathrm{VaR}_\alpha^{\max}.$$

If we want to compute the cdf. for the sum of the random variables X_i with the joint distribution described by the copula C, we can use formula (1). In this case we obtain:

$$A(s) = \{(u,v) : v \leq \varphi_s(u)\},$$

where $\varphi_s(u) = 1 - \frac{1}{1-u}e^{-s}$ and the formula (1) takes the form:

$$F_C(s) = \int_0^{u_0}\int_0^{\varphi_s(u)} dC(u,v),$$

where u_0 is a solution of equation $\varphi_s(u) = 0$.

For countermonotonic random variables X_1, X_2, the density of copula W, treated as the cumulative distribution function on $[0, 1]^2$, is uniformly focused on the line $v = 1 - u$. So the cumulative distribution function of the sum is done by the following formula:

$$F_W(s) = \begin{cases} 0 & s < \ln 4 \\ \sqrt{1-4e^{-s}} & s \geq \ln 4 \end{cases}.$$

When the dependence structure of random variables X_i is done by the copula C_β, the distribution of sum S takes the form:

$$F_{C_\beta}(s)=\begin{cases}1-e^{-s/2} & s<-2\ln(1-\beta)\\ \beta & -2\ln(1-\beta)\le s\\ \beta+\sqrt{(1-\beta)^2-4e^{-s}} & -2\ln((1-\beta)/2)\le s\end{cases}.$$

For $\beta < \alpha$, the VaR is a solution of the equation $\beta+\sqrt{(1-\beta)^2-4e^{-s}}=\alpha$ and equals:

$$\mathrm{VaR}_\alpha\left(S;C_\beta\right)=-ln\left(((1-\beta)^2-(\alpha-\beta)^2\right)/4),$$

but for $\beta = \alpha$ we obtain:

$$\mathrm{VaR}_\alpha\left(S;C_\beta\right)=-2\ln\left(1-\alpha\right)<\lim_{\beta\to\alpha}\mathrm{VaR}_\alpha\left(S;C_\beta\right)=\mathrm{VaR}_\alpha^{\max}.$$

We will model the middle degrees of dependence between the random variables X_1 and X_2 using the Clayton copula:

$$Cl_\alpha(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha},$$

where $-1 < \alpha$. The parameter α reflects the degree of dependence. For the limit case $\alpha = -1$, we obtain perfect negative dependence, for $\alpha = 0$ independence and for $\alpha = \infty$ we have perfect positive dependence. This parameter is strictly connected with Kendall coefficient of correlation τ (Nelsen, 1999; Heilpern, 2007):

$$\tau=\frac{\alpha}{\alpha+2}.$$

Figures 1 and 2 contain the graphs of the cumulative distribution function of the sum S for different cases of the dependent structure between the random variables X_1 and X_2. The graph in Figure 2 is the upper segment of the graph from Figure 1. It shows us the values of these cumulative distribution functions strictly connected with the real values of the confidence levels of VaR.

We see that the graphs of the presented cumulative distribution functions cross each other. But we have some regular situation on the upper segment of such graph (see Figure 2). The greater value of degree of dependence gives us the smaller value of cumulative distribution function. Only artificial copula C0.94, which does not occur in practice, and Clayton copula with the greater value of the degree of dependence damage such regularity.

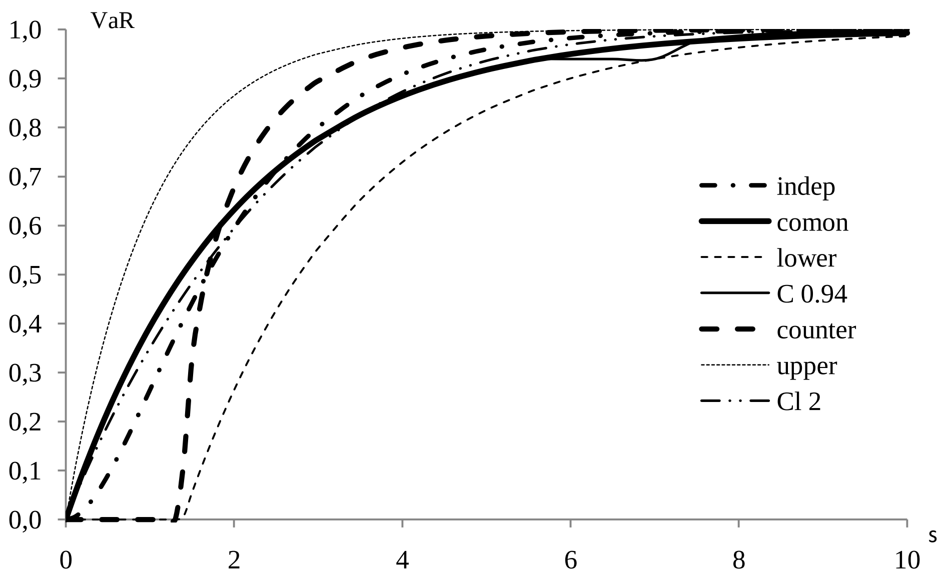

Fig. 1. The graph of the cumulative distribution functions of sum *S* for different dependent structures and exponential marginals

Source: own elaboration.

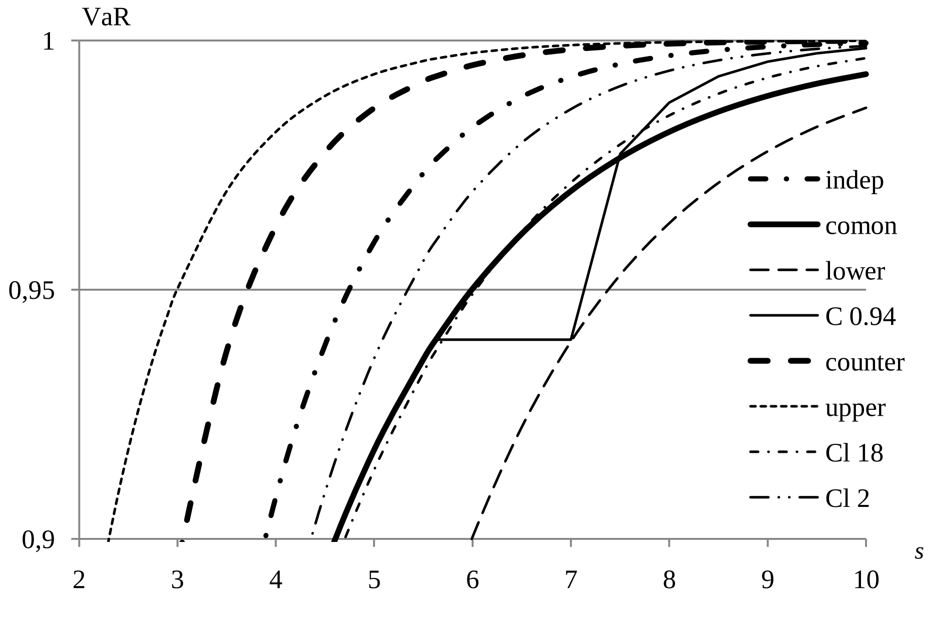

Fig. 2. The upper segment of the graph from Fig. 1

Source: own elaboration.

Now we derive the Value-at-Risk and expected shortfall at the confidence level $\alpha = 0.95$ of the sum $S = X_1 + X_2$ for different dependence structure between random variables X_i. First, we compute the bound of the VaR:

$$\mathrm{VaR}_{0.95}^{\min} = 2.9957, \quad \mathrm{VaR}_{0.95}^{\max} = 7.3778.$$

The values of VaR and ES for countermonotonic, independent, Clayton copulas Cl_2, Cl_{18} for $\alpha = 2$ and $\alpha = 18$, copula $C_{0.94}$ and comonotonic cases are shown in Table 1. The values of the *Clayton* copula parameters α correspond with values 0.5 and 0.9 of Kendal τ coefficient of correlation.

Table 1. The values of VaR and ES for the sum of exponential random variables

	countermon.	independent	Cl_2	Cl_{18}	$C_{0,94}$	comonotonic
VaR	3.7142	4.7439	5.3340	6.0316	**7.0413**	5.9915
ES	4.7015	5.9180	6.6083	7.6091	7.7477	**7.9915**

Source: own elaboration.

We see that the greatest value of VaR is obtained for the copula $C_{0.94}$, not for the comonotonic case. We can increase such a value, changing the values of parameter β. But we never obtain the maximal value of VaR in this case. For $C_{0.95}$ we have $\mathrm{VaR}_{0.95} = 5.9915$ (Figure 2). The Clayton copula Cl_{18} gives us the greater value of VaR than for the perfect positive dependence too. The greater degree of dependence implies the greater value of expected shortfall, on the other hand, and we obtain the greatest value of ES for the comonotonic case.

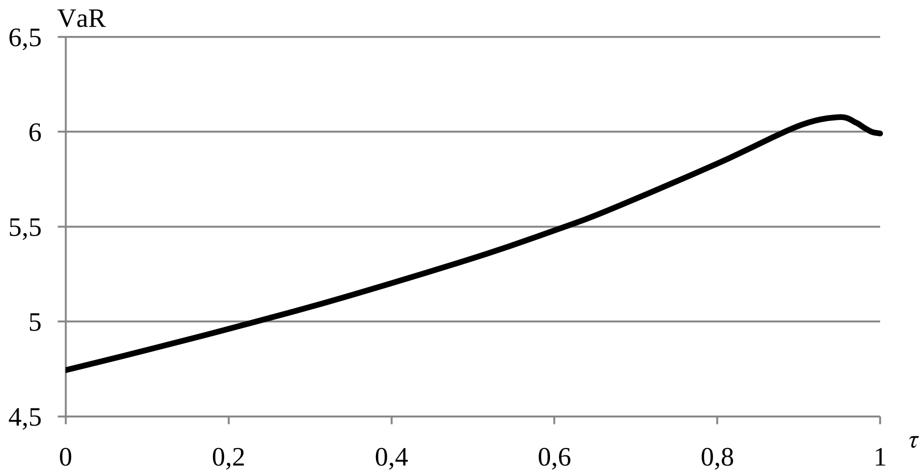

Fig. 3. The graph of the values of VaR for different values of Kendal τ

Source: own elaboration.

In Figure 3 the graph of values of VaR for sum S and for different values of Kendal τ coefficient of correlation is given. The dependent structure is described by the Clayton copula in this case. We see first that the bigger values of degree of dependence gives us the bigger value of VaR. We obtain the greatest value for $\tau = 0.95$ and next we observe a little decline of VaR.

b) **Pareto distribution**

We obtain another, more nonregular situation when the random variables X_1 and X_2 have heavy-tailed Pareto distribution. First, we assume that $F_i(x) = 1 - \frac{1}{x}$ for $x \geq 0$, $i = 1, 2$. This is Pa(1,1,1) distribution. Then for the independent random variables X_i, the sum $S = X_1 + X_2$ has the cdf. done by formula:

$$F_I(s) = \frac{s-2}{s} - \frac{2}{s^2}\ln(s-1) \quad \text{for} \quad s \geq 2.$$

The comonotonic random variables give us the sum with the Pareto distribution Pa(1; 2; 2):

$$F_M(s) = 1 - \frac{2}{s} \quad \text{for} \quad s \geq 2,$$

and the sum with counter monotonic marginals has the cumulative distribution function:

$$F_W(s) = \sqrt{\frac{s-4}{s}} \quad \text{for} \quad s \geq 4.$$

The lower and upper bound of the cdf. of sum S have Pareto distributions: Pa(1; 4; 4) and Pa(1; 1; 2), i.e. we have

$$F_{\min}(s) = 1 - \frac{4}{s} \quad \text{for} \quad s \geq 4,$$

$$F_{\max}(s) = 1 - \frac{1}{s-1} \quad \text{for} \quad s \geq 2.$$

The graphs of the cumulative distribution functions of sum S for different dependence structure are shown in Figures 4 and 5. We see that the cdf. of the sum for comonotonic case takes greater values than the countermonotonic and independent cases. We have

$$F_W(s) < F_M(s) \text{ and } F_I(s) < F_M(s),$$

for every $s > 0$. The graphs of the cdf. of the countermonotonic and independent cases cross each other. For $s > 5.995$ we obtain $F_I(s) < F_W(s)$.

Table 2. The values of VaR and ES for the sum of Pa(1,1,1) distributed marginals

	upper	comonotonic	countermonot.	independent	*Cl* 2	lower
VaR	21	40	41.025	43.451	45.677	80

Source: own elaboration.

The values of VaR of the sum S of the marginal with the Pareto distribution for different dependence structure are shown in Table 2. We cannot compute the expected shortfall in this case because the expected value of marginals are equal infinity. We see that the independent case gives us the greater value of VaR than perfect dependence cases. The values of VaR of the sum for the different dependent structure done by Clayton copula are shown in Figure 6.

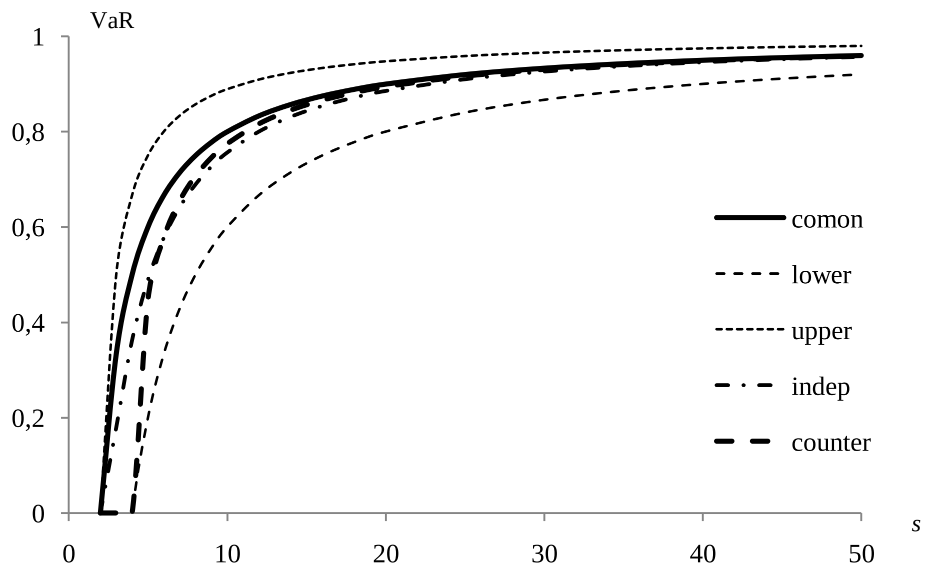

Fig. 4. The graph of the cumulative distribution functions of sum S for different dependent structures and Pa(1,1,1) marginals

Source: own elaboration.

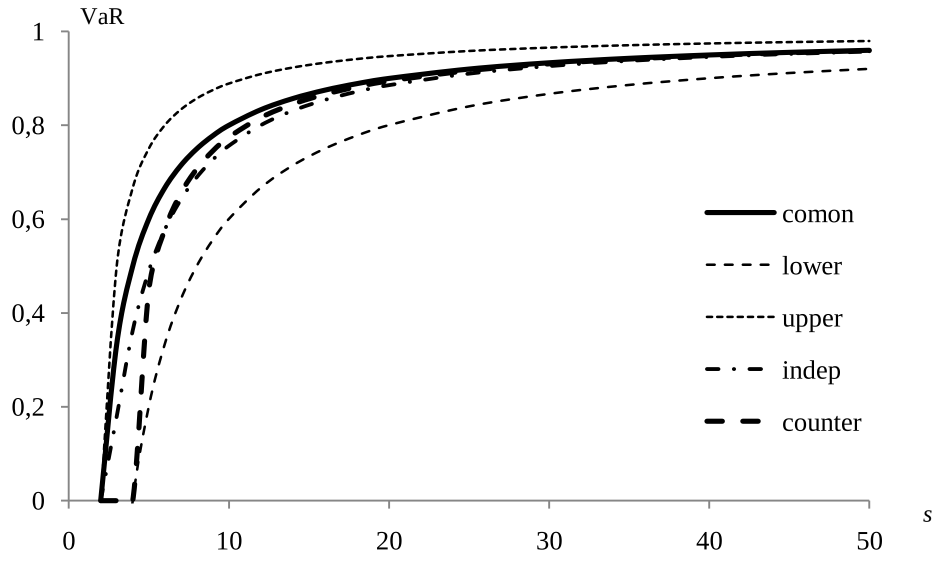

Fig. 5. The upper segment of the graph from Fig. 4

Source: own elaboration.

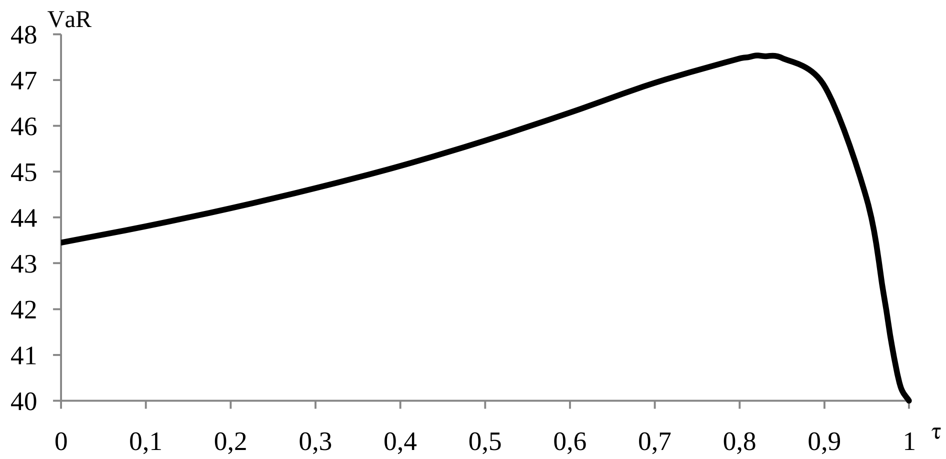

Fig. 6. The graph of the values of VaR for different values of Kendal τ and Pa(1,1,1) marginals

Source: own elaboration.

We see first that the values of VaR slowly increases when the Kendal τ increases. They obtain the maximum for $\tau = 0.82$ and rapidly decrease for the greater degrees of dependence.

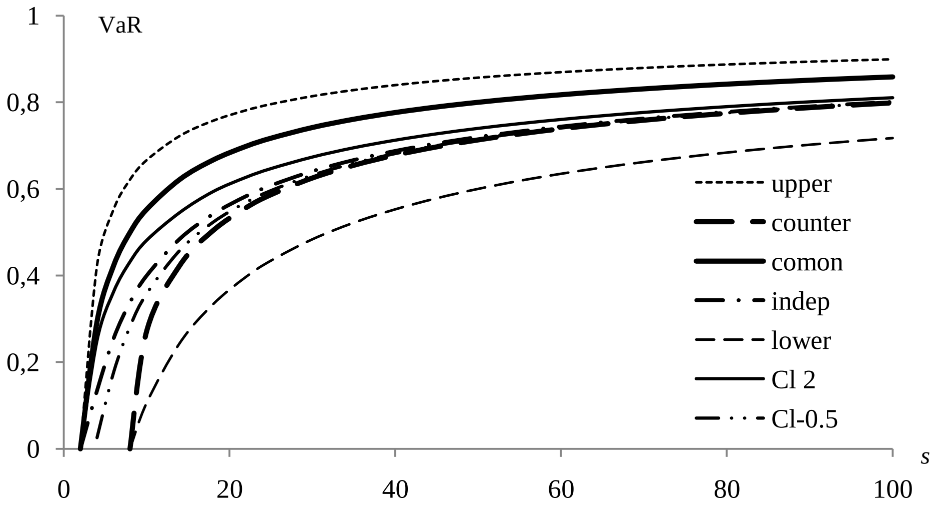

Fig. 7. The graph of the cumulative distribution functions of sum *S* for different dependent structures and Pa(2,1,1) marginals

Source: own elaboration.

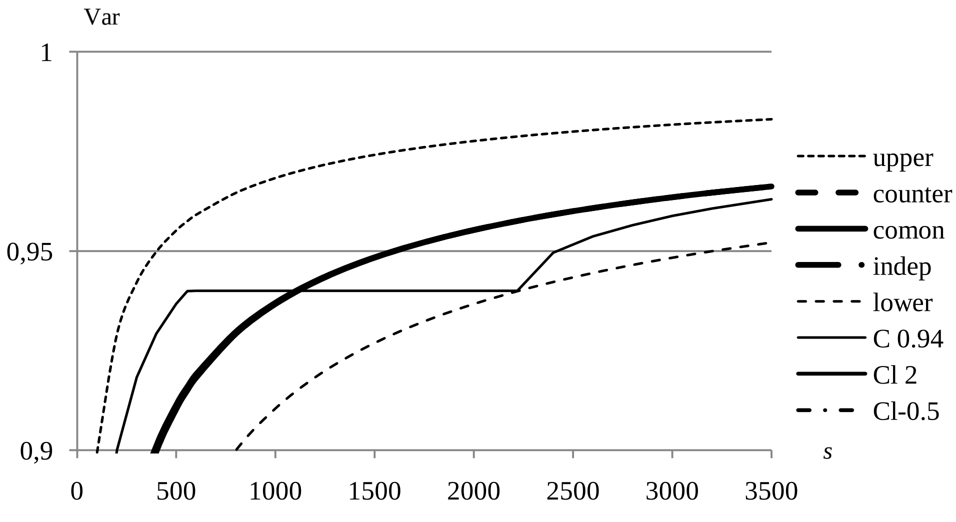

Fig. 8. The graph of the cumulative distribution functions of sum *S* for different dependent structures and Pa(2,1,1) marginals for greater values of arguments *s*

Source: own elaboration.

We obtain the more regular case when the margins X_i have Pa(2,1,1) distribution, i.e. $F_i(x) = 1-1/\sqrt{x}$ for $x \geq 0$. The tail of such distribution is more heavy than in the former case. We see that the cumulative distribution functions presented in Figures 7 and 8 do not cross each other. The countermonotonic case gives us the greatest value of VaR, but the differences between the independent, countermonotonic and Clayton $Cl_{-0.5}$ and Cl_2 cases are small. The copula $C_{0.94}$ and lower border $F_{\min}$ give us the considerably greater value of VaR. The Clayton copula $C_{-0.5}$ reflects the negative dependence: $\tau = -1/3$.

5. Conclusion

In this paper we study the risk measure, mainly VaR, of the sum of dependent random variables. We investigate two cases when the marginals have the exponential and Pareto distribution. We show that the most value of VaR is not obtained for extreme, perfect dependent cases. The value of VaR of such sums depends on the distribution of marginals and on the degree of dependence of them.

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