

THE EFFECT OF COMMON RISK ON GROUP INSURANCE¹

Anna Nikodem-Słowikowska

Abstract. Group life insurance is a type of life insurance in which a single contract covers an entire group of people. In the contract a status that determines the termination of the policy is described. For example, in the case of the joint-life status the sum insured is paid because of the first death in the group. To compute the net single premium for a life insurance contract or for a life annuity, we have to calculate the probability that the status is still intact. In the classical group life insurance, it is assumed that the future life times of the insured are independent. This assumption is unrealistic in many practical situations. In this paper two statuses will be considered: the joint-life status and the last-survivor status. The common risk will be taken into account during premium calculation. The effect of the common risk on the group life insurance will be demonstrated in numerical examples.

Keywords: group life insurance, joint-life status, last-survivor status, net single premium, common risk.

JEL Classification: C10, G22.

1. Introduction

Group life insurance is a type of life insurance in which a single contract covers an entire group of people. In the contract a status that determines the termination of the policy is described. In this paper two statuses will be considered: the joint-life status and the last-survivor status (see (Błaszczyszyn, Rolski, 2004; Skalba, 1999)).

The joint-life status exists as long as they all live; hence, it fails with the first death. This status is denoted by

$$u := x_1 : x_2 : \dots : x_m, \quad (1)$$

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where x_k is the age of the k -th insured person in the group. The failure time $T(u)$ of this status is equal to minimum of the random variables $T(x_k)$, which are the future lifetime of the k -th life. Hence:

$$T(u) = \min\{T(x_1), T(x_2), \dots, T(x_m)\}. \quad (2)$$

To calculate the net single premium for an insurance or for an annuity, the probability distribution of the failure time of status, the probability distribution that status is in failure at time t and the force of failure can be computed. The survival probability of the status is given by

$$\begin{aligned} {}_t p_u &= P(T(u) > t) = P(\min\{T(x_1), T(x_2), \dots, T(x_m)\} > t) = \\ &= P(T(x_1) > t, T(x_2) > t, \dots, T(x_m) > t). \end{aligned} \quad (3)$$

If all the future lifetimes $T(x_k)$ are independent, then

$${}_t p_u = \prod_{k=1}^m P(T(x_k) > t) = \prod_{k=1}^m {}_t p_{x_k}, \quad (4)$$

where ${}_t p_{x_k}$ is the probability that a life aged x_k will survive at least t years. The probability that the status is in failure at time t is given by

$${}_t q_u = 1 - {}_t p_u, \quad (5)$$

and for independent random variables $T(x_k)$ is equal to

$${}_t q_u = 1 - \prod_{k=1}^m {}_t p_{x_k} = 1 - \prod_{k=1}^m (1 - {}_t q_{x_k}), \quad (6)$$

where ${}_t q_{x_k}$ denoted the probability that a life aged x_k will die within t years. The force of failure is given by

$$\mu_{u+t} = -\frac{d}{dt} \ln({}_t p_u). \quad (7)$$

Assuming the independence, we have

$$\mu_{u+t} = -\frac{d}{dt} \ln({}_t p_u) = -\frac{d}{dt} \ln\left(\prod_{k=1}^m {}_t p_{x_k}\right) = \sum_{k=1}^m -\frac{d}{dt} \ln({}_t p_{x_k}) = \sum_{k=1}^m \mu_{x_k+t}, \quad (8)$$

where μ_{x_k+t} is the force of mortality of a life aged x_k at the age $x_k + t$.

The last-survivor status exists as long as one person is still alive, it fails with the last death. The failure time $T(z)$ of this status is equal to the maximum of the random variables $T(x_k)$, i.e.

$$T(z) = \max \{T(x_1), T(x_2), \dots, T(x_m)\},$$

where

$$z := \overline{x_1 : x_2 : \dots : x_m}$$

denotes this status. The survival probability of the status z is given by

$$\begin{aligned} {}_t p_z &= P(T(z) > t) = P(\max \{T(x_1), T(x_2), \dots, T(x_m)\} > t) = \\ &= P((T(x_1) > t) \cup (T(x_2) > t) \cup \dots \cup (T(x_m) > t)) = \\ &= S_1^t - S_2^t + S_3^t - \dots + (-1)^{m-1} S_m^t, \end{aligned} \quad (9)$$

where S_k^t denotes the symmetric sum

$$S_k^t = \sum {}_t P_{x_{j_1} : x_{j_2} : \dots : x_{j_k}}.$$

The probability that the status is in failure at time t is given by

$$\begin{aligned} {}_t q_z &= P(T(z) < t) = P(\max \{T(x_1), T(x_2), \dots, T(x_m)\} < t) = \\ &= P(T(x_1) < t, T(x_2) < t, \dots, T(x_m) < t) \end{aligned} \quad (10)$$

and assuming independence this probability is equal to

$${}_t q_z = \prod_{k=1}^m {}_t q_{x_k}. \quad (11)$$

Hence, the survival probability can be calculated also in this way

$${}_t p_z = 1 - {}_t q_z = 1 - \prod_{k=1}^m {}_t q_{x_k} = 1 - \prod_{k=1}^m (1 - {}_t p_{x_k}). \quad (12)$$

In this status the force of failure cannot be calculated in a simple way like in the joint-life status.

In the classical theory of multiple life insurance, it is assumed that the future lifetime of the k -th life are mutually independent. This assumption, which simplifies the computations, is not appropriate in practical situations. In the next section the common risk random variable will be introduced.

2. Common risk

Consider the random variable Z defined as the common risk that affects all future lifetimes of lives in the group. This random variable is associated with the time of the catastrophe. Let Z have an exponential distribution with a parameter λ , which is defined as the common risk parameter. The new future lifetime $J(x_k)$ of k -th life is equal to minimum of natural death and death as a result of catastrophe, i.e. (see (Elliott, 2008))

$$J(x_k) = \min\{T(x_k); Z\}, \quad (13)$$

where $T(x_k)$ is the future lifetime to the natural death. It is assumed that the random variables $T(x_k)$ and Z are independent. The random variables $J(x_k)$ are dependent because of the common risk. In the next section the statuses with the common risk will be presented.

2.1. The joint-life status

The failure time $J(u)$ of status u is equal to minimum of random variables (13)

$$J(u) = \min\{J(x_1), J(x_2), \dots, J(x_m)\}. \quad (14)$$

The survival probability in this status is given by

$$\begin{aligned} {}_t\tilde{p}_u &= P(J(u) > t) = P(\min\{J(x_1), J(x_2), \dots, J(x_m)\} > t) = \\ &= P(\min\{T(x_1); Z\} > t \cap \min\{T(x_2); Z\} > t \cap \dots \cap \min\{T(x_m); Z\} > t) = \\ &= P(T(x_1) > t, T(x_2) > t, \dots, T(x_m) > t, Z > t). \end{aligned} \quad (15)$$

Using the assumption of the random variables $T(x_k)$ and Z we have

$${}_t\tilde{p}_u = \prod_{k=1}^m P(T(x_k) > t) \cdot e^{-\lambda t} = \prod_{k=1}^m {}_t p_{x_k} \cdot e^{-\lambda t} = {}_t p_u \cdot e^{-\lambda t}. \quad (16)$$

The probability that the status is in failure at time t is given by

$${}_t\tilde{q}_u = 1 - {}_t\tilde{p}_u.$$

Hence, from (16) we obtain

$${}_t\tilde{q}_u = 1 - \prod_{k=1}^m {}_t p_{x_k} \cdot e^{-\lambda t} = 1 - \prod_{k=1}^m (1 - {}_t q_{x_k}) \cdot e^{-\lambda t}. \quad (17)$$

From definition of the force of failure, we get

$$\begin{aligned} \tilde{\mu}_{u+t} &= -\frac{d}{dt} \ln({}_t \tilde{p}_u) = -\frac{d}{dt} \ln\left(\prod_{k=1}^m {}_t p_{x_k} \cdot e^{-\lambda t}\right) = \\ &= \sum_{k=1}^m -\frac{d}{dt} \ln({}_t p_{x_k}) + \left(-\frac{d}{dt} \ln(e^{-\lambda t})\right) = \sum_{k=1}^m \mu_{x_k+t} + \lambda = \mu_{u+t} + \lambda. \end{aligned} \quad (18)$$

2.2. The last-survivor status

In the last-survivor status the failure time $J(z)$ of the status is equal to the maximum of random variables (13)

$$J(z) = \max\{J(x_1), J(x_2), \dots, J(x_m)\}. \quad (19)$$

The survival probability of the status z with the common risk is given by

$$\begin{aligned} {}_t \tilde{p}_z &= P(J(z) > t) = P(\max\{J(x_1), J(x_2), \dots, J(x_m)\} > t) = \\ &= P(\min\{T(x_1); Z\} > t \cup \min\{T(x_2); Z\} > t \cup \dots \cup \min\{T(x_m); Z\} > t) = \\ &= \tilde{S}_1^t - \tilde{S}_2^t + \tilde{S}_3^t - \dots + (-1)^{m-1} \tilde{S}_m^t, \end{aligned}$$

where \tilde{S}_k^t denotes the symmetric sum, which is equal to

$$\tilde{S}_k^t = \sum {}_t \tilde{p}_{x_{j_1} : x_{j_2} : \dots : x_{j_k}} = \sum ({}_t p_{x_{j_1} : x_{j_2} : \dots : x_{j_k}} \cdot e^{-\lambda t}) = e^{-\lambda t} \sum {}_t p_{x_{j_1} : x_{j_2} : \dots : x_{j_k}} = S_k^t \cdot e^{-\lambda t}.$$

Hence:

$${}_t \tilde{p}_z = e^{-\lambda t} (S_1^t - S_2^t + S_3^t - \dots + (-1)^{m-1} S_m^t) = {}_t p_z \cdot e^{-\lambda t}, \quad (20)$$

and from (12) we obtain

$${}_t \tilde{p}_z = \left(1 - \prod_{k=1}^m (1 - {}_t p_{x_k})\right) \cdot e^{-\lambda t}. \quad (21)$$

The probability that the status z is in failure at time t is given by

$${}_t \tilde{q}_z = 1 - {}_t \tilde{p}_z,$$

where from (21) we have

$${}_t \tilde{q}_z = 1 - e^{-\lambda t} + \prod_{k=1}^m {}_t q_{x_k} \cdot e^{-\lambda t}. \quad (22)$$

The force of failure in this case can be expressed as

$$\begin{aligned} \tilde{\mu}_{z+t} &= -\frac{d}{dt} \ln({}_t \tilde{p}_z) = -\frac{d}{dt} [\ln({}_t p_z) + \lambda t] = \\ &= -\frac{d}{dt} \ln({}_t p_z) + \lambda = \mu_{z+t} + \lambda. \end{aligned} \quad (23)$$

2.3. Gompertz Model in the joint-life status

In the Gompertz Model the force of mortality has the following form:

$$\mu_{x+t} = Bc^{x+t} \text{ for } t \geq 0, \quad (24)$$

where $B > 0$, $c > 1$ are the parameters for population (see (Balicki, 2006)).

If all lives are subject to the same Gompertz mortality law, then from (18) and (24) the force of failure in the joint-life status is equal to

$$\tilde{\mu}_{u+t} = \sum_{k=1}^m \mu_{x_k+t} + \lambda = \sum_{k=1}^m Bc^{x_k+t} + \lambda = Bc^t \sum_{k=1}^m c^{x_k} + \lambda. \quad (25)$$

After solving the equation:

$$c^{x_1} + c^{x_2} + \dots + c^{x_m} = c^w, \quad (26)$$

for w , the force of failure (25) can be expressed by the sum of the force of mortality of a life aged w at the age $w+t$ and a common risk parameter:

$$\tilde{\mu}_{u+t} = Bc^t c^w + \lambda = \mu_{w+t} + \lambda. \quad (27)$$

The probability distribution of the failure time of status u is then given by

$$\begin{aligned} {}_t\tilde{p}_u &= \exp\left[-\int_0^t \tilde{\mu}_{u+s} ds\right] = \exp\left[-\int_0^t \mu_{w+s} + \lambda ds\right] = \\ &= \exp\left[-\int_0^t \mu_{w+s} ds\right] \cdot \exp\left[-\int_0^t \lambda ds\right] = {}_t p_w \cdot e^{-\lambda t}, \end{aligned} \quad (28)$$

where

$$\begin{aligned} {}_t p_w &= \exp\left[-\int_0^t \mu_{w+s} ds\right] = \exp\left[-\int_0^t Bc^{w+s} ds\right] = \exp\left[-Bc^w \int_0^t c^s ds\right] = \\ &= \exp\left[-Bc^w \frac{c^s}{\ln c} \Big|_0^t\right] = \exp\left[-Bc^w \frac{c^t - 1}{\ln c}\right] = \exp\left[-\frac{B}{\ln c} \cdot (c^{w+t} - c^w)\right]. \end{aligned} \quad (29)$$

Hence:

$${}_t\tilde{p}_u = {}_t p_w \cdot e^{-\lambda t} = \exp\left[-\frac{B}{\ln c} \cdot (c^{w+t} - c^w) - \lambda t\right]. \quad (30)$$

If all lives are subject to the same Gompertz mortality law, then all calculations of the probabilities and the net single premium can be performed in terms of the single life aged w .

3. Estimation of the parameters in the Gompertz Model

The survival probability can be expressed by the force of mortality, i.e.

$${}_t p_x = \exp \left[- \int_0^t \mu_{x+u} du \right]. \quad (31)$$

Hence, using (24) the probability that a life aged x will survive at least one year is equal to

$$p_x = \exp \left[- \int_0^1 Bc^{x+u} du \right] = \exp \left[-Bc^x \int_0^1 c^u du \right] = \exp \left[-\frac{B}{\ln c} (c-1)c^x \right]. \quad (32)$$

Taking the logarithm of equation (32), we obtain

$$\ln p_x = -\frac{B}{\ln c} (c-1)c^x. \quad (33)$$

Let $\alpha_x = -\ln p_x$ and $\beta = \frac{B}{\ln c} (c-1)$, then expression (33) can be written the following way:

$$\ln(\alpha_x) = \ln \beta + x \ln c. \quad (34)$$

Using the least squares method, we can find the estimator of parameters β_0 and c_0 , which minimize the function:

$$L(\beta; c) = \sum_{x=0}^{100} [\ln(\alpha_x) - \ln \beta - x \ln c]^2.$$

Hence, we have to solve

$$\begin{cases} \frac{\partial L}{\partial \beta} = 2 \sum_{x=0}^{100} [\ln(\alpha_x) - \ln \beta - x \ln c] \left(-\frac{1}{\beta} \right) = 0, \\ \frac{\partial L}{\partial c} = 2 \sum_{x=0}^{100} [\ln(\alpha_x) - \ln \beta - x \ln c] \left(-\frac{x}{c} \right) = 0, \end{cases}$$

and after simplification we obtain

$$\begin{cases} \sum_{x=0}^{100} \ln(\alpha_x) - 101 \cdot \ln \beta - \ln c \sum_{x=0}^{100} x = 0, \\ \sum_{x=0}^{100} x \ln(\alpha_x) - \ln \beta \sum_{x=0}^{100} x - \ln c \sum_{x=0}^{100} x^2 = 0. \end{cases}$$

The estimations of the parameters β_0 and c_0 are given by

$$\ln \beta_0 = \frac{\sum_{x=0}^{100} \ln(\alpha_x) - \ln c_0 \sum_{x=0}^{100} x}{101}, \quad (35)$$

$$\ln c_0 = \frac{\sum_{x=0}^{100} x \sum_{x=0}^{100} \ln(\alpha_x) - 101 \sum_{x=0}^{100} x \ln(\alpha_x)}{\left(\sum_{x=0}^{100} x \right)^2 - 101 \sum_{x=0}^{100} x^2}, \quad (36)$$

where $\alpha_x = -\ln p_x$ are determined on the basis of the life table.

Example 1. Using formulas (35), (36) and Life Table of Poland 2009 (see (Central Statistical Office)), we have obtained the estimations of the parameter

$$\ln c_0 = 0.08092,$$

$$\ln \beta_0 = -8.87038.$$

Hence, the parameters of the force of mortality (24) are equal to

$$c = 1.084284202,$$

$$B = 0.000134881.$$

4. Effect of common risk on the single net premium

In this section the net single premiums with and without common risk are compared. The following type of the premium will be considered (see (Balicki, 2006; Skafba, 1999)):

– the net single premium of an insured benefit of 1 unit, payable immediately upon the failure of status u :

$$\bar{A}_u = \int_0^{\infty} v^t p_u \mu_{u+t} dt; \quad (37)$$

– the net single premium for an insurance with the sum insured which is payable immediately on the first death if this occurs within the first n years:

$$\bar{A}_{u:\overline{n}|} = \int_0^n v^t {}_t p_u \mu_{u+t} dt ; \quad (38)$$

– the net single premium for an insurance payable at the end of the year of the first death:

$$A_u = \sum_{k=0}^{\infty} v^{k+1} {}_k p_u q_{u+k} ; \quad (39)$$

– the net single premium for an insurance with the sum insured which is payable at the end of year of the first death if this occurs within the first n years:

$$A_{u:\overline{n}|}^1 = \sum_{k=0}^{n-1} v^{k+1} {}_k p_u q_{u+k} ; \quad (40)$$

– the net single premium of a life annuity-due:

$$\ddot{a}_z = \sum_{k=0}^{\infty} v^k {}_k p_z ; \quad (41)$$

– the net single premium of an n -year temporary life annuity-due:

$$\ddot{a}_{z:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_k p_z . \quad (42)$$

The effect of the common risk is presented in the examples. The relative errors have been calculated according to formula:

$$\frac{NSP^* - NSP}{NSP} ,$$

where NSP^* is the net single premium for the common risk parameter $\lambda > 0$ and NSP denotes the net single premium without common risk.

Example 2. A group of four men, ages 39, 40, 42 and 45, take out a four-life insurance policy which pays benefits of 1 unit at the time of the first death. Using the Life Table of Poland 2009 and formula (39) the net single premium is calculated. The interest rate is equal to 2%. In Figure 1 the relative errors for the net premium are presented.

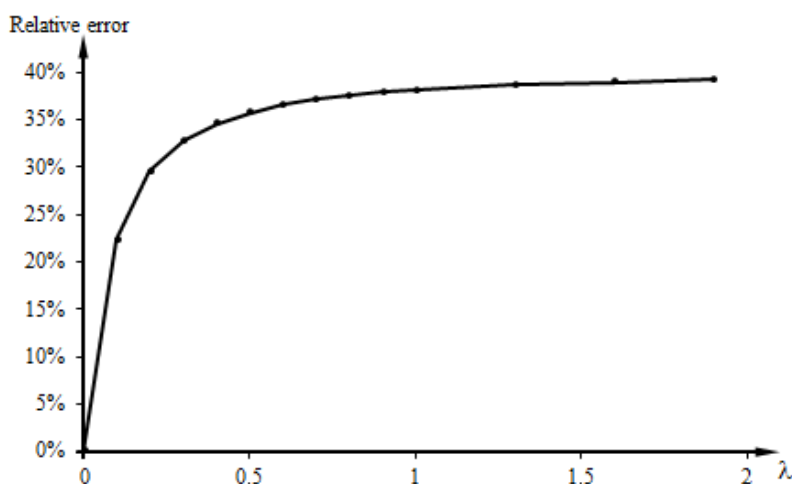


Fig. 1. The relative error of the net single premium for different common risk parameters λ
Source: author's own study

In Figure 1 we can see that the relative error increases with λ . For $\lambda > 1$ the errors are larger than 35% and they are at the same level.

Example 3. Consider the same group of people as in Example 2. In this example the men take out the term insurance policy. Using the Life Table of Poland 2009 and formula (40), the net single premium is calculated. The interest rate is equal to 2%. In Table 1 the net single premium $A_{u:\overline{n}|}^1$ and relative error (r.e.) are given for the different parameter λ and the insurance term n .

In Table 1 we can see that the relative error increases with common risk parameter λ . Moreover, for a larger insurance term the relative errors are smaller, which we can see in Figure 2.

Table 1. The net single premium and the relative error for term insurance

λ	$n = 5$		$n = 10$		$n = 20$		$n = 40$	
	$A_{u:\overline{n} }^1$	r.e. (%)	$A_{u:\overline{n} }^1$	r.e. (%)	$A_{u:\overline{n} }^1$	r.e. (%)	$A_{u:\overline{n} }^1$	r.e. (%)
0	0.0858	0.00	0.1969	0.00	0.4553	0.00	0.6981	0.00
0.002	0.0943	9.96	0.2109	7.11	0.4704	3.33	0.7043	0.88
0.004	0.1027	19.82	0.2246	14.08	0.4850	6.54	0.7102	1.73
0.006	0.1111	29.58	0.2381	20.92	0.4992	9.64	0.7160	2.55
0.008	0.1194	39.25	0.2513	27.64	0.5128	12.63	0.7215	3.35
0.01	0.1276	48.83	0.2643	34.22	0.5259	15.52	0.7269	4.12

Source: author's own study.

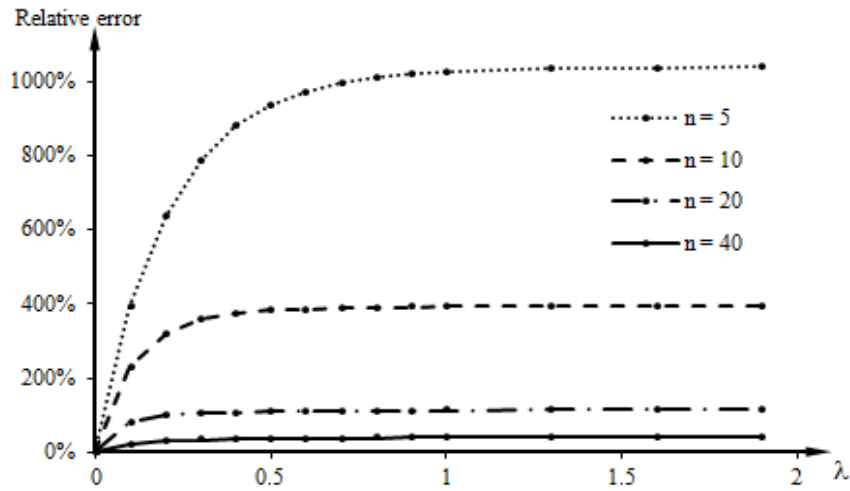


Fig. 2. The relative error of the net single premium for different parameters λ and insurance term n

Source: author's own study.

Example 4. A group of men, the same as in Example 2, take out the life annuity-due contract. This contract provides for annual payments of 1 unit as long as the last survivor lives. Using the Life Table of Poland 2009 and formula (41), the net single premium is calculated. The interest rate is equal to 2%.

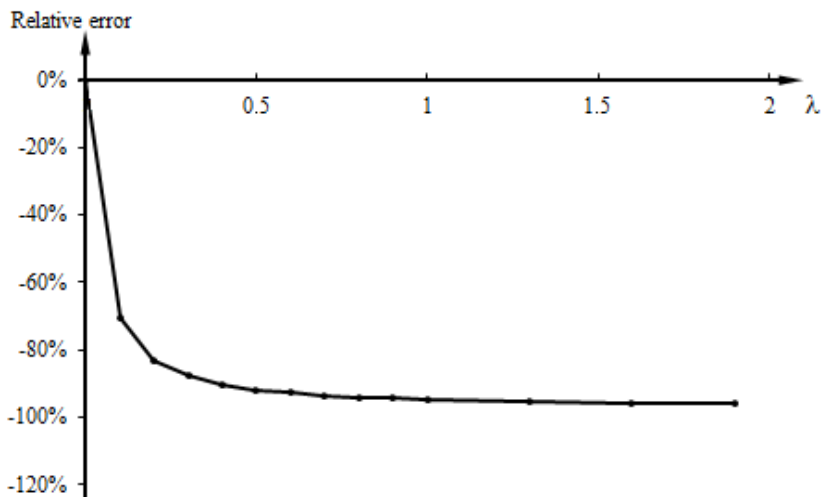


Fig. 3. The relative error of the net single premium for different parameters λ

Source: author's own study.

In Figure 3 we can see that the relative error decreases with λ . For $\lambda > 0.4$ the errors are larger than 90% and for $\lambda > 1$ they are at the same level.

Example 5. Consider the same group of men as in Example 4. In this case the men take out the temporary life annuity-due contract. To calculate the net single premium, the Life Table of Poland 2009 and formula (42) are used. The interest rate is equal to 2%. In Table 2 the net single premium $\ddot{a}_{z:\overline{n}|}$ and the relative error (r.e.) are given for the different parameter λ and the insurance term n .

Table 2. The net single premium and the relative error for temporary life annuity-due

λ	$n = 5$		$n = 10$		$n = 20$		$n = 40$	
	$\ddot{a}_{z:\overline{n} }$	r.e. (%)	$\ddot{a}_{z:\overline{n} }$	r.e. (%)	$\ddot{a}_{z:\overline{n} }$	r.e. (%)	$\ddot{a}_{z:\overline{n} }$	r.e. (%)
0	4.8077	0.00	9.1622	0.00	16.6767	0.00	27.4376	0.00
0.002	4.7889	-0.39	9.0833	-0.86	16.3854	-1.75	26.5493	-3.24
0.004	4.7703	-0.78	9.0053	-1.71	16.1014	-3.45	25.7025	-6.32
0.006	4.7517	-1.17	8.9282	-2.55	15.8243	-5.11	24.8951	-9.27
0.008	4.7332	-1.55	8.8521	-3.38	15.5541	-6.73	24.1248	-12.07
0.01	4.7149	-1.93	8.7770	-4.20	15.2905	-8.31	23.3898	-14.75

Source: author's own study.

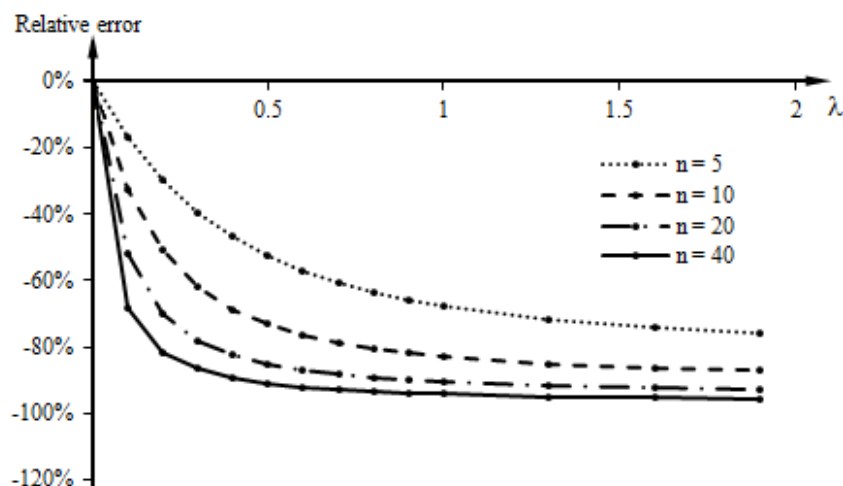


Fig. 4. The relative error of the net single premium for different parameters λ and insurance term n

Source: author's own study.

In Table 2 we can see that the difference between the net single premium with the common risk and the net single premium with $\lambda = 0$ increases with parameter λ . The longer insurance term is, the larger relative error gets. We can see that in Figure 4.

In the next example the Gompertz model is used to calculate the net single premium.

Example 6. Consider the group of men from Example 2. They take out the life insurance with the sum insured, which is payable immediately on the first death. Assume that all lives are subject to the same Gompertz mortality law. In Example 1 we obtain the estimation of the parameters of the force mortality (24). After solving equation (26), we have $w = 58.8498$. Using formulas (37), (27), (30), the net single premium and the relative error (r.e.) are calculated for a different common risk parameter. Some outcomes are presented in Table 3.

Table 3. The net single premium and the relative error for life insurance

λ	\bar{A}_w	r.e. (%)
0	0.7036	0.00
0.1	0.8670	23.23
0.2	0.9185	30.54
0.3	0.9418	33.87
0.4	0.9549	35.72
0.5	0.9632	36.91
0.6	0.9690	37.72
0.7	0.9732	38.32
0.8	0.9763	38.77
0.9	0.9789	39.13
1	0.9809	39.42

Source: author's own study.

In Figure 5 the relative error is presented for larger values of the parameter λ than in Table 3. We can see that for a larger value of the common risk parameter the relative error is larger. A similar situation is in Example 2.

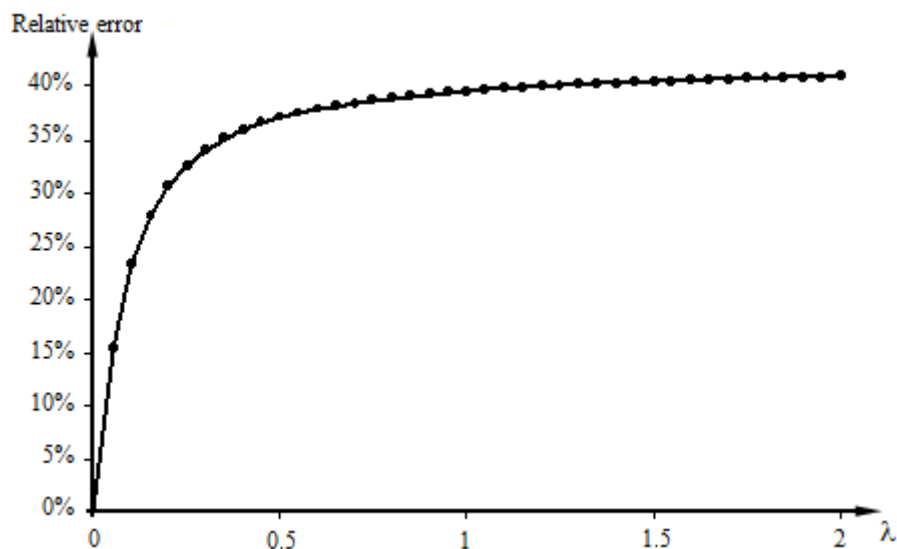


Fig. 5. The relative error of the net single premium for different parameters λ

Source: author's own study.

Summing up, the value of the common risk parameter influences the value of the net single premium. The larger the value of the parameter λ is, the larger the value of the premium for insurance gets. In the case of annuity, the larger the value of parameter λ is, the smaller the value of the premium for life annuity-due gets. The insurance term also influences the net single premium, as we can see in Table 1 and Table 2.

Literature

- Balicki A. (2006). *Analiza przeżycia i tablice wymieralności*. Polskie Wydawnictwo Ekonomiczne. Warszawa.
- Błaszczyszyn B., Rolski T. (2004). *Podstawy matematyki ubezpieczeń na życie*. Wydawnictwo Naukowo-Techniczne. Warszawa.
- Central Statistical Office. Official website. www.stat.gov.pl.
- Elliott C. (2008). *Multiple-Life Theory*. <http://www.scribd.com/doc/48882333/000453534r>.
- Skalba M. (1999). *Ubezpieczenia na życie*. Wydawnictwa Naukowo-Techniczne. Warszawa.