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Zagadnienia statystyki aktuarialnej ISSN 1899-3192

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THE EFFECT OF DEPENDENCE ON LIFE INSURANCE*

Summary: In the classical life insurance it is assumed that the risks are independent. This assumption is not appropriate in many practical situations. For example the lifelengths of two insured persons (such as a husband and a wife) are dependent because they share a common way of life and more or less are exposed to the same risks. In the case of the group life insurance for persons that work for the same company, the mortality is dependent on a certain event (i.e. explosion, breakdown). In this paper models for modelling the dependence will be presented. The influence of the dependence on the aggregate claims distribution and the net premium in the life insurance will be demonstrated on numerical examples.

Keywords: individual risk model, dependent risks, aggregate claims distribution, recursion formula, net premium.

1. Introduction

The individual risk model can be used in the group of life insurance. Usually it is assumed that the risks of the insurance portfolio are independent. This assumption is not appropriate in many practical situations. For example a husband and a wife shares a common way of life and more or less are exposed to the same risk. The persons that work for the same company are exposed to a certain event (i.e. explosion, breakdown). Therefore a risk that is common in every life will be considered in the model.

Let the insurance portfolio consist of $n + m$ risks. The m married couples have policies in this portfolio. Then the aggregate claim amount denoted by S can be written as (see [3])

$$S = \sum_{i=1}^{m} (X_{i1} + X_{i2}) + \sum_{i=m+1}^{n} X_i \, , \tag{1}$$

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where $X_{i1} + X_{i2}$ is the claim amount for policies of couple i, X_i is the claim amount for the policy i. The claim amount is expressed as $X_i = b_i I_i$, where

$$I_{il} = \min(J_{il} + J_0; 1) \qquad \text{for } i = 1, \ldots, m, \ l = 1, 2, \tag{2}$$

$$I_i = J_i \qquad \text{for } i = m + 1, \ldots, n. \tag{3}$$

The random variables J_i, J_{il}, J_0 are independent and have the Bernoulli distribution:

$$P(J_i = 1) = q_i, \qquad P(J_i = 0) = 1 - q_i = p_i \qquad \text{for } i = m + 1, \ldots, n, \tag{4}$$

$$P(J_{il} = 1) = q_{il}, \qquad P(J_{il} = 0) = 1 - q_{il} = p_{il} \qquad \text{for } i = 1, \ldots, m, \ l = 1, 2, \tag{5}$$

$$P(J_0 = 1) = q_0, \qquad P(J_0 = 0) = 1 - q_0 = p_0. \tag{6}$$

Random variables $X_{i1} + X_{i2}$ and X_i for $i = 1, \ldots, n$ are mutually independent, but X_{i1} and X_{i2} are dependent because of the common risk factor J_0.

The next model can be used in the group life insurance, where m persons are exposed to the same risk. Let S be the aggregate claim amount given by

$$S = X_1 + X_2 + \ldots + X_m + \sum_{i=m+1}^{n} X_i , \tag{7}$$

where the claim amount for the policy i is expressed by $X_i = bI_i$. The random variable I_i is defined by

$$I_i = \min(J_i + J_0; 1) \qquad \text{for } i = 1, \ldots, m, \tag{8}$$

$$I_i = J_i \qquad \text{for } i = m + 1, \ldots, n, \tag{9}$$

where J_i and J_0 are independent Bernoulli random variables with

$$P(J_i = 1) = q, \qquad P(J_i = 0) = 1 - q = p \qquad \text{for } i = 1, \ldots, n, \tag{10}$$

$$P(J_0 = 1) = q_0, \qquad P(J_0 = 0) = 1 - q_0 = p_0. \tag{11}$$

Therefore random variables X_i are independent for $i = m + 1, \ldots, n$ and for $i = 1, \ldots, m$ are dependent because of dependence between the occurrence random variables I_i.

In actuarial science, we are interested in the computation of the distribution function of the aggregate claim amount S. In many papers we can find methods of calculating the aggregate claim amount with independent risks (see e.g. [2; 4; 5; 6]). In [4] the recursive method for individual risk model is shown. Various papers have also addressed the subject of dependency between risks (see e.g. [1; 3; 7]). In [1] the common shock model is described.

In this paper, a model proposed by Dhaene and Goovaerts [3] is considered. The dependence between two risks is introduced via the occurrence of a random variable (similarly as in [1]). In the next section the recursion for the aggregate claims distribution is presented. In the last section the influence of the dependence on the aggregate claims distribution and the net premium is presented by numerical examples.

2. Calculation of the aggregate claims distribution

An exact recursion formula for computing the aggregate claims distribution in the individual model with dependent risks described in (1) is given in the following theorem:

Theorem 1. The aggregate claims distribution for model described in (1) can be calculated by

$$P(S=0)=\prod_{i=1}^{m}\left(p_0 p_{i1} p_{i2}\right)\cdot\prod_{i=m+1}^{n}p_i, \tag{12}$$

$$sP(S=s)=\sum_{i=1}^{m}v_i(s)+\sum_{i=m+1}^{n}w_i(s), \tag{13}$$

where the coefficients $w_i(s)$, $v_i(s)$ are given by

$$w_i(s)=\frac{q_i}{p_i}\left[b_i\cdot P(S=s-b_i)-w_i(s-b_i)\right], \tag{14}$$

$$\begin{aligned} v_i(s)=&\frac{q_{i1}}{p_{i1}}\cdot\left[b_{i1}\cdot P(S=s-b_{i1})-v_i(s-b_{i1})\right]+\frac{q_{i2}}{p_{i2}}\cdot\left[b_{i2}\cdot P(S=s-b_{i2})-v_i(s-b_{i2})\right]+\\ &+\frac{\left(p_0 q_{i1} q_{i2}+q_0\right)}{p_0 p_{i1} p_{i2}}\cdot\left[\sum_{l=1}^{2}b_{il}\cdot P(S=s-\sum_{l=1}^{2}b_{il})-v_i(s-\sum_{l=1}^{2}b_{il})\right]\end{aligned} \tag{15}$$

for $s=\min\{b_1,\ldots,b_n\},\ldots,\sum_{i=1}^{n}b_i$ and $w_i(s)=v_i(s)=0$ elsewhere.

Proof. The proof is based on Theorem 3 in [4]. Let $Y_i=X_{i1}+X_{i2}$, then the probability generating function of the aggregate claim amount is given by

$$\begin{aligned} g_S(t)=E\left[t^S\right]=E\left[t^{\sum_{i=1}^{m}Y_i+\sum_{i=m+1}^{n}X_i}\right]=E\left[t^{\sum_{i=1}^{m}Y_i}\right]\cdot E\left[t^{\sum_{i=m+1}^{n}X_i}\right]=\\ =\prod_{i=1}^{m}E\left[t^{Y_i}\right]\cdot\prod_{i=m+1}^{n}E\left[t^{X_i}\right]=\prod_{i=1}^{m}g_{Y_i}(t)\cdot\prod_{i=m+1}^{n}g_{X_i}(t),\end{aligned} \tag{16}$$

where the generating function of the random variable X_i and Y_i are respectively equal to

$$g_{X_i}(t)=E\left[t^{X_i}\right]=E\left[t^{b_i I_i}\right]=E\left[t^{b_i J_i}\right]=p_i+q_i t^{b_i} \tag{17}$$

and

$$g_{Y_i}(t)=E\left[t^{Y_i}\right]=E\left[t^{(X_{i1}+X_{i2})}\right]=E\left[t^{b_{i1}I_{i1}}\cdot t^{b_{i2}I_{i2}}\right]=E\left[t^{b_{i1}\min(J_{i1}+J_0\,;\,1)}\cdot t^{b_{i2}\min(J_{i2}+J_0\,;\,1)}\right]=$$

$$=E\left[\prod_{l=1}^{2}t^{b_{il}\min(J_{il}+J_0\,;\,1)}\right]=p_0\cdot E\left[\prod_{l=1}^{2}t^{b_{il}\min(J_{il}+0;\,1)}\right]+q_0\cdot E\left[\prod_{l=1}^{2}t^{b_{il}\min(J_{il}+1;\,1)}\right]=$$

$$=p_0\cdot E\left[\prod_{l=1}^{2}t^{b_{il}J_{il}}\right]+q_0\cdot E\left[\prod_{l=1}^{2}t^{b_{il}}\right]=p_0\left(\prod_{l=1}^{2}E\left[t^{b_{il}J_{il}}\right]\right)+q_0\cdot t^{\sum_{l=1}^{2}b_{il}}=$$

$$=p_0\cdot\prod_{l=1}^{2}\left(p_{il}+q_{il}t^{b_{il}}\right)+q_0\cdot t^{\sum_{l=1}^{2}b_{il}}\,. \tag{18}$$

If the death benefit b_i is a positive and integer number, then the random variable takes value $s=0,\min\{b_1,\ldots,b_n\},\ldots,\sum_{i=1}^{n}b_i$ and the probability $P(S=s)$ can be calculated by the following way

$$P(S=s)=\frac{g_S^{(s)}(0)}{s!}\quad\text{for}\quad s=0,\min\{b_1,\ldots,b_n\},\ldots,\sum_{i=1}^{n}b_i\,. \tag{19}$$

Therefore

$$P(S=0)=\frac{g_S^{(0)}(0)}{0!}=g_S(0)=\prod_{i=1}^{m}g_{Y_i}(0)\cdot\prod_{i=m+1}^{n}g_{X_i}(0)=$$

$$=\prod_{i=1}^{m}\left(p_0\cdot\prod_{l=1}^{2}\left(p_{il}+q_{il}0^{b_{il}}\right)+q_0\cdot 0^{\sum_{l=1}^{2}b_{il}}\right)\cdot\prod_{i=m+1}^{n}\left(p_i+q_i0^{b_i}\right)=$$

$$=\prod_{i=1}^{m}\left(p_0p_{i1}p_{i2}\right)\cdot\prod_{i=m+1}^{n}p_i\,. \tag{20}$$

In order to calculate the other probabilities the derivative of order s will be taken. The logarithm of (16) is equal to

$$\ln g_S(t)=\sum_{i=1}^{m}\ln g_{Y_i}(t)+\sum_{i=m+1}^{n}\ln g_{X_i}(t) \tag{21}$$

and derivative of is

$$\frac{g_S'(t)}{g_S(t)}=\sum_{i=1}^{m}\frac{g_{Y_i}'(t)}{g_{Y_i}(t)}+\sum_{i=m+1}^{n}\frac{g_{X_i}'(t)}{g_{X_i}(t)},$$

$$g_S'(t)=\sum_{i=1}^{m}\underbrace{\frac{g_{Y_i}'(t)\cdot g_S(t)}{g_{Y_i}(t)}}_{V_i(t)}+\sum_{i=m+1}^{n}\underbrace{\frac{g_{X_i}'(t)\cdot g_S(t)}{g_{X_i}(t)}}_{W_i(t)}\,.$$

Taking the auxiliary function $V_i(s)$ and $W_i(s)$ we obtain

$$g'_S(t) = \sum_{i=1}^{m} V_i(t) + \sum_{i=m+1}^{n} W_i(t),$$

where $V_i(s)$, $W_i(s)$ are defined by

$$V_i(t) = \sum_{x=0}^{\infty} v_i(x+1)t^x,$$

$$W_i(t) = \sum_{x=0}^{\infty} w_i(x+1)t^x.$$

Therefore the derivative of order s of $g_s(t)$ is equal to

$$g_S^{(s)}(t) = \sum_{i=1}^{m} V_i^{(s-1)}(t) + \sum_{i=m+1}^{n} W_i^{(s-1)}(t),$$

where

$$V_i^{(s-1)}(t) = \sum_{x=s-1}^{\infty} v_i(x+1)x!t^{x-(s-1)},$$

$$W_i^{(s-1)}(t) = \sum_{x=s-1}^{\infty} w_i(x+1)x!t^{x-(s-1)}.$$

Thus for $t = 0$ we obtain

$$s!P(S=s) = g_S^{(s)}(0) = \sum_{i=1}^{m} V_i^{(s-1)}(0) + \sum_{i=m+1}^{n} W_i^{(s-1)}(0) =$$

$$= \sum_{i=1}^{m} v_i(s)(s-1)! + \sum_{i=m+1}^{n} w_i(s)(s-1)!,$$

which implies that

$$sP(S=s) = \sum_{i=1}^{m} v_i(s) + \sum_{i=m+1}^{n} w_i(s).$$

In order to find functions $v_i(s)$ and $w_i(s)$, we take the $(s-1)$ order derivative of

$$W_i(t) = \frac{g'_{X_i}(t) \cdot g_S(t)}{g_{X_i}(t)},$$

that is

$$W_i(t) \cdot g_{X_i}(t) = g'_{X_i}(t) \cdot g_S(t).$$

Using the Leibnitz formula for $(s-1)$ th derivative of a product we obtain

$$D^{(s-1)}\left(W_i(t)\cdot g_{X_i}(t)\right)=D^{(s-1)}\left(g'_{X_i}(t)\cdot g_S(t)\right),$$

$$\sum_{k=0}^{s-1}\binom{s-1}{k}D^kW_i(t)\cdot D^{s-1-k}g_{X_i}(t)=\sum_{k=0}^{s-1}\binom{s-1}{k}D^kg'_{X_i}(t)\cdot D^{s-1-k}g_S(t).$$

Inserting $t=0$ we have

$$\binom{s-1}{s-1}D^{s-1}W_i(0)\cdot D^0g_{X_i}(0)+\binom{s-1}{s-1-b_i}D^{s-1-b_i}W_i(0)\cdot D^{b_i}g_{X_i}(0)=$$

$$=\binom{s-1}{b_i-1}D^{b_i-1}g'_{X_i}(0)\cdot D^{s-b_i}g_S(0).$$

Hence

$$D^{s-1}W_i(0)\cdot D^0g_{X_i}(0)=D^{b_i}g_{X_i}(0)\left[\binom{s-1}{b_i-1}\cdot D^{s-b_i}g_S(0)-\binom{s-1}{s-1-b_i}D^{s-1-b_i}W_i(0)\right].$$

From (17) we have

$$g_{X_i}^{(0)}(0)=g_{X_i}(0)=p_i.$$

The n'th derivative of the probability generating function of X_i equals

$$g_{X_i}^{(n)}(t)=q_ib_i(b_i-1)(b_i-2)...(b_1-n+1)t^{b_i-n},$$

where

$$\begin{cases} g_{X_i}^{(n)}(0)=0 & \text{for } n\neq b_i,\\ g_{X_i}^{(b_i)}(0)=q_i(b_i!) & \text{for } n=b_i. \end{cases}$$

Therefore we get the following recursion formula for the function $w_i(s)$

$$w_i(s)(s-1)!\cdot p_i=$$

$$=q_i(b_i!)\left[\frac{(s-1)!}{(s-b_i)!(b_i-1)!}\cdot(s-b_i)!P(S=s-b_i)-\frac{(s-1)!}{b_i!(s-1-b_i)!}w_i(s-b_i)(s-1-b_i)!\right],$$

$$w_i(s)=\frac{q_i}{p_i}\left[b_i\cdot P(S=s-b_i)-w_i(s-b_i)\right].$$

The recursion formula for the function $v_i(s)$ is obtained like for function $w_i(s)$. Using the Leibnitz formula we calculate the $(s-1)$ order derivative of

$$V_i(t)\cdot g_{Y_i}(t)=g'_{Y_i}(t)\cdot g_S(t),$$

that is

$$D^{(s-1)}\left(V_i(t)\cdot g_{Y_i}(t)\right)=D^{(s-1)}\left(g'_{Y_i}(t)\cdot g_S(t)\right),$$

$$\sum_{k=0}^{s-1}\binom{s-1}{k}D^kV_i(t)\cdot D^{s-1-k}g_{Y_i}(t)=\sum_{k=0}^{s-1}\binom{s-1}{k}D^kg'_{Y_i}(t)\cdot D^{s-1-k}g_S(t). \qquad (22)$$

Form (18) we have

$$g_{Y_i}^{(0)}(0)=g_{Y_i}(0)=p_0p_{i1}p_{i2}.$$

Taking the auxiliary function $U_i(t)$ and $Z_i(t)$ we obtain

$$g_{Y_i}(t)=p_0\cdot\underbrace{\prod_{l=1}^{2}\left(p_{il}+q_{il}t^{b_{il}}\right)}_{U_i(t)}+q_0\cdot\underbrace{t^{\sum_{l=1}^{2}b_{il}}}_{Z_i(t)},$$

that is

$$g_{Y_i}(t)=p_0\cdot U_i(t)+q_0\cdot Z_i(t).$$

Than the n'th derivative of the probability generating function of Y_i is equal to

$$g_{Y_i}^{(n)}(t)=p_0\cdot U_i^{(n)}(t)+q_0\cdot Z_i^{(n)}(t).$$

For the function $Z_i(t)=t^{b_{i1}+b_{i2}}$ we obtain

$$Z_i^{(n)}(t)=m(m-1)...(m-n+1)t^{m-n},\ m=b_{i1}+b_{i2},$$

where

$$\begin{cases}Z_i^{(n)}(0)=0 & \text{for } n\neq b_{i1}+b_{i2},\\ Z_i^{(n)}(0)=\left(b_{i1}+b_{i2}\right)! & \text{for } n=b_{i1}+b_{i2}.\end{cases}$$

For the function $U_i(t)=\prod_{l=1}^{2}\left(p_{il}+q_{il}t^{b_{il}}\right)$ we have

$$U_i(t)=\left(p_{i1}+q_{i1}t^{b_{i1}}\right)\left(p_{i2}+q_{i2}t^{b_{i2}}\right)=p_{i1}p_{i2}+p_{i1}q_{i2}t^{b_{i2}}+p_{i2}q_{i1}t^{b_{i1}}+q_{i1}q_{i2}t^{b_{i1}+b_{i2}}. \qquad (23)$$

The n'th derivative of (23) has the following form

$$\begin{aligned}U_i^{(n)}(t)=&\,p_{i1}q_{i2}\left[b_{i2}(b_{i2}-1)...(b_{i2}-n+1)t^{b_{i2}-n}\right]+\\&+p_{i2}q_{i1}\left[b_{i1}(b_{i1}-1)...(b_{i1}-n+1)t^{b_{i1}-n}\right]+\\&+q_{i1}q_{i2}\left[\sum_{l=1}^{2}b_{il}(\sum_{l=1}^{2}b_{il}-1)...(\sum_{l=1}^{2}b_{il}-n+1)t^{\sum_{l=1}^{2}b_{il}-n}\right]\end{aligned}$$

and is equal to

$$U_i^{(b_{i1})}(0) = p_{i2}q_{i1}\left(b_{i1}!\right),$$

$$U_i^{(b_{i2})}(0) = p_{i1}q_{i2}\left(b_{i2}!\right),$$

$$U_i^{(b_{i1}+b_{i2})}(0) = q_{i1}q_{i2}\left(\left(b_{i1}+b_{i2}\right)!\right)$$

and $U_i^{(n)}(0) = 0$ elsewhere.
Therefore the probability generating function of Y_i is equal to

$$g_{Y_i}^{(b_{i1})}(0) = p_0 p_{i2} q_{i1}\left(b_{i1}!\right), \tag{24}$$

$$g_{Y_i}^{(b_{i2})}(0) = p_0 p_{i1} q_{i2}\left(b_{i2}!\right), \tag{25}$$

$$g_{Y_i}^{(b_{i1}+b_{i2})}(0) = p_0 q_{i1} q_{i2}\left(\left(b_{i1}+b_{i2}\right)!\right) + q_0\left(\left(b_{i1}+b_{i2}\right)!\right) \tag{26}$$

and $g_{Y_i}^{(n)}(0) = 0$ elsewhere.

Putting $t = 0$ into (22) we obtain

$$D^{s-1}V_i(0)\cdot D^0 g_{Y_i}(0) =$$

$$= D^{b_{i1}} g_{Y_i}(0)\left[\binom{s-1}{b_{i1}-1}\cdot D^{s-b_{i1}} g_S(0) - \binom{s-1}{s-1-b_{i1}} D^{s-1-b_{i1}} V_i(0)\right] +$$

$$+ D^{b_{i2}} g_{Y_i}(0)\left[\binom{s-1}{b_{i2}-1}\cdot D^{s-b_{i2}} g_S(0) - \binom{s-1}{s-1-b_{i2}} D^{s-1-b_{i2}} V_i(0)\right] +$$

$$+ D^{\sum_{l=1}^{2} b_{il}} g_{Y_i}(0)\left[\binom{s-1}{\sum_{l=1}^{2} b_{il}-1}\cdot D^{s-\sum_{l=1}^{2} b_{il}} g_S(0) - \binom{s-1}{s-1-\sum_{l=1}^{2} b_{il}} D^{s-1-\sum_{l=1}^{2} b_{il}} V_i(0)\right].$$

Using (24) – (26) we have

$$v_i(s) =$$

$$= \frac{q_{i1}}{p_{i1}}\cdot\left[b_{i1}\cdot P(S = s-b_{i1}) - v_i(s-b_{i1})\right] +$$

$$+ \frac{q_{i2}}{p_{i2}}\cdot\left[b_{i2}\cdot P(S = s-b_{i2}) - v_i(s-b_{i2})\right] +$$

$$+ \frac{\left(p_0 q_{i1} q_{i2} + q_0\right)}{p_0 p_{i1} p_{i2}}\cdot\left[\sum_{l=1}^{2} b_{il}\cdot P(S = s - \sum_{l=1}^{2} b_{il}) - v_i(s - \sum_{l=1}^{2} b_{il})\right].$$

This completes the proof.

Theorem 2. The aggregate claims distribution for the model described in (7) is satisfied by

$$P(S=0)=p_0 \cdot p^n, \tag{27}$$

$$sP(S=s)=v(s)+\sum_{i=m+1}^{n} w_i(s)=v(s)+(n-m)\cdot w(s), \tag{28}$$

where the coefficients $v(s)$ and $w(s)$ are given by

$$v(s)=\sum_{r=1}^{m-1}\left[\left(\binom{m}{r}\left(\frac{q}{p}\right)^r\right)\left(br\cdot P(S=s-br)-v(s-br)\right)\right]+$$
$$+\left(\frac{p_0 q^m+q_0}{p_0\cdot p^m}\right)\left(bm\cdot P(S=s-bm)-v(s-bm)\right), \tag{29}$$

$$w(s)=\frac{q}{p}\left[b\cdot P(S=s-b)-w(s-b)\right]. \tag{30}$$

for $s=b, 2b..., nb$ and $w(s)=v(s)=0$ elsewhere.

Proof. This proof is similar to proof of Theorem 1. Let $Y=X_1+X_2+...+X_m$, then the probability generating function of *S* is equal to

$$g_S(t)=E\left[t^S\right]=g_Y(t)\cdot\prod_{i=m+1}^{n} g_{X_i}(t), \tag{31}$$

where

$$g_{X_i}(t)=E\left[t^{X_i}\right]=E\left[t^{bI_i}\right]=E\left[t^{bJ_i}\right]=p+qt^b,$$
$$g_Y(t)=E\left[t^Y\right]=p_0\cdot\left(p+qt^b\right)^m+q_0\cdot t^{bm}. \tag{32}$$

Putting $t = 0$ in (31) lead to (27). Taking the logarithm and the derivative of $g_s(t)$ we obtain

$$g'_S(t)=V(t)+\sum_{i=m+1}^{n} W_i(t), \tag{33}$$

where

$$V(t)=\sum_{x=0}^{\infty} v(x+1)t^x=\frac{g'_Y(t)\cdot g_S(t)}{g_Y(t)},\quad W_i(t)=\sum_{x=0}^{\infty} w_i(x+1)t^x=\frac{g'_{X_i}(t)\cdot g_S(t)}{g_{X_i}(t)}.$$

Taking the *s*'th derivative of (33) and inserting $t=0$ we have

$$sP(S=s)=v(s)+\sum_{i=m+1}^{n} w_i(s).$$

The recursion formula for $w_i(s)$ has the same form like in the previous theorem, that is

$$w_i(s) = \frac{q}{p}\left[b \cdot P(S = s - b) - w_i(s - b)\right].$$

The recursion formula for $v(s)$ is obtained by differentiating the following expression $(s - 1)$ times:

$$V(t) \cdot g_Y(t) = g_Y'(t) \cdot g_S(t).$$

Using the Leibnitz formula we have

$$D^{(s-1)}\left(V(t) \cdot g_Y(t)\right) = D^{(s-1)}\left(g_Y'(t) \cdot g_S(t)\right),$$

$$\sum_{k=0}^{s-1}\binom{s-1}{k} D^k V(t) \cdot D^{s-1-k} g_Y(t) = \sum_{k=0}^{s-1}\binom{s-1}{k} D^k g_Y'(t) \cdot D^{s-1-k} g_S(t). \quad (34)$$

From (32) we obtain

$$g_Y^{(0)}(0) = g_Y(0) = p_0 \cdot p^m. \quad (35)$$

If we express the probability generating function of Y by means of the auxiliary function $U(t)$ and $Z(t)$, that is

$$g_Y(t) = p_0 \cdot \underbrace{\left(p + qt^b\right)^m}_{U(t)} + q_0 \cdot \underbrace{t^{bm}}_{Z(t)} = p_0 \cdot U(t) + q_0 \cdot Z(t),$$

then the n'th derivative is equal to

$$g_Y^{(n)}(t) = p_0 \cdot U^{(n)}(t) + q_0 \cdot Z^{(n)}(t).$$

Taking the derivative of order n we have

$$Z^{(n)}(t) = bm(bm - 1)\ldots(bm - n + 1)t^{bm-n},$$

where

$$\begin{cases} Z^{(n)}(0) = 0 & \text{for } n \neq bm, \\ Z^{(n)}(0) = (bm)! & \text{for } n = bm. \end{cases} \quad (36)$$

Consider the function $U(t)$:

$$U(t) = \left(p + qt^b\right)^m =$$

$$= p^m + mp^{m-1}\left(qt^b\right) + \frac{m(m-1)}{2!} p^{m-2}\left(qt^b\right)^2 + \frac{m(m-1)(m-2)}{3!} p^{m-3}\left(qt^b\right)^3 + \ldots +$$

$$+ \frac{m(m-1)\ldots(m-r+1)}{r!} p^{m-r}\left(qt^b\right)^r + \ldots + mp\left(qt^b\right)^{m-1} + \left(qt^b\right)^m =$$

$$= \sum_{r=1}^{m} \binom{m}{r} p^{m-r} \left(qt^{b}\right)^{r}$$
.

The *n*'th derivative of $U(t)$ is equal to

$$D^{n}U(t) = \sum_{r=1}^{m} D^{n}\left[\binom{m}{r} p^{m-r}\left(qt^{b}\right)^{r}\right] = \sum_{r=1}^{m}\binom{m}{r} p^{m-r} q^{r} D^{n}\left(t^{br}\right) =$$

$$= \sum_{r=1}^{m}\binom{m}{r} p^{m-r} q^{r}\left(br(br-1)...(br-n+1)t^{br-n}\right) = \sum_{r=1}^{m}\binom{m}{r} p^{m-r} q^{r}\left(\frac{(br)!}{(br-n)!}t^{br-n}\right).$$

Inserting $t = 0$ we have

$$\begin{cases} U^{(n)}(0) = 0 & \text{for } n \neq br, \\ U^{(br)}(0) = \binom{m}{r} p^{m-r} q^{r}(br)! & \text{for } r = 1, ..., m. \end{cases}$$

Therefore

$$g_{Y}^{(br)}(0) = p_{0}\cdot U^{(br)}(0) + q_{0}\cdot Z^{(br)}(0) =$$

$$= \begin{cases} p_{0}\binom{m}{r} p^{m-r} q^{r}(br)! & \text{for } r = 1, ..., m-1, \\ p_{0}q^{m}(bm)! + q_{0}(bm)! & \text{for } r = m \end{cases} \qquad (37)$$

and $g_{Y}^{(n)}(0) = 0$ elsewhere.
Putting $t = 0$ into (34) we obtain

$$\binom{s-1}{s-1} D^{s-1}V(0)\cdot D^{0}g_{Y}(0) =$$

$$= \sum_{r=1}^{m-1} D^{br}g_{Y}(0)\left[\binom{s-1}{br-1}\cdot D^{s-br}g_{S}(0) - \binom{s-1}{s-1-br} D^{s-1-br}V(0)\right] +$$

$$+ D^{bm}g_{Y}(0)\left[\binom{s-1}{bm-1}\cdot D^{s-bm}g_{S}(0) - \binom{s-1}{s-1-bm} D^{s-1-bm}V(0)\right]$$

Using (35), (36), (37) we have

$$v(s)=\sum_{r=1}^{m-1}\left[\left(\binom{m}{r}\left(\frac{q}{p}\right)^{r}\right)\left(br\cdot P(S=s-br)-v(s-br)\right)\right]+$$

$$+\left(\frac{p_0 q^m+q_0}{p_0\cdot p^m}\right)\left(bm\cdot P(S=s-bm)-v(s-bm)\right).$$

This completes the proof.

3. Influence of the dependence on the aggregate claims distribution and the premium

In this section, influence of the dependence on the aggregate claims distribution and the net premium is presented by numerical examples.

Example 1

Consider a portfolio consisting of 80 policies purchased by 40 married couples and additionally by 39 men and 12 women. The probability of natural death for men is

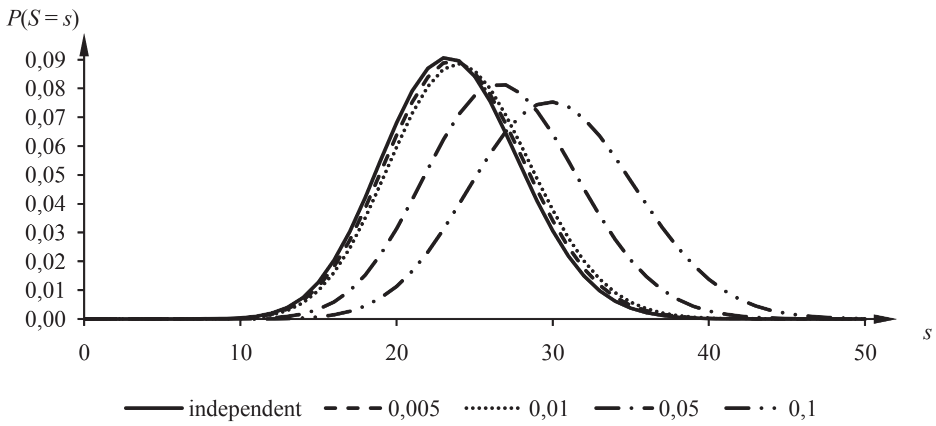

Fig. 1. The distribution function of *S*

Source: autor's own study.

equal to 0.2 and for women 0.15. The probability of death as a result of the common risk is equal to 0.005, 0.01, 0.05 and 0.1. In Fig. 1. the exact values of the distribution function of *S* for different values of q_0 can be found. Those values determine the degree of dependence between the risks. For $q_0 = 0$ we have the case where the risks are independent.

From Fig. 1. one can see that for small values of q_0 (such as 0.005) the curves are very close to the curve in the case of independent risks. For larger values the difference is significant, the distribution function is more flattened and modal values are larger.

Example 2

Consider the portfolio consisting of 98 policies. The 20 persons are exposed to the same risk. The probability of natural death is equal to 0.2. The probability of death as a result of the common risk takes value of 0.005, 0.01, 0.05 and 0.1. The aggregate claims distribution of *S* is shown in Fig. 2.

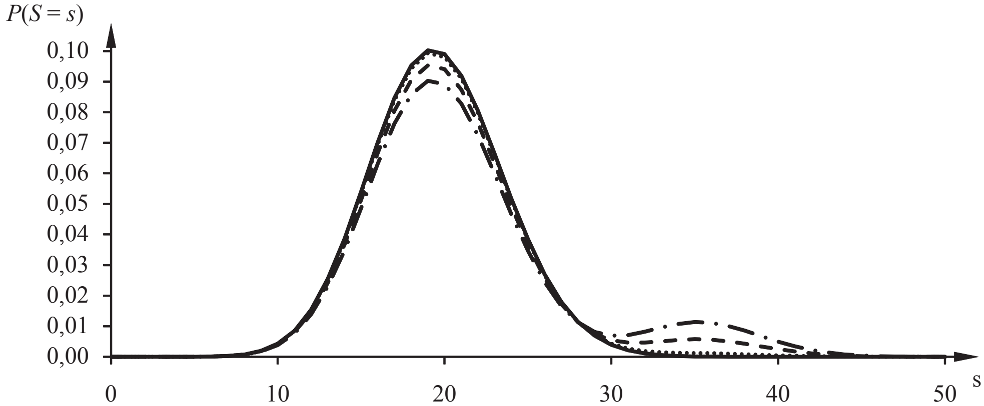

Fig. 2. The distribution function of *S*

Source: autor's own study.

From Fig. 2. one can see that the curve for small value of q_0 (such as 0.01) is very close to the curve in case of independent risk. We also observe that for larger values of q_0 the distribution function of *S* became the bimodal distribution.

In the next examples there is considered the influence of the dependence on the net premium in the presented models. In the model the net premium for the portfolio is given by

$$E(S) = \sum_{i=1}^{m} E(X_{i1} + X_{i2}) + \sum_{i=m+1}^{n} E(X_i) = \sum_{i=1}^{m}\sum_{l=1}^{2} E(X_{il}) + \sum_{i=m+1}^{n} E(X_i),$$

where

$$E(X_{il}) = E(b_{il} I_{il}) = b_{il} E\left(I_{il}\right) = b_{il} E\left(\min(J_{il} + J_0, 1)\right) =$$

$$= b_{il}\left[p_0 E\left(\min(J_{il} + 0, 1)\right) + q_0 E\left(\min(J_{il} + 1, 1)\right)\right] =$$

$$= b_{il}\left[p_0 E\left(J_{il}\right) + q_0\right] = b_{il}\left[p_0 q_{il} + q_0\right] \text{ for } i = 1, \ldots, m$$

and

$$E(X_i) = E(b_i I_i) = b_i E\left(J_i\right) = b_i q_i \text{ for } i = m + 1, \dots, n.$$

For model (7) the net premium $E(S)$ for portfolio is equal to

$$E(S) = E(X_1) + \dots + E(X_m) + \sum_{i=m+1}^{n} E(X_i),$$

where

$$E(X_i) = E(bI_i) = bE\left(I_i\right) = bE\left(\min(J_i + J_0, 1)\right) =$$

$$= b\left[p_0 E\left(\min(J_i + 0, 1)\right) + q_0 E\left(\min(J_i + 1, 1)\right)\right] =$$

$$= b\left[p_0 E\left(J_i\right) + q_0\right] = b\left[p_0 q + q_0\right] \text{ for } i = 1, \dots, m,$$

$$E(X_i) = E(bI_i) = bE\left(J_i\right) = bq \text{ for } i = m + 1, \dots, n.$$

Hence the net premium $E(S)$ depends on the probability of natural death and also probability of death as a result of the common risk.

Example 3

Consider the portfolio described in Example 1. The value of the net premium for dependent risks $E(S)$ are compared with the premium for independent model $E(S)^*$. In Table 1. the net premium and the relative error are given for small values of q_0. The relative error is equal to $(E(S) - E(S)^*) / E(S)^*$.

Table 1. The relative error for the net premium

q_0	p_0	$E(S)$	$\frac{E(S) - E(S)^*}{E(S)^*}$
0.001	0.999	10.8116	0.11%
0.002	0.998	10.8233	0.22%
0.003	0.997	10.8350	0.32%
0.004	0.996	10.8468	0.43%
0.005	0.995	10.8587	0.54%
0.006	0.994	10.8706	0.65%
0.007	0.993	10.8825	0.76%
0.008	0.992	10.8945	0.88%
0.009	0.991	10.9066	0.99%

Source: autor's own study.

In Fig. 3 the relative error for the net premium is shown for q_0 from 0 to 1.

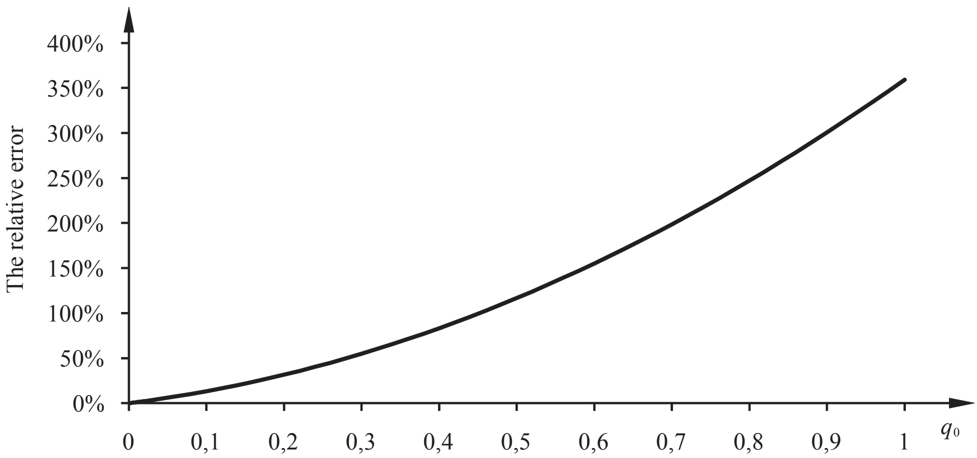

Fig. 3. The relative error for the net premium

Source: autor's own study.

Example 4

Consider the portfolio described in Example 2. The value of the net premium for dependent risks $E(S)$ is compared with the premium $E(S)^*$. In Table 2 the net premium and the relative error are given for small q_0. In Fig. 4 the relative error for the net premium is shown for q_0 varying from 0 to 1.

Table 2. The relative error for the net premium

q_0	p_0	$E(S)$	$\frac{E(S)-E(S)^*}{E(S)^*}$
0.001	0.999	19.616	0.08%
0.002	0.998	19.632	0.16%
0.003	0.997	19.648	0.24%
0.004	0.996	19.664	0.33%
0.005	0.995	19.680	0.41%
0.006	0.994	19.696	0.49%
0.007	0.993	19.712	0.57%
0.008	0.992	19.728	0.65%
0.009	0.991	19.744	0.73%

Source: autor's own study.

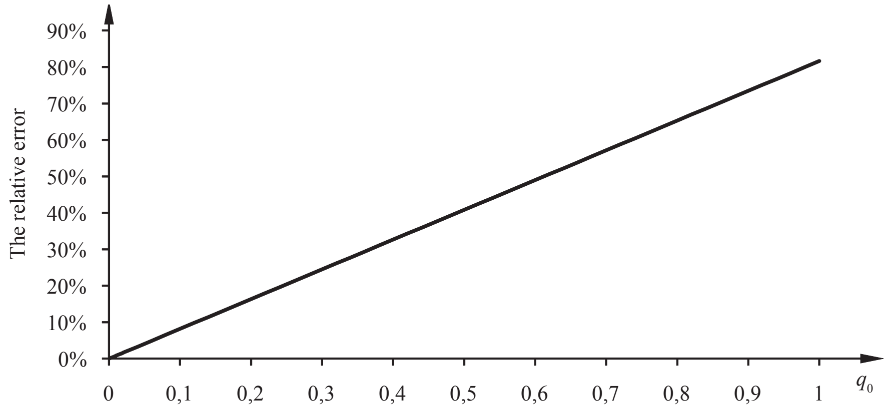

Fig. 4. The relative error for the net premium

Source: autor's own study.

From Fig. 1-4 we can see that the dependence influences the probability distribution of S and the net premium $E(S)$. The value of the net premium increases with q_0 (see Table 1, 2 and Fig. 3, 4). Example 3 and 4 show that for small value of q_0 the relative error does not exceed one percent. If the value of the probability of the natural death is less than the value in example 3 and 4, the relative error is larger. It is shown in the next example.

Example 5

Let the aggregated claim amount have form (7). The probability of death as a result of common risk is fixed. In Fig. 5 the relative error for the net premium is shown for the different values of the probability of natural death.

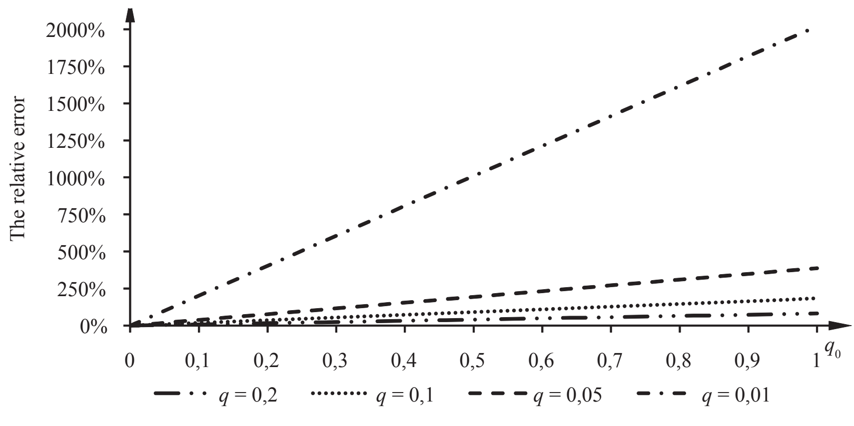

Fig. 5. The relative error for the net premium with fixed q_0

Source: autor's own study.

From Fig. 5 we can see that for smaller values of the probability of natural death q the relative errors are larger. For the larger values of probability of death, as a result of the common risk, the difference between the curves is larger.

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WPŁYW ZALEŻNOŚCI NA UBEZPIECZENIA NA ŻYCIE

Streszczenie: W klasycznych ubezpieczeniach na życie zakłada się, że wielkości szkód są niezależne. Założenie to w wielu sytuacjach jest niewłaściwe. Na przykład długości życia męża i żony są zależne, gdyż dzielą oni wspólne życie oraz mniej lub bardziej narażeni są na to samo ryzyko. W grupie ubezpieczonych osób, które pracują w tej samej fabryce, śmiertelność zależy od pewnych wydarzeń (tj. wybuch, zawalenie hali). W artykule przedstawione będą modele uwzględniające zależność między szkodami. Zbadany zostanie wpływ zależności na rozkład zagregowanych szkód i na składkę netto.

Słowa kluczowe: model indywidualnego ryzyka, rozkład zagregowanej wielkości szkód, wzór rekurencyjny, składka netto.