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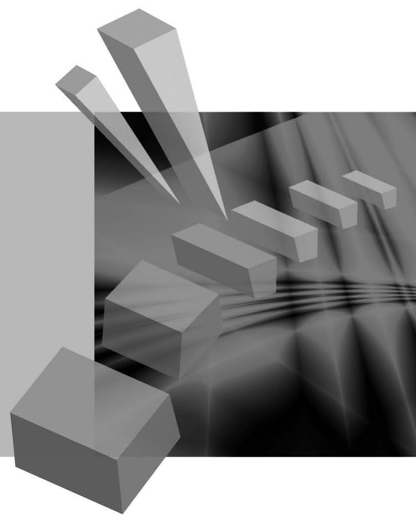

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Zastosowanie badań operacyjnych
Zarządzanie projektami, decyzje finansowe, logistyka ISSN 1899-3192

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A NEW RANKING METHOD FOR FUZZY NUMBERS AND ITS APPLICATION TO THE FUZZY KNAPSACK PROBLEM

Summary: In the paper the knapsack problem with fuzzy parameters is considered. In case of fuzzy characteristics of the elements which may be put into the knapsack it is not unequivocal which solution is either optimal (i.e. guarantees the highest value in the knapsack) or which solutions can be considered as those which fulfill the knapsack capacity constraint. The evaluation and choice of the elements to be put into the knapsack depends on the method we use to compare fuzzy numbers. In the paper a new method of comparing fuzzy numbers is proposed and applied to the fuzzy knapsack problem. The method is based on discrete approximations of fuzzy numbers using the Hausdorf metrics.

Keywords: fuzzy knapsack, fuzzy number, distance between fuzzy numbers, fuzzy numbers norm.

1. Introduction

Fuzzy numbers arise in many fields of science, because of the intrinsic inaccuracy of measurement. Generally speaking the fuzzy number represents a signal, a piece of information, quantity that cannot be represented in a crisp way. One fuzzy number by itself, understood as a real function (like a wave function), conveys information about an isolated phenomenon. However, we are also able to operate on these numbers in a way consistent to what we are used to in real numbers. Thus fuzzy numbers do not only carry information but also allow to operate on this information in accordance with the rules of algebra.

In this paper fuzzy numbers will represent the constraint right hand side and the objective function and constraint coefficients of the knapsack problem. The constraint right hand side represents the amount of space we have in the knapsack at our disposal, the constraint coefficients represent the amount of space each element occupies and the objective function coefficients represent the value each element represents to the knapsack holder. All these values may be imprecise in the moment of decision making. That is why they will be represented by fuzzy numbers.

Fuzzyfying the knapsack problem parameters may be useful in all planning problems, where the question of packing and a sufficient capacity of transportation means arises. Weights, prices and volumes of transported good may often be known only imprecisely. For example in a Polish warehouse, where merchandise is stored for many vendors, the problem arises which goods should be transported in the first place, in the very vans that are accessible in the very moment. The planning of the individual transport takes place before the capacity of available vans are known, before the actual order (thus the volume of individual products to be transported) is known and before their current sales prices are known. In this case the knapsack problem with fuzzy parameters can be applied.

The problem with such a knapsack problem formulation is that it contains a priori undefined expressions: maximization of a fuzzy number and a comparison of two fuzzy numbers. There exist many approaches to the comparison and ranking of fuzzy numbers [Lin, Yao 2001; Martello et al. 2000; Okada, Gen 1994 with the modification of Kuchta 2002, Kasperski and Kulej 2007]. They differ among each other, as each of them represents a special attitude of the decision maker and each of them takes into account some and disregards the other aspects of fuzzy numbers. In this paper we propose a new method of comparing (and in fact ranking) fuzzy numbers which has some advantages over the other methods and we apply it to the knapsack problem. Thus the main goal of the paper is to apply a different ranking method of fuzzy numbers in order to interpret the fuzzy knapsack problem in a different way which would correspond to other decision maker preferences than in the fuzzy knapsack models known in the literature. The ranking method itself is a second product of the paper. It is presented in Section 3, where all the definitions stem from us – to our knowledge, they are not even similar to any definition from the literature.

In Section 2 we give basic information about fuzzy numbers. In Section 3 we discuss the problem of comparing and ranking them, we criticize the existing approaches and suggest a new one. In Section 4 we formulate the fuzzy knapsack problem and propose an algorithm for solving it, based on the method of comparing fuzzy numbers proposed in Section 3. In Section 5 we show a numerical example illustrating the proposed approach.

We will denote real numbers by *a,b,c,..*, real variables by *x,y,z..*, fuzzy numbers by a^*, b^*, c^*, fuzzy variables by x^*, y^*, z^*.

2. Essentials about fuzzy numbers

The definitions and notions presented in this paper are based on [Viertl 2011].

Definition 1: We say that the function $\xi_{a^*}(x)$ is the characterizing function of a fuzzy number a^* if:

(1) There exists $x_0 \epsilon \mathbb{R}$ such that $\xi_{a^*}(x_0) = 1$;.

(2) The function $\xi_{a^*}(x)$ is fuzzy convex, i.e. for every $x, y \epsilon \mathbb{R}$ and $\lambda \epsilon [0,1]$ we have

$$\xi_{a^*}(\lambda x + (1-\lambda)y) \geq min\{\xi_{a^*}(x), \xi_{a^*}(y)\}.$$

(3) The function $\xi_{a^*}(x)$ is semi-continuous from above, i.e. for every $x_0 \epsilon \mathbb{R}$ and for every sequence x_n, if $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} \xi_{a^*}(x_n) \leq \xi_{a^*}(x_0)$.

(4) $\lim_{x\to-\infty} \xi_{a^*}(x) = 0$ and $\lim_{x\to\infty} \xi_{a^*}(x) = 0$.

Thus each fuzzy number a^* can be identified with its characterizing function, while $\xi_{a^*}(x)$ expresses e.g. the possibility degree that an unknown value, characterized by a^*, will take on value x, $x \epsilon \mathbb{R}$.

The set of all fuzzy numbers will be denoted by $\mathcal{F}(\mathbb{R})$.

Definition 2: Two (or more) fuzzy numbers $\xi_{a_1^*}(x)$, $\xi_{a_2^*}(x)$ are combined into one fuzzy vector $min\{a_1^*, a_2^*\}$, characterized by the function $\xi_{min\{a_1^*,a_2^*\}}(x,y)$, $\xi_{min\{a_1^*,a_2^*\}}: \mathbb{R}^2 \to [0,1]$, by the minimum combination rule:

$$\xi_{min\{a_1^*,a_2^*\}}(x,y) = min\{\xi_{a_1^*}(x), \xi_{a_2^*}(y)\}.$$

Zadeh's extension principle is often cited here:

Definition 3: For a function $f: \mathbb{R}^2 \to \mathbb{R}$ and fuzzy numbers a_1^*, a_2^* with the characterizing functions $\xi_{a_1^*}(x)$, $\xi_{a_2^*}(x)$ we define $y^* = f(a_1^*, a_2^*)$ by its characterizing function $\xi_{y^*}(x)$ as follows:

$$\xi_{y^*}(z) = \begin{cases} sup\{\xi_{min\{a_1^*,a_2^*\}}(x,y), f(x,y) = z \text{ for } z \text{ that } \exists (x,y) \in \mathbb{R}^2 \; f(x,y) = z\} \\ 0 \text{ otherwise} \end{cases}.$$

The function $\xi_{y^*}(z)$ need not necessarily correspond to a fuzzy number in the sense that one or more parts of Definition 1 might not be met, but we have the following:

Theorem 1 [Viertl 2011]: For $f \epsilon C^0(\mathbb{R}^2)$ $y^* = f(a_1^*, a_2^*)$ is always a fuzzy number in the sense of Definition 1.

Let us now discuss basics of fuzzy arithmetic. On the basis of Definition 3 we can define the addition and multiplication of two fuzzy numbers:

$$\xi_{a_1^* \oplus a_2^*}(z) = sup\left\{min\{\xi_{a_1^*}(x), \xi_{a_2^*}(y)\}: x + y = z\right\}, \tag{1}$$

$$\xi_{a_1^* \odot a_2^*}(z) = sup\left\{min\{\xi_{a_1^*}(x), \xi_{a_2^*}(y)\}: x \cdot y = z\right\}. \tag{2}$$

Crisp numbers and intervals can be seen as special cases of fuzzy numbers. We have thus $\xi_{a^*}(x) = \chi_{\{a\}}(x)$ for a^* being a crisp number a, where $\chi_{\{a\}}$ is a characteristic function of the one element set $\{a\}$ and $\xi_{a^*}(x) = \chi_{[\underline{a},\overline{a}]}(x)$ for a^* being a closed interval $[\underline{a}, \overline{a}]$.

If we have two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, their sum will be defined according to (1). It can be easily shown that than we have:

$$\xi_{[\underline{a},\overline{a}] \oplus [\underline{b},\overline{b}]}(z) = \chi_{[\underline{a}+\underline{b}, \overline{a}+\overline{b}]}(z). \tag{3}$$

Each fuzzy number can be identified with a family of closed intervals. Namely, it can be easily shown that Definition 4 is the equivalent of Definition 1 [Viertl 2011].

Definition 4: We say that the function $\xi_{a^*}(x)$ is the characterizing function of a fuzzy number a^* iff:

(1) $\xi_{a^*}: \mathbb{R} \to [0,1]$

(2) $\forall \alpha \epsilon (0,1]$ the set $C_\alpha(a^*) = \{x \epsilon \mathbb{R}: \xi_{a^*}(x) \geq \alpha\}$ is a closed connected interval.

The set $C_\alpha(a^*)$ is called α-cut of the fuzzy number a^*. Both ends of the interval $C_\alpha(a^*)$ will be denoted as $\underline{a}_\alpha, \overline{a}_\alpha$ respectively, thus we have $C_\alpha(a^*) = [\underline{a}_\alpha, \overline{a}_\alpha]$. It is obvious that we have $C_{\alpha_1}(a^*) \subseteq C_{\alpha_2}(a^*)$ for $\alpha_1 > \alpha_2$.

The following theorem is central in the theory and especially crucial to this paper:

Theorem 2 (The Representation Theorem): The characterizing function of a fuzzy number a^* is uniquely determined by the family of its α-cuts $C_\alpha(a^*), \alpha \epsilon (0,1]$ and we have:

$$\xi_{a^*}(x) = \max\{\alpha \cdot \chi_{C_\alpha(a^*)}(x), \alpha \epsilon (0,1]\}. \quad (4)$$

Now, using the representation of crisp numbers as fuzzy numbers and equalities (1) and (2), we can express the two arithmetic operations we will use in this paper, multiplication of a fuzzy number with a crisp numbers and addition of two fuzzy numbers, as operations on the α-cuts:

$$C_\alpha(b \odot a^*) = [b \cdot \underline{a}_\alpha, b \cdot \overline{a}_\alpha], \alpha \epsilon (0,1], b > 0, \quad (5)$$

$$C_\alpha(a_1^* \oplus a_2^*) = [\underline{a}_{1\alpha} + \underline{a}_{2\alpha}, \overline{a}_{1\alpha} + \overline{a}_{2\alpha}], \alpha \epsilon (0,1]. \quad (6)$$

3. Comparing and ranking fuzzy numbers

There have been a lot of approaches in the literature to establish some form of linear order on the set of fuzzy numbers (a review can be found in [Ramli and Mohamad 2009]). Most of those approaches have focused their attention on special classes of fuzzy numbers, such as triangle and trapezoidal or L-R fuzzy numbers. In this paper we apply no such a restriction. Another constantly used approach has been the attempt to "defuzzify" those fuzzy numbers by attaching a universal crisp index (one of the best known approaches is the Yager index [Yager 1980] to each number, thus stripping the concept of fuzziness of its very essence.

We believe that the comparison of any two fuzzy numbers must be relative with respect to the specific actual problem being investigated currently. It seems impossible to establish a mathematically and practically meaningful order independent of a tangible context.

One major drawback in the majority of ranking methods of triangle numbers is the indiscernibility of numbers sharing the same "right leg", i.e . the values $\xi_{a^*}(x)$ for x $\geq sup\{y\epsilon\mathbb{R}, \xi_{a^*}(y) = 1\}$, or sometimes, correspondingly, the left leg (i.e. the values $\xi_{a^*}(x)$ for $x \leq inf\{y\epsilon\mathbb{R}, \xi_{a^*}(y) = 1\}$). This is not the case here. The right leg / left leg problem is in our view due to the futile attempt to rank the whole family of fuzzy numbers, which by their nature is not really suitable for that. Usually only partial orders will make sense.

Here we propose a new method of comparing fuzzy numbers, utilizing the metrics developed by B.B. Chaudhuri and Azriel Rosenfeld [Chaudhuri, Rosenfeld 1996] which are defined for discrete-valued fuzzy numbers. Let us define a special class of fuzzy numbers, staircase fuzzy numbers, which can be used to approximate any fuzzy number by the representation theorem (Theorem 2) and will allow to use the metrics defined for discrete-valued fuzzy numbers to compare arbitrary fuzzy numbers.

Definition 5: Let $\{\alpha_n\}$, $n = 0,1, \ldots, N$, be any sequence of numbers such that $\alpha_0 = 0$, $\alpha_N = 1$, $\alpha_n > \alpha_{n-1}$ for $n = 1, \ldots, N$ (called tuning of the staircase fuzzy number). Let $A_n = [\underline{a}_n, \overline{a}_n]$ (we shall call them "base sets" of the staircase fuzzy number) be a collection of closed intervals of the real line, such that $A_{n-1} \subset A_n$ (stricty) for $n = 1,\ldots,N$. Then the following function is a characterizing function of the staircase fuzzy number a^*:

$$\xi_{a^*}(x) = \max\{\alpha_n \chi_{A_n}(x)\}_{n=0}^{N}. \quad (7)$$

Corollary 1: For staircase fuzzy numbers we have $C_0(a^*) = A_0$, $C_\alpha(a^*) = C_{\alpha_n}(a^*) = A_n$ for $\alpha_n > \alpha > \alpha_{n-1}$, $n = 1, \ldots, N$ (in particular $C_1(a^*) = A_N$).

The representation theorem (Theorem 2) tells us that any fuzzy number can be approximated with deliberate accuracy by a staircase fuzzy number.

Corollary 2: Let us suppose that two staircase fuzzy numbers a_1^*, a_2^* share the same tuning $\{\alpha_n\}$, $n = 0,1,\ldots, N$. From Definition 5 and (5) and (6) we have:

$$\xi_{a_1^* \oplus a_2^*}(x) = \max\left\{\alpha_n \chi_{[A_n^1 \oplus A_n^1]}(x)\right\}_{n=0}^{N} = \max\left\{\alpha_n \chi_{\left[\underline{a}_n^1 + \underline{a}_n^2, \overline{a}_n^1 + \overline{a}_n^2\right]}(x)\right\}_{n=0}^{N}.$$

And for each staircase fuzzy number a^* we have

$$\xi_{b \oplus a^*}(x) = \max\{\alpha_n \chi_{b \oplus A_n}(x)\}_{n=0}^{N} = \max\left\{\alpha_n \chi_{[b\underline{a}_n, b\overline{a}_n]}(x)\right\}_{n=0}^{N} \text{for } b\epsilon\mathbb{R}, b > 0.$$

Definition 6: Let us consider an arbitrary fuzzy number a^*. Its approximation for a given tuning $\{\alpha_n\}$, $n = 0,1, \ldots, N$ will be denoted as $a^*_{\{\alpha_n\}}$ and defined as a the staircase fuzzy number determined by the tuning $\{\alpha_n\}$, $n = 0,1, \ldots, N$ and the base sets $C_{\alpha_n}(a^*)$.

Corollary 2': The Hausdorff distance between two closed intervals A,B of the real line can be expressed as $max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$.

Now we are in position to define the distance between two arbitrary fuzzy numbers a^*, b^*. It will be an adaptation of the metrics of [Chaudhuri, Rosenfeld 1996] to fuzzy numbers of non-discrete value range and will simply be dependent on the selected tuning.

Definition 7 (distance): Let us consider two arbitrary fuzzy numbers a^*, b^*. Their distance for a given tuning $\{\alpha_n\}$, n = 0,1,…,N will be denoted as $H_{\{\alpha_n\}}(a^*, b^*)$ and defined as follows:

$$H_{\{\alpha_n\}}(a^*, b^*) = H\left(a^*_{\{\alpha_n\}}, b^*_{\{\alpha_n\}}\right) = \frac{1}{\sum_{n=0}^{N}\alpha_n}\sum_{n=0}^{N}\alpha_n \cdot max\{|\underline{a}_{\alpha_n} - \underline{b}_{\alpha_n}|, |\overline{a}_{\alpha_n} - \overline{b}_{\alpha_n}|\}.$$

The idea of the weighted sum in Definition 7 is that the difference between the corresponding α_n-cuts for higher values of α_n weight more (the weight is equal to α_n) as they are linked to higher possibility degree. However, the whole range of α_n-cuts is taken into account and it represents the whole fuzzy numbers a^*, b^* the better the finer the tuning is. It is important that all the α_n-cuts can be placed according to personal preference. The more α_n-cuts the decision maker chooses next to a given value $\alpha \epsilon [0,1]$, the more preference is given to that level in the solution process.

We can also introduce the following definition:

Definition 8 (norm): For each fuzzy number a^* and given the tuning $\{\alpha_n\}$, n = 0,1,…,N we can define a norm as the distance in the sense of Definition 7 of the fuzzy number a^* from the special case of fuzzy numbers, the crisp 0:

$$\|a^*\|_{\{\alpha_n\}} = \left\|a^*_{\{\alpha_n\}}\right\| = H\left(a^*_{\{\alpha_n\}}, 0\right) = \frac{1}{\sum_{n=0}^{N}\alpha_n}\sum_{n=0}^{N}\alpha_n \cdot \overline{a}_{\alpha_n}.$$

The following important corollary is easy to prove:

Corollary 3: For each two fuzzy numbers a^*, b^* , a positive real number s and a given tuning $\{\alpha_n\}$, n = 0,1,…,N we have:

$$\|sa^*\|_{\{\alpha_n\}} = s\|a^*\|_{\{\alpha_n\}}$$

$$\|a^* \oplus b^*\|_{\{\alpha_n\}} = \|a^*\|_{\{\alpha_n\}} + \|b^*\|_{\{\alpha_n\}}.$$

This means, that the norm (Definition 7) is a linear functional on the space of $\{\alpha_n\}$ -fuzzy numbers and the relevant theorems apply.

Now we will pass to the knapsack problem itself.

4. Fuzzy knapsack problem

A crisp knapsack problem is defined as follows:

We have a knapsack of capacity b and K elements which have each two attributes: a_k and $c_{k,}$ $k = 1,..,K$ where a_k stands for the volume of the k-th element and c_k for its value. The problem consists in choosing such elements to be put into the knapsack that their total volume does not exceed the capacity of the knapsack and their total value is as big as possible. This problem can be formulated as a binary linear programming problem with one constraint:

$$\sum_{k=1}^{K} c_k \cdot x_k \to max$$

$$\sum_{k=1}^{K} a_k \cdot x_k \leq b \tag{8}$$

$$x_k = 0{,}1 \text{ for } k = 1,..,K.$$

There are methods, like a modification of the branch and bound method, which can determine all the alternative optimal solutions, thus all the subsets of the set of all the K elements which fit into the knapsack and have the maximal total value. In case alternative solutions exist, we may prefer those solutions which do not fill up the knapsack completely, which gives some additional room in the knapsack for some other content.

Now a fuzzy generalization of any crisp problem consists in fuzzifying all or some of the parameters of the crisp problem. Let us assume that the knapsack capacity, the volumes and the values of the elements are given in an imprecise form, as fuzzy numbers. We get thus the following fuzzy knapsack problem:

$$\oplus_{k=1}^{K} \sum c_k^* \cdot x_k \to max$$

$$\oplus_{k=1}^{K} \sum \quad a_k^* \cdot x_k \leq b^* \tag{9}$$

$$x_k = 0{,}1 \text{ for } k = 1,..,K.$$

The interpretation of the problem will be different and will depend on the method which we use to compare fuzzy numbers. We propose here to choose a tuning $\{\alpha_n\}$, $n = 0,1, \ldots, N$, which can be made finer or generally changed according to the preferences of the decision maker, and for a fixed tuning to reformulate problem (9) as follows:

$$\left\| \sum_{k=1}^{K} c_k^* \cdot x_k \right\|_{\{\alpha_n\}} \to max$$

$$\left\| \sum_{k=1}^{K} a_k^* \cdot x_k \right\|_{\{\alpha_n\}} \leq \|b^*\|_{\{\alpha_n\}} \tag{10}$$

$$x_k = 0{,}1 \text{ for } k = 1,..,K.$$

Problem (10), according to corollary 3, can be reformulated as a crisp knapsack problem:

$$\sum_{k=1}^{K} \|c_k^*\|_{\{\alpha_n\}} \cdot x_k \to max$$

$$\sum_{k=1}^{K} \|a_k^*\|_{\{\alpha_n\}} \cdot x_k \le \|b^*\|_{\{\alpha_n\}} \tag{11}$$

$$x_k = 0,1 \text{ for } k = 1,..,K.$$

Problem (11) can be solved using the standard algorithms for the crisp knapsack problem. In case there are alternative solutions, with the same value $\left\|\sum_{k=1}^{K} c_k^* \cdot x_k\right\|_{\{\alpha_n\}}$, we choose those for which $H_{\{\alpha_n\}}\left(\sum_{k=1}^{K} a_k^* \cdot x_k , b^*\right)$ is maximal (of course apart from comparing, using various methods, all the fuzzy numbers $\sum_{k=1}^{K} c_k^* \cdot x_k$ such that $\left\|\sum_{k=1}^{K} c_k^* \cdot x_k\right\|_{\{\alpha_n\}}$ is the optimal objective function value of problem (11). In case the preceding measures do not lead to a single optimal solution we refine our initial partition $\{\alpha_n\}$ and start anew.

An algorithmic notation of the above idea might look as follows:

Step 0: Formulate the initial fuzzy knapsack problem (9).

Step 1: Choose a tuning $\{\alpha_n\}$, $n = 0,1, \ldots, N$.

Step 2: Reformulate the problem in the form (10).

Step 3: Solve the crisp equivalent (11) – looking for all the alternative solutions. If there are no alternative solutions, go to Step 5.

Step 4: The solution has been found – STOP.

Step 5: Choose the solution with the maximal value of $H_{\{\alpha_n\}}\left(\sum_{k=1}^{K} a_k^* \cdot x_k , b^*\right)$. If the decision is not unequivocal, go to Step 6. Otherwise STOP – solution has been found.

Step 6: Choose one of the solutions arbitrarily and go to Step 4 or choose a finer tuning $\{\alpha_n\}$, $n = 0,1, \ldots, N$ and go to Step 2.

The proposed approach and the process of refining the tuning and its consequences will be illustrated by the following numerical example.

5. A numerical example

In the example we are making use of three frequently met types of fuzzy numbers: crisp, triangular, (piecewise) linear and naturally continuous (sinus), and a mixed crisp-linear type.

Let the knapsack problem be defined as follows:

Table 1. Parameters in example (9)

Fuzzy knapsack problem (9) parameter	Membership function	Comment
a_1^*	$\chi_{\{1/4\}}$	Crisp number
a_2^*	$\chi_{\{1/4\}}$	Crisp number
a_3^*	$\begin{cases} 1-x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$	
a_4^*	$\begin{cases} x-1 & \text{if } x \in [1,2] \\ -x+3 & \text{if } x \in (2,3) \\ 0 & \text{otherwise} \end{cases}$	
c_1^*	$\begin{cases} x-1 & \text{if } x \in [1,2] \\ -x+3 & \text{if } x \in (2,3) \\ 0 & \text{otherwise} \end{cases}$	Triangular fuzzy number
c_2^*	$\begin{cases} \sin x \text{ if } x \in [0,2\pi] \\ 0 \quad \text{otherwise} \end{cases}$	
c_3^*	$\begin{cases} 4x & \text{if } x \in [0,1/4] \\ -4x+2 & \text{if } x \in \left(1/4, 1/2\right) \\ 0 & \text{otherwise} \end{cases}$	Triangular fuzzy number
c_4^*	$\begin{cases} 8x & \text{if } x \in [0,1/8] \\ -8x+2 & \text{if } x \in (2,3) \\ 0 & \text{otherwise} \end{cases}$	Triangular fuzzy number
b^*	$\begin{cases} \frac{5}{4} - \text{x} & \text{if} \quad \text{x} \in \left(\frac{1}{2}, \frac{5}{4}\right] \\ 1 & \text{if} \quad \text{x} = \frac{1}{2} \\ 0 & \text{else} \end{cases}$	

Source: own work

Let us start with the following tuning: $\{\alpha_n\} = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$. In order to solve the corresponding problem (11), we have to calculate the respective norms (Definition 8). We start with the upper bound b^*, whose membership function is not continuous.

By Definition 8 we have $\|b^*\|_{\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}.} = \frac{4}{10}.\left(1.\frac{2}{4} + \frac{3}{4}.\frac{2}{4} + \frac{2}{4}.\frac{3}{4} + \frac{1}{4}.\frac{4}{4}\right) = \frac{1}{2} + \frac{1}{10}$ =0,6.

Then we have $\|a_1^*\|_{\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}} = \|a_2^*\|_{\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}} = 0{,}25$, as both numbers are crisp, and $\|a_3^*\|_{\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}} = 0{,}25$, because $\|a_3^*\|_{\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}} = \frac{4}{10}.\left(1.0 + \frac{3}{4}.\frac{1}{4} + \frac{2}{4}.\frac{2}{4} + \frac{1}{4}.\frac{3}{4}\right) = = \frac{4}{10}.\frac{10}{16} = 0{,}25$. Again following Definition 8 we get $\|a_4^*\|_{\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}} = \frac{5}{16}$ = 0,3125.

For the objective function coefficients we get in the same way:

$\|c_1^*\|_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}} = \|c_3^*\|_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}} =$ 2,5, $\|c_2^*\|_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}} = \frac{4}{10}(\sin^{-1}(1).1 + \sin^{-1}(\frac{3}{4}).\frac{3}{4} + \sin^{-1}\left(\frac{2}{4}\right).\left(\frac{2}{4}\right) + \sin^{-1}(\frac{1}{4}).\frac{1}{4})) = 1{,}74$, $\|c_4^*\|_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}} = \frac{1}{8} + \frac{1}{32} = 0{,}16.$

Solving problem (11), which in our case becomes:

$$2{,}5x_1 + 2{,}5x_2 + 1{,}74x_3 + 0{,}16x_4 \to max$$

$$0{,}25x_1 + 0{,}25x_2 + 0{,}25x_3 + 0{,}3{,}125x_4 \leq 0{,}6$$

$$x_k = 0{,}1 \text{ for } k = 1,..,K$$

we come to the conclusion that two solutions:

- $x_1 = x_2 = 1, x_3 = x_4 = 0$
- $x_2 = x_3 = 1, x_1 = x_4 = 0$

have the maximal value of the objective function of problem (10), equal to 2,56.

Now, trying to differentiate the solutions with respect to the distance to the knapsack capacity, thus considering $H_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}}(a_1^* \oplus a_2^*, b^*)$ and $H_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}}(a_2^* \oplus a_3^*, b^*)$, we get no result, as both Hausdorff distances are equal 0,1. Thus no solution has been selected and we need a finer tuning.

Thus we refine our initial tuning $\{\alpha_n\} = \left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$ in two distinct ways:

1. $\{\alpha_n\}' = \left\{\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$
2. $\{\alpha_n\}'' = \left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{7}{8},1\right\}$

An analogous calculation shows that:

1. $H_{\{\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}}(a_1^* \oplus a_2^*, b^*) = H_{H_{\{\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}}}(a_2^* \oplus a_3^*, b^*)$,

2. $H_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{7}{8},1\}}(a_1^* \oplus a_2^*, b^*) < H_{\{\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{7}{8},1\}}(a_2^* \oplus a_3^*, b^*)$.

Thus it is only the second refinement that allows to differentiate the two alternative solutions. We choose $x_2 = x_3 = 1, x_1 = x_4 = 0$, as this solution leaves more space in the knapsack with the same norm of the objective function, equal 2,56. With the first refinement the procedure would have to be continued, because no unequivocal solution can be chosen.

6. Conclusions

We have proposed a method of solving the fuzzy knapsack problem. The method is interactive and allows the decision maker to emphasize more certain aspects of fuzzy knapsack problem parameters and to take into account more uncertainty types than it is usually done. The method allows for fuzzy parameters of an arbitrary type – not just triangular or trapezoidal fuzzy numbers (which is not the case in the fuzzy knapsack models known in the literature). The decision maker may choose himself the "tuning" – the exactness with which the shape of the fuzzy numbers are taken into account in comparing them, and thus in comparing various knapsack problem solutions. This is – to our knowledge – an innovative way of ranking fuzzy numbers, dissimilar from any of the numerous ranking methods known in the literature. This ranking method of fuzzy numbers is a byproduct of the present paper. Further research may concern the multiple choice knapsack problem as well as any other optimalization problem where it makes sense to fuzzify the problem parameters. We intend to apply our fuzzy numbers ranking methods there in order to test its behavior and usefulness.

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NOWA METODA RANKINGOWA DLA LICZB ROZMYTYCH I JEJ ZASTOSOWANIE DLA PROBLEMU ROZMYTEGO PLECAKA

Streszczenie: W artykule rozpatrywany jest problem plecakowy z rozmytymi parametrami. W przypadku, kiedy obiekty które mają być umieszczane w plecaku mają rozmyte charakterystyki, optymalne rozwiązane problemu plecakowe (tzn. gwarantujące najwyższą sumaryczną wartość włożoną do plecaka), nie jest jednoznaczne. Również niejednoznaczne jest stwierdzenie, czy wybrane obiekty mieszczą się w plecaku, zwłaszcza jeśli pojemność plecaka jest również podana w postaci liczby rozmytej. Wybór rozwiązania zależy od metody porównywania liczb rozmytych. W artykule nowa metoda porównywania liczb rozmytych została zaproponowana i zastosowana do rozwiązania rozmytego problemu plecakowego. Metoda opiera się na dyskretnej aproksymacji liczb rozmytych przy zastosowaniu metryki Hausdorfa.

Słowa kluczowe: rozmyty plecak, liczba rozmyta, odległość między liczbami rozmytymi, norma liczby rozmytej.