Faculty of Computer Science and Management

# PHD THESIS

## Grounding of modal conditionals in agent systems

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keywords:
statement grounding, cognitive agent, indicative conditionals, modal conditionals, semantics of conditionals

abstract:
A theory for extracting of conditional sentences from agent's empirical knowledge is presented. A class of indicative conditionals with three modal operators of possibility, belief or knowledge is analysed. Theory ensures common-sense natural language understanding of the conditionals is preserved. Proposed theory is based on and extends existing theory for grounding of a class of simple and complex modal formulas.

Wrocław 2013

# Contents

# 1. Introduction

Until the nineties most of the research in artificial intelligence (AI) concentrated on symbolic systems. The systems where symbolic representations of world properties and their syntax based processing rules played a key role in modelling of the intelligence. In such systems reasoning was performed using formal processing rules, without a connection to semantic interpretations of predefined symbols. Outputs of such systems were generated using formal syntactic transpositions of symbols and as such were unbound to physical objects of the environment. They were simply a result of formal reasoning.

A noticeable shift in thinking came with (Brooks 1990, 1991) who proposed behavioural approach to intelligence. Brooks neglected the need of symbolic representation in modelling of AI systems. He proposed an idea of intelligence arising from interactions between an agent and an environment. Behavioural approach assumed there is no need to process the rules syntactically and intelligent response of a system can be a reaction resulting from previously performed world interactions.

The proposal of Brooks is closely related to the idea of embodied cognition (Anderson 2003), a position that intelligence is a result of interaction between a mind, a body and an environment. Embodied cognition is treated as a situated activity resulting from actions of beings. On the contrary to behavioural approach of Brooks embodied cognition does not neglect the need for abstract syntactic reasoning, but it acknowledges it to be only a part of broad spectrum of cognitive processes. Embodied cognition assumes most of cognitive processes should be bound to the mental representations often also called as internal reflections or mental images residing in a mind of an intelligent being. This assumption inspired some of the researchers to change the way of modelling of natural language. In the new approach natural language is treated only as a part (a surface representation) of a set of cognitive processes and can not be analysed without them. Language symbols should be related to their mental representations and indirectly to physical objects. Such approach gained popularity in case of robotic systems, where sensory data (perceptions) should play the key role in determining agent's actions. It became important to model relations between the language symbol, the speaker and the physical environment.

It has been ancient Greece, where we can find first discussions on the nature and the role of symbols (Cuypere and Willems 2008) but today's science usually starts with a work on the nature of signs from De Saussure (1983). He proposed a two-part model of the sign composed of the signifier and the signified (fig. 1.1). The signifier stands for the form which the sign takes. The signified stands for the concept the sign represents.

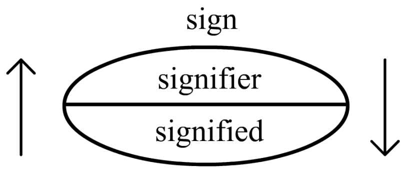


Figure 1.1. Sign according to De Saussure (1983)

There are a few more detailed interpretations of the form and the concept of the sign. Here I assume the form represents some part of the perception of the environment: the sound wave (as we hear it), the image (as we see it). The concept of the sign is the part of a mental state in the intelligent being, activated or associated with the occurrence of the sign.

According to De Saussure the sign must have both the signifier and the signified. Like two sides of a paper the both parts must coexist in order to call them a sign. One always requires the other but they can be separated for analytical purposes. Arrows in the model were used to show interaction between both parts. Either the signifier activates the signified (when we percept the sign) or the signified activates the signifier (when we utter the sign).

At about the same time as De Saussure, Peirce proposed a different model of sign consisting of three elements: an representamen, an interpretant and an object (Peirce 1931). The representamen and the interpretant can be treated more or less as respectively the signifier and the signified from the De Saussure's theory. The key difference lies in the addition of the object. The object is something the sign stands for in respect to some sort of idea. Inclusion of the object allowed to model a connection between the sign and the external world. In the simplest case the object may be the physical object such as a ball, a lamp, a cow etc. and the sign refers to some property of that object such as its colour, being lit, being alive etc..

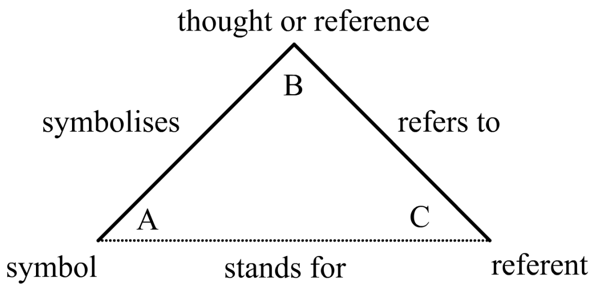


Figure 1.2. Semiotic Triangle according to (Richards and Ogden 1989)

Peirce's model can be illustrated using a semiotic triangle (fig. 1.2) from (Richards and Ogden 1989). The 'symbol' (A) is analogous to the representamen, the 'thought or reference' (B) to the interpretant and the 'referent' (C) to the object. In the terms of De Saussure's model, the symbol is analogous to the signifier and 'thought of reference' to the signified.

De Saussure stressed that the connection between the signifier and the signified is arbitrary. Arbitrariness of a sign meant that the way the signifier looks or sounds is independent from the signified. In other words, any name is a good name for a particular signifier. De Saussure referred to the language system as a non-negotiable 'contract' into which one is born. *Semantics of the sign are arbitrary to us and we must obey them.* Peirce had a different idea on the arbitrariness. He called interactions between the representamen, the interpretant and the object as *semiosis*. He was aware of the dynamic structure of a sign system and treated semiosis as a process binding the three elements of the sign. Natural languages are not, of course, arbitrarily established but once the sign has come into historical existence it cannot be arbitrarily changed (Lévi-Strauss 2008). For well established symbols, the semantics stay unchanged and are given to us in advance.

In terms from the semantic triangle, De Saussure tried to model the edge A-B. De Saussure was particularly interested in connections between signs, rather than between their references to external world or particular properties of signifiers. He claimed that a system of signs such as natural language is parallel and much independent to external world and signs take their meanings from their relations to other signs. He seemed to ignore the importance of empirical experiences. The addition of object in Peirce's model allowed for some sort of connection between the sign (the signified and the signifier) and the part of world being described by that sign.

Classical approaches to AI from the nineties were based on boolean logic with Tarski interpretations of truth based on truth tables (Tarski 1944, 1969). According to classical notion of truth: 'To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true' (from Aristotle's Metaphysics). Let $M$ be a model structure and $\phi$ be a language symbol (for example a natural language statement). Semantics of $\phi$ are given a priori. In terms of classical definition of truth the occurence of a relation $M \models \phi$ should be treated as 'what is that it is'. Such approach relates directly to edge A-C between the symbol and the referent. Vertex B (the interpretant or the signified) is omitted in this model. In result the role of the bearer of the interpretant (the intelligent being) is also omitted. This has lead to a series of interpretative problems with logical statements such as alternative or implication (See (Ingarden 1949; Pelc 1986; Jadacki 1986) for a few examples). Many of such problems seem to result from the lack of proper references to speaker's subjective knowledge. Furthermore, in classical model, it is impossible to include different moods of the speaker. According to Ingarden (1949) different modis (potentialis, irrealis and realis) should not be omitted in logics.

It became quite clear that the role of the intelligent being as a interpreter of the sign, that itself forms a complex structure, should no longer be neglected. Works from De Saussure and Peirce have offered important theoretical grounds for modelling of signs and their role in artificial intelligent beings, but they lacked in formal mathematical solutions. The problems of modelling the relation between the signifier (or representamen), signified (or interpretant) and the object have later been reinterpreted and addressed in the works in the fields of cognitive science and AI such as (Harnad 1990; Vogt 2003; Roy and Reiter 2005).

This work also addresses one of such sub-problems, namely the symbol grounding problem. This problem is assumed to be one of the hardest and most important problems in AI and robotics (Vogt 2003). The symbol grounding problem has been introduced in the famous work of Harnad (1990). To explain the problem author presented an example of Chinese/Chinese Dictionary-Go-Round. Suppose one has to learn Chinese. The only source of information he has is a Chinese to Chinese dictionary. He can learn symbols by looking on other meaningless symbols. In result the person ends in an infinite merry-go-round over the dictionary never learning anything. When it comes to a computer program, it can reposition the symbols (for example using syntactic rules), never understanding their meaning[1]. Harnad suggested grounding symbols in sensory data gathered by an artificial system. In such a way, a link between empirical experiences from current or past moments and the symbol itself is built. In order to understand symbols, they have to be somehow connected to physical objects or more precisely perceptions of these objects. Construction of such a connection is a key task in solving the grounding problem.

One of sub-problems of the symbol grounding problem is the anchoring problem (Coradeschi and Saffiotti 2000; Vogt 2003). It concerns joining the symbols to their representations held within sensory information. In terms of semiotic triangle anchoring allows for construction of a link between the symbol and the referent (edge A-C). Unlike in classical approach, the link is not directly provided, but constructed with the use of sensory information gathered by an intelligent being (a robot). The problem seems to be at least partially solved in some simple domains of objects and properties. Please refer to (Coradeschi and Saffiotti 2000; Steels and Belpaeme 2005) for exemplary solutions. The grounding problem seems simpler when one possesses sensory data where a representation of object or feature can be easily extracted.

Eco (1996) and others noted that the link between the symbol and the physical object is not always existent. For example a unicorn has no representation in the real world. On the other hand we have no problem with imagining what a unicorn is. It is simply a horse with a horn. At least some of the symbols must be defined *only* in terms of other symbols. Furthermore, it is often impossible to construct a direct link between the sensory data and the symbol because the symbol is currently not perceived or has an abstract meaning. In particular, complex statements such as: 'P or Q', 'If P, then Q', 'I believe that P', 'It is possible that P' can not be directly associated to sensory data. Such statements possess 'a meta-meaning' that extends beyond simple objects and properties.

[1] De Saussure was criticised for neglecting the importance of empirical experiences in his theory. Chinese/Chinese Dictionary-Go-Round is a perfect example proving that at least some of the symbols must be anchored to the sensory data.

In order to solve the grounding problem for complex sentences, one has to properly model all concepts lying in the vertices of the semiotic triangle and their connections represented by triangle's edges. It is crucial to construct internal structures of an intelligent being, so that they allow for proper modelling of signified / interpretant associated to complex sentences. Contents of these internal structures should depend on empirical experiences (perceptions) of the environment. In the end a relation between the symbol and the internal structures must provide intuitive meaning of the complex sentence compliant with its natural language understanding.

The entry point of this thesis is in the work of Katarzyniak published as a book in Polish (2007) and partially in a series of articles in English (Katarzyniak 2001, 2003, 2005, 2006). Katarzyniak addressed the grounding problem for the case of complex symbols such as modal sentences. Author proposed models for all elements of semiotic triangle to allow for grounding of a given class of modal formulas. In his work the grounding process is considered from a perspective of an autonomous cognitive agent located in a not necessarily physical environment. Grounding itself is understood as a construction of a indirect link between empirical knowledge and a symbol. The link is constructed through a mental representation built autonomously by the agent. The grounding process is performed from the empirical knowledge to the symbol (not the other way), so the agent is treated as a potential speaker, not a listener.

Author defined a formal language, that covers simple statements '$o$ exhibits $p$' ($p(o)$) and complex statements with conjunctions such as: 'and' ($p(o) \wedge q(o)$), 'or' ($p(o) \vee q(o)$) and 'either ... or' ($p(o) \veebar q(o)$). Formulas can be extended to their modal forms by one of three modal operators: 'I know that' ($Know$), 'I believe that' ($Bel$) and 'It is possible that' ($Pos$). Proposed simple and modal formulas should not be confused with classical formulas or classical modal formulas with Tarski or Kripke interpretations. Author assumed that the semantics of formulas and modal operators are arbitrary and compliant with their intuitive and conventional understanding in natural language. Such assumption is consistent with the arbitrariness of symbols proposed by De Saussure. Further author proposed a set of formal common-sense constraints on the usage of formulas resulting from conventional denotations of modal statements. These constraints do not refer to truth conditions (as in classical approaches), but to subjective knowledge of the cognitive agent. For example one of such constraints implied that 'It is possible that $o$ exhibits $p$' ($Pos(p(o))$) denotes also that the speaker (the agent) does not know if $o$ exhibits $p$.

The referents (vertex C) from the semiotic triangle are represented by a simple environment model consisting of sets of recognizable atomic objects and their binary properties. The agent observes this world and doing so builds its empirical knowledge base. Agent is limited in its perceptive abilities so obtained data consists of incomplete reflections of the environment states. Proposed model of the environment and empirical data does not refer directly to sensory data. It has been assumed that obtained data is already processed so that objects and their properties are indicated and recognised.

To model the signified (vertex B) in the semiotic triangle a two-layer mental model was proposed. A bottom layer models unconscious area and a top layer models conscious area of an artificial mind state. Such approach is consistent with fundamental assumptions on other mind models from non-technical literature such as (Paivio 1990; Freeman 1999, 2000). Proposed model further consists of grounding sets that contain previously gathered observations of the environment (including the current one) forming an empirical grounding material for various modal formulas.

Finally author defined a key component: a set of epistemic relations that model edge A-B of the semiotic triangle. These epistemic relations are validated against grounding sets being the elements of the mental representation. Only a proper distribution of grounding material allows for grounding of the given formulas. Formula is grounded only when epistemic relation holds. In such case formula, together with the mental representation, constructs, what I call, a *mental sign*. I use term 'mental' because both the formula (the signifier) and the mental representation (the signified) exist only in the mind of the agent. The grounded formula (sign) is 'activated' in the mind but does not have to be externalised (i.e. uttered). For an example: One may imagine a dog (mental representation) and doing so activate a symbol 'dog' (formula) but he does not need to literally utter a word 'dog'. The symbol stays properly grounded even if it is not uttered.

Epistemic relations rely on a set of parameters called grounding thresholds. Only a some of possible settings of these parameters ensure the grounding process meets a set of common-sense postulates. Author has formulated and proven a set of theorems that in turn defined proper settings of the grounding thresholds. When a setting of grounding thresholds meets criteria outlined in the set of theorems, then epistemic relations meet previously formulated common-sense constraints resulting from conventional natural language understanding of modal formulas. This implies that all assumed denotations of signs in forms of complex modal statements hold.

Katarzyniak's grounding theory is described in detail in chapter 2. Formal models of the environment, the empirical knowledge base and the grounding sets are presented. Formal language of modal formulas and their respective epistemic relations are defined. The whole process of grounding is explained. Some computational examples have also been provided.

Katarzyniak's grounding theory may be further developed to allow for grounding of new types of complex statements. This thesis does so by adding support for conditional sentences and their modal extensions. Addition of conditionals to the grounding theory seems to be the next natural step, after conjunctions and alternatives, in its development process.

Conditionals are the sentences of the form: 'If $P$, then $Q$'. The $P$ phrase is called an antecedent and the $Q$ phrase is a consequent. These statements can be later divided into two main groups: indicatives and subjunctives. Indicatives refer to real plausible possibilities while subjunctives express hypothetical, counter-factual claims. Usually subjunctives can be easily distinguished from indicatives because they contain 'would/could/should' phrase in the consequent. Please refer to (Bennett 2003) for a detailed analysis of this classification. Conditionals are particularly interesting because they are widely used in everyday life and their usage circumstances can significantly differ from situation to situation. It seems there is no perfect theory on conditionals. Most of older theories directly refer to environment states often inaccessible to the speaker. Such approach seems to be a dead end resulting in various interpretative problems (Pelc 1986; Jadacki 1986; Bennett 2003; Ajdukiewicz 1956; Edgington 1995). Recent theories on conditionals often discard absolute truth conditions and tend to model these statements with respect to the subjective knowledge of the speaker (Oberauer 2010). For example a theory from Johnson-Laird and Byrne (2002) builds the meaning of conditionals from mental models. This thesis fits into the new stream, as conditionals grounding conditions are validated against and in accordance to the subjective mental representations. Hopefully such an approach shall solve at least some of well known problems that harass known theories on conditionals.

The fundamental claim of the thesis is that:
*It is possible to design an agent, who, for a provided empirical knowledge base structure, shall meet a series of common-sense constraints imposed on the process of grounding messages in form of modal conditional formulas.*

This thesis extends the grounding theory, so almost all of theoretical basis and assumptions are directly transferred from it. Addition of modal conditionals to the grounding theory requires a series of steps to be made. Firstly the language has to be extended with formal formulas representing various kinds of modal conditional sentences. Secondly a model for mental representations (grounding sets) needs to be reconsidered. Thirdly a common-sense constraints on their usage need to be proposed. These constraints need to take into consideration speaker's subjective knowledge. Proposed constraints must ensure conventional natural language denotations of conditionals. Constraints need to be formulated also for various modal extensions of conditionals. Finally epistemic relations must be formulated, so that they allow for proper grounding of conditionals. A series of theorems must be formulated and analytically proven to ensure previously proposed constraints can be met for the correctly chosen parameters (grounding thresholds).

Conditional sentences can be classified in may ways depending on chosen criterion. There are many possible usage patterns and meaning of conditionals. Chapter 3 presents a general discussion on conditional statements. Some of classifications of conditionals are given. In particular a distinction to indicatives and subjunctives is considered. There is also a short notice on the 'then' adverb that seems innocent but its occurrence significantly changes the meaning of a conditional. A subset of conditionals is separated to further analysis. Proposed extended theory deals only with indicative conditionals, leaving subjunctives for future research. Afterwards an extension to the formal language of formulas is defined. Existing language from the grounding theory is extended with conditional sentences. Considered conditionals can contain one of three possible modal operators ($Pos, Bel, Know$). These modal operators can be fitted into two possible places. Operator can be put either at the beginning of the conditional forming of what I call a *modal conditional* or in the consequent forming a *conditional modality.* The distinction between the two types of modal conditionals is sketched. Finally semantics of conditionals are defined. These semantics are meant to comply with conventional understanding of conditionals. Unlike in classical approaches, proposed semantics also refer to the speaker's subjective and partial knowledge. At this stage explanation of assumed semantics is based on readers intuitive understanding of conditionals. Simple usage examples are presented and explained to signalize assumed meaning of conditionals.

Chapter 4 briefly describes the most known approaches to conditionals. It is shown that material implication, despite its undoubted importance in proof systems, is not a good model for natural language conditional sentences. The problems arise from the fact that the falsity of an antecedent is enough for the material implication to be true. This feature of material implication has been widely criticized. Further famous Ramsey test is introduced and discussed. Objective and subjective approaches to conditionals are compared. There are many arguments against treating conditionals as truth-functionals (a statements whose truth may be defined by truth of its compounds). Finally probabilistic approaches are explained. Some of the typical pitfalls are presented and discussed.

Chapter 5 starts with a series of references to previous works on conditionals from Ajdukiewicz, Jadacki, Bogusławski, Clark, Woods, Edgington, Bennet and others. Author tries to prove that many problems with conditionals tend to arise from the ignorance of the language commons. Once conditional statement is said, it tells not only about the world but also about the speaker himself. Namely about the speaker's knowledge (or actually the lack of it) on the antecedent and the consequent. A epistemic, subjective, context dependent, connection between the antecedent and the consequent is postulated. Further a rational, common-sense criteria of conditionals usage are searched. In particular Gricean theory on implicatures is referred. Typical conclusions made by a listener of a message in form of a conditional are presented. Conventional denotations and implicatures of conditionals are formulated. These implicatures refer not only to the world state but also to the speaker's knowledge. Further the role of the modal operators is discussed. Their Influence on the meaning of conditionals is carefully analysed. Two different positions of the modal operators in conditionals are considered. All that leads to a series of postulates forming conventional assumptions on the meaning of conditionals and their modal extensions. These common-sense postulates are later transformed into three groups of formal constraints: C1, C2 and C3. First group of constraints reflects a broad meaning of conditionals that in many ways resembles material implication. Groups 2 and 3 further constrain the meaning of conditionals hopefully forming their conventional understanding in natural language.

Formal extension to the grounding theory is presented in chapter 6. Grounding process is formally specified for conditional formulas. Grounding sets modelling mental representation of modal formula are defined. Further a measure called grounding strength is defined. The grounding strength numerically measures the distribution of empirical material used to ground conditional formula. Depending on this measure's value different modal operators can be added to conditionals. A pragmatic filter in form of a conditional relation is defined. This relation is associated to constraints groups 2 and 3.

Finally epistemic relations are defined. There are 3 separate propositions for epistemic relations: normal epistemic relation, pragmatic epistemic relation and strictly pragmatic epistemic relation (definitions: 6.8-6.17). Each of the propositions is associated to one of previously formulated common-sense constraints.

The behaviour of epistemic relations depends on the choice of parameters called grounding thresholds. It is important to choose a proper setting of these parameters as not all grounding thresholds ensure rational grounding results. In chapter 7, through a series of theorems, rational settings of grounding thresholds are defined. Any setting meeting theorems 7.12, 7.32 and 7.58 ensures meeting of simultaneous usage constraints defined in table 5.2. In such a way a rational behaviour, in accordance with common-sense constraints, of the grounding process of conditionals is analytically proven.

In chapter 8 the grounding theory is compared to other known theories on conditionals. The grounding theory, although not perfect and quite narrow, seems to posses a series of qualities not present in other theories. The grounding theory has many common features with the proposition of (Johnson-Laird and Byrne 2002). Both are based on mental representations and both suggest mental representations are only partially explicit (conscious). Theory from Johnson is more general, but doesn't define many of the technical details present in the grounding theory.

In the end some usage examples of the extended grounding theory have been provided. Chapter 9 starts with a computational example presenting the work-flow of the grounding theory. Various features of the theory are exemplified and shortly discussed. Second example utilizes grounding theory of conditionals to summarize the transaction base. Finally there is an example based on well known Mushrooms dataset from (Bache and Lichman 2013) where the grounding theory is utilized to provide a mushroom picker with context dependent tips in form of conditional sentences.

# 2. The existing grounding theory

This thesis extends the existing theory of grounding of modal formulas in agent systems by Radosław Katarzyniak. The grounding theory has been published as in book in Polish (2007) and partially in a series of articles in English (Katarzyniak 2001, 2003, 2005, 2006). Katarzyniak addresses the grounding problem for a given class of modal formulas. Modality is expressed in form of three modal operators for possibility, belief and knowledge. Semantics of modal formulas are assumed to be consistent with conventional understanding of respective natural language statements.

In this chapter the theory is briefly summarized to outline its key features. Formal mathematical definitions of agent structures and grounding process components have been provided. Most of these definitions are later directly transferred to the thesis.

The grounding process addressed by the grounding theory is modelled within a cognitive agent. It is hence important to at least sketch what is meant by the term cognitive agent and how it is modelled in the grounding theory.

Defining of the terms 'agent', 'intelligent agent' and 'cognitive agent' from computer science is not easy. Every definition seems to be either too narrow or too broad (Franklin and Graesser 1997). One of the most general definitions states that:

> "An agent is anything that can be viewed as perceiving its environment through sensors and acting upon that environment through actuators." (Russell 2009)

It is a very broad definition that catches only the most general aspects of being an agent. Agent should be located in some environment (real or artificial). It should observe this environment (gather some data from it) and finally take actions or make decisions.

The definition states that something may be only viewed as an agent. This shifts the problem from being an agent to interpreting something as an agent. This way even a thermostat can be an agent (Jennings et al. 1998). It observes the environment (measures temperature) and changes the heating level.

In computer science the agent is usually a software that interacts with other agents, performs some data processing and makes decisions to achieve predefined goals.

It is usually required that the agent is intelligent, hence the term 'intelligent agent'. This term is also hard to define, as it is hard to define what intelligence is. It is informally understood that the agent performs some non-trivial reasoning.

One usually requires many additional features from the intelligent agent (Wooldridge et al. 1995; Jennings et al. 1998). It should be *autonomous*, meaning it makes some decisions or takes actions on its own. It should possess *social abilities*, meaning it is able to communicate with other agents (possibly humans). It should be *rational*, roughly meaning its actions may be justified in the context of asserted goal. It should also be: reactive, pro-active, adaptive, possess learning skills etc.. On the other hand lacking of some of these features does not necessarily mean a computer program is not an agent.

In computer science a 'cognitive agent' (Huhns and Singh 1998) is a kind of intelligent agent that implements some of cognitive processes of humans, i.e. some of thinking mechanisms. Cognitive agent tries to directly model some of humans mind components or is designed to reflect some of humans cognitive processes. One of the best known models of the cognitive agent is BDI (Belief, Desire, Intention) agent (Rao et al. 1995). The belief stands for agent's knowledge. The desire may be interpreted as agent's long term goals. The intentions are currently pursued aims.

The cognitive agent considered within the grounding theory is meant to possess language skills i.e. it is able to utter a communicate in form of a modal formula. Agent is assumed to be *autonomous* as it decides on its own which formulas to utter. Agent is also *rational* as it may utter only formulas whose meaning is consistent with our conventional understanding of respective natural language proposition.

In the grounding theory the cognitive agent is the intelligent being realizing the grounding of symbols in form of modal formulas. Formulas are grounded within the agent, so their semantics are analysed from a perspective of the agent as the messages source. The agent is responsible for constructing the links between the mental representation, the symbol and the object and hence grounding the symbol in the perceptions of the object. In that sense the agent realizes the semiotic triangle. The grounding theory defines agent's components that are crucial in the context of formula grounding process. Other agent's aspects such as: predefined goals, possible actions, decision models, reasoning processes, etc. are not considered.

## 2.1. The grounding process

The symbol grounding problem is one of most important problems in artificial intelligence and cognitive sciences (Vogt 2003). Symbol grounding is a task of finding meanings of symbols and in narrower sense understanding the meaning itself. The problem has been described in the broadly cited work of Harnad (1990). Harnad suggests grounding symbols in sensory data gathered by an artificial system. In such a way, a link between empirical experiences and the symbol itself is built. Unfortunately this link is indirect and for more complex symbols it requires links not only to most recent sensory data but also to previous empirical experiences gathered by the artificial system.

The grounding process is a trip from sensory data to a natural language statement (or the other way round). What makes the grounding process hard is the knowledge bearer who stands in the middle of the road. The knowledge bearer is a living being that observes, remembers, reinterprets and finally externalises its thoughts using natural language. A language so complex, that it seems a miracle we are able to use it.

To ground a symbol intelligent being forms a structure that can be formally represented as a semiotic triangle (Richards and Ogden 1989) (see fig. 1.2). The semiotic triangle (also called the semiotic triad or the semantic triangle) links an external object (vertex C), a symbol occurrence (vertex A) and mental representation (vertex B). From a perspective of a developer of the cognitive agent, the grounding process can be described in a series of steps. In the first step agent obtains new sensory data from the environment. The external object is perceived by the agent. The agent stores observations of the object in its internal empirical knowledge base. In the next step various cognitive processes use data stored in the knowledge base to construct a mental representation of the object. This mental representation doesn't have to necessarily depend on current observation. It can (and often must) depend on past observations of the considered object. This way a link between the object and its mental representation is built (edge B-C). In the last step, when the mental representation is coherent with arbitrary semantics of some language symbol it can (but doesn't have to) construct a link between itself and the symbol (edge A-B). This link shall later be modelled by an *epistemic relation*. In such a way the mental symbol comes to existence in artificial or natural mind. The mental symbol is grounded in (bound to) the mental representation (edge A-B) and indirectly to the external object (edge A-C). Later the mental symbol can (but again doesn't have to) be externalised (uttered, written down, told, shown etc.) to form the physical symbol in the environment.

The statement (the modal formula) grounding problem addressed by the grounding theory from Katarzyniak is a sub-problem of the symbol grounding problem. The grounding theory analyses the process in a direction described above, starting from the object (when the agent sees or thinks of the object). The grounding process can be analysed the other way round, starting from the symbol. Such situation happens when one notices a symbol which in turn forces forming of coherent mental representation. This direction is not considered by the grounding theory. Figure 2.1 visualises all steps of the grounding process. Formal mathematical symbols used inside the diagram shall be defined and explained later.

Some of the sub-steps in the grounding process are difficult tasks themselves. A shift from the sensory data to objects and properties requires specialized object recognition algorithms. Cognitive processes that form mental representations from empirical material need to be modelled by various clustering, classification, reasoning, data mining, pattern recognition, etc. algorithms. A step from properly grounded modal formula to uttered natural language sentence needs sophisticated pragmatic filters and translation to natural language. All these elements are outside the scope of the grounding theory. Aspects covered by the theory have been marked by the grey area on figure 2.1. The theory itself puts most effort on the construction of the link between mental representation and the symbol (edge A-B of the semiotic tirangle).

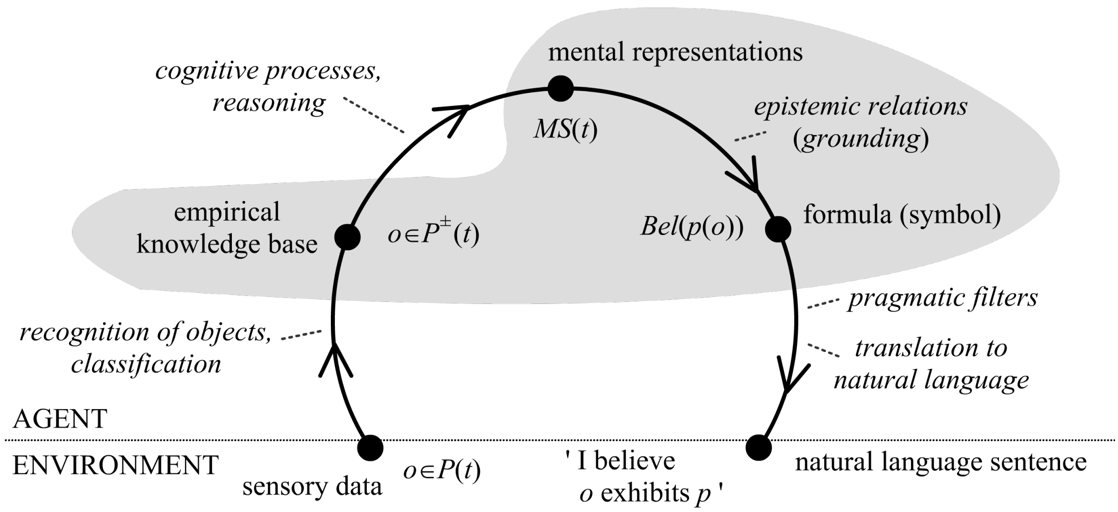


Figure 2.1. The Grounding process in the context of agent and its environment. Grey area marks topics covered by the grounding theory. Mathematical notations are defined later in the thesis.

## 2.2. Environment and empirical knowledge

To model the grounding process one needs to define all vertices and edges of the semiotic triangle. The grounding theory starts with a simple model of the environment, that is observed by the agent.

### 2.2.1. Environment model

The external environment of the cognitive agent (the external world) is a dynamic system of atomic objects. A line of time points $\mathcal{T} = \{t_1, t_2, ...\}$ is assigned to this world. Each state of the world is always related to one and only one time point. At each time point objects of the world exhibit or do not exhibit particular properties $\mathcal{P} = \{P_1, P_2, ..., P_K\}$. Formally such a world can be captured as follows (Katarzyniak and Nguyen 2000; Katarzyniak 2005):

**Definition 2.1.** *Each state of the external environment related to the time point t is called t-related world profile and is represented by the following relational system:*

$$WP(t) =< O, P_1(t), P_2(t), ..., P_K(t) >$$

The following interpretation of elements of $WP(t)$ is assumed:

— Set $O = \{o_1, o_2, ..., o_M\}$ is the set of all atom objects of the external world.
— Set $\mathcal{P} = \{P_1, P_2, ..., P_K\}$ is the set of unique properties that can be attributed to the objects from $O$.

Each object $o \in O$ may or may not exhibit a particular property $P \in \{P_1, P_2, ..., P_K\}$ (at a particular time point $t$).

— For $t \in \mathcal{T}$, the symbol $P(t)$ denotes a unary relation $P(t) \subseteq O$
— The condition $o \in P(t)$ holds if and only if the object $o$ exhibits the property $P$ at the time point $t$.
— The condition $o \notin P(t)$ holds if and only if the object $o$ does not exhibit the property $P$ at the time point $t$.

### 2.2.2. Empirical knowledge base

The cognitive agent observes the environment (obtains sensory data from it). This sensory data is transformed and preprocessed to recognize world's objects and their properties. The processing of the sensory data is outside the scope of the grounding theory. It is assumed agent can internally store representations of particular states of properties $P_1, P_2, ..., P_K$ in individual objects $o_1, o_2, ..., o_M$.

Each individual perception of the environment realizes as internal reflections of objects (not) exhibiting properties. These internal reflections are held within a formal data structure called a base profile, which is related to the concept of the world profile introduced above (see definition 2.1). The content of each base profile is always associated to this part of the external world which was covered by the related perception. This property of base profiles corresponds to the constrained cognitive capabilities of natural and artificial agents which are never able to observe the overall current state of all external objects at one time point $t$. The internal reflection of an individual observation realized by the cognitive agent at a moment $t$ is called $t$-related base profile and is given by the following definition:

**Definition 2.2.** *The internal reflection of an observation of the world (usually partial) realized at a time point t is called t-related base profile and is given by the relational system:*

$$BP(t) = < O, P_1^+(t), P_1^-(t), P_2^+(t), P_2^-(t), ..., P_K^+(t), P_K^-(t) >$$

For each $k = 1, 2, ..., K$ and $o \in O$, the following interpretations and constraints are assumed for $t$-related base profiles:

— The set $O = \{o_1, o_2, ..., o_M\}$ consists of all representations of atom objects $o \in O$, where the symbol $o$ (used in the context of this base profile) denotes a unique internal reflection of the related atomic object located in the external world.
— $P_k^+(t) \subseteq P_k(t)$, $P_k^-(t) \subseteq O \setminus P_k(t)$ and $P_k^+(t) \cap P_k^-(t) = \emptyset$ hold.
— the relation $o \in P_k^+(t)$ holds if and only if the agent observed at the time point $t$ that the object $o$ exhibited property $P_k$.

— the relation $o \in P_k^-(t)$ holds if and only if the agent observed at the time point $t$ that the object $o$ did not exhibit the property $P_k$.

In relation to each $t$-related base profile $BP(t)$ the idea of knowledge ignorance $P^{\pm}(t)$ is defined as regards to the observed state of particular property $P \in \{P_1, P_2, ..., P_K\}$:

**Definition 2.3.** *The $t$-related $P$-ignorance is defined as this set of atomic objects which members were not covered by any observation of the world carried out by the agent at the time point $t$. The content of $t$-related $P$-ignorance is given as follows:*

$$P^{\pm}(t) = O \setminus (P^+(t) \cup P^-(t))$$

While world state $WP(t)$ contains representations of real physical objects, base profile $BP(t)$ contains only their reflections obtained from perceptions made by the agent. An internal reflection of an object held within base profile $BP(t)$ is only a surface representation of the physical external object in $WP(t)$. The perception is greatly constrained by temporal, spacial and physical limitations.

The process of constructing of internal reflections from the perceptions of real world (from sensory data) is not considered within the grounding theory. It is simply acknowledged, such process takes place and it is faultless. To keep notation simple, the same symbols are used to denote external world objects, properties and their internal reflections. A reader should be aware that they are not the same.

At each time point $t \in \mathcal{T}$ the overall state of basic empirical knowledge collected and stored by cognitive agent in its internal knowledge base is given as a temporally ordered set of base profiles (Katarzyniak and Nguyen 2000; Katarzyniak 2005). The related definition is given as follows:

**Definition 2.4.** *The overall state of empirical knowledge collected by the cognitive agent up to the time point $t$ is given as the following temporally ordered collection of base profiles:*

$$KS(t) = \{BP(\hat{t}) : \hat{t} \in T \wedge \hat{t} \leqslant t\}$$

Set $KS(t)$ holds all empirical knowledge resulting from environment perceptions gathered by an agent up to time moment $t$. The moment $t$ is usually interpreted as the current moment.

## 2.3. Formal language of modal formulas

The grounding theory considers the grounding problem for a given class of formulas and their modal extensions. These modal formulas are constructed from a formal language $L$. This section defines syntax and intuitive semantics of this language.

### 2.3.1. Language syntax

**Definition 2.5.** *Alphabet of language $L$ consists of:*

— $O = \{o_1, o_2, ..., o_N\}$ *a set of perceptually recognizible objects*[1]
— $\Gamma = \{p_1, p_2, ..., p_K\}$ *a set of perceptually recognizible unary properties*[2]
— $\neg$ *symbol of negation*
— $\wedge, \vee, \veebar$ *symbols of a conjunction, disjunction and exclusive disjunction.*
— $Pos, Bel, Know$ *modal operators of possibility, belief and knowledge*
— $(, )$ *brackets*

**Definition 2.6.** *Let sets $L^b, L^c, L^M$ be defined as:*

— $L^b$ *is a set of all simple formulas of the forms:* $p_k(o_n)$, $\neg p_k(o_n)$, *where* $p_k \in \Gamma$ *and* $o_n \in O$.[3]
— $L^c$ *is a set of all complex non-modal formulas of the forms:* $\phi\ \delta\ \psi$ *where* $\phi \in \{p_i(o_n), \neg p_i(o_n)\}$, $\psi \in \{p_j(o_n), \neg p_j(o_n)\}$, $i \neq j$ *and* $\delta \in \{\wedge, \vee, \veebar\}$.
— $L^M$ *is a set of all modal formulas of the forms:* $Pos(\theta), Bel(\theta), Know(\theta)$, *where* $\theta \in L^b \cup L^c$.

*Any formula from set $L = L^b \cup L^c \cup L^M$ is a proper formula of language $L$. No other formula is allowed.*

Please notice the language is not extensible. Multiple conjunctions or nested modal operators are not allowed. Given exemplary formulas $Pos(Bel(p_k(o_n)))$, $p_1(o_n) \wedge p_2(o_n) \vee p_3(o_n)$ are NOT proper formulas of the language $L$.

### 2.3.2. Intuitive language semantics

Formulas of language $L$ are treated as formal representations of natural language sentences. The grounding theory assumes that the semantics of the considered language are arbitrarily given. The semantics are meant to be compliant with conventional denotations of respective natural language sentences. Table 2.1 defines intuitive semantics of formulas of language $L$ considered within the grounding theory.

Provided semantics are compliant with common-sense interpretations of natural language sentences. These semantics are neither formal, classical interpretations with truth tables from Tarski nor formal Kripke semantics of modal logic.

The interpretation of formulas of language $L$ provides that property $P_k$ has already assigned language symbol $p_k$ and that the perception of object $o_n$ is directly related to real environment object $o_n$. Atomic objects and their binary properties possess fixed language labels. This effectively implies that the grounding process of simple non-modal

[1] I shall also use symbol $o$ marking chosen object $o_n \in O$.
[2] I shall also use symbols $p, q$. Each symbol is marking some chosen property $p_k \in \Gamma$.
[3] When it is clear, I shall use shorthand notation: $p$ and $q$ to denote some fixed properties and objects $p(o)$ and $q(o)$

Table 2.1. Formulas and their semantics ($p \in \Gamma$, $o \in O$, $\phi, \psi \in L^b$, $\theta \in L^b \cup L^c$)

| formula | intuitive semantics |
|---|---|
| | simple statements |
| $p(o)$ | Object $o$ exhibits property $P$. |
| $\neg p(o)$ | Object $o$ does not exhibit property $P$. |
| | complex statements |
| $\phi \wedge \psi$ | $\phi$ and $\psi$. |
| $\phi \vee \psi$ | $\phi$ or $\psi$. |
| $\phi \veebar \psi$ | Either $\phi$ or $\psi$. |
| | simple and complex modal statements |
| $Pos(\theta)$ | It is possible that $\theta$. |
| $Bel(\theta)$ | I believe that $\theta$. |
| $Know(\theta)$ | I know that $\theta$. |

statements $p_k(o_n)$ is assumed to be already done. The grounding theory addresses the grounding problem only on modal and complex statement levels.

Formulas formally represent agent's empirical knowledge state. Denotations of formulas are considered from the perspective of the agent as a speaker. It is assumed the formula's denotation is constructed with respect to the knowledge of the speaker (the agent). This implies that agent's mental model should comply with intuitive understanding of the statement treated as being uttered by him. The grounding of a formula in agent's mental model must sustain the intuitive understanding of the associated natural language statement. This in turn enforces a series of common-sense constraints on the grounding of these formulas. The grounding theory must not allow for simultaneous grounding of formulas that are intuitively conflicting. For example a formula $Pos(p(o))$ conventionally denotes that the speaker (the agent) does not know whether $o$ exhibits $P$ or not. Hence simultaneous grounding of $Pos(p(o))$ and formula $Know(p(o))$ should be disallowed. For a second example a formula $Know(p(o) \vee q(o))$ denotes the speaker knows that $o$ exhibits one of the properties. But it also denotes that the speaker does not know whether $o$ exhibits $P$ and he does not know whether $o$ exhibits $Q$. In result the grounding theory must disallow for simultaneous grounding of the $Know(p(o) \vee q(o))$ and for example $Know(p(o))$.

## 2.4. The mental model

The language of modal formulas and the empirical knowledge base formally define contents of vertices A and C of the semiotic triangle respectively. The only missing vertex is C which represents agent's thought image of the object.

From the perspective of cognitive linguistics the sign is correlated with a structure in a mind called a mental representation. It can be assumed that the mental representation contains empirical material that is 'activated' by the symbol. The choice of the 'activated' material is made autonomously (and at least partially unconsciously) by the agent. Such choice can be made from previous interactions with the environment and other agents. This process is often described as semiosis (Peirce 1931) that is controlled by various cognitive processes. Activated material forms the meaning of the symbol. Mental representations are one of the elementary theoretical structures used in the grounding theory.

Language production is a very complex cognitive process that involves multilevel actions of the mind. In natural agents, such as humans, the processing of language is not limited to conscious phenomena. In contrary, there are many important levels of cognitive activities, which influence the process of producing 'linguistic' labels of cognitive states.

According to linguists, spoken language is only a surface representation of a deeper sense and the choice of a particular representation is strictly defined by the whole empirical experience held within natural or artificial agents. The whole gathered material plays important role on every level of language related activities. In particular it is assumed that all gathered empirical knowledge, including latest perceptions, determines the choice of a surface representation.

Unconscious processes play important role in the choice of final surface (language) representation. A level of awareness, has impact on the form of external expression. The awareness itself produces different levels of conviction, expressed within a surface representation. It is claimed empirical material buried within unconscious area determines awareness level, forcing agents such as humans, to express uncertainty within uttered statements.

Different awareness levels may result from incapability to thoroughly process vast empirical material. The more agents think on particular issue, the greater the awareness and the more certain they may become.

The empirical knowledge in form of internal reflections is used to construct mental representation (Pitt 2012) of object and its property. This mental representation is supported by appropriate and relevant empirical material chosen autonomously by the agent. Later mental representation can be associated with an external surface language representation (Grice 1957; Searle 1983).

Suppose the agent directly observes the object $o$. He perceives that $o$ exhibits $P$ ($o \in P^{+}(t)$). The direct observation leads to activation of associated language formula $p(o)$. In such case grounding process of a language description is trivial with respect to definitions proposed by the grounding theory. The theory assumes atomic objects and their binary properties are already grounded.

Such simple situation does not happen when the object is NOT directly observed. When object is not observed agent has to refer to its previous observations of the object. Such situation has been exemplified in figure 2.2. This figure visualises the fundamental

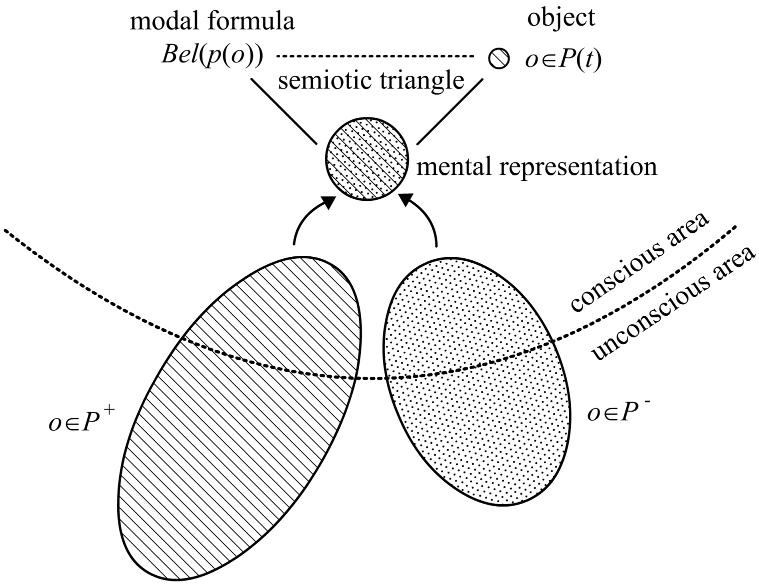


Figure 2.2. The role of empirical material in the construction of mental representation for currently unobserved object $o$.

assumptions on the role of the mental model in the grounding of modal formula. Various cognitive processes activate empirical material adequate for the description of object $o$. One can distinguish between two situations: where object was observed to exhibit $P$ ($o \in P^+$) and where it didn't exhibit $P$ ($o \in P^-$). Cognitive processes result in activation of two complementary mental representations. Some parts of them can emerge in the conscious area while most of them stay in the unconscious part of agent's mind. One can say that observations in the conscious area provide explicit examples of $o$ (not) exhibiting $P$. One is aware of explicit observations as he can point and describe them. Other observations (in the unconscious area) stay hidden from direct introspection but are felt and influence agent's state of mind.

### 2.4.1. The cognitive state

At each time moment agent is in some cognitive state where some part of knowledge is activated to introspection. This structure is modelled in the grounding theory as a partition of empirical material (definition 2.7).

**Definition 2.7.** *At each time point $t \in T$ agent is in a cognitive state whose contents are modelled by a binary partition of the set $KS(t)$:*

$$MS(t) = \{\overline{MS}(t), \underline{MS}(t)\}$$

*where $\overline{MS}(t) \cup \underline{MS}(t) = KS(t)$ and $\overline{MS}(t) \cap \underline{MS}(t) = \emptyset$.*

Empirical material is partitioned into two areas: conscious $\overline{MS}(t)$ and unconscious area $\underline{MS}(t)$ forming a two level structure. The distribution to conscious and unconscious parts depends on agent's mental capabilities. For humans the division results from focussing on some phenomenon, where most adequate empirical material plays the key role in situation's evaluation, while the rest of it stays deep in mind but is internally felt. This feeling results in an awareness level influencing the choice of mental representations and defining the states of mind and further cognitive processes.

In computer systems this distribution may be understood as a division into thoroughly processed data and data partially processed or awaiting to be processed. Katarzyniak in his book also called these areas using technical terms: working memory and permanent memory. Where working memory is meant to contain data currently being processed. Such approach can be aligned with fundamental assumptions of mind models from non-technical literature (Paivio 1990; Freeman 1999, 2000). For details please refer to (Katarzyniak 2007). The division of empirical material into two parts is also much consistent with theory of mental models and possibilities (Johnson-Laird and Savary 1999). Johnson proposed that some models should be divided into two types: explicit and implicit. His explicit models can be treated as residing in conscious area and implicit models reside in unconscious area.

Proposed cognitive state model is obviously a cruel simplification of real human cognitive states. This model mirrors only the most crucial properties of the cognitive state. The properties that are important in the context of the grounding process.

### 2.4.2. The grounding sets

The final division between conscious and unconscious levels of awareness depends not only on mental capabilities but also on agent's point of focus. This point of focus includes considered context and in result also considered utterances. If agent focuses on property $P$, the resulting cognitive state shall be different than when it considers property $Q$. Mental representation of object $o$ (not) being $P$ is a part of mental model associated to internal reflections where $o$ and its $P$ were known. Mental representation of two properties of some object consists of internal reflections where both of these properties were known. There can be more than two complementary mental representations.

Cognitive state model proposed by definition 2.7 can be further partitioned into complementary representations according to valuations of the properties. For one property $P$ the model can be divided into two sets. First set contains empirical material with observations where object $o$ exhibited property $P$. These observations support a statement $p(o)$. The second set contains empirical material where object $o$ did not exhibit property $P$ and it supports statement $\neg p(o)$. These two sets have been visualized on figure 2.2 by two ovals, the first is filled with lines and the second one is filled with dots. These two sets represent two competing representations of object $o$ (not) exhibiting $P$.

Within the grounding theory these two representations are called *grounding sets* and contain empirical material for respective valuations of properties.

**Definition 2.8.** *Grounding sets associated with property $P \in \mathcal{P}$ of object $o \in O$ define a division of the grounding material into two mutually disjoint sets according to valuations of the property $P$:*

$$\begin{aligned} C^{p(o)}(t) =& \{BP(\hat{t}) \in KS(t) : o \in P^{+}(\hat{t})\} \\ C^{\neg p(o)}(t) =& \{BP(\hat{t}) \in KS(t) : o \in P^{-}(\hat{t})\} \end{aligned}$$

*where $\hat{t} \in \mathcal{T}$ denotes any time moment $\hat{t} \leqslant t$.*

Grounding sets presented in definition 2.8 are divided according to one property and object. These sets formally model mental representation of object $o$ (not) exhibiting $P$. Each of them supports one of two competing formulas: $p(o)$ and $\neg p(o)$. Such grounding sets are constructed for simple modal formulas such as: $Pos(p(o))$, $Bel(p(o))$, $Know(p(o))$ and $Pos(\neg p(o))$, $Bel(\neg p(o))$, $Know(\neg p(o))$.

The agent can also simultaneously consider two properties $P$ and $Q$ of the object $o$. In this case the grounding material covers observations where both of these properties have been observed. It can be divided into four mutually exclusive sets according to valuations: $o \in P^{+} \cap Q^{+}$, $o \in P^{+} \cap Q^{-}$, $o \in P^{-} \cap Q^{+}$ and $o \in P^{-} \cap Q^{-}$. These sets have been presented in definition 2.9. In such case each set supports one of four competing formulas: $p(o) \wedge q(o)$, $p(o) \wedge \neg q(o)$, $\neg p(o) \wedge q(o)$ and $\neg p(o) \wedge \neg q(o)$.

**Definition 2.9.** *Grounding sets associated with a pair of properties $P, Q \in \mathcal{P}$ of object $o \in O$ define a division of the grounding material into four mutually disjoint sets according to valuations of the properties $P$ and $Q$:*

$$\begin{aligned} C^{p(o)\wedge q(o)}(t) =& \{BP(\hat{t}) \in KS(t) : o \in P^{+}(\hat{t}) \wedge o \in Q^{+}(\hat{t})\} \\ C^{p(o)\wedge \neg q(o)}(t) =& \{BP(\hat{t}) \in KS(t) : o \in P^{+}(\hat{t}) \wedge o \in Q^{-}(\hat{t})\} \\ C^{\neg p(o)\wedge q(o)}(t) =& \{BP(\hat{t}) \in KS(t) : o \in P^{-}(\hat{t}) \wedge o \in Q^{+}(\hat{t})\} \\ C^{\neg p(o)\wedge \neg q(o)}(t) =& \{BP(\hat{t}) \in KS(t) : o \in P^{-}(\hat{t}) \wedge o \in Q^{-}(\hat{t})\} \end{aligned}$$

*where $\hat{t} \in \mathcal{T}$ denotes any time moment $\hat{t} \leqslant t$.*

Only a proper distribution of the grounding material between the two (or four) sets allows for grounding of particular simple (or complex) modal formula. Intuitively the more material in set $C^{p(o)}(t)$, the more willing we are to accept a statement $Bel(p(o))$ or even a statement $Know(p(o))$.

The grounding sets can be later divided according to conscious and unconscious areas of the cognitive state $MS(t)$. Such subsets shall be denoted with additional upper and lower lines respectively:

Let $\phi \in \{p(o), \neg p(o)\}, \psi \in \{q(o), \neg q(o)\}$:

$$\overline{C}^{\phi} = C^{\phi} \cap \overline{MS}(t),\ \underline{C}^{\phi} = C^{\phi} \cap \underline{MS}(t),\ C^{\phi} = \overline{C}^{\phi} \cup \underline{C}^{\phi} \tag{2.1}$$

$$\overline{C}^{\phi\wedge\psi} = C^{\phi\wedge\psi} \cap \overline{MS}(t),\ \underline{C}^{\phi\wedge\psi} = C^{\phi\wedge\psi} \cap \underline{MS}(t),\ C^{\phi\wedge\psi} = \overline{C}^{\phi\wedge\psi} \cup \underline{C}^{\phi\wedge\psi} \tag{2.2}$$

According to the provided definitions of the cognitive state and the grounding sets, it may seem that a modal formula enforces construction of the grounding material. Such situation would take place if the agent was treated as a listener. In the grounding theory the agent plays the role of the speaker and the grounding material precedes the formula. Only if agent's cognitive state and grounding sets form a proper distribution, a particular formula can be grounded (not the other way).

A more sophisticated agent could (and should) simultaneously construct many grounding sets associated to various modal formulas on different properties and objects. Such a construction is outside the scope of the grounding theory but it has been employed in various works utilizing it (see (Skorupa and Katarzyniak 2012; Skorupa et al. 2012; Popek 2012) for examples).

## 2.5. Grounding of modal formulas

The key element modelled by the grounding theory is the edge A-B between the cognitive state and the modal formula of the semiotic triangle. A series of constraints imposed on agent's cognitive state need to be met in order to sustain intuitive semantics of a modal formula. These constraints have been included in *epistemic satisfaction relations*. Given modal formula is acknowledged to be properly grounded only if its epistemic relation holds. If this relation does not hold, link A-B is not constructed and formula can't be grounded.

In this section epistemic relations for simple modal formulas and modal conjunctions are presented. For epistemic relations for other types of formulas (alternatives and exclusive alternatives) please refer to (Katarzyniak 2007).

There are two theoretical concepts associated with epistemic relations that need introduction. First of them is the *relative grounding strength*. It is a measure over the grounding strengths that models agents certainty level. The second concept is related to the choice of particular modal operator. This choice is modelled with the use of *grounding thresholds*, being numeric parameters defining certainty intervals related to respective modal operators.

### 2.5.1. The grounding strengths

Grounding of a modal formula depends on the distribution of grounding material divided between the grounding sets. Dennett proposed that: 'multiple exposure to $x$ - that is, sensory confrontation with $x$ over suitable period of time - is the normally sufficient condition for knowing (or having true beliefs) about $x$' (Dennett 1997). Intuitively the more numerous $C^{p(o)}$ is, the more willing the agent to utter $Bel(p(o))$ or even $Know(p(o))$. This simple idea incorporates common-sense requirements for uttering different levels of certainty, i.e. possibility, belief and knowledge. The grounding theory defines a measure over the grounding sets, called the *relative grounding strength* to numerically express the influence of the grounding material on agent's certainty levels (definitions 2.10 and 2.11).

**Definition 2.10.** *Relative grounding strength at time moment $t \in \mathcal{T}$ for a simple formula $p(o)$ is calculated as:*

$$\lambda^{p(o)}(t) = \frac{\mathrm{card}(C^{p(o)}(t))}{\mathrm{card}(C^{p(o)}(t) \cup C^{\neg p(o)}(t))}$$

The relative grounding strength takes values from the interval $[0, 1]$. When the grounding strength is 0, there were no observations where $o$ exhibited $P$. When the grounding strength is equal to 1, $o$ exhibited $P$ in all past observations of the property.

For a conjunction $p(o) \wedge q(o)$ the grounding material is divided into four grounding sets and the grounding strength is defined as:

**Definition 2.11.** *Relative grounding strength at time moment $t \in \mathcal{T}$ for a conjunction $p(o) \wedge q(o)$ is calculated as:*

$$\lambda^{p(o)\wedge q(o)}(t) = \frac{\mathrm{card}(C^{p(o)\wedge q(o)}(t))}{\mathrm{card}(C^{p(o)\wedge q(o)}(t) \cup C^{p(o)\wedge \neg q(o)}(t) \cup C^{\neg p(o)\wedge q(o)}(t) \cup C^{\neg p(o)\wedge \neg q(o)}(t))}$$

Grounding strengths for various negations of properties, such as $\lambda^{\neg p(o)}(t)$ or $\lambda^{p(o)\wedge\neg q(o)}(t)$, are calculated symmetrically to the ones provided in definitions 2.10 and 2.11. One should respectively change negations of properties.

### 2.5.2. The grounding thresholds

The grounding strength measures the distribution of the grounding material between complementary competing experiences. In this sense the grounding strength $\lambda^{p(o)}(t)$ measures agent's certainty level of $o$ exhibiting $P$. Similarly the grounding strength $\lambda^{p(o)\wedge q(o)}(t)$ measures agent's conviction in $o$ exhibiting $P$ and $Q$. Some values of the grounding strength can be associated with particular modal operators of the possibility and the belief. The choice of the modal operator is constrained by the grounding thresholds. The grounding thresholds are parameters in form of real numbers from interval $[0, 1]$. For simple modal formulas the grounding thresholds are defined as:

$$0 \leqslant \lambda^b_{minPos} < \lambda^b_{maxPos} \leqslant \lambda^b_{minBel} < \lambda^b_{maxBel} \leqslant 1 \text{ [4]} \tag{2.3}$$

[4] In the grounding theory $0 < \lambda^b_{minPos}$ was proposed. It has been later suggested to change it to $0 \leqslant \lambda^b_{minPos}$, so that $\lambda^b_{minPos} = 0$ can be chosen.

and for the modal conjunctions they are similarly defined as:

$$0 \leqslant \lambda^{\wedge}_{minPos} < \lambda^{\wedge}_{maxPos} \leqslant \lambda^{\wedge}_{minBel} < \lambda^{\wedge}_{maxBel} \leqslant 1 \ ^{5} \tag{2.4}$$

The modal operator of possibility ($Pos$) is associated with grounding thresholds $\lambda^{b}_{minPos}$, $\lambda^{b}_{maxPos}$ ($\lambda^{\wedge}_{minPos}$, $\lambda^{\wedge}_{maxPos}$ for conjunctions). The modal operator of belief ($Bel$) is associated with the grounding thresholds $\lambda^{b}_{minBel}$, $\lambda^{b}_{maxBel}$ ($\lambda^{\wedge}_{minBel}$, $\lambda^{\wedge}_{maxBel}$ for conjunctions). When the grounding strength is in interval: $\lambda^{b}_{minPos} < \lambda^{p(o)} < \lambda^{b}_{maxPos}$ the modal operator of possibility can be chosen. Similarly for the modal operator of belief and the grounding thresholds $\lambda^{b}_{minBel}$, $\lambda^{b}_{maxBel}$. Intuitively values of $\lambda^{p(o)}(t)$ close to zero should allow for grounding of $Pos(p(o))$. Values close to one, should result in the grounding of $Bel(p(o))$. The modal operator of possibility should be grounded for lower values of the grounding strength $\lambda^{p(o)}(t)$. Additionally it has been assumed that at most one of modal operators should be simultaneously grounded. Agent should be unable to concurrently ground two statements like $Pos(p(o))$ and $Bel(p(o))$. That is why given inequalities between the grounding thresholds were proposed.

The final choice of the grounding thresholds can be done by an expert. In the grounding theory it has been proposed that these thresholds can be learned from social interactions made by agents and hopefully humans. The final choice of the exact values should be a result of semiosis.

Constraints provided by inequalities 2.3 and 2.4 are only initial. As it is later shown these constraints are not strict enough to ensure meeting of common-sense postulates on the grounding process.

### 2.5.3. Epistemic relations

Epistemic relations are the key components of the grounding theory. These relations bind agent's cognitive state with respective modal formulas. Each formula type has its own epistemic relation. Only if the epistemic relation holds, the formula can be grounded. Otherwise agent's cognitive state can not be properly described with the given formula. Epistemic relations for simple modal formulas are defined as:

**Definition 2.12.** *Let $t \in \mathcal{T}$ and cognitive state $MS(t)$ be given. For every property $P \in \mathcal{P}$ and object $o \in O$:*

— *Epistemic relation $MS(t) \models^{\mathrm{E}} Pos(p(o))$ holds iff*

$$o \in P^{\pm}(t) \ \wedge \ \overline{C}^{p(o)}(t) \neq \emptyset \ \wedge \ \lambda^{b}_{minPos} < \lambda^{p(o)}(t) < \lambda^{b}_{maxPos} \ ^{6}$$

[5] In the grounding theory $0 < \lambda^{\wedge}_{minPos}$ was proposed. It has been later suggested to change it to $0 \leqslant \lambda^{\wedge}_{minPos}$, so that $\lambda^{\wedge}_{minPos} = 0$ can be chosen.

[6] Previously in (Katarzyniak 2007) the lower bound $\lambda^{b}_{minPos} < \lambda^{p(o)}(t)$ has been set to $\lambda^{b}_{minPos} \leqslant \lambda^{p(o)}(t)$. It has been later suggested to change it to strict inequality.

— *Epistemic relation* $MS(t) \models^{\mathrm{E}} Bel(p(o))$ *holds iff*

$$o \in P^{\pm}(t) \;\wedge\; \overline{C}^{p(o)}(t) \neq \emptyset \;\wedge\; \lambda^{b}_{minBel} \leqslant \lambda^{p(o)}(t) < \lambda^{b}_{maxBel}$$

— *Epistemic relations* $MS(t) \models^{\mathrm{E}} Know(p(o))$ *and* $MS(t) \models^{\mathrm{E}} p(o)$ *hold iff*

$$\textit{either } o \in P^{+}(t) \quad \textit{or} \quad o \in P^{\pm}(t) \;\wedge\; \overline{C}^{p(o)}(t) \neq \emptyset \;\wedge\; \lambda^{p(o)}(t) = 1$$

*where* $\lambda^{b}_{minPos}, \lambda^{b}_{maxPos}, \lambda^{b}_{minBel}, \lambda^{b}_{maxBel}$ *are fixed parameters called the grounding thresholds.*

The epistemic relations for simple modal formulas have been constructed from the perspective of the agent as a message source. Although it has not been directly stated in the grounding theory, the epistemic relations ground statements that speak about the current time moment. They describe the currently observed situation. The conditions $o \in P^{\pm}(t)$ refer directly to the time moment $t$ and clearly suggest that statements $Pos(p(o))$ or $Bel(p(o))$ are describing currently observed situation.

The proposed form of epistemic relations already suggests some fundamental intuitive meaning of the grounded formulas. Statement $Pos(p(o))$ can be grounded if three conditions hold. Firstly, the agent can not currently observe the object $o$ to measure the property $P$. This is ensured by the condition $o \in P^{\pm}(t)$. In such state the agent must refer to its past empirical experiences. The agent must recall past observations of $o$ exhibiting $P$. This is modelled by the condition $\overline{C}^{p(o)}(t) \neq \emptyset$. Finally the agent has to refer (partially unconsciously) to its empirical material to measure the state of conviction into $o$ exhibiting $P$. Only if there were enough observations of $o$ (not) exhibiting $P$ in the past, the agent can use the possibility modal operator. This is ensured by the condition $\lambda^{b}_{minPos} < \lambda^{p(o)}(t) < \lambda^{b}_{maxPos}$.

Constraints $o \in P^{\pm}(t)$ and $\lambda^{p(o)}(t) < \lambda^{b}_{maxPos}$ directly suggest a physical meaning of the possibility operator. This meaning is different from the logical meaning (as in Kripke semantics). In the logical meaning one can say $p(o)$ is possible even if he directly knows that $p(o)$ holds. Such interpretation is not intuitive as one is misleading the listener. Possibility operator, in its physical interpretation, is allowed only when one is uncertain of the factual state of the environment. It is not true in the grounding theory that $Know(p(o))$ implies also $Pos(p(o))$. In fact the opposite is true. A more accurate state of knowledge grounding $Know(p(o))$, excludes less accurate state grounding $Pos(p(o))$.

The belief operator posesses interpretation similar to interpretation of the possibity operator. For belief modal operator the agent requires that $o$ has exhibited $P$ more often in the past. This follows from inequalities 2.3 and the condition $\lambda^{b}_{minBel} \leqslant \lambda^{p(o)}(t) < \lambda^{b}_{maxBel}$.

Formulas $p(o)$ and $Know(p(o))$ have the same grounding conditions and as such are equivalent in the grounding theory. They are equivalent according to assumed interpretation where only agent's cognitive state and knowledge is considered. Statement '$o$ exhibits $p$', stated by the agent, denotes that she knows $p(o)$ holds. Formula $Know(p(o))$

can be grounded in two separate cases. Either the agent has directly observed $o$ to exhibit $P$ at the moment $t$ or in the past it has always been that $o$ exhibited $P$. In the first case, where condition $o \in P^{+}(t)$ is met, there is no need to refer to past observations.

The grounding theory defines similar epistemic relations for modal conjunctions:

**Definition 2.13.** *Let $t \in \mathcal{T}$ and cognitive state $MS(t)$ be given. For every property pair $P, Q \in \mathcal{P}$, $P \neq Q$ and object $o \in O$:*

— *Epistemic relation $MS(t) \models^{\mathrm{E}} Pos(p(o) \wedge q(o))$ holds iff:*

$$o \in P^{\pm}(t) \wedge o \in Q^{\pm}(t) \ \wedge \ \overline{C}^{p(o)\wedge q(o)}(t) \neq \emptyset \ \wedge \\ \lambda^{\wedge}_{minPos} < \lambda^{p(o)\wedge q(o)}(t) < \lambda^{\wedge}_{maxPos} \ {}^{7}$$

— *Epistemic relation $MS(t) \models^{\mathrm{E}} Bel(p(o) \wedge q(o))$ holds iff:*

$$o \in P^{\pm}(t) \wedge o \in Q^{\pm}(t) \ \wedge \ \overline{C}^{p(o)\wedge q(o)}(t) \neq \emptyset \ \wedge \\ \lambda^{\wedge}_{minBel} \leqslant \lambda^{p(o)\wedge q(o)}(t) < \lambda^{\wedge}_{maxBel}$$

— *Epistemic relations $MS(t) \models^{\mathrm{E}} Know(p(o) \wedge q(o))$ and $MS(t) \models^{\mathrm{E}} p(o) \wedge q(o)$ hold iff:*

$$\textit{either } o \in P^{+}(t) \wedge o \in Q^{+}(t) \\ \textit{or } \ o \in P^{\pm}(t) \wedge o \in Q^{\pm}(t) \ \wedge \ \overline{C}^{p(o)\wedge q(o)}(t) \neq \emptyset \ \wedge \\ \lambda^{p(o)\wedge q(o)}(t) = 1$$

*where $\lambda^{\wedge}_{minPos}, \lambda^{\wedge}_{maxPos}, \lambda^{\wedge}_{minBel}, \lambda^{\wedge}_{maxBel}$ are fixed parameters called the grounding thresholds.*

Interpretation of possibility and belief operators in modal conjunctions stays the same as for simple modal statements. The agent can ground the formula $Pos(p(o) \wedge q(o))$ only if she has not observed any of the properties. If at least one of them was observed the agent must resign from the usage of the possibility and beliefs operators.

To ground formulas with various negations such as $Know(\neg p(o))$ or $Pos(\neg p(o) \wedge q(o))$ one should symmetrically apply negations to conditions of definitions 2.12 and 2.13.

For other types of modal formulas (alternatives and exclusive alternatives) please refer to (Katarzyniak 2007).

### 2.5.4. Common-sense constraints

The epistemic relations form a formal system of requirements based on the grounding strengths and thresholds. Definitions themselves impose preliminary constraints on the grounding of modal formulas but do not ensure rational behaviour. For example for a purposely bad setting of the grounding thresholds such that $\lambda^{b}_{minBel} = 0.3$ and $\lambda^{b}_{maxBel} =$

0.9 it is possible to simultaneously ground two formulas: $Bel(p(o))$ and $Bel(\neg p(o))$. Such simultaneous grounding should be disallowed as the two formulas conflict each other according to our intuitive understanding of belief.

The grounding thresholds need to meet a series of constraints to provide common-sense behaviour of the grounding theory. Katarzyniak proposed such constraints in form of formal theorems. These theorems define what settings of the grounding thresholds allow (or disallow) concurrent grounding of given subsets of modal formulas. Exemplary theorems for simple modal formulas are:

**Theorem 2.1.** *If epistemic relation $MS(t) \models^{\mathrm{E}} Pos(p(o))$ is met, then $MS(t) \models^{\mathrm{E}} Bel(p(o))$ is not met.*

**Theorem 2.2.** *A necessary condition to allow for simultaneous grounding two statements $Pos(p(o))$ and $Pos(\neg p(o))$ in the same cognitive state $MS(t)$ is:*

$$\lambda^b_{minPos} < 0.5 < \lambda^b_{maxPos}$$

Exemplary theorems for modal conjunctions are:

**Theorem 2.3.** *Let $\alpha, \beta \in \{p(o) \wedge q(o), p(o) \wedge \neg q(o), \neg p(o) \wedge q(o), \neg p(o) \wedge \neg q(o)\}$ and $\alpha \neq \beta$. A necessary condition to disallow of simultaneous grounding of two formulas $Bel(\alpha)$ and $Bel(\beta)$ is:*

$$0.5 < \lambda_{minBel} \quad or \quad \lambda_{maxBel} < 0.5$$

**Theorem 2.4.** *Let $\alpha, \beta, \gamma, \delta \in \{p(o) \wedge q(o), p(o) \wedge \neg q(o), \neg p(o) \wedge q(o), \neg p(o) \wedge \neg q(o)\}$ be four different conjunctions. It is possible to set grounding thresholds, so that all four formulas: $Bel(\alpha), Pos(\beta), Pos(\gamma), Pos(\delta)$ can be simultaneously met in the same cognitive state.*

For formal proofs and more theorems please refer to (Katarzyniak 2007). All such theorems have been formulated with respect to common-sense and intuitive natural language understanding of modal formulas according to previously assumed interpretations. Many of theorems provide additional constraints on possible settings of the grounding thresholds. See 2.2 and 2.3 for two examples. Such constraints from all theorems for simple modalities can be joined together. One obtains inequality 2.5.

$$0 < \lambda^b_{minPos} \leqslant 0.5 \leqslant \lambda^b_{maxPos} \leqslant \lambda^b_{minBel} < 1 - \lambda^b_{minPos} \leqslant \lambda^b_{maxBel} \leqslant 1 \tag{2.5}$$

Similarly all theorems for modal conjunctions lead to one set of constraints given by inequality 2.6.

$$0 < \lambda^{\wedge}_{minPos} < \frac{1}{6} < 0.5 < \lambda^{\wedge}_{maxPos} \leqslant \lambda^{\wedge}_{minBel} < 1 - 3\lambda^{\wedge}_{minPos} \leqslant \lambda^{\wedge}_{maxBel} \leqslant 1 \tag{2.6}$$

Any setting of the grounding thresholds meeting those constraints also meets all properties described in a series of theorems. This implies that all common-sense postulates can be met by the grounding theory.

## 2.6. Computational example

To visualise the work-flow of the grounding theory let me introduce a simple computational example.

Let the environment consist of three perceptually recognizable objects $O = \{o_1, o_2, o_3\}$ and three properties $\mathcal{P} = \{P_1, P_2, P_3\}$. Language symbols associated with the properties are respectively $\Gamma = \{p_1, p_2, p_3\}$.

Agent is situated in this environment and observes it. Observations gathered by the agent up to time moment $t = 6$ are encapsulated in 6 base profiles $KS(6) = \{BP(1), BP(2), ..., BP(6)\}$ whose contents have been presented in table 2.2. For example at time moment $\hat{t} = 4$ agent has observed that objects $o_1$ and $o_3$ do not exhibit property $P_1$ and object $o_2$ exhibits $P_2$ and does not exhibit $P_3$. For the current time moment $t = 6$ the agent has observed $o_1$ does not exhibit $P_3$ and $o_3$ exhibits $P_1$ and $P_3$. The agent has not observed whether $o_1$ exhibits $P_1$ or $P_2$ and has not observed any of the properties of $o_2$. Formally the current base profile can be written as:

$$
\begin{aligned}
BP(6) =< O, P_1^+(6), P_1^-(6), P_2^+(6), P_2^-(6), P_3^+(6), P_3^-(6) > \\
O = \{o_1, o_2, o_3\}, \\
P_1^+(6) = \{o_3\}, \quad P_1^-(6) = \emptyset, \\
P_2^+(6) = \emptyset, \quad P_2^-(6) = \emptyset, \\
P_3^+(6) = \{o_3\}, \quad P_3^-(6) = \{o_1\}.
\end{aligned}
$$

Table 2.2. Agent's knowledge state up to current time moment $t = 6$

| $\hat{t}$ | $P_1^+$ | $P_1^-$ | $P_2^+$ | $P_2^-$ | $P_3^+$ | $P_3^-$ |
|---|---|---|---|---|---|---|
| 6 | $o_3$ | | | | $o_3$ | $o_1$ |
| 5 | $o_1, o_3$ | $o_2$ | | $o_2$ | $o_2$ | |
| 4 | | $o_1, o_3$ | $o_2$ | | | $o_2$ |
| 3 | $o_1, o_3$ | $o_2$ | | $o_2$ | $o_2$ | $o_1$ |
| 2 | $o_1, o_3$ | $o_2$ | $o_2$ | $o_1$ | | $o_3$ |
| 1 | $o_1$ | $o_2$ | | $o_3$ | $o_3$ | $o_1$ |

Agent's knowledge is distributed between conscious and unconscious areas (working and permanent memory). The division is a result of various cognitive processes. Let us assume agent's cognitive state (definition 2.7) is set to:

$$
\begin{aligned}
\overline{MS}(6) = \{BP(6), BP(5), BP(4)\} \\
\underline{MS}(6) = \{BP(3), BP(2), BP(1)\}
\end{aligned}
$$

Where $\overline{MS}(6)$ is the conscious and $\underline{MS}(6)$ is the unconscious area. Such division can result from focusing on last observations.

Let grounding thresholds be set to:

$$\lambda^b_{minPos} = \lambda^\wedge_{minPos} = 0.1, \quad \lambda^b_{maxPos} = \lambda^\wedge_{maxPos} = 0.6,$$
$$\lambda^b_{minBel} = \lambda^\wedge_{minBel} = 0.6, \quad \lambda^b_{maxBel} = \lambda^\wedge_{maxBel} = 1.$$

Such setting can result from social semiosis. This setting meets constraints 2.5 and 2.6. Meeting of common-sense postulates is analytically ensured.

Now let us focus on exemplary grounding sets and strengths that can be obtained from agent's knowledge. These sets and strengths are further used to ground modal formulas with the use of epistemic relations. Grounded formulas describe current time moment according to agent's empirical knowledge.

For a base formula $p_1(o_1)$ the grounding sets are:

$$C^{p_1(o_1)}(6) = \{BP(5), BP(3), BP(2)\}, \quad C^{\neg p_1(o_1)}(6) = \{BP(4)\}$$

the grounding strength, according to definition 2.10, is $\lambda^{p_1(o_1)}(6) = \frac{3}{4}$. Similarly for $\neg p_1(o_1)$ the grounding strength is $\lambda^{\neg p_1(o_1)}(6) = \frac{1}{4}$.

All conditions for the epistemic relation for $Bel(p_1(o_1))$ hold (definition 2.12). Similarly all conditions for $Pos(\neg p_1(o_1))$ hold. These two formulas can be simultaneously grounded. It means 'I believe $o_1$ exhibits $P_1$' and 'It is possible $o_1$ does not exhibit $P_1$' properly (intuitively) describe agent's knowledge state. Agent does not know whether $o_1$ currently exhibits $P_1$ as he has not observed it. Agent also thinks, that $o_1$ rather exhibits $P_1$. This knowledge comes from previous experiences.

For object $o_2$ and property $P_2$, the grounding sets are:

$$C^{p_2(o_2)}(6) = \{BP(4), BP(2)\}, \quad C^{\neg p_2(o_2)}(6) = \{BP(5), BP(3)\}$$

and the grounding strengths are $\lambda^{p_2(o_2)}(6) = \lambda^{\neg p_2(o_2)}(6) = 0.5$. In such a case both formulas $Pos(p_2(o_2))$ and $Pos(\neg p_2(o_2))$ can be grounded.

The agent has observed that at the current time moment $t = 6$ object $o_3$ exhibits $P_3$. Condition $o \in P_3^+(6)$ holds and both formulas $p_3(o_3)$, $Know(p_3(o_3))$ can be grounded. They can be grounded regardless of the distribution of the grounding material between the grounding sets.

For an exemplary modal conjunction let us consider object $o_2$ and its properties $P_2$ and $P_3$. The grounding sets are:

$$C^{p_2(o_2)\wedge p_3(o_2)}(6) = \emptyset, \quad C^{p_2(o_2)\wedge\neg p_3(o_2)}(6) = \{BP(4)\},$$
$$C^{\neg p_2(o_2)\wedge p_3(o_2)}(6) = \{BP(5), BP(3)\}, \quad C^{\neg p_2(o_2)\wedge\neg p_3(o_2)}(6) = \emptyset$$

Other observations $BP(6), BP(2), BP(1)$ do not participate in the grounding process, as at least one of the properties was unknown to the agent. For such a setting formulas $Bel(\neg p_2(o_2) \wedge p_3(o_2))$, $Pos(p_2(o_2) \wedge \neg p_3(o_2))$ can be grounded and formulas $Pos(p_2(o_2) \wedge p_3(o_2))$, $Pos(\neg p_2(o_2) \wedge \neg p_3(o_2))$ can NOT be grounded.

All formulas that can be grounded intuitively describe agent's knowledge about the current time moment.

## 2.7. Summary

Presented grounding theory provides formal criteria for grounding of a given class of modal formulas. The grounding process is modelled with respect to the semiotic triangle. The definitions of the empirical knowledge base, the formal language of modal formulas and the cognitive state model the three vertices of the semiotic triangle. To model all the vertices some necessary simplifications had to be made. The system of atomic objects and their binary properties has been formulated for the empirical knowledge base. As for the mental model, a two layer representation of the cognitive state partitioned into the grounding sets has been proposed. It has been assumed that the mental representation results from various cognitive processes that themselves are outside the scope of the grounding theory. Proposed representations simplify perceptive and cognitive abilities of living beings such as humans but they contain all the features necessary for the grounding of modal formulas.

The epistemic relations (definitions 2.12, 2.13), together with the formal constraints on the grounding thresholds (equations 2.5, 2.6) define the grounding conditions for simple modal formulas and modal conjunctions. These conditions ensure common-sense, intuitive, conventional understanding of respective natural language sentences is sustained. The meaning of sentences is constructed with respect to agent's subjective knowledge and treating the agent as a potential message source.

In the grounding theory, the grounding process ends when the sign in form of a modal formula is grounded in the mind of the agent. Many formulas can be simultaneously grounded forming mental signs. Some of these signs can, but do not have to be later externalised (uttered). The grounding theory does not define which signs should be uttered. This task depends on agent's and listener's aims or needs.

Book (Katarzyniak 2007) contains numerous examples utilising the grounding theory in implementations of BDI agents. Author proposed a few contextualization techniques and methods for choosing the most informative utterance out of a set of properly grounded formulas. The grounding theory has been also extended in various directions and utilized in some tasks (Lorkiewicz et al. 2011; Skorupa et al. 2012; Popek 2012). Some minor improvements and changes have also been suggested (Lorkiewicz et al. 2012).

This thesis extends the grounding theory by addition of a new type of formulas in form of indicative conditionals with modal operators. Fundamental building blocks of the grounding theory are directly transferred to the constructed extension. Only a minor change to the cognitive state shall be proposed to allow for more flexible contextualization of the grounded sentences.

To allow for grounding of conditionals, a formal language has to be extended with adequate modal formulas. Conventional semantics of conditional sentences and their modal extensions must be defined. Furthermore formal constraints resulting from conventional usage patterns and denotations of conditionals need to be formulated. Finally epistemic relations must be proposed.

The next two chapters concentrate on conditional sentences and their meaning. Some of well known theories on conditionals are briefly discussed. Formal extension to the language is defined in section 3.2. Epistemic relations are proposed in three forms, differently restricting the meaning of conditionals, in chapter 6.

# 3. Modal conditional statements

Conditional statements are the statements of a form: “If $A$, then $B$”. The $A$ phrase is called a cause (an antecedent) and the $B$ phrase is an effect (a consequent). Provided definition is a simplification, please refer to (Bennett 2003) for a broad discussion on the definition of conditionals. Informally, one may say the antecedent influences, or more strongly: causes, the consequent. Both the antecedent and the consequent may take different grammatical forms changing sentence meaning. For some simple examples:

— If she loves him, she will marry him.
— If she loved him, she would marry him.
— If she had loved him, she would have married him.

Conditional statements are broadly used in everyday language. Their meaning differs based on usage patterns, context, speaker’s mood etc. In order to maintain clarity and avoid misunderstandings of this technical work, I am forced to constrain conditionals meaning. This requires a short review of conditionals’ classifications and usage patterns.

## 3.1. On the classification of conditionals

There are many ways to classify conditionals. Furthermore there are many special types of conditionals and usage patterns. Conditionals may be used as claims, propositions, commands, offers, questions etc.. See (Pelc 1986; Jadacki 1986; Bennett 2003) for broad reviews.

The most common division is into indicative and subjunctive or counter-factual conditionals. Indicatives are the conditionals that refer to real plausible situations. Situations where the antecedent and the consequent are seen as factual possibilities. On the contrary the subjunctive conditionals refer to unrealistic situations that did not happen or are very unlikely to happen. Usually subjunctive conditionals have ‘would’ phrase in the consequent while indicatives do not. For example (from (Clark 1971)):

— ‘If it rains, the match will be cancelled.’ - is an indicative conditional
— ‘If it were to rain, the match would be cancelled.’ - is a subjunctive conditional

Within the first sentence a speaker thinks it may rain, because maybe it is windy and cloudy. In the second sentence a speaker does not think that a rain is a real possibility and in result he claims the match won’t be cancelled.

Woods (Woods 2003) claims the distinction between indicatives and subjunctives is not entirely correct and in my opinion he is right, but for our purposes the explanation presented above should be good enough.

Conditionals may be used for different purposes and to express many moods. There are conditional statements, claims, commands, questions ... similarly there are conditional beliefs, desires, fears ... (Edgington 2008).

Further, conditionals may be classified based on a relation type between an antecedent and a consequent. Examples of such distinction:

- Cause-effect relation: If your parachute does not open, you will die. (Pelc 1986)
- Symbol relation: If flags are left at half of a mast, the ruler is dead. (Pelc 1986)
- Common practice: If you are caught stealing, you will go to prison. (Pelc 1986)
- To mark a general relation understood as a formal material implication ($\forall_x F(x) \rightarrow G(x)$) (Pelc 1986)
- Structural relation: If today is Monday, tomorrow will be Tuesday.
- Analytical relation: If Alice is a mother of Bob, then Bob is a son of Alice.

The conditional statements may have 'then' adverb or not. Existence of 'then' marks a relation between the antecedent and the consequent (Davis 1983). Not every intuitively acceptable conditional without word 'then' is acceptable when 'then' is added. Consider: '(Even) if war breaks out tomorrow, the tides will (still) continue to rise and fall'. When we insert 'then', we obtain 'If war breaks out tomorrow, then the tides will continue to rise and fall' that suggests dependence between tides and war, which is absurd (Bennett 2003).

## 3.2. Formal language of conditionals

In the grounding theory natural language statements have been represented by formal language of formulas with fixed conventional semantics. To allow for grounding of conditionals this language needs to be extended. Adequate formulas representing conditional sentences with modal operators need to be added. At least intuitive semantics need to be provided. There are numerous usage patterns and different meanings conditionals can take. These patterns and meanings shall be limited to some 'typical' situations.

### 3.2.1. Language syntax

**Definition 3.1.** *Let the alphabet of the extended language $L'$ consist of all symbols of the language $L$ provided in definition 2.5 and additionally a symbol '$\rightarrow$' of a conditional statement.*

**Definition 3.2.** *Let the extended language $L'$ consist of all formulas of the language $L$ provided in definition 2.6 and additionally of three sets of formulas $L^{\rightarrow}, L^{\rightarrow M}, L^{M\rightarrow}$ for conditionals with modal operators:*

- *$L^{\rightarrow}$ is a set of all conditionals of the form: $\phi \rightarrow \psi$, where $\phi, \psi \in L^b$.*
- *$L^{\rightarrow M}$ is a set of all conditional modalities: $\phi \rightarrow Pos(\psi), \phi \rightarrow Bel(\psi), \phi \rightarrow Know(\psi)$, where $\phi, \psi \in L^b$.*
- *$L^{M\rightarrow}$ is a set of all modal conditionals: $Pos(\phi \rightarrow \psi), Bel(\phi \rightarrow \psi), Know(\phi \rightarrow \psi)$, where $\phi, \psi \in L^b$.*

*Any formula from set $L' = L \cup L^{\rightarrow} \cup L^{\rightarrow M} \cup L^{M\rightarrow}$ is a proper formula of the extended language $L'$. No other formula is allowed.*

Any formula of the form $\phi \rightarrow \Pi(\psi)$, where $\Pi \in \{Pos, Bel, Know\}$ is called a *conditional modality* and formula of the form $\Pi(\phi \rightarrow \psi)$ is called a *modal conditional.*

Defined language $L$ is not extensible. It means that formulas $Bel(p(o)) \rightarrow Pos(p(o))$, $p(o) \rightarrow Pos(p(o) \rightarrow q(o))$ are NOT proper.

Where it is clear, a shorthand notation: $p$ and $q$ to denote $p(o)$ and $q(o)$ shall be used. For example $Pos(p \rightarrow q)$ denotes $Pos(p(o) \rightarrow q(o))$ where object $o$ is assumed to be known and fixed.

### 3.2.2. Language semantics

In further chapter 4 I shall more broadly discuss on the meaning of conditional statements and their usage constraints. Here I refer to reader's intuitive understanding of conditionals. I focus mostly on assumed meaning according to previously proposed classifications (see section 3.1). I also impose technical restrictions on the structure of antecedent and the consequent and provide examples of usage patterns to constrain assumed semantics.

Table 3.1 presents intuitive semantics for formulas of language $L$. It is assumed, each statement is referring to exactly specified objects. Suppose agent utters formula $Pos(p(o) \rightarrow q(o))$. It should be understood as: "It is possible (from agent's perspective) that if object $o$ is $p$ (one fixed object), then $o$ is $q$ (one fixed object)". For example: "I believe that, if the apple is red, then it is ripe" is understood as referring to one fixed instance of the apple being now somewhere in the environment, not a general rule for all apples.

I assume formulas represent typical indicative conditionals. By typical I mean simple conditionals used in casual situations. I assume these are used without complex conversational context. Furthermore, conditionals are used to state some noticed fact or answer a simple question related to the consequent. I am considering only conditional statements (not questions, commands, etc.) joined with conditional beliefs.

Table 3.1. Formulas and their semantics ($\phi, \psi \in L^b$)

| formula | intuitive semantics |
|---|---|
| | conditional statement |
| $\phi \rightarrow \psi$ | If $\phi$, then $\psi$. |
| | conditional modalities |
| $\phi \rightarrow Pos(\psi)$ | If $\phi$, then it is possible that $\psi$. |
| $\phi \rightarrow Bel(\psi)$ | If $\phi$, then I believe that $\psi$. |
| $\phi \rightarrow Know(\psi)$ | If $\phi$, then I know that $\psi$. |
| | modal conditionals |
| $Pos(\phi \rightarrow \psi)$ | It is possible that if $\phi$, then $\psi$. |
| $Bel(\phi \rightarrow \psi)$ | I believe that if $\phi$, then $\psi$. |
| $Know(\phi \rightarrow \psi)$ | I know that if $\phi$, then $\psi$. |

I am also not considering any metaphorical or special types of conditionals, like for example so called Thomason conditionals[1].

Within all considered conditionals I assume 'then' adverb is present, so some kind of dependence between the antecedent and the consequent is required.

Modal operators *Pos*, *Bel* and *Know* understanding is consistent with their natural language interpretation (Hintikka 1962). These operators should not be mistakenly understood as formal operators of modal logic with their relational interpretation based on worlds (Kripke 1963).

Conditional statements and phrases 'It is possible that', 'I believe that', 'I know that' shall be evaluated from agent's perspective as an autonomous knowledge bearer and utterance source. Statement's 'truth' shall be evaluated based on agent's knowledge, not as an absolute, generally accepted, truth. I am studying, whether an agent can (or can't) utter a given statement basing only on its knowledge. What I am introducing here, is an *auto-epistemic* approach to statement's truth. I am trying to answer a question: If I were an agent and knew only what it knows, would I agree that given statements' usage is proper or not. In other words: would I (as an agent) feel allowed to utter given statement. I am evaluating subjective notion of truth, different than absolute truths or Kripke relational models.

Conditional modalities (statements of the form $p(o) \rightarrow \Pi(q(o))$) refer to agent's knowledge state, where:

— Agent thinks that $p(o)$ is possible (doesn't know that $\neg p(o)$).
— Agent is considering $q(o)$ in the context of $p(o)$.

[1] An example of a Thomason conditional is: 'If Sally is deceiving me, I do not believe it.' (Willer 2010).

— Agent doesn't explicitly know that $q(o)$ does not hold.
— Agent is aware of a similar situation where $p(o)$ holds and $q(o)$ does not - knows neglecting example (only for $Pos$ and $Bel$ operators).
— Agent evaluates a subjective chance of $q(o)$ holding, assuming $p(o)$ holds.

Knowledge state description is provided here only to clarify assumed semantics. Requirements on knowledge state shall be broadly discussed later.

To illustrate such a knowledge state, consider an example to establish assumed conventional meaning of a conditional modality.

**Example:**
A: If $P$, then it is possible that $Q$.
B: Why do you think so?
A: Because $Q$ sometimes holds, when $P$ holds. It happened that $P$ and $Q$ held. It also happened that $P$ held and $Q$ did not.
B: And if $P$ does not hold?
A: Then $Q$ also can't hold.

Speaker A utters conditional modality $p(o) \rightarrow Pos(q(o))$. When B asks about details, A reveals that $Q$ may happen assuming $P$. The uncertainty is applied to $Q$ happening (assuming $P$), not the conditional itself. A knows conditional $p(o) \rightarrow q(o)$ is not true, as he possesses example denying it. Speaker's mental state strictly points to the consequent as the uncertainty source. Assuming $P$, $Q$ may happen or not. $Q$ is related to $P$, but $P$ does not determine it. The example presents a typical situation where the conditional modality may be used and supplies the meaning of a statement we are assuming.

Modal conditionals ($\Pi(p(o) \rightarrow q(o))$) refer to agent's knowledge state, where:

— Agent thinks that $p(o)$ is possible (doesn't know that $\neg p$).
— Agent is considering $q(o)$ in the context of $p(o)$.
— Agent doesn't explicitly know that $q(o)$ does not hold.
— Agent is unaware of a similar situation where $p(o)$ holds and $q(o)$ does not - has not found a neglecting example yet.
— Agent is not sure (has some intuitive doubts) whether conditional between $p(o)$ and $q(o)$ holds (only for $Pos$ and $Bel$).

Knowledge state description is provided here only to clarify assumed semantics. Requirements on knowledge state shall be broadly discussed later.

Again we shall consider an example to illustrate such knowledge state and establish statement's meaning:

**Example:**
A: It is possible that if $P$, then $Q$.
B: Why do you think so?
A: Because in all situations I can currently remember, whenever was $P$, there was also $Q$.

B: So, why do you think it is only possible?
A: Because it seems to me, there may exist other situations where it isn't so.

Here A utters a modal conditional $Pos(p(o) \rightarrow q(o))$. Speaker A has no explicit examples denying the conditional itself but he feels something is not known yet. Something may still 'go wrong'. Statement $p(o) \rightarrow q(o)$ may be true, but A is unable to prove it yet, he is not convinced. Speaker A unconsciously feels that not all possibilities have been thoroughly analysed (at least not yet). Some empirical material influences his certainty level denying a stronger statement $Bel(p(o) \rightarrow q(o))$. This material possesses examples denying the conditional but is hidden somewhere in a unconscious knowledge. Material has not migrated to conscious area because, for example, A had no time to thoroughly consider it. Unconscious material may or may not be relevant to a conversational context. Speaker feels that, hence he is uncertain.

The remaining question is: Where does uncertainty applied to modal conditionals come from and how to model such 'intuition' in computer systems? Within this work we assume that gathered empirical knowledge supplies such uncertainty but does not supply an explicit example denying the conditional (at least not at the moment of the utterance). Such example has not been retrieved / discovered within knowledge base. This may happen due to: real-time requirements. Agent may also be unable to decide whether gathered empirical knowledge is suitable within currently considered context.

# 4. On some approaches to conditionals

Within this chapter I try to explain how conditionals are understood in natural language and when they are used. I explain why their meaning significantly differs from material implication. Some of the most known existing approaches are briefly discussed. I also mention which of conditional's usage aspects are analyzed.

## 4.1. Conditionals and material implication

The most known approach to taming conditionals is a material implication. Here we denote the material implication of predicates $p$ and $q$ by: $p \Rightarrow q$. Its semantics are defined by truth table 4.1. The only situation where material implication is false happens when an antecedent is true and a consequent is false. From all possible truth valuations of $p \Rightarrow q$, table 4.1 seems the only feasible candidate for conditionals. If truth conditions of conditional can be defined by truth valuations of the antecedent and the consequent, then this solution must be right (Edgington 2008). In such a form the material implication is equivalent to the alternative $\neg p \vee q$.

Table 4.1. Truth table of material implication. 1 means true, 0 means false

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Unfortunately, despite its undoubted importance in logical systems, the material implication does not model the meaning of conditionals well. Let us look at a few natural language examples:

— If the moon is a piece of cheese, then I can jump 100 meters high.
— If birds can fly, then Roosevelt was a president of United States.
— If I am a snail, then Earth is round.

All the statements above are true when treated as material implications, although no one would use such statements in typical, everyday situations. The problem comes from the fact that the falsity of an antecedent is enough for the material implication to be true. Similarly the truth of the consequent is enough for the material implication to be true. Statements being true as material implications are not necessarily acceptable as conditional statements.

Consider a statement from (Clark 1971): ‘If Hitler hadn’t committed suicide, Germany would have won the war’. This statement is true as a material implication but fails our understanding of a conditional. Committing suicide is simply not enough to win the war. Unfortunately almost any counter-factual conditional is true according to material implication. This happens simply because the antecedent is known to be false.

The problem gets even bigger when we try to analyze different grammatical forms of counter-factual conditionals. ‘If Oswald hadn’t killed Kennedy, somebody else would have’. seems at least weird, especially when compared to: ‘If Oswald didn’t kill Kennedy, somebody else did’ (Adams 1970). If we assume Oswald in fact killed Kennedy, both implications are true. But the former is clearly wrong and the latter seems good. The problem comes from different attitudes expressed within a conditional statement. A slight change in grammar may have huge impact on statements meaning. According to Pelc (Pelc 1986) it is impossible to put various conditional statements into one material implication. According to Ingarden (Ingarden 1949) (p. 312) different modis (potentialis, irrealis and realis) should not be omitted in logics. Speakers knowledge on the subject plays a crucial role in a decision whether a conditional is correct or wrong.

Defenders of the material implication as a proper model of a conditional statement eventually fail to strong arguments against it (Ingarden 1949; Pelc 1986; Jadacki 1986; Edgington 1995). Even if material implication models a conditional statement, it fails to model its proper usage patterns because subjective knowledge is ignored. Although material implication plays a key role in formal logics it simply does not model conditionals well.

## 4.2. Ramsey test

If the material implication is not a good model of a conditional statement, then what is? There are numerous works on that subject and it seems there is no perfect candidate. Going deeper into conditionals analysis leads to new unexpected problems. When it comes to understanding conditionals, probably the most discussed declaration is the Ramsey test (Ramsey 1929):

> “If two people are arguing ‘If $p$, will $q$?’ and are both in doubt as to $p$, they are adding $p$ hypothetically to their stock of knowledge, and arguing on that basis about $q$ ... We can say they are fixing their degrees of belief in $q$ given $p$.” based on (Bradley 2007)

Ramsey put this sentence in the footnote of his paper but it has had a significant influence on many of future works on conditionals. Statement probably describes the most important aspect of conditional sentences: assume that $p$ and argue whether $q$. This is a big step when compared to material implication. When the antecedent is false, conditionals truth should be evaluated by assuming the antecedent and arguing about the consequent. In such a way we may reason on statements like: "If moon is a piece of cheese, then I can fly." or "If moon is a piece of cheese, then we will be able to eat it.". The former is false, because assuming moon being cheese does not make me able to fly. The later is true, because at some time point one can go to the moon and eat its surface.

Unfortunately Ramsey test's contents have ambiguous meaning leading to a lot of interpretative problems, when going into technical details (Read and Edgington 1995; Lindström and Rabinowicz 1998; Bradley 2007).

Most discussed problems occur in the case of counter-factual conditionals. These problems are related to a way of adding $p$ to stock of knowledge. It is unclear how to change beliefs when $p$ is contradictory with it.

Secondly, Ramsey test does not explain how to 'argue about $q$'. Does the chance for $q$ have to change in any way when adding $p$? Can $q$ be guaranteed to hold regardless of $p$?

Ramsey test, although generally correct, is very unclear and does not explain all required aspects, even for the most simple conditionals. Firstly it is not clear how to add $p$ to stock of knowledge: Does it mean that $p$ can't already be in our beliefs? Can it be contradictory to our beliefs? Secondly: How do you argue about $q$, how do you fix your degree of belief? Is it enough for $q$ to have high probability? Thirdly: Don't situations where not $p$ matter?

## 4.3. Objectivity and conditionals

Defining conditionals truth conditions requires assumption that they have absolute truth valuation. It means that they can always be true or false regardless of the speaker and time. There are many arguments against treating conditionals as truth-functionals (a statements whose truth may be defined by truth of its compounds).

One of such arguments comes from Gibbard (Gibbard 1980). He claims that if two statements are consistent, a person may believe them both. For consistent $P$, and any $Q$, people do not simultaneously believe both 'If $P$, $Q$' and 'If $P$, $\neg Q$' (this is not in accordance with the material implication). Although it may happen that one person believes 'If $P$, $Q$' while the other 'If $P$, $\neg Q$'. They both are rational and basing their beliefs on known facts. Gibbard provided a Sly Pete story to support his example, but I shall use an example from Dorothy Edgington (Edgington 1995) (page 294) as it does not suffer from critics to Gibbard's example:

> In a game, (1) all red square cards are worth 10 points, and (2) all large

square cards are worth nothing. X caught a glimpse as Z picked a card and saw that it was red. Knowing (1), he believes “If Z picked a square card, it is worth 10 points”. Y, seeing it bulging under Z’s jacket, where Z is keeping it out of view, knows it is large. Knowing (2), he believes “If Z picked a square card, it is worth nothing”. (Someone who knows all the relevant facts knows it isn’t square, and has no use for a conditional beginning “If it is square”.)

Both X and Y have reasons to state their statements. Furthermore X has reasons to reject Y’s statement and the opposite. From godlike perspective we simply know the conditional has no use here because $P$ is false. But neither of speakers knows that, hence both of their conditionals were used rationally. A reasonable observer cannot say one is wrong and the other is good. On the other hand he would not utter both of them. In fact he would use neither of them. It seems that looking from a omniscient godlike perspective truth conditions of a conditional are those of material implication. But it is not how conditionals are used. The need for conditionals comes from a lack of exact knowledge, from subjective attitude. We need conditional claims to suppose on unknown situations. God has no need for conditionals because he simply knows whether antecedent and consequent hold (this is consistent with Dorothy Edgington’s claims).

I do not wish to argue whether conditionals are truth functionals or not. Edgington’s (or Gibbard’s) example does not neglect material implication, but it shows that the applicability of a conditional depends on speakers subjective view and knowledge. In my opinion we should speak not about the truth of conditionals, but a usage consistent with the speakers mental attitude. If we are trying to decide whether a conditional can be uttered or not, we have to rely on agent’s subjective knowledge. It does not matter to me whether a conditional is true or not, or even if it can have assigned a truth value. It matters, whether it is rational to state such a statement. As the example shows, this analysis can be done only from agent’s subjective perspective. Its knowledge determines whether a conditional is well or badly used.

## 4.4. Probabilistic approach

Some of well known approaches to conditionals are related to the idea of conditional probability. The preliminary works on the subject come from Ramsey. The idea has been extended and studied by Jeffrey, Adams and Stalnaker (based on (Edgington 1995) pp. 259-262).

The conditional probability itself comes from Bayes equation:

$$P(P \wedge Q) = P(Q|P)p(Q) \tag{4.1}$$

The first move towards conditionals requires a shift from probability to belief. Instead of assuming some objective probability distribution ($P$) we assume existence of subjective probable belief ($b$) in a speaker. In such a way we avoid interpretative problems related to subjectivity described in section 4.3. Probable belief ($b$) meets all basic arithmetical properties characteristic for probability (like: sum to one, probability of disjoint sets, conditional probability, etc.). The thesis used to define conditionals in terms of conditional belief ((Edgington 1995) p. 263):

$$b(Q\ if\ P) = b(Q \wedge P)/b(P) \tag{4.2}$$

Equation 4.2 states that our belief in a conditional 'If $P$, $Q$' is equal to our belief in both $P$ and $Q$ divided by belief in $P$. In such a way we may have some partial belief in a conditional. Let's say I am 90% sure that if the lights are on, they are at home. Assuming equation 4.2 states that my belief in a conditional is 0.9.

Such approach would be a big step towards taming at least indicative conditionals unless Lewis proof (Lewis 1976). He has proven, there is no proposition at all such that your degree of belief in its truth systematically matches your degree of belief in $Q$ given $P$. In other words: there is no $X$ such that $p(X) = p(Q|P)$ in all probability distributions in which these are defined (Edgington 1995). In result one cannot treat conditional belief as a belief in a conditional.

Additionally the theories based on thesis 4.2 do not work well for absurd conditionals like: 'If you don't smoke, you will get cancer.' or 'If birds can fly, then Roosevelt was a president of United States.'

# 5. Common-sense usage criteria

Within this chapter I try to explain how and when conditionals are used in situations analysed within this work. I shall refer to works on pragmatics, rationality and natural language usage patterns. Based on these I will formulate common-sense postulates on the usage of conditionals. In the end I shall formulate formal constraints, resulting from the common-sense postulates, required for the proper grounding of conditionals.

## 5.1. Rational and epistemic view on conditionals

One of interesting works on the matter of material implication relation to conditional statements comes from Ajdukiewicz (Ajdukiewicz 1956), who defends material implication by shifting the problem to language common. He claims that interpretation of a conditional as a material implication is correct but the problem lies in conditionals' usage patterns. Although his approach has been later criticized (Jadacki 1986; Bogusławski 1986), he notices an interesting feature: conditionals applicability depends on the language common. Some conditional statements are simply not used, because it is unreasonable to do so.

According to Pelc (Pelc 1982) "people think a conditional is well built (and true) if: (a) they see a semantic relation between an antecedent and a consequent, (b) they are not convinced of falsity of the antecedent and the consequent, (c) they are not sure of their truth, (d) see a conditional connection between the antecedent and the consequent." Pelc, not without a reason, refers here to peoples' opinion on a well built conditional. He considers criteria a-d in the context of subjective knowledge and language commons.

Similarly Clark (Clark 1971) claims that "the standard natural language conditional implies some connection between antecedent and consequent beyond the truth-value relations required by the material implication ... . And such a conditional also implies uncertainty about, or disbelief in, the antecedent and the consequent.". Going further he says[1]: "implications may arise, not from the meaning of 'if', but in virtue of some general conversational requirements or conventions about relevance, point, etc.".

One of the tempting suggestions is that there must exist some sort of connection, relation or dependence between the antecedent and the consequent. Defining such connection is a challenge itself (Barwise 1985; Braine and O'Brien 1991; Johnson-Laird and Byrne

[1] Clark refers to Grice's implicature (Grice 1957)

2002). This connection seems difficult to describe as it can be very subtle and context dependent. Yet a proper evaluation of this connection could filter out many of absurd conditionals like ‘If I am a snail, then Earth is round’.

Woods (Woods 2003) claims connection between the antecedent and the consequent exists, but is only epistemic in nature. I must agree with him. It exists only in speakers mind and is highly dependent on circumstances and subjective knowledge. This relation realizes in the form of possibilities within speaker’s mind. This way one can explain, there is a relation between the antecedent and the consequent that does not have to exemplify itself as a general dependence between the two. The word ‘then’ seems to play a crucial role in defining of this connection. If you omit it, you can know the consequent regardless of the antecedent (Davis 1983; Bennett 2003).

All mentioned authors claim that a distinction between the material implication and natural language conditional is required. All give hints on how conditionals should be analysed. Their works often focus on a discussion on how to compare the material implication and the conditional statement. I do not wish to get into the middle of this discussion. Instead I focus on *common-sense, rational criteria of statements grounding within agent’s empirical knowledge.* These criteria come from natural language usage patterns of conditionals, not from formal truth valuations of the material implication. Furthermore these criteria should be evaluated against speaker’s (agent’s) subjective knowledge state.

## 5.2. Implicatures for a conditional statement

When it comes to rational understanding and reasonable usage of language one cannot omit works of Grice (Grice 1957). He introduced a term *implicature* to distinguish what is meant by a statement from what is meant by a speaker. Suppose an example (Davis 2012):

Alan: Are you going to Paul’s party?
Barb: I have to work.

We know Barb’s statement in this conversational context answers Alan’s question well, although it does not answer it directly. Barb, by telling that she has to work, also states that she is not going to the party. The statement itself doesn’t tell anything about a party but the answer to the stated question can be deduced from the context. This is what Grice calls an *implicature.*

An easier example is: “Some athletes smoke”. It is natural to conclude that the speaker also means that not all athletes smoke (or at least he does not know that). Otherwise the speaker would use a stronger statement: “All athletes smoke”. The infered fact that ‘not all athletes smoke’ is an implicature.

There are various types of implicatures. Two most general kinds are: *conversational implicatures* and *conventional implicatures.* Conversational implicatures refer to the meaning that is not included in statement's conventional meaning, but comes from conversational context. Alan and Barb is an example of a conversational implicature. Conventional implicatures refer to things that are conventionally assumed about a statement[2]. Exemplary 'Some athletes smoke' implies there are athletes that smoke and additionally 'Not all athletes smoke' and 'More than one athlete smokes'.

Depending on context and our knowledge, implicatures may be added or removed from a statement. For a simple example: 'Some athletes smoke. In fact all do.' removes implicature 'Not all athletes smoke'. Conditionals similarly have implicatures that may be removed.

Grice developed a general theory on how to handle implicatures. This theory is based on a few principles, all of which roughly can be described as rational principles of language usage. I do not wish to go into details of his theory. I will just mark that it is important for the speaker to obey implicatures, if he doesn't want to mislead a listener or be wrongly understood.

I formulate conventional implicatures for conditional statements. In other words: I try to find assumptions a rational listener makes upon hearing a conditional statement and hence the speaker should consider them when uttering such a statement. These implicatures shall be later used to formulate formal common-sense constraints.

Suppose a speaker utters an indicative conditional proposition 'If $p(o)$, then $q(o)$'. A reasonable listener may conclude that:

**I1.1** Speaker does not know that not $p(o)$. $p(o)$ is at least possible.
**I1.2** Speaker does not know that not $q(o)$. $q(o)$ is at least possible.
**I1.3** Speaker either knows $q(o)$ or thinks that if he knew $p(o)$ holds, he would also know or conclude $q(o)$.

In my opinion these conclusions are the most general conventional implicatures of an indicative conditional statement. All of these conclusions roughly cover remarks made by (Ajdukiewicz 1956; Clark 1971; Pelc 1982) (see section 5.1). It is in the nature of implicatures that there are specific situations, where some of them may be removed.

Implicatures I1.1 and I1.2 refer to speakers knowledge about the antecedent and the consequent. When the speaker utters an indicative conditional statement, he mustn't know that not $p(o)$ and he mustn't know that not $q(o)$. If he knew that not $p(o)$, the uttered conditional would be a subjunctive. If he knew that not $q(o)$, he would be lying. These implicatures are not valid for subjunctive conditionals.

Implicature I1.3 states that either $q(o)$ is already known to hold or $p(o)$ implies $q(o)$. It means that situation where $p(o)$ and not $q(o)$ is impossible, at least according to the speaker. This implicature can be removed when the speaker is in a motivational mood. Consider:

— 'If you don't wear a hat, you will catch a cold' - mother speaking to her child.

[2] I am assuming a wider definition of a conventional implicature than Grice does. Here it means any implicature that can (without a context) be inferred from a statement by default

— ‘If you smoke, you will get a lung cancer.’ - speaking to a smoker.
— ‘If we support them louder, they will win’ - on a football stadium.

Not wearing a hat doesn’t necessarily mean that one will catch a. A mother is protective and wants to warn her child on a possibly dangerous situation. Similarly smoking does not mean that one will surely get lung cancer. Supporting a team is a matter of motivation and faith. All the three examples could be extended with modal operators to improve the precision of the utterance:

— If you don’t wear a hat, then it is possible that you will catch a cold.
— If you smoke, then it is possible that you will get a lung cancer.
— I believe that if we support them louder, then they will win.

Unfortunately, in this form, the statements would loose their motivational mood.

It is important to mark that we often use conditional of the form ‘If $P$, then $Q$’ when we are almost certain of the consequent[3]. We very rarely can be absolutely certain. Suppose terrorists have taken hostages and threaten to detonate a bomb. Two police officers stand outside the building and one of them says: ‘If you go in there, you will die’. He can’t be 100% sure that going into the building implies death. Terrorists may not detonate the bomb; the bomb may be a fake; the fuse may not work, etc.. There are hundreds of reasons why one won’t die. But the policeman is not considering them. When uttering a conditional ‘If you go in there, you will die’, the only viable option he thinks of is a detonation. In his mind, the consequence of death is a certainty. When referring to speakers state of knowledge, I1.3 is almost always a reasonable implicature. Although it is usually not true, when referring of all the possibilities, including very minor and improbable ones. It is not that the speaker removes the implicature. The speaker’s mind has constructed only mental representations for $P \wedge Q$ and $\neg P \wedge \neg Q$. At the moment of speaking: ‘If we support them louder, they will win.’ the ‘support louder’ and ‘not win’ is not an option in the speakers mind. Constructed mental representation is only partial and subjective to the speaker.

Implicatures I1.1-I1.3 are not the only ones a reasonable hearer may apply. If they were, a conditional: ‘If birds can fly, then Roosevelt was the president of United States’ would be perfectly good for us. We usually apply stronger implicatures of the form:

**I2.1** Speaker does not know whether $p(o)$ (both $p(o)$ and $\neg p(o)$ are possible).
**I2.2** Speaker does not know whether $q(o)$ (both $q(o)$ and $\neg q(o)$ are possible).
**I2.3** $q(o)$ is not guaranteed without $p(o)$.

Implicatures I2.1 and I2.2 are stronger forms of implicatures I1.1 and I1.2. When a speaker utters the conditional statement (as an answer to question on $q(o)$, to state a relation between $p(o)$ and $q(o)$, as a propositional statement) he shouldn’t know neither the antecedent, nor the consequent. If he knew the antecedent, he would also know the

[3] Refer to the idea of conditional belief in section 4.4 that was an interesting candidate to solve this problem.

consequent. If he knew the consequent, there would be no need to analyse whether it holds in the context of the antecedent.

Again implicatures I2.1 and I2.2 can be removed depending on a context and speaker's intentions. For example when we are explaining some phenomenon or our reasoning process. Examples of conditionals *not* meeting the implicatures are:

- 'If I heat water to 100 degrees, it will boil' - said by a teacher on a physics lesson.
- 'I know she stayed at home. If she stayed at home, then she will not come' - said by a friend, talking to his fellow, about a girl.

It is not that the speaker removes the implicature. In his mind appropriate mental models for $P$ and not $P$ have been constructed. In the first case the teacher may be performing an experiment where water is heated. The teacher knows he will heat the water to 100 degrees. But the conditional is said in a more general context, namely for every heated water, not just the one at the lesson. Similarly a statement 'If she stayed at home, then she will not come' is constructed with respect to a hypothetical situation where she has not stayed at home.

Implicature I2.3 is related to implicatures I2.1 and I2.2. If $q(o)$ was known to be true without $p(o)$ the speaker would know $q(o)$ and in the consequence would have no point in stating a conditional. It does not mean that $p(o)$ is the required knowledge to ensure $q(o)$. There may be other factors determining $q(o)$ but the speaker has no knowledge of them as well.

Proposed conventional implicatures refer strongly to speakers subjective mood. I think, this is a strong evidence that conditionals in their casual usage cannot be analysed without referring to speakers knowledge and his subjective mind state.

All of the implicatures may be removed from a conditional in a particular context. This is one of the reasons that makes conditionals so hard to define.

## 5.3. Understanding uncertainty - modal operators

Adding uncertainty to conditionals in form of modal operators of possibility, belief or knowledge makes conditionals even harder to understand. Following (Katarzyniak 2007) I am assuming natural language understanding of modal operators. Possibility operator (It is possible that $p(o)$) means that the agent finds $p(o)$ possible to hold, but does not know that $p(o)$ holds. I am again referring to agent's knowledge. Belief operator (I believe that $p(o)$) means that the agent thinks that $p(o)$ rather is, but is uncertain of it. Knowledge operator (I know that $p(o)$) means that agent with no doubt knows $p(o)$. Agent can know $p(o)$, if she has directly observed ($o \in P^+(t)$) or inferred it.

Due to natural language understanding of modal operators it is necessary for a speaker to always choose a modal operator describing his mental state (uncertainty level) most precisely. This implies two general common-sense constraints:

— Always choose at most one operator to describe your knowledge.
— Choose a modal operator that describes your knowledge precisely.

According to these general constraints, it should be impossible for example to: Tell that something is (only) possible, when the speaker knows it. Simultaneously use two statements that differ only on modal operator i.e. 'I believe $p(o)$' and 'I know $p(o)$'. These constraints are in accordance with assumptions made in the grounding theory (Katarzyniak 2007).

Analysing modality within conditional statements (for example $p(o) \rightarrow Pos(q(o))$, $Pos(p(o) \rightarrow q(o)))$ can be very tricky, unless we realize that in everyday usage it is often omitted. Going back to the example: 'If you smoke, you will get a lung cancer' compared to 'If you smoke, then it is possible that you will get a lung cancer'. Second one is more precise, but looses its motivational mood. In everyday usage we often omit modalities to: make statement simpler, stress the consequent or motivate the listener. I assume analysed agent doesn't possess such capabilities and apply strict criterion to always add modal operator when it fits the statement. This implies, designed agent will always use a modal operator within a conditional statement unless it is certain of the consequent (assuming the antecedent).

### 5.3.1. Modal operator's influence on a meaning

For conditionals without modal operators ($p(o) \rightarrow q(o)$) and with knowledge operator ($p(o) \rightarrow Know(q(o))$) it is assumed that $q(o)$ should always hold when $p(o)$ does. When one adds a modal operator of possibility or belief, the assumption does not hold. Modal operator adds certain uncertainty to conditional. Modal conditional can be also used, where $q(o)$ doesn't have to hold in all cases where $p(o)$ does.

On the other hand one does not want to resign from influence of $p(o)$ on $q(o)$ entirely, as this would allow for many absurd conditionals. Exemplary statement: 'If he is tall, then it is possible that the apple is red.' has no reasonable sense (at least not in any typical circumstances). The antecedent has some chance of being true, so does the consequent. Both are possible, but they are clearly unrelated. One can again refer to the connection between antecedent and consequent discussed earlier.

Dependence between the antecedent and the consequent can be tricky. Let us have a look at two examples of similar conversations:

Example 1:[4]
A: Can this apple be ripe?
B1: If the apple is green, then it is possible that it is ripe.
B2: If the apple is red, then it is ripe.

[4] In examples 1 and 2 is is assumed: all red apples are ripe, some green apples can be ripe, most green apples are not ripe yet.

Example 2:
A: Can the green apple be ripe?
B1: If the apple is green, then it is possible that it is ripe.
B2: A green apple can be ripe.

B1 and B2 represent alternative answers B can give. Within the example 1, B2 seems to be the correct answer. If we use B1, we are misleading A. From B1 A may infer that red apples can't be ripe. On the other hand, within example 2, answer B1 seems correct but B2 would be better. Stated question makes the difference. Within example 1, A requires a more detailed knowledge on the relation between a color and being ripe. Answer B2 may suggest to A that the red apple (at which he may be looking at) is not ripe. We are dealing with a conventional implicature of a conditional that tells us something not only about apple being red and ripe but also about it being green (not being red) and ripe. Within example 2, the question removes this implicature from conditional and answer B1 is correct. Both A and B know, nothing was said about apple not being ripe. Meaning of the conditional is degraded to that of statement B2. This example suggests that conditionals generally tell us not only about $p(o) \wedge q(o)$ but also about $\neg p(o) \wedge q(o)$. However it is possible to remove declaration on $\neg p(o) \wedge q(o)$ from conditional's sense. When this declaration is removed, I shall say the meaning of a conditional is *degraded*. Otherwise I tell the meaning is *full*.

If You agree with me that conventionally the conditional tells us about both situations where $p(o)$ and $\neg p(o)$ and the modal operator must be chosen according to the assumed natural language usage constraints (see introductory section of this chapter), then there remains a question on how those situations may be modelled. To answer it, I shall use a smoker and a cancer example:

Within the example given by table 5.1 a modal operator is 'good' when it is well suited to speaker's certainty level. Modal operator is 'bad' when a different operator should be used - when different operator suits better. Meaning is 'degraded' if conditional says only on situations when antecedent holds (see apple examples above). Meaning is 'full' if whole meaning of the conditional is sustained.

Belief in table is understood as a speaker's subjective believe in the consequent. Generally, if a consequent is very likely to happen (for the speaker), then belief in it is high. If it is unlikely to happen, belief is low. Belief (in the consequent) is understood to 'rise', when the consequent seems more probable assuming the antecedent holds. It means that when the antecedent doesn't hold the consequent seems less probable than in hypothetical situations where the antecedent holds. Occurrence of the antecedent raises belief of the consequent. Opposite situation happens when belief 'falls', then the consequent is less probable assuming the antecedent.

There is a strict correlation between 'full'/'degraded' meaning of a conditional and 'rise'/'fall' of the belief in the consequent. In my opinion this correlation holds for all conditionals and is a key to understanding how we use them. It is not enough for our belief in the consequent to be high (see section 4.4).

Table 5.1. A smoker and a cancer example

| | | | |
|---|---|---|---|
| If he is a smoker, then it is possible that he has a cancer. | | | |
| $p(o) \to Pos(q(o))$ | belief: *rises* | meaning: *full* | modal operator: *good* |
| If he is a smoker, then it is possible that he doesn't have a cancer. | | | |
| $p(o) \to Pos(\neg q(o))$ | belief: *falls* | meaning: *degraded* | modal operator: *bad* |
| If he is not a smoker, then it is possible that he has a cancer. | | | |
| $\neg p(o) \to Pos(q(o))$ | belief: *falls* | meaning: *degraded* | modal operator: *good* |
| If he is not a smoker, then it is possible that he doesn't have a cancer. | | | |
| $\neg p(o) \to Pos(\neg q(o))$ | belief: *rises* | meaning: *full* | modal operator: *bad* |
| If he is a smoker, then I believe that he has a cancer. | | | |
| $p(o) \to Bel(q(o))$ | belief: *rises* | meaning: *full* | modal operator: *bad* |
| If he is a smoker, then I believe that he doesn't have a cancer. | | | |
| $p(o) \to Bel(\neg q(o))$ | belief: *falls* | meaning: *degraded* | modal operator: *good* |
| If he is not a smoker, then I believe that he has a cancer. | | | |
| $\neg p(o) \to Bel(q(o))$ | chance: *falls* | meaning: *degraded* | modal operator: *bad* |
| If he is not a smoker, then I believe that he doesn't have a cancer. | | | |
| $\neg p(o) \to Bel(\neg q(o))$ | chance: *rises* | meaning: *full* | modal operator: *good* |

The belief for the consequent changes (doesn't stay the same) in every statement from a smoker and a cancer example. This happens because these two things are related. In my opinion the change in belief is required for us, to be able to even consider a conditional. If it doesn't change, we do not consider a conditional statement at all. Following (Woods 2003) the connection between the antecedent and the consequent can be only epistemic in nature, relative to speakers knowledge and circumstances, but it always exists. Following (Davis 1983) such dependence always exists with conditionals containing 'then' conjunct.

This leads to a conclusion that the change in belief is a conventional implicature suitable both for full and degraded meanings of conditionals:

**I2.4** Speaker's belief in the consequent changes, depending on whether antecedent is assumed to hold or not to hold.

We can use conditionals in wider (degraded) sense, like: 'If he is a smoker, then I believe that he doesn't have a cancer', but only if this sense is known to be assumed both by a speaker and a listener. Otherwise the speaker is misleading the listener.

A rise of belief in the consequent must happen to use conditional in its full meaning. *When hearing a conditional statement with modality we conventionally assume that speaker's belief in a consequent rises, when he assumes an antecedent holds.* This rise of belief is also a conventional implicature:

**I3.1** Speaker's belief in the consequent is greater, when antecedent is assumed to hold, compared to situations where antecedent is assumed not to hold.

I2.4 and I3.1, like other implicatures, can be removed from a statement in specific situations. Consider an example:

Example 3:
A: Can this person be a girl, if the ball is red?
B1: If the ball is red, it is possible that this person is a girl.
B2: This person can be a girl regardless of the ball's colour.

Within example 3, the ball's colour and person's sex are unrelated. Despite that, B1 seems a reasonable answer. Within the dialogue, speaker A forces a conditional on B. In result B can answer with B1, although B2 would probably be a more informative and precise answer.

When it comes to the choice between modal operators, *we tend to use operator that mirrors our level of belief well.* The belief operator ($Bel$) shall be used, when assuming the antecedent makes the consequent likely to happen. The possibility operator ($Pos$) shall be used, when (assuming the antecedent) the consequent is rather unlikely to happen. At the same time, without the antecedent, the consequent is usually even more unlikely to happen or impossible (see implicature 3.1). The knowledge operator ($Know$) shall be used, if assuming the antecedent makes us certain about the consequent.

### 5.3.2. Modal operator's position within a conditional

When it comes to conditionals, a modal operator can be situated in different places in a sentence. Here two cases are considered:

— *Conditional modality*: a modal operator is placed in a consequent ($p(o) \rightarrow Pos(q(o))$, $p(o) \rightarrow Bel(q(o))$, $p(o) \rightarrow Know(q(o))$ ).
— *Modal conditional*: a modal operator is placed at the beginning of a statement ($Pos(p(o) \rightarrow q(o))$, $Bel(p(o) \rightarrow q(o))$, $Know(p(o) \rightarrow q(o))$ ).

Interpreting those two types of statements requires analysis of conventional implicatures of conditionals together with modal operators. Let us look at two statements: 'If $p(o)$, then I believe that $q(o)$' and 'I believe that if $p(o)$, then $q(o)$'. Do these two differ in meaning? If yes, then how? And finally: How to formulate this difference formally?

Let us compare two examples with belief operator:

— 'I believe that if it is a bird, then it can fly.' compared to 'If it is a bird, then I believe that it can fly'.
— 'I believe that if they love each other, then they will get married.' compared to 'If they love each other, then I believe that they will get married'.

Within the former pair, the second statement seems slightly better. Within the later, the first sentence seems slightly better. But why? In my opinion, conditional modality stresses uncertainty at the consequent, while modal conditional stresses uncertainty for the whole statement.

The modality in a consequent means that the speaker: Knows the conditional $p(o) \rightarrow q(o)$ is not true. Knows that $p(o)$ significantly changes the chance of $q(o)$. Even if he found out $p(o)$ holds and $q(o)$ does not, he would not be wrong. $q(o)$ is not guaranteed by $p(o)$. Returning to the example: the speaker knows that there are birds that can not fly, but most of them can.

Placing modality before the whole statement means that the speaker has some doubts whether the conditional as a whole holds. He is not sure whether $p(o) \rightarrow q(o)$ is as the whole sentence true or not. If he found out that $p(o)$ holds and $q(o)$ does not, he would feel he was wrong (his belief was wrong). Returning to the example: we tend to believe that true love should lead to a wedding. If this doesn't happen, we ask questions if it was really a true love. It seems that an eventual wedding is written somewhere in the definition of true love. On the other hand, we know that not every love leads to a wedding. That's why we use the modal operator of belief at the beginning of the conditional. We can't (or don't want to) deny it and we can't prove it.

In my opinion we are dealing with two types of beliefs here. One is of statistical nature: I know most examples in the past supported my claim but some did not. Another one is of metaphysical nature, something like: I am not familiar with the domain I am speaking about, but I believe it may be so (it either is so or not, but I am not in a position to be sure which one it is). We can notice such distinction also for simple statements ($Bel(p(o))$). A statement 'I believe that this bird can fly' (I don't know the exact species) represents a statistical possibility. On the contrary a statement 'I believe all birds can fly.' (said by a child unfamiliar with birds that can't fly, like ostriches or penguins) is of metaphysical nature.

Similar situation happens for possibility operator. When it is applied to the beginning of a conditional, it applies uncertainty to the whole statement. Applied to the consequent, stresses uncertainty of the consequent, given the antecedent.

In case of the modal operator in front of a statement (for example $Pos(p(o) \rightarrow q(o)))$ the uncertainty applies to whole conditional, speaker claims the conditional is true but is uncertain of that. Applying modal operator to the consequent (for example $p(o) \rightarrow Pos(q(o)))$ stresses that the speaker sees some distribution between situations $p(o) \wedge q(o)$ and $p(o) \wedge \neg q(o)$. To explain this phenomenon more, let me use a bulb example:

— $Know(p(o) \rightarrow q(o))$: I know that, if the bulb shines, then you turned the light switch on earlier.
— $Pos(q(o) \rightarrow p(o))$: It is possible that, if you turned the light switch on, then the bulb does not shine.
— $q(o) \rightarrow Pos(p(o))$ If you turned the light switch on, then it is possible that the bulb does not shine.

If the bulb lights it surely must have been turned earlier on. Yet it is possible that the bulb does not light, although the switch was turned on. This happens when the bulb is broken. When speaking of the one specific bulb, we can accept simultaneously statements 1 and 2. Within statement 2, the speaker does not dispute whether there is some probabilistic distribution between turning the switch and lighting the bulb. The speaker tells, simply that either the bulb lights or not (either he is right or wrong). Within statement 3, the speaker tells the bulb may lit or not lit at some chance. He marks that there is some possibility of (not) lighting.

The choice between possibility, belief and knowledge operators depends strictly on our certainty levels. We tend to use a modal operator that is well fitted to the certainty level.

Conventional implicatures for conditionals with modal operators are:

**I1.4** For 'If $p(o)$, then I it is possible that $q(o)$'
Speaker thinks that $q(o)$ may hold, when $p(o)$ is assumed to hold.

**I1.5** For 'If $p(o)$, then I belive that $q(o)$'
Speaker thinks that $q(o)$ rather holds, when $p(o)$ is assumed to hold.

**I1.6** For 'It is possible that if $p(o)$, then $q(o)$'
Speaker is uncertain, whether $p(o)$ implies $q(o)$.

**I1.7** For 'I believe that if $p(o)$, then $q(o)$'
Speaker is somewhat uncertain, whether $p(o)$ implies $q(o)$.

Implicatures I1.4 and I1.5 state that $q(o)$ may hold or not. Antecedent $p(o)$ does not have to imply $q(o)$. On the other hand, implicatures I1.6 and I1.7 state that the speaker is uncertain whether $p(o)$ causes $q(o)$. Speaker wishes to be so, but feels he may be wrong.

## 5.4. Common-sense constraints

Thesis considers a problem of grounding modal conditional statements within the cognitive agent. Proper grounding is understood here as a relation between a statement and agent's knowledge state that exists only when it meets common-sense constraints related to natural language usage and understanding of conditionals and modal operators. Within this section I formulate such common-sense constraints. These constraints are related to conventional implicatures I1.1 - I3.1 and additional conclusions underlined in sections earlier. Constraints are formulated to suit a model of the cognitive agent that is using indicative conditionals to answer simple questions or to describe some observed environment. Agent formulates conditionals basing only on its empirical knowledge consisting of past and current observations of the environment.

### 5.4.1. Formal constraints

Implicatures provided earlier have been divided into three groups: I1.* (I1.1 - I1.7), I2.* (I2.1, I2.2, I2.3, I2.4) and I3.* (I3.1). Group I1.* provides most general implicatures a reasonable message recipient can make. Groups I2.* and I3.* provide additional implicatures that further constrain conditional's meaning. Provided implicatures are related to conventional situations where conditionals are used in natural language communication.

Most of proposed implicatures can be removed from a statement in particular circumstances. To avoid interpretative problems further reasoning on conditionals is limited to indicative conditionals uttered to answer simple questions related to the consequent or to describe observed environment. In accordance with formal language specifications the antecedent and the consequent refer to binary properties an object may posses or lack.

Constraints proposed below define properties of agent's cognitive state required for proper grounding of a conditional statement. Cognitive state is constructed from empirical knowledge and has been formally defined in section 2.4. Proposed constraints are formulated from implicatures, but refer directly to agent's empirical knowledge. Constraints are divided into three groups. Group C1 is related to implicatures I1.*. Group C2 is related to implicatures I2.*. Adding C2 to C1 further filters conditionals to ensure rational usage of conditionals. Finally adding C3 (related to I3.1) makes the most restrictive group of constraints ensuring conditionals obey all conventional implicatures and hence sustain the most typical understanding of a modal conditional statement.

Consider a conditional with $p(o)$ as an antecedent and $q(o)$ as a consequent. Relevant empirical material is accessible to an agent via cognitive state $MS(t)$. It contains base profiles that support the antecedent ($o \in P^+$) or neglect it ($o \in P^-$)[5]. Similarly the consequent is supported ($o \in Q^+$) or neglected ($o \in Q^-$). A specific distribution of empirical material between these disjoint cases allows for proper grounding of conditional statements. For the conditional to be properly grounded the following constraints must be met:

### Constraints group C1

**C1.1** For all statements:
Agent possesses empirical material where $o \in P^+$. $o \in P^+$ is at least possible.

**C1.2** For all statements:
Agent possesses empirical material where $o \in Q^+$. $o \in Q^+$ is at least possible.

**C1.3** For $p(o) \to q(o)$, $p(o) \to Know(q(o))$ and $Know(p(o) \to q(o))$:
In the cognitive state there is no observation where $o \in P^+$ and $o \in Q^-$.

**C1.4** For $p(o) \to Pos(q(o))$:
Agent possesses empirical material, where $o \in Q^+$ and $o \in P^+$. There is also a lot of data, where $o \in P^+$ and $o \in Q^-$.

**C1.5** For $p(o) \to Bel(q(o))$:
Agent possesses empirical material, where $o \in P^+$ and $o \in Q^+$ held. There is also some data, where $o \in P^+$ and $o \in Q^-$.

[5] The time index was intentionally removed as I am referring to any time moment

**C1.6** For $Pos(p(o) \rightarrow q(o))$ and $Bel(p(o) \rightarrow q(o))$:
Within conscious area, there is no empirical material, where $o \in P^+$ and $o \in Q^-$.

**C1.7** For $Pos(p(o) \rightarrow q(o))$:
Agent possesses a lot of empirical material (in unconscious area), where $o \in P^+$ and $o \in Q^-$.

**C1.8** For $Bel(p(o) \rightarrow q(o))$:
Agent possesses some empirical material (in unconscious area), where $o \in P^+$ and $o \in Q^-$.

Constraints C1.1 and C1.2 are a result of implicatures I1.1 and I1.2 respectively. These constraints limit usage to indicative conditionals. If agent knew $\neg p(o)$ or $\neg q(o)$, the conditional would be a counter-factual conditional.

Constraint C1.3 is a result of implicature I1.3. It means $P$ must imply $Q$. According to constraint C1.3, for all conditionals with modal operator of knowledge ($Know$) the position of modal operator does not matter, i.e. they have the same usage restrictions. Constraint C1.3 refers directly to material implication, whose truth is defined by inductive reasoning: For all experienced $P$ and $Q$, whenever was $P$, there was also $Q$. Based on inductive reasoning, material implication: $p(o) \Rightarrow q(o)$ is true. Similarly $p(o) \rightarrow q(o)$, $p(o) \rightarrow Know(q(o))$ and $Know(p(o) \rightarrow q(o))$ are true (according to this constraint).

Constraints C1.4 and C1.5 have been deduced from implicatures I1.4 and I1.5 respectively. These constraints impose natural language understanding of modal operators of possibility and belief. Constraints C1.3, C1.4 and C1.5 are assumed to be disjoint. This is a result of common-sense assumptions on modal operators, where at most one of them can be used.

Constraint C1.6 is a joined conclusion on implicatures I1.6 and I1.7. It states that agent's uncertainty can't be applied to the consequent. Agent is uncertain about the conditional sentence as a whole. There is no relevant, explicitly accessible empirical material that would deny the conditional sentence.

Constraints C1.7 and C1.8 provide conditions for uncertainty in accordance with implicatures I1.6 and I1.7 respectively. This phenomenon is modelled by the division of agent's empirical material into conscious and unconscious areas. Unconscious area of cognitive state is the source of this uncertainty. Similarly to constraints C1.3-C1.5 it is assumed C1.7 and C1.8 can't be simulatenously met.

Constraints' Group C1 does not filter statements like:

- If he is tall, then birds can fly. (no constraints to ensure dependence between $P$ and $Q$)
- If the ball is red, then it is possible that this person is a girl. (no constraints to ensure dependence between $P$ and $Q$)
- If $2 + 2 = 4$, then a word 'apple' is a noun. (no constraints to ensure the speaker doesn't know $P$ or $Q$)

This group does not constrain formulas enough to ensure every uttered (and previously properly grounded) conditional possesses its conventional natural language meaning.

#### Additional group C2

Group C2 refers to rational usage of conditionals and is related to implicatures I2.1-4.

For all conditionals:
**C2.1** Agent possesses empirical material supporting both $o \in P^{+}$ and $o \in P^{-}$.
**C2.2** Agent possesses empirical material supporting both $o \in Q^{+}$ and $o \in Q^{-}$.
**C2.3** The distribution of empirical material between $o \in Q^{+}$ and $o \in Q^{-}$ is significantly different, between observations where $o \in P^{+}$ and where $o \in P^{-}$.

Together constraints C1 and C2 filter conditional according to common, rational usage of conditionals. Usually, the speaker mustn't know neither the antecedent, nor the consequent. If he knew at least one of them, he would mislead the listener. Such assumption is valid for the considered usage of a conditional. A epistemic connection between the antecedent and the consequent is required (see implicature I2.4). Constraints still do not filter out conditionals in their 'degraded' sense (see section 5.3.1) like:

— If he is a smoker, then I believe he doesn't have a lung cancer. (no constraints on the rise of belief in the consequent)
— If the apple is green, then it is possible that it is ripe.

#### Additional group C3

Constraint C3 is deduced from implicature I3.1. It limits conditionals usage to situations where the belief in the consequent is higher when one assumes the antecedent. Adding constraint C3 to groups C1 and C2 forces the most restrictive understanding of conditionals. It ensures 'full' meaning of conditionals is sustained (see section 5.3.1).

**C3** For all conditionals:
Empirical material supports the thesis that $o \in Q^{+}$ is significantly more probable, when $o \in P^{+}$, compared to observations where $o \in P^{-}$.

A conditional sentence obeying all constraints C1 + C2 + C3 can always be understood as a typical indicative conditional statement. The statement said by a speaker who doesn't know neither $p(o)$, nor $q(o)$. Additionally the statement said to underline that $p(o)$ increases speakers belief in $q(o)$.

### 5.4.2. Simultaneous usage constraints

Basing on constraints from section 5.4.1 one can outline pairs of conditionals that can or can not be used simultaneously. It means that a reasonable speaker, obeying constraints C1, C2 and C3 can / can't be in a position to utter (at the same time moment and cognitive state) both statements.

Table 5.2. Some non-trivial pairs of modal conditionals and their acceptability according to constraints C1-C3. Columns C1, +C2 and +C3 mean usage of only constraints C1, C1+C2 and C1+C2+C3 respectively. Symbol ✓ means simultaneous usage of statements is allowed and ✕ means it is disallowed. Numbers C$n$ refer to constraints that disallow simultaneous usage of given pairs.

| no. | pair | C1 | +C2 | +C3 |
|---|---|---|---|---|
| S1 | $p \rightarrow Know(q)$, $\neg p \rightarrow Know(q)$ | ✓ | ✕ C2.2 | ✕ |
| S2 | $p \rightarrow Know(q)$, $\neg p \rightarrow Know(\neg q)$ | ✓ | ✓ | ✓ |
| S3 | $p \rightarrow Know(q)$, $p \rightarrow Know(\neg q)$ | ✕ C1.1[4] | ✕ | ✕ |
| S4 | $p \rightarrow Know(q)$, $p \rightarrow Bel(q)$ | ✕ C1.3, C1.5[1] | ✕ | ✕ |
| S5 | $p \rightarrow Know(q)$, $p \rightarrow Pos(q)$ | ✕ C1.3, C1.4[1] | ✕ | ✕ |
| S6 | $p \rightarrow Bel(q)$, $p \rightarrow Pos(q)$ | ✕ C1.4, C1.5[1] | ✕ | ✕ |
| S7 | $p \rightarrow Know(q)$, $p \rightarrow Pos(\neg q)$ | ✕ C1.3, C1.4 | ✕ | ✕ |
| S8 | $p \rightarrow Bel(q)$, $p \rightarrow Bel(\neg q)$ | ✕ C1.5[5] | ✕ | ✕ |
| S9 | $p \rightarrow Bel(q)$, $p \rightarrow Pos(\neg q)$ | ✓ | ✓ | ✕ C3 |
| S10 | $p \rightarrow Know(q)$, $q \rightarrow Pos(\neg p)$ | ✓ [3] | ✓ | ✕ C3[2] |
| S11 | $p \rightarrow Bel(q)$, $q \rightarrow Bel(\neg p)$ | ✓ | ✓ | ✕ C3[2] |
| S12 | $p \rightarrow Bel(q)$, $q \rightarrow Pos(\neg p)$ | ✓ | ✓ | ✕ C3[2] |
| S1' | $Know(p \rightarrow q)$, $Know(\neg p \rightarrow q)$ | ✓ | ✕ C2.2 | ✕ |
| S2' | $Know(p \rightarrow q)$, $Know(\neg p \rightarrow \neg q)$ | ✓ | ✓ | ✓ |
| S3' | $Know(p \rightarrow q)$, $Know(p \rightarrow \neg q)$ | ✕ C1.3[4] | ✕ | ✕ |
| S4' | $Know(p \rightarrow q)$, $Bel(p \rightarrow q)$ | ✕ C1.3, C1.8[1] | ✕ | ✕ |
| S5' | $Know(p \rightarrow q)$, $Pos(p \rightarrow q)$ | ✕ C1.3, C1.7[1] | ✕ | ✕ |
| S6' | $Bel(p \rightarrow q)$, $Pos(p \rightarrow q)$ | ✕ C1.7, C1.8[1] | ✕ | ✕ |
| S7' | $Know(p \rightarrow q)$, $Pos(p \rightarrow \neg q)$ | ✕ C1.3, C1.7 | ✕ | ✕ |
| S8' | $Bel(p \rightarrow q)$, $Bel(p \rightarrow \neg q)$ | ✕ C1.8[5] | ✕ | ✕ |
| S9' | $Bel(p \rightarrow q)$, $Pos(p \rightarrow \neg q)$ | ✓ | ✓ | ✕ C3 |
| S10' | $Know(p \rightarrow q)$, $Pos(q \rightarrow \neg p)$ | ✓ [3] | ✓ | ✕ C3[2] |
| S11' | $Bel(p \rightarrow q)$, $Bel(q \rightarrow \neg p)$ | ✓ | ✓ | ✕ C3[2] |
| S12' | $Bel(p \rightarrow q)$, $Pos(q \rightarrow \neg p)$ | ✓ | ✓ | ✕ C3[2] |

[1] Statements are unacceptable simultaneously due to an assumption that only one modal operator can be used. We assume constraints mentioned in the table are disjoint.

[2] It is impossible for both beliefs for $Q$ (assuming $P$) and $\neg P$ (assuming $Q$) to rise.

[3] see the bulb example

[4] This pair can't be used as indicative conditionals. $p$ is at least possible.

[5] It is unwise to believe both in $q$ and $\neg q$. 'Some data' in constraints C1.5 and C1.8 mean there is not much data.

# 6. Grounding of conditional statements

Whole grounding process should be evaluated simultaneously on a word, a phase and a sentence levels. An agent figures out the meaning of words, then phrases and in the end whole statements. All levels may influence each other. Within the grounding theory and this thesis *the grounding process is constrained only to the sentence level. It is assumed simple words and phases have been previously grounded by the agent. Grounding shall be considered only on a modal formula level.* A fixed and known meaning of objects and properties language representations is assumed, so agent is able to associate internal reflection $o \in P^{+}(t)$ with a language representation $p(o)$.

Grounding on a sentence level requires a construction of a relation between the cognitive state and the external representation in form of modal conditional formula. In order for such relation to be proper, it must stand in compliance with common-sense constraints proposed earlier. Only then an uttered formula sustains it conventional meaning. In result a recipient may reason not only on the environment but also on speaker's subjective knowledge state. Depending on a conditional's usage, recipient's conclusions can be limited according to constraints C1, C1+C2 or C1+C2+C3. Solution to the grounding problem is obtained by proposing of formal grounding criteria in form of episemic relations for conditionals.

The proposed empirical knowledge and cognitive state model provide information necessary to solve the grounding problem for typical conditionals. In the simplest case, when speaking of a directly perceived object, the grounding process can be realized in the following steps:

1. A physical object $o$ has property $P$ in environment ($o \in P(t)$).
2. Agent perceives this feature, what results in a realization of an internal reflection $o \in P^{+}(t)$.
3. Agent forms a cognitive model that holds only one base profile: $BP(t)$.
4. Mental representation consists only of one reflection $o \in P^{+}(t)$.
5. Agent assigns a language representation to mental representation and utters $p(o)$ ('Object $o$ is $p$').

Realization of step 5 requires the grounding process on word and phrase levels. In order to choose proper language representation, one has to construct a link between language symbol $p(o)$ and its mental representation $o \in P^{+}(t)$, so symbol $p(o)$ must be previously grounded. It is assumed such process has already taken place and this problem is outside the scope of the grounding theory.

Process described in steps 1-5 above is the simplest case, where described object is directly observed. The relation between internal reflections and mental representations usually is not so straightforward. It is never straightforward, when described objects are not perceived directly or when considering complex statements. For a conditional statement $p(o) \rightarrow Bel(q(o))$, the grounding process can be realized in the following steps:

1. A physical object $o$ has properties $P$ and $Q$ ($o \in P(t)$, $o \in Q(t)$).
2. Agent is unable to directly observe these properties ($o \in P^{\pm}(t)$, $o \in Q^{\pm}(t)$).
3. Agent constructs a cognitive state. It consists of base profiles from empirical knowledge $KS(t)$ relevant to current circuimstances.
4. Mental representation of a conditional statement is formed from the cognitive state. Mental representation is a result of past observations of $o$ (not) being $P$ and (not) being $Q$.
5. Agent assigns a matching language representation from language $L'$ describing obtained mental representation and utters $p(o) \rightarrow Bel(q(o))$ ('If $o$ exhibits $p$, then I believe $o$ exhibits $q$').

The transition from physical object to an external language representation is realized through the cognitive state. The cognitive state in turn is constructed from past empirical knowledge, not from a direct observation. Assignment of the language representation to the mental representations performed in step 5 is crucial for proper grounding of a modal conditional formula. It is necessary to formulate criteria that allow grounding of conditionals consistent with the common-sense constraints. This is the key problem addressed in the thesis.

Next section slightly changes definition of one component of the grounding theory, the cognitive state. This slight change allows for more flexibility in the choice of utterances context.

The layout of further sections is similar to the one used to present the grounding theory in chapter 2. Firstly grounding sets that distribute empirical material to four mutual cases of $o$ (not) being $P$ and (not) being $Q$ are defined. After that (section 6.3) a measure called grounding strength is defined. It quantifies the empirical material distribution with respect to a conditional. Section 6.4 proposes an conditional relation that is used to filter out irrational usage patterns of conditionals. Finally, in sections 6.5-6.7, formal definitions of epistemic relations for conditionals are formulated. Epistemic relations are proposed in three variants, each related to one group of previously presented constraints.

## 6.1. Cognitive state redefined

Mental representation is a part of the cognitive state (the state of all object representations and processes realized in mind). Cognitive state model is understood in the grounding theory as a structure, ordering preprocessed empirical material $MS(t)$, taking part in grounding of modal statements. Cognitive state model holds observations from the past. These observations are distributed into two levels (conscious and unconscious).

In its raw form, the grounding theory allows for grounding of statements about the current time moment. It is easy to change respective definitions of epistemic relations 2.12, 2.13 to allow for grounding of statements generally describing agents knowledge. To do so one can simply remove requirements related to the current observation: $o \in P^{\pm}(t)$, $o \in Q^{\pm}(t)$ or $o \in P^{+}(t)$, $o \in Q^{+}(t)$. There are a few papers that have employed this approach.

Although the grounded formulas are about the current time moment, dependencies between the properties are not taken into consideration by the grounding theory. In the computational example (section 2.6), one can notice that $o_1$ and $o_3$ always together exhibit or lack the property $P_1$. One can discover this knowledge and use it to conclude $o_1$ exhibits $P_1$ at time moment $t = 6$. Such reasoning is outside the scope of the grounding theory. The grounding theory, in its raw form, always uses whole empirical material (see definition 2.7 for the cognitive state) as the conscious and unconscious areas always sum to $KS(t)$. In result utterances are not contextualized.

Mental representation depends greatly on the choice of empirical material supporting it. This empirical material should be relevant to considered context i.e. to the context of an utterance. When the agent is speaking of some object $o$ and its property $P$ in general (simply summaries its knowledge base), all the empirical material, where these were observed, takes part in the construction of mental representation.

Katarzyniak, later in his book (Katarzyniak 2007), proposed a few strategies that allow for contextualisation. A few solutions to this problem can be also found in (Popek 2012). All of these approaches usually have two common features. Firstly the empirical material is filtered to discard observations (base profiles) that are irrelevant to considered circumstances. This implies that only relevant empirical material takes part in the grounding process. Secondly some distance measure between base profiles is presented to divide the empirical material between conscious and unconscious areas. Base profiles that are similar to the assumed context (for example to the current observation) are placed into the conscious area.

To allow for more flexibility in contextualisation the definition of the cognitive state is slightly changed:

**Definition 6.1.** *Agent's cognitive state model at moment $t$ denoted as $CS(t)$ is defined as a division of knowledge state $KS(t)$ into $\overline{CS}(t)$ and $\underline{CS}(t)$:*

$$CS(t) = \overline{CS}(t) \cup \underline{CS}(t) \subseteq KS(t) \text{ and } \overline{CS}(t) \cap \underline{CS}(t) = \emptyset$$

*Sets $\overline{CS}(t)$ and $\underline{CS}(t)$ are called conscious and unconscious areas of knowledge state at moment $t$.*

In comparison to the original definition 2.7, there may be some non-empty area $KS(t) \setminus CS(t)$ excluded from the cognitive state. Empirical material held within this area is meant to be irrelevant to the considered context. Such a minor change does not neglect the assumptions made in the grounding theory. It simply allows for more flexibility in the choice of the context. Empirical material taking part in the grounding process can now be freely filtered depending on particular needs.

This change allows to switch the referent of the statement (for example the described time) without redefining the cognitive state model or the epistemic relations. When the agent is speaking on current time moment only relevant, most similar, empirical material should be chosen. Suppose the agent has observed that at moment $t$ object $o$ exhibits $P$. Further assume the agent wants to utter a statement about $o$ exhibiting $P$ now (at the current time moment). In such case only the direct observation matters. Past empirical material is irrelevant as it is not important how many times this property has (not) been observed. In such case the cognitive state should hold only the current observation resulting in a partition $\overline{CS}(t) = \{BP(t)\}$, $\underline{CS}(t) = \emptyset$ and the grounding of a formula $Know(p(o))$. Now suppose the agent has not observed at time $t$ whether the object $o$ exhibits $P$. In order to utter a statement about $o$ exhibiting $P$ (now, not generally) the agent has to refer to its past experiences. In the grounding theory the agent has to use all of the empirical knowledge. But the intelligent agent should choose only the empirical material that is somehow relevant. The changed definition of the cognitive state simply allows for choosing of the relevant material. The relevance of the empirical material is a separate matter outside the bounds of this work.

*From now all components of the grounding theory such as the grounding sets and grounding strengths use the newly defined cognitive state $CS(t)$ (definition 6.1) instead of the original version $MS(t)$ (definition 2.7).*

Obviously the proposed cognitive state model is still a cruel simplification of real human cognitive states. This model mirrors only the most crucial properties of real cognitive states. The properties that are important in the context of the grounding process.

The distribution to conscious and unconscious parts depends on agent's mental capabilities. For humans the division results from focussing on some phenomenon, where most adequate empirical material plays the key role in situation's evaluation, while the rest of it stays deep in mind but is internally felt. This feeling results in an awareness level influencing the choice of mental representations and defining the states of mind and further mental processes. In computer systems this distribution may be understood as a division into thoroughly processed data and data partially processed or awaiting to be processed.

The final division between conscious and unconscious levels of awareness depends not only from the mental capabilities but also from agent's point of focus. This point of focus includes considered context (for example time bounds) and considered properties. If the agent focuses on property $P$, the resulting cognitive state shall be different than when she considers property $Q$.

One can think of many possible solutions to the problem of constructing the cognitive state partitions according to the proposed model. As this is a separate problem, the grounding theory does not focus on it. Within simulations presented in chapter 9 some simple exemplary solutions are proposed. In further paragraphs it is simply assumed that the partition of the empirical material in the cognitive state is fixed for given time moment.

## 6.2. The grounding sets

When considering the grounding of a complex statement i.e. a conditional formula $p(o) \rightarrow q(o)$, mental representations of both properties $P$ and $Q$ have to be considered within the cognitive state. All base profiles, where both properties were observed form a mental representation of an interrelation between these properties. This mental representation forms the whole relevant grounding material of a conditional formula. The grounding sets for a conditional are defined similarly as the grounding sets for a conjunction.

**Definition 6.2.** *Let grounding material for a given cognitive state and a formula expressing information about object $o$ and its properties $P$ and $Q$ be defined as a set $C$:*

$$C(t, o, P, Q) = \{BP(\hat{t}) \in CS(t) : \ o \in P^+(\hat{t}) \cup P^-(\hat{t}) \wedge o \in Q^+(\hat{t}) \cup Q^-(\hat{t})\}$$

*where $\hat{t} \in \mathcal{T}$ denotes any time moment $\hat{t} \leqslant t$.*

The grounding material can be divided into conscious and unconscious areas according to the cognitive state:

$$\overline{C}(t, o, P, Q) = C(t, o, P, Q) \cap \overline{CS}(t), \quad \underline{C}(t, o, P, Q) = C(t, o, P, Q) \cap \underline{CS}(t)$$

Grounding material $C$ can also be divided into the grounding sets, $C^{p(o)\wedge q(o)}$, $C^{p(o)\wedge\neg q(o)}$, $C^{\neg p(o)\wedge q(o)}$, $C^{\neg p(o)\wedge\neg q(o)}$. Each grounding set contains past observations supporting one of four possible situations[1].

**Definition 6.3.** *Grounding sets define a partition of grounding material into four mutually disjoint cases according to valuations of properties $P$ and $Q$:*

$$\begin{aligned}
C^{p(o)\wedge q(o)} =& \{BP(\hat{t}) \in C : o \in P^+(\hat{t}) \wedge o \in Q^+(\hat{t})\} \\
C^{p(o)\wedge\neg q(o)} =& \{BP(\hat{t}) \in C : o \in P^+(\hat{t}) \wedge o \in Q^-(\hat{t})\} \\
C^{\neg p(o)\wedge q(o)} =& \{BP(\hat{t}) \in C : o \in P^-(\hat{t}) \wedge o \in Q^+(\hat{t})\} \\
C^{\neg p(o)\wedge\neg q(o)} =& \{BP(\hat{t}) \in C : o \in P^-(\hat{t}) \wedge o \in Q^-(\hat{t})\}
\end{aligned}$$

[1] To simplify notation, object $o$ and time $t$ are often omitted. We consider one fixed object $o$ and time moment $t$. Symbols $p$ and $q$ are short notations of $p(o)$ and $q(o)$ respectively. Symbols $C, \overline{CS}$, $\underline{CS}$ etc. are short forms of $C(o, P, Q, t)$, $\overline{CS}(t)$, $\underline{CS}(t)$ etc.

*where $\hat{t} \in \mathcal{T}$ denotes any time moment $\hat{t} \leqslant t$.*

For example set $C^{p(o)\wedge\neg q(o)}$ contains observations where object $o$ exhibited property $P$ but didn't exhibit $Q$. Intuitively, the more grounding material in $C^{p(o)\wedge q(o)}$ and the less material in $C^{p(o)\wedge\neg q(o)}$, the more willing we are to utter a conditional $p(o) \to q(o)$.

Grounding material where object $o$ was $P$ ($Q$) regardless of $Q$ ($P$) can be calculated as an union of sets:

$$C^{p(o)} = C^{p(o)\wedge q(o)} \cup C^{p(o)\wedge\neg q(o)} \quad (6.1)$$

$$C^{q(o)} = C^{p(o)\wedge q(o)} \cup C^{\neg p(o)\wedge q(o)} \quad (6.2)$$

Eventually each grounding set can be divided into conscious and unconscious partitions. Let $\phi \in \{p(o), \neg p(o)\}, \psi \in \{q(o), \neg q(o)\}$:

$$\overline{C}^{\phi\wedge\psi} = C^{\phi\wedge\psi} \cap \overline{C}, \quad \underline{C}^{\phi\wedge\psi} = C^{\phi\wedge\psi} \cap \underline{C} \quad \text{and} \quad C^{\phi\wedge\psi} = \overline{C}^{\phi\wedge\psi} \cup \underline{C}^{\phi\wedge\psi} \quad (6.3)$$

Together grounding sets form a grounding material that can be freely divided into types of situations residing in conscious or unconscious parts of agent's mind:

$$C = C^{p(o)\wedge q(o)} \cup C^{p(o)\wedge\neg q(o)} \cup C^{\neg p(o)\wedge q(o)} \cup C^{\neg p(o)\wedge\neg q(o)} \quad (6.4)$$

$$\overline{C} = \overline{C}^{p(o)\wedge q(o)} \cup \overline{C}^{p(o)\wedge\neg q(o)} \cup \overline{C}^{\neg p(o)\wedge q(o)} \cup \overline{C}^{\neg p(o)\wedge\neg q(o)} \quad (6.5)$$

$$\underline{C} = \underline{C}^{p(o)\wedge q(o)} \cup \underline{C}^{p(o)\wedge\neg q(o)} \cup \underline{C}^{\neg p(o)\wedge q(o)} \cup \underline{C}^{\neg p(o)\wedge\neg q(o)} \quad (6.6)$$

## 6.3. The relative grounding strength

To quantify a interrelation between object's properties $P$ and $Q$ one requires a numeric measure. Such measure, similarly as in the grounding theory, is called a *relative grounding strength.* The relative grounding strength is a value between 0 and 1. Intuitively, when it is close to 1, we are willing to accept a conditional statement.

**Definition 6.4.** *Let the cognitive state $CS(t)$ be given. Let the relative grounding strength of a base formula $p(o) \to q(o)$ be defined as*[2]*:*

$$\lambda^{p(o)\to q(o)} = \frac{\mathrm{card}(C^{p(o)\wedge q(o)})}{\mathrm{card}(C^{p(o)\wedge q(o)} \cup C^{p(o)\wedge\neg q(o)})}$$

Since the sets $C^{p(o)\wedge q(o)}$ and $C^{p(o)\wedge\neg q(o)}$ are disjoint:

$$\mathrm{card}(C^{p(o)\wedge q(o)} \cup C^{p(o)\wedge\neg q(o)}) = \mathrm{card}(C^{p(o)\wedge q(o)}) + \mathrm{card}(C^{p(o)\wedge\neg q(o)}).$$

[2] Relative grounding strengths for $\neg p(o)$ or $\neg q(o)$ are calculated similarly

To be able to calculate grounding strength one requires that the denominator is not zero. If the denominator were equal to zero, considered conditional statement would not be an indicative conditional. If it were that always not $p$, we would be dealing with a subjunctive conditional, which is not considered within the thesis.

One can find similarities between the relative grounding strength and conditional probability or more precisely an estimator for conditional probability. The main difference is that grounding strength is subjective and calculated from the cognitive state. In opposition to conditional probability or belief we are not searching for a formal conditional probability distribution. Relative grounding strength is a measure quantifying impact of the grounding material on $Q$ provided that $P$.

## 6.4. The conditional relation - a pragmatic filter

The knowledge that the antecedent holds (or doesn't hold) must impact the subjective chance of holding of the consequent. Strawson (Strawson 1952) claims that natural language conditional implies some kind of connection between antecedent and consequent. Ajdukiewicz (Ajdukiewicz 1956) shifts the problem to language common. Grice (Grice 1957) calls it implicature. It is difficult to deny that there must be some kind of relation that joins antecedent and consequent yet it can be very difficult to grasp. According to (Woods 2003) this connection can be only epistemic in its nature. It can be valid only in particular circumstances for a particular person. If there is no such connection, there is no point in stating an indicative conditional. This property of conditionals has been already more throughly discussed in section 5.1.

Here conditionals are evaluated from agent's perspective basing on its knowledge. It seems reasonable to model such an epistemic connection between the antecedent and the consequent. To model this phenomenon a *conditional relation* (Skorupa and Katarzyniak 2011) is introduced.

**Definition 6.5.** *Let* $\underline{f} : [0,1] \to [0,1]$ *and* $\overline{f} : [0,1] \to [0,1]$ *are lower and upper boundary functions respectively iff they meet the following criteria:*

1. $\underline{f}(x) \leqslant x$ *and* $\overline{f}(x) \geqslant x$ *for all* $x$
2. $\underline{f}, \overline{f}$ *are monotonically increasing (non-decreasing)*
3. $\underline{f}, \overline{f}$ *are continuous on* $[0,1]$

For a given upper and lower boundary functions one can define conditional and strict conditional relations:

**Definition 6.6.** *We say* $q(o)$ *is conditionally related to* $p(o)$ *if and only if:*

$$\lambda^{p(o)\to q(o)} > \overline{f}(\lambda^{\neg p(o)\to q(o)}) \quad or \quad \lambda^{p(o)\to q(o)} < \underline{f}(\lambda^{\neg p(o)\to q(o)})$$

**Definition 6.7.** *We say $q(o)$ is strictly conditionally related to $p(o)$ if and only if:*

$$\lambda^{p(o)\rightarrow q(o)} > \overline{f}(\lambda^{\neg p(o)\rightarrow q(o)})$$

For $Q$ to be conditionally related with $P$, there must be a significant difference in the occurences of $Q$ between situations where $P$ holds and the ones where it does not. The required absolute value of that difference changes depending on various $\lambda^{p(o)\rightarrow q(o)}$ and $\lambda^{\neg p(o)\rightarrow q(o)}$ settings. Strict conditional relation requires the chance for $Q$ to *rise* when $P$ holds. In other words: agent must notice a significant increase in the occurrences of $o \in Q^{+}(t)$ in situations, where $o \in P^{+}(t)$.

The main advantage of conditional relation is that it can be used to filter out lots of unreasonable conditionals like: 'If he is tall, then birds can fly', 'If the Moon is a piece of cheese, then I can jump 100 meters.'. Furthermore conditional relation works well with modalities. Conditional relation does not hold for a statement: 'If birds can fly, then it is possible you will die on a day with an even date'. It is possible you will die on a day with an even date, but it is possible regardless of birds abilities.

The main disadvantage is that it can filter out some roughly related antecedents and consequents. Upper and lower boundary functions must be chosen carefully. One can choose any upper boundary function $\overline{f}$ as long as it meets provided criteria. Let us propose a few exemplary upper boundary functions:

$$\overline{f}_c(x) = \sqrt{r^2 - (x - x_0)^2} - x_0 + 1, \text{ where } r \geqslant 1,\ x_0 = \frac{1 + \sqrt{2r^2 - 1}}{2} \tag{6.7}$$

$$\overline{f}_s(x) = \sqrt[n]{x}, \text{ where } n \geqslant 2 \tag{6.8}$$

$$\overline{f}_q(x) = -(1 - x)^n + 1, \text{ where } n \geqslant 2 \tag{6.9}$$

Function $\overline{f}_c$ is a circle fragment crossing points (0,0) and (1,1). Parameter $r$ is a radius of that circle. The smaller the radius, the more significant difference between $\lambda^{p(o)\rightarrow q(o)}$ and $\lambda^{\neg p(o)\rightarrow q(o)}$ is required. Figure 6.1 presents exemplary $f_c$ function. Grey area contains all points meeting strict conditional relation (see definition 6.7).

Theorem 6.1 states that a lower boundary function can be easily constructed from an upper boundary function and vice versa.

**Theorem 6.1.** *Function $\overline{f}(x)$ is a proper upper boundary function iff $\underline{f}(x) = 1 - \overline{f}(1 - x)$ is a proper lower boundary function.*

*Proof.* Obviously $\underline{f}(x) = 1 - \overline{f}(1 - x)$ is a continous function over $[0, 1]$. Proper lower boundary function has to meet condtions 1 and 2.

Condition 1: We know that $\overline{f}(x) \geqslant x$.

$$\begin{aligned} \overline{f}(x) &\geqslant x && / \; x = 1 - t \\ \overline{f}(1-t) &\geqslant 1 - t \\ t &\geqslant 1 - \overline{f}(1-t) = \underline{f}(t) \end{aligned}$$

Condition 2: We know that for any $a > 0$: $\overline{f}(x + a) \geqslant \overline{f}(x)$

$$\begin{aligned} \overline{f}(x+a) &\geqslant \overline{f}(x) && / \; x = 1 - t \\ \overline{f}(1-t+a) &\geqslant \overline{f}(1-t) \\ 1 - \overline{f}(1-t+a) &\leqslant 1 - \overline{f}(1-t) \\ \underline{f}(t-a) &\leqslant \underline{f}(t) \end{aligned}$$

The proof in the opposite direction is similar. □

Required absolute difference between $\lambda^{p(o)\to q(o)}$ and $\lambda^{\neg p(o)\to q(o)}$ varies based on their values. Let us analyse four exemplary points in the context of strict conditional relation and upper boundary function given by figure 6.1.

Assume $\lambda^{p(o)\to q(o)} = 1$ and $\lambda^{\neg p(o)\to q(o)} = 0.2$. This setting meets conditional relation. In case of $P$ we are sure of $Q$. Without $P$, $Q$ is only possible. It is the most typical case, where conditionals are used.

Secondly assume a setting: $\lambda^{p(o)\to q(o)} = 0.5$, $\lambda^{\neg p(o)\to q(o)} = 0.6$. Such point doesn't lie within grey area presenting strict conditional relation. On the contrary, the point might be accepted by a conditional relation. When $P$ holds, chance for $Q$ diminishes. Please compare with constraints C2, C3 and section 5.3.1.

Thirdly assume $\lambda^{p(o)\to q(o)} = 0.2$, $\lambda^{\neg p(o)\to q(o)} = 0.05$. Overall chance of $Q$ is small. This point (for reasonable radius $r$) lies within grey area and the conditional relation holds. Although the absolute probability difference is low (equal to 0.15), the occurrence of $P$ increases chance for $Q$ four times.

In the end, let us assume a point $\lambda^{p(o)\to q(o)} = 1$, $\lambda^{\neg p(o)\to q(o)} = 1$. In such case $\lambda^{p(o)\to q(o)} = \lambda^{\neg p(o)\to q(o)}$ and conditional relation doesn't hold for any upper boundary function. We know that $Q$ always occurs regardless of $P$, so there is no point in stating a conditional. The $n$th root function $\overline{f}_s$ requires little difference between $\lambda^{p(o)\to q(o)}$ and $\lambda^{\neg p(o)\to q(o)}$, when $\lambda^{\neg p(o)\to q(o)}$ is close to one and a significant difference, when it is close to zero. $n$th power function $\overline{f}_q$ has the opposite property: significant difference, when $\lambda^{\neg p(o)\to q(o)}$ probability is low and little difference, when it is high.

## 6.5. (Normal) epistemic relations

In this section the epistemic satisfaction relations for: $p(o) \to Know(q(o))$, $p(o) \to Bel(q(o))$ and $p(o) \to Pos(q(o))$ is defined. If the epistemic relation holds, the state-

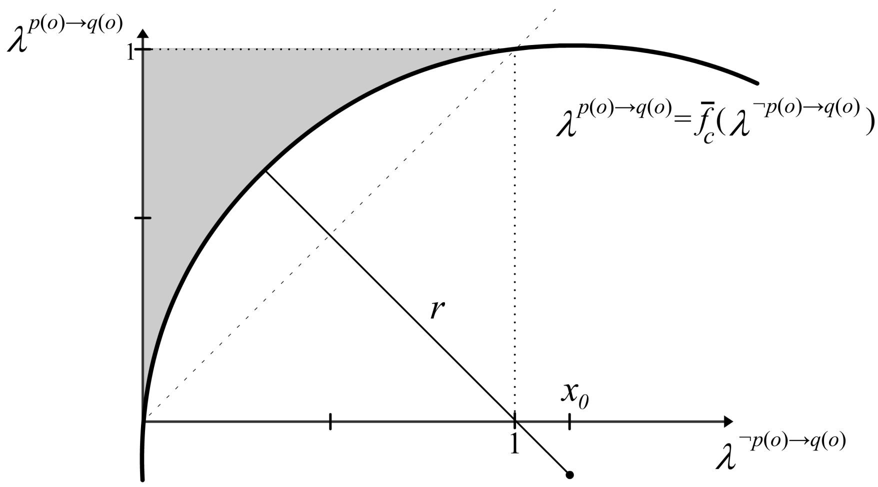


Figure 6.1. Exemplary upper boundary function $\overline{f}_c$. Grey area marks allowed combinations of the grounding strengths according to the strict conditional relation definition 6.7.

ment can be properly grounded (has conventional meaning). The definitions tell whether agent's cognitive state allows for grounding of the conditional statements.

There are three variants of the definitions. Definitions from variant 1, given by 6.8-6.13, are consistent with constraints group C1 but do not meet constraints C2 and C3. Definitions 6.14 and 6.15 additionally meet pragmatic constraints from group C2. Definitions 6.16 and 6.17 meet all constraints from group C1, C2 and C3. The former definitions enforce the most restrictive and conventional understanding of conditional sentences.

In definitions $CS$ is used as a short notation for $CS(t)$. Current time moment $t \in \mathcal{T}$ is assumed to be fixed and known.

### 6.5.1. Epistemic relations for conditional modalities

Definitions 6.8-6.10 present proper grounding conditions for the conditional modalities, where the modal operator is placed in the consequent. These epistemic relations shall be also called *normal epistemic relations* and have been designed to meet constraint group C1.

**Definition 6.8.** *Epistemic relation* $CS \models^{\mathrm{E}} p(o) \to Pos(q(o))$ *holds iff*
*1.* $C^{p(o)} \neq \emptyset$
*2.* $\lambda_{minPos} < \lambda^{p(o)\to q(o)} \leqslant \lambda_{maxPos}$

**Definition 6.9.** *Epistemic relation* $CS \models^{\mathrm{E}} p(o) \to Bel(q(o))$ *holds iff*
*1.* $C^{p(o)} \neq \emptyset$
*2.* $\lambda_{minBel} < \lambda^{p(o)\to q(o)} < \lambda_{maxBel}$

**Definition 6.10.** *Epistemic relation $CS \models^{E} p(o) \rightarrow Know(q(o))$ holds iff*
*1.* $C^{p(o)} \neq \emptyset$
*2.* $\lambda^{p(o) \rightarrow q(o)} = 1$

In order to obtain epistemic relations for various negations of $p(o)$ or $q(o)$ one should symmetrically apply negations in the definitions 6.8-6.10. For example a formula $\neg p(o) \rightarrow Bel(q(o))$ can be grounded iff $C^{\neg p(o)} \neq \emptyset$ and $\lambda_{minBel} < \lambda^{\neg p(o) \rightarrow q(o)} < \lambda_{maxBel}$.

Provided definitions for normal epistemic relations already suggest some intuitions on the previously assumed meaning of conditionals. Condition $C^{p(o)} \neq \emptyset$ requires that agent's cognitive state contains observations where $o$ exhibited $p$. This means that the agent must not know that $\neg p(o)$. If she new that, the statement would be a subjunctive conditional. This condition is directly related to implicature I1.1 and constraint C1.1.

The conditions limiting possible values of the grounding strength $\lambda^{p(o) \rightarrow q(o)}$ constrain agent's levels of belief in the consequent. For $p(o) \rightarrow Know(q(o))$ this level of belief must be equal to 1. This means that whenever was $p(o)$ there also was $q(o)$. Parameters $\lambda_{minPos}$, $\lambda_{maxPos}$, $\lambda_{minBel}$, $\lambda_{maxBel}$ are called the *grounding thresholds*. These thresholds have to be set in advance by an expert (or evaluated by the agent, see (Lorkiewicz et al. 2011)). We wish the belief modal operator to denote higher level of belief than the possibility operator. It is hence initially assumed that:

$$\lambda_{minPos} \leqslant \lambda_{minBel} \quad \text{and} \quad \lambda_{maxPos} \leqslant \lambda_{maxBel} \tag{6.10}$$

Inequalities 6.10 form only *initial* restrictions as they do not guarantee meeting of simultaneous usage constraints (see table 5.2). For example for an intentionally badly chosen setting $\lambda_{minBel} = 0.4$ and $\lambda_{maxBel} = 0.9$ it is possible to simultaneously ground two statements: $p(o) \rightarrow Bel(q(o))$ and $p(o) \rightarrow Bel(\neg q(o))$. Both statements intuitively contradict each other and must not be grounded together. To meet the simultaneous usage constraints it is crucial to provide further limitations for the grounding thresholds. For example implicature I1.2 and the respective constraint C1.2 tells that the agent must not know $q(o)$. This constraint shall be met only when $\lambda_{minPos} \geqslant 0$. The final limitations on the grounding thresholds are presented in chapter 7.

Normal epistemic relations 6.8-6.10 allow for grounding of indicative conditionals in their broad meaning. A dependence between the antecedent and the consequent is not required. The agent may already know that $p(o)$ (or $q(o)$) holds and still ground a conditional using normal epistemic relation. In that sense, the conditional can be uttered as an answer to a question: 'Is it possible that $q(o)$ when $p(o)$?'. The answer may be 'Yes. If $p(o)$ then it is possible that $q(o)$'. The agent says nothing about situations where $\neg q(o)$. The agent does not indicate any connection / dependence between $p(o)$ and $q(o)$. In the sense defined by the normal epistemic relation it is simply stated that $q(o)$ may hold when $p(o)$ holds.

### 6.5.2. Epistemic relations for modal conditionals

Definitions 6.11-6.13 present proper grounding conditions for the modal conditionals, where the modal operator is placed before the whole conditional statement.

**Definition 6.11.** *Epistemic relation* $CS \models^{\mathrm{E}} Pos(p(o) \rightarrow q(o))$ *holds iff*
*1.* $\overline{C}^{p(o) \wedge \neg q(o)} = \emptyset$, $C^{p(o)} \neq \emptyset$
*2.* $\beta_{minPos} < \lambda^{p(o) \rightarrow q(o)} \leqslant \beta_{maxPos}$

**Definition 6.12.** *Epistemic relation* $CS \models^{\mathrm{E}} Bel(p(o) \rightarrow q(o))$ *holds iff*
*1.* $\overline{C}^{p(o) \wedge \neg q(o)} = \emptyset$, $C^{p(o)} \neq \emptyset$
*2.* $\beta_{minBel} < \lambda^{p(o) \rightarrow q(o)} < \beta_{maxBel}$

**Definition 6.13.** *Epistemic relation* $CS \models^{\mathrm{E}} Know(p(o) \rightarrow q(o))$ *holds iff*
*1.* $\overline{C}^{p(o) \wedge \neg q(o)} = \emptyset$, $C^{p(o)} \neq \emptyset$
*2.* $\lambda^{p(o) \rightarrow q(o)} = 1$

The parameters $\beta_{minPos}, \beta_{maxPos}, \beta_{minBel}, \beta_{maxBel}$ are the *grounding thresholds* for the modal conditionals. The thresholds have the same role as $\lambda_{minPos}, \lambda_{maxPos}, \lambda_{minBel}$, $\lambda_{maxBel}$ used for conditional modalities. Similarly it is initially assumed that:

$$\beta_{minPos} \leqslant \beta_{minBel} \text{ and } \beta_{maxPos} \leqslant \beta_{maxBel} \tag{6.11}$$

Definitions 6.11-6.13 differ from 6.8-6.10 in the condition 1. For modal conditionals it is required that there are no observations within the conscious area of the cognitive state that deny the conditional $p(o) \rightarrow q(o)$. The agent has not found any 'explicit' occurrences denying the conditionals, but such experiences may be present in the unconscious area of the cognitive state. This way the uncertainty regarding the whole conditional is modelled. The agent 'feels' that she is uncertain but has no explicit example for the uncertainty. The uncertainty is hidden in the unconscious area.

For such an interpretation the modal conditional may be used to express agent's uncertainty when the empirical material is only partially processed. The agent may be asked about $q(o)$. She is required to answer quickly and says $Bel(p(o) \rightarrow q(o))$. The agent only partially processed the empirical knowledge what resulted in a cognitive state where most of the material stays in the unconscious area. The conscious area holds only few most adequate past observations referring to the described situation. Neither of those observations denies the conditional. In that sense, agent's mind state can be explained as: 'It may be that $p(o) \rightarrow q(o)$ but the agent is uncertain'. The agent needs more time (or data) to gain certainty.

## 6.6. Pragmatic epistemic relations

Below pragmatic epistemic relations 6.14 and 6.15 are defined. These relations further constrain the meaning of conditionals. Definitions of pragmatic epistemic relations have been designed to meet constraint groups C1 and C2.

**Definition 6.14.** *Let* $\Pi \in \{Pos, Bel, Know\}$*. Pragmatic epistemic relation* $CS \models^{\mathrm{PE}} p(o) \rightarrow \Pi(q(o))$ *holds iff epistemic relation for* $CS \models^{\mathrm{E}} p(o) \rightarrow \Pi(q(o))$ *holds and:*
*1.* $C^{\neg p(o)} \neq \emptyset$
*2.* $\lambda^{p(o)\rightarrow q(o)} > \overline{f}(\lambda^{\neg p(o)\rightarrow q(o)})$ *or* $\lambda^{p(o)\rightarrow q(o)} < \underline{f}(\lambda^{\neg p(o)\rightarrow q(o)})$
*(the conditional relation holds).*

**Definition 6.15.** *Let* $\Pi \in \{Pos, Bel, Know\}$*. Pragmatic epistemic relation* $CS \models^{\mathrm{PE}} \Pi(p(o) \rightarrow q(o))$ *holds iff epistemic relation for* $CS \models^{\mathrm{E}} \Pi(p(o) \rightarrow q(o))$ *holds and:*
*1.* $C^{\neg p(o)} \neq \emptyset$
*2.* $\lambda^{p(o)\rightarrow q(o)} > \overline{f}(\lambda^{\neg p(o)\rightarrow q(o)})$ *or* $\lambda^{p(o)\rightarrow q(o)} < \underline{f}(\lambda^{\neg p(o)\rightarrow q(o)})$
*(the conditional relation holds).*

It is required that $C^{\neg p(o)} \neq \emptyset$. This means that the antecedent can't be already known to hold. This condition together with the condition $C^{p(o)} \neq \emptyset$ for the normal epistemic relation implies that the agent must not know whether the antecedent holds or not. Both of these conditions intuitively ensure the meeting of implicature I2.1 and related constraint C2.1.

It is also additionally required that the conditional relation between $p(o)$ and $q(o)$ holds. This ensures that there is a dependence between the antecedent and the consequent. The conditional relation intuitively ensures the meeting of implicature I2.3 and related constraint C2.3.

It is required for the agent not to know whether the consequent holds or not (implicature I2.2 and constraint C2.3). When the consequent is known to hold (not hold) in advance the grounding strength $\lambda^{p(o)\rightarrow q(o)}$ is equal to 1 (0 respectively). The constraint is met because of the required conditional relation as it is impossible that $1 = \lambda^{p(o)\rightarrow q(o)} > \overline{f}(\lambda^{\neg p(o)\rightarrow q(o)}) = \overline{f}(0)$ or $0 = \lambda^{p(o)\rightarrow q(o)} < \underline{f}(\lambda^{\neg p(o)\rightarrow q(o)}) = \underline{f}(0)$.

When the pragmatic epistemic relation is used to ground a conditional it imposes a meaning that is more constrained than by the normal epistemic relation. The agent must not know neither the antecedent, nor the consequent. Furthermore the agent must have noticed some dependence between the antecedent and the consequent. The chance for the consequent must be significantly different between the situations where the antecedent holds and the consequent holds. When the agent utters a conditional $p(o) \rightarrow Bel(q(o))$, grounded with the pragmatic epistemic relation 6.9, she also means that: 'I do not know if the antecedent holds', 'I do not know if the consequent holds', 'The antecedent influences the consequent', 'If the antecedent holds then the consequent rather holds' Agent's conclusions are based on previous observations of the environment. The antecedent may increase or decrease the chance for the consequent. The direction of the change of this chance is not specified.

Intuitively the pragmatic epistemic relation allows for example for grounding of a conditional 'If you do not smoke, then it is possible you will get a lung cancer'. Not smoking does not cause cancer, but it is still possible one will get it. The pragmatic epistemic relation does not allow for grounding of not related features such as: 'If the Earth is round, then I can jump 1 meter high'.

Functions $\underline{f}$ and $\overline{f}$ are called the *boundary functions* and should meet the criteria provided in definition 6.5. These criteria are only *initial* as they do not guarantee meeting of all the simultaneous usage constraints (see table 5.2). Similarly to the grounding thresholds, the boundary functions must meet a series of limitations in order to meet all of the simultaneous usage constraints. Final limitations on the grounding thresholds are presented in chapter 7.

## 6.7. Strictly pragmatic epistemic relations

Definitions 6.16 and 6.17 have been designed to meet the most demanding group of grounding conditions C1, C2 and C3.

**Definition 6.16.** *Let* $\Pi \in \{Pos, Bel, Know\}$. *Strictly pragmatic epistemic relation* $CS \models^{\mathrm{SPE}} p(o) \to \Pi(q(o))$ *holds iff epistemic relation for* $CS \models^{\mathrm{E}} p(o) \to \Pi(q(o))$ *holds and:*
*1.* $C^{\neg p(o)} \neq \emptyset$
*2.* $\lambda^{p(o)\to q(o)} > \overline{f}(\lambda^{\neg p(o)\to q(o)})$ *(strict conditional relation holds).*

**Definition 6.17.** *Let* $\Pi \in \{Pos, Bel, Know\}$. *Strictly pragmatic epistemic relation* $CS \models^{\mathrm{SPE}} \Pi(p(o) \to q(o))$ *holds iff epistemic relation for* $CS \models^{\mathrm{E}} \Pi(p(o) \to q(o))$ *holds and:*
*1.* $C^{\neg p(o)} \neq \emptyset$
*2.* $\lambda^{p(o)\to q(o)} > \overline{f}(\lambda^{\neg p(o)\to q(o)})$ *(strict conditional relation holds).*

The strictly pragmatic epistemic relations differ from the pragmatic epistemic relations in the condition on the conditional relation. In strictly pragmatic epistemic relations the strict conditional relation is required. This means that the chance for the consequent must be higher when the antecedent holds than when it does not hold.

The strictly pragmatic epistemic relations are meant to impose conventional and 'full' (see the smoker and cancer example given by table 5.1) understanding of the conditional. An exemplary conditional $p(o) \to Bel(q(o))$ grounded according to the conditional relation means that: 'I do not know neither the antecedent, nor the consequent', 'The antecedent influences the consequent', 'If the antecedent holds then the consequent rather holds', 'If the antecedent does not hold then the consequent also rather does not hold'.

The antecedent, if it holds, increases the chance for the consequent. If the antecedent does not hold that chance is much smaller.

The strictly pragmatic epistemic relations provide very constrained and much specified meaning of conditionals. This meaning possesses all of conventionally assumed implicatures upon hearing an indicative conditional. Intuitively the strictly epistemic relation allows for grounding of a statement ‘If you smoke then it is possible you will get a lung cancer’. The strictly pragmatic epistemic relation does not allow for grounding of statements like: ‘If the apple is green then it is possible it is ripe’, ‘If the Earth is round then I can jump 2 meters’. These statements are allowed respectively by the pragmatic epistemic relation and the normal epistemic relation.

The upper boundary function $\overline{f}$ should meet criteria provided in definition 6.5. Again these criteria are only *initial* as they do not guarantee meeting of all the simultaneous usage constraints (see table 5.2). Similarly to the grounding thresholds, the upper boundary function must meet a series of limitations in order to meet all of the simultaneous usage constraints. Final limitations on the grounding thresholds are presented in chapter 7.

# 7. Properties of epistemic relations for conditionals

Proposed definitions of epistemic relations for conditional formulas (definitions 6.8-6.17) require proper setting of grounding thresholds and boundary functions. The choice of these parameters has a crucial influence on the behavior of epistemic relations. Badly chosen parameters can lead to uttering of unreasonable or even contradictory sets of statements.

Within this chapter a series of theorems that provide limitations on the grounding thresholds and the boundary functions are presented. Meeting of these limitations ensures meeting of the simultaneous usage constraints provided in table 5.2.

In the end it is proven that these parameters can be chosen so that all of common-sense requirements are met. Choosing a grounding threshold setting meeting 7.5 ensures agent rationally uses conditionals according their simple meaning defined by normal epistemic relation (see constraints group C1). Similarly, choosing a pragmatic epistemic relation, grounding thresholds and boundary functions meeting 7.5, 7.7, 7.8 ensures that agent rationally uses conditionals according to their pragmatic meaning (see constraints C2). Finally, choosing a strictly pragmatic epistemic relation, grounding thresholds and boundary functions with accordance to 7.5, 7.15, guarantees conditionals are used in their most restrictive meaning (see constraint C3).

In further sections 7.1, 7.2 and 7.3, theorems related to the normal, pragmatic and strictly pragmatic epistemic relations respectively are formulated and proven. Section 7.4 defines conditions that bind conditional modalities and modal conditionals. In the last section 7.6 all of the theorems and the constraints are summarized and compared to common-sense usage criteria provided earlier.

Within all theorems a fixed and constant agent's cognitive state model $C$ is assumed. If not stated differently, variables $\Pi, \Xi \in \{Pos, Bel, Know\}$ denote any of the modal operators. For clarity each theorem is marked with: (Normal), (Pragmatic), (Strict) or (ALL) meaning it applies to normal, pragmatic, strictly pragmatic epistemic relation or all of them respectively.

## 7.1. Theorems for normal epistemic relation

**Theorem 7.1.** *(Normal) Epistemic relation* $CS \models^{\mathrm{E}} p \rightarrow Know(q)$ *can be met.*

*Proof.* Grounding sets and strength setting such that $\lambda^{p \rightarrow q} = 1$ allows for epistemic relation for knowledge operator. □

**Theorem 7.2.** *(Normal) Epistemic relation* $CS \models^{\mathrm{E}} p \rightarrow Bel(q)$ *can be met iff* $\lambda_{minBel} < \lambda_{maxBel}$*,* $\lambda_{minBel} < 1$ *and* $0 < \lambda_{maxBel}$*.*

*Proof.* Grounding strength $\lambda^{p \rightarrow q}$ belongs to $[0, 1]$. If $\lambda_{minBel} < 1$ or $0 < \lambda_{maxBel}$ is not met, then grounding strength requirement $\lambda_{minBel} < \lambda^{p \rightarrow q} < \lambda_{maxBel}$ can't be met. Similarly for $\lambda_{minBel} < \lambda_{maxBel}$.

If for some $\lambda^{p \rightarrow q}$, $\lambda_{minBel} < \lambda^{p \rightarrow q} < \lambda_{maxBel}$, all three inequalities must hold. □

**Theorem 7.3.** *(Normal) Epistemic relation* $CS \models^{\mathrm{E}} p \rightarrow Pos(q)$ *can be met iff* $\lambda_{minPos} < \lambda_{maxPos}$*,* $\lambda_{minPos} < 1$ *and* $0 \leqslant \lambda_{maxPos}$*.*

*Proof.* Proof of theorem 7.3 is similar to the proof of theorem 7.2. □

**Theorem 7.4.** *(Normal) Let conditions given by theorems 7.1-7.3 be met and let* $\Pi \neq \Xi$*. It is possible to set grounding thresholds, so that meeting of epistemic relation* $CS \models^{\mathrm{E}} p \rightarrow \Pi(q)$ *excludes meeting of epistemic relation* $CS \models^{\mathrm{E}} p \rightarrow \Xi(q)$*.*
*A sufficient and necessary condition is:*

$$\lambda_{maxPos} \leqslant \lambda_{minBel} < \lambda_{maxBel} \leqslant 1.$$

*Proof.* Definitions 6.8, 6.9, 6.10 differ only on grounding strengths requirements. We have:

definition 6.8: $\lambda_{minPos} < \lambda^{p \rightarrow q} \leqslant \lambda_{maxPos}$
definition 6.9: $\lambda_{minBel} < \lambda^{p \rightarrow q} < \lambda_{maxBel}$
definition 6.10: $\lambda^{p \rightarrow q} = 1$

Condition provided by theorem 7.4 ensures that grounding strength requirements given above are disjoint. This proves sufficiency.

Proof of necessity can be performed by contradiction.

If $\lambda_{maxBel} \leqslant 1$ is not met, then epistemic relations for $p \rightarrow Bel(q)$ and $p \rightarrow Know(q)$ can be met simultaneously for $\lambda^{p \rightarrow q} = 1$.

If $\lambda_{maxPos} < 1$ is not met, then epistemic relations for $p \rightarrow Pos(q)$ and $p \rightarrow Know(q)$ can be met simultaneously for $\lambda^{p \rightarrow q} = 1$.

If $(\lambda_{minPos}, \lambda_{maxPos}] \cap (\lambda_{minBel}, \lambda_{maxBel}) \neq \emptyset$, then there exists $\lambda^{p \rightarrow q}$ that meets both $p \rightarrow Pos(q)$ and $p \rightarrow Bel(q)$. Due to initial restriction 6.10, intersection of both sets can be empty only when $\lambda_{maxPos} \leqslant \lambda_{minBel}$. □

**Theorem 7.5.** *(Normal) Let the condition given by theorem 7.4 be met. It is possible to set grounding thresholds so that meeting of epistemic relation $CS \models^{\mathrm{E}} p \to Know(q)$ excludes meeting of normal epistemic relation $CS \models^{\mathrm{E}} p \to \Pi(\neg q)$. A sufficient and necessary condition is:*

$$0 \leqslant \lambda_{minPos}$$

*Proof.* Suppose by contradiction that both epistemic relations are met. They can be met only if $\lambda^{p\to q} = 1$ and $\lambda^{p\to\neg q} = 1 - \lambda^{p\to q} = 0$. Grounding threshold requirement:

— $\lambda_{minPos} < \lambda^{p\to\neg q} \leqslant \lambda_{maxPos}$ is met only if $\lambda_{minPos} < 0$.
— $\lambda_{minBel} < \lambda^{p\to\neg q} < \lambda_{maxBel}$ is met only if $\lambda_{minBel} < 0$.
— $\lambda^{p\to q} = 1$ is never met.

This implies both epistemic relations can be met only if $\lambda_{minPos} < 0$ or $\lambda_{minBel} < 0$. From initial requirement 6.10 we already know $\lambda_{minPos} \leqslant \lambda_{minBel}$. In result one obtains $\lambda_{minPos} < 0$. This proves sufficiency.

If by contradiction $0 > \lambda_{minPos}$, then a grounding strength setting $\lambda^{p\to q} = 1$, $\lambda^{\neg p\to q} = 0.5\lambda_{maxPos}$ meets epistemic relations for both formulas. This proves necessity. □

Conditions provided by theorems 7.4 and 7.5 together lead to an initial constraints on grounding thresholds 7.1. These are necessary and sufficient conditions that ensure epistemic relations for conditional formulas can be met at all. Furthermore, constraints ensure that at most one modal operator can be used. I.e. It is impossible to use two different modal operators for the same conditional statement in the same situation.

**Initial constraints for grounding thresholds:**

$$0 \leqslant \lambda_{minPos} < \lambda_{maxPos} \leqslant \lambda_{minBel} < \lambda_{maxBel} \leqslant 1 \tag{7.1}$$

Most of theorems provided later assume this constraint setting.

**Theorem 7.6.** *(Normal) Assume a grounding threshold setting meeting inequalities 7.1. Normal epistemic relations $CS \models^{\mathrm{E}} p \to \Pi(q)$ and $CS \models^{\mathrm{E}} \neg p \to \Xi(q)$ can be both met.*

*Proof.* Required grounding strength for $p \to \Pi(q)$ is:

$$\lambda^{p\to q} = \frac{\mathrm{card}(C^{p\wedge q})}{\mathrm{card}(C^{p\wedge q}) + \mathrm{card}(C^{p\wedge\neg q})}$$

respective grounding strength for $\neg p \to \Xi(q)$ is:

$$\lambda^{\neg p\to q} = \frac{\mathrm{card}(C^{\neg p\wedge q})}{\mathrm{card}(C^{\neg p\wedge q}) + \mathrm{card}(C^{\neg p\wedge\neg q})}$$

Pairs of sets $C^{p\wedge q}, C^{p\wedge\neg q}$ and $C^{\neg p\wedge q}, C^{\neg p\wedge\neg q}$ can be fixed independently. Grounding strength $\lambda^{p\to q}$ does not depend on $\lambda^{\neg p\to q}$. Both statements can be concurrently grounded regardless of chosen modal operators and grounding thresholds. □

**Theorem 7.7.** *(Normal) Assume a grounding threshold setting meeting inequalities 7.1. Epistemic relations $CS \models^{\mathrm{E}} p \rightarrow \Pi(q)$ and $CS \models^{\mathrm{E}} \neg p \rightarrow \Xi(\neg q)$ can be both met.*

*Proof.* Inequalities 7.1 ensure all statement types can be grounded according to normal epistemic relation. Grounding strengths $\lambda^{p\rightarrow q}$ and $\lambda^{\neg p \rightarrow \neg q}$ are calculated from disjoint grounding sets: $C^{p\wedge q} \cup C^{p\wedge\neg q}$ and $C^{\neg p\wedge\neg q} \cup C^{\neg p\wedge q}$. One can obtain any desired values of these grounding strengths. □

**Theorem 7.8.** *(Normal) Assume a grounding threshold setting meeting inequalities 7.1. It is possible to set grounding thresholds, so that meeting of epistemic relation $CS \models^{\mathrm{E}} p \rightarrow Bel(q)$ excludes meeting of epistemic relation $CS \models^{\mathrm{E}} p \rightarrow Bel(\neg q)$. A necessary and sufficient condition is:*

$$\lambda_{minBel} \geqslant 0.5 \;\vee\; 0.5 \leqslant \lambda_{maxBel}$$

*Proof.* Assume by contradiction that condition from theorem 7.8 is met and both statements can be grounded. Then, there exist $\lambda^{p\rightarrow q}$, $\lambda^{p\rightarrow\neg q}$ such that:

$$\lambda_{minBel} < \lambda^{p\rightarrow q} < \lambda_{maxBel} \text{ and } \lambda_{minBel} < \lambda^{p\rightarrow\neg q} < \lambda_{maxBel}$$

Adding two inequalities leads to:

$$2\lambda_{minBel} < \lambda^{p\rightarrow q} + \lambda^{p\rightarrow\neg q} < 2\lambda_{maxBel}$$

Both grounding strengths are dependent ($\lambda^{p\rightarrow\neg q} = 1 - \lambda^{p\rightarrow q}$):

$$2\lambda_{minBel} < 1 < 2\lambda_{maxBel}$$

what is contradictory to the condition from theorem 7.8. This proves sufficiency.

Now assume by contradiction that condition from theorem 7.8 is not met and both statements can't be grounded. When the condition is not met, then a contradictory condition $\lambda_{minBel} < 0.5 < \lambda_{maxBel}$ is met.

Let $\lambda^{p\rightarrow q} = \lambda^{p\rightarrow\neg q} = 0.5$ and $\lambda^{\neg p\rightarrow q} = 0$, $\lambda^{\neg p\rightarrow\neg q} = 1 - \lambda^{\neg p\rightarrow q} = 1$. For such setting both formulas meet normal epistemic relation for belief operator:

$$\lambda_{minBel} < \lambda^{p\rightarrow q} = \lambda^{p\rightarrow\neg q} = 0.5 < \lambda_{maxBel}$$

We obtain a contradiction, what proves necessity. □

**Theorem 7.9.** *(Normal) Assume a grounding threshold setting meeting inequalities 7.1. It is possible to set grounding thresholds so that normal epistemic relations $CS \models^{\mathrm{E}} p \rightarrow Pos(q)$ and $CS \models^{\mathrm{E}} p \rightarrow Bel(\neg q)$ can be both met. A necessary and sufficient condition is:*

$$\lambda_{minPos} + \lambda_{minBel} < 1 < \lambda_{maxPos} + \lambda_{maxBel}$$

*Proof.* From definition 6.8 for $p \rightarrow Pos(q)$ we know that:

$$\lambda_{minPos} < \lambda^{p \rightarrow q} \leqslant \lambda_{maxPos} \tag{7.2}$$

and from definition 6.9 for $p \rightarrow Bel(\neg q)$ and definition of $\lambda^{p \rightarrow q}$ we know that:

$$\lambda_{minBel} < \lambda^{p \rightarrow \neg q} = 1 - \lambda^{p \rightarrow q} < \lambda_{maxBel} \tag{7.3}$$

Meeting of both conditions requires that their sums are also met:

$$\lambda_{minPos} + \lambda_{minBel} < 1 < \lambda_{maxPos} + \lambda_{maxBel}$$

This proves condition's necessity.

To prove condition's sufficiency we have to show that there always exists a setting of $\lambda^{p \rightarrow q}$ $(0 \leqslant \lambda^{p \rightarrow q} \leqslant 1)$ and $\lambda^{p \rightarrow \neg q} = 1 - \lambda^{p \rightarrow q}$ meeting inequalities 7.2 and 7.3. We shall consider four mutually exclusive cases:

**Case 1:** Assumption: a. $\lambda_{minBel} + \lambda_{maxPos} < 1$ and b. $\lambda_{maxBel} + \lambda_{minPos} > 1$

In such case we assume[1]:

$$\lambda^{p \rightarrow q} = \frac{\lambda_{minPos} + \lambda_{maxPos}}{2}$$

Proving inequality 7.2 is straightforward (see restriction 7.1):

$$\lambda_{minPos} < \lambda^{p \rightarrow q} = \frac{\lambda_{minPos} + \lambda_{maxPos}}{2} < \lambda_{maxPos}$$

Now we will prove inequality 7.3:

$$\begin{aligned}
\lambda^{p \rightarrow \neg q} = 1 - \frac{\lambda_{minPos} + \lambda_{maxPos}}{2} &> \quad \text{(assumption a)} \\
1 - \frac{\lambda_{minPos} + 1 - \lambda_{minBel}}{2} &> \quad \text{(condition from theorem 7.9)} \\
1 - \frac{1 - \lambda_{minBel} + 1 - \lambda_{minBel}}{2} &= \quad \lambda_{minBel}
\end{aligned}$$

and

$$\begin{aligned}
\lambda^{p \rightarrow \neg q} = 1 - \frac{\lambda_{minPos} + \lambda_{maxPos}}{2} &< \quad \text{(assumption b)} \\
1 - \frac{1 - \lambda_{maxBel} + \lambda_{maxPos}}{2} &< \quad \text{(condition from theorem 7.9)} \\
1 - \frac{1 - \lambda_{maxBel} + 1 - \lambda_{maxBel}}{2} &= \quad \lambda_{maxBel}
\end{aligned}$$

[1] All settings assumed within this proof meet requirement: $0 \leqslant \lambda^{p \rightarrow q} \leqslant 1$

This finishes the proof for case 1.

**Case 2:** Assumption: a. $\lambda_{minBel} + \lambda_{maxPos} < 1$ and b. $\lambda_{maxBel} + \lambda_{minPos} \leqslant 1$

In such case we assume:

$$\lambda^{p \to q} = \frac{\lambda_{maxPos} + 1 - \lambda_{maxBel}}{2}$$

First we will prove inequality 7.2:

$$\begin{aligned} \lambda^{p \to q} = \frac{\lambda_{maxPos} + 1 - \lambda_{maxBel}}{2} &\geqslant \quad \text{(assumption b)} \\ \frac{\lambda_{maxPos} + 1 - 1 + \lambda_{minPos}}{2} &= \\ \frac{\lambda_{minPos} + \lambda_{maxPos}}{2} &> \lambda_{minPos} \end{aligned}$$

and

$$\begin{aligned} \lambda^{p \to q} = \frac{\lambda_{maxPos} + 1 - \lambda_{maxBel}}{2} &< \quad \text{(condition from theorem 7.9)} \\ \frac{\lambda_{maxPos} + 1 - 1 + \lambda_{maxPos}}{2} &= \lambda_{maxPos} \end{aligned}$$

Now we will prove inequality 7.3:

$$\begin{aligned} \lambda^{p \to \neg q} = 1 - \frac{\lambda_{maxPos} + 1 - \lambda_{maxBel}}{2} &> \quad \text{(assumption a)} \\ 1 - \frac{1 - \lambda_{minBel} + 1 - \lambda_{maxBel}}{2} &= \\ \frac{\lambda_{minBel} + \lambda_{maxBel}}{2} &> \lambda_{minBel} \end{aligned}$$

and

$$\begin{aligned} \lambda^{p \to \neg q} = 1 - \frac{\lambda_{maxPos} + 1 - \lambda_{maxBel}}{2} &< \quad \text{(condition from theorem 7.9)} \\ 1 - \frac{1 - \lambda_{maxBel} + 1 - \lambda_{maxBel}}{2} &= \lambda_{maxBel} \end{aligned}$$

This finishes the proof for case 2.

**Case 3:** Assumption: a. $\lambda_{minBel} + \lambda_{maxPos} \geqslant 1$ and b. $\lambda_{maxBel} + \lambda_{minPos} > 1$

In such case we assume:

$$\lambda^{p \to q} = \frac{\lambda_{minPos} + 1 - \lambda_{minBel}}{2}$$

First we will prove inequality 7.2:

$$\begin{aligned} \lambda^{p \to q} = \frac{\lambda_{minPos} + 1 - \lambda_{minBel}}{2} &> \quad \text{(condition from theorem 7.9)} \\ \frac{\lambda_{minPos} + 1 - 1 + \lambda_{minPos}}{2} &= \quad \lambda_{minPos} \end{aligned}$$

and

$$\begin{aligned} \lambda^{p \to q} = \frac{\lambda_{minPos} + 1 - \lambda_{maxBel}}{2} &\leqslant \quad \text{(assumption a)} \\ \frac{\lambda_{minPos} + 1 - 1 + \lambda_{maxPos}}{2} &< \quad \lambda_{maxPos} \end{aligned}$$

Now we will prove inequality 7.3:

$$\begin{aligned} \lambda^{p \to \neg q} = 1 - \frac{\lambda_{minPos} + 1 - \lambda_{minBel}}{2} &> \quad \text{(condition from theorem 7.9)} \\ 1 - \frac{1 - \lambda_{minBel} + 1 - \lambda_{minBel}}{2} &= \quad \lambda_{minBel} \end{aligned}$$

and

$$\begin{aligned} \lambda^{p \to \neg q} = 1 - \frac{\lambda_{minPos} + 1 - \lambda_{minBel}}{2} &> \quad \text{(assumption b)} \\ 1 - \frac{1 - \lambda_{maxBel} + 1 - \lambda_{minBel}}{2} &= \\ \frac{\lambda_{minBel} + \lambda_{maxBel}}{2} &> \quad \lambda_{minBel} \end{aligned}$$

This ends the proof for case 3.

**Case 4:** Assumption: $\lambda_{minBel} + \lambda_{maxPos} \geqslant 1$ and $\lambda_{maxBel} + \lambda_{minPos} \leqslant 1$

In such case we assume:

$$\lambda^{p \to q} = \frac{1 - \lambda_{maxBel} + 1 - \lambda_{minBel}}{2}$$

First we will prove inequality 7.2:

$$\lambda^{p\to q} = \frac{1-\lambda_{maxBel}+1-\lambda_{minBel}}{2} > \quad \text{(condition from theorem 7.9)}$$
$$\frac{1-\lambda_{maxBel}+1-1+\lambda_{minPos}}{2} \geqslant \quad \text{(assumption b)}$$
$$\frac{1-1+\lambda_{minPos}+\lambda_{minPos}}{2} = \lambda_{minPos}$$

and

$$\lambda^{p\to q} = \frac{1-\lambda_{maxBel}+1-\lambda_{minBel}}{2} < \quad \text{(condition from theorem 7.9)}$$
$$\frac{1-1+\lambda_{maxPos}+1-\lambda_{minBel}}{2} \leqslant \quad \text{(assumption a)}$$
$$\frac{\lambda_{maxPos}+1-1+\lambda_{maxPos}}{2} = \lambda_{maxPos}$$

Proving inequality 7.3 is straightforward (see restriction 7.1):

$$\lambda_{minBel} < \lambda^{p\to\neg q} = 1-\lambda^{p\to q} = \frac{\lambda_{maxBel}+\lambda_{minBel}}{2} < \lambda_{maxBel}$$

This ends the proof for case 4. Proofs for all four cases together form a proof of condition's sufficiency in theorem 7.9. □

**Theorem 7.10.** *(Normal) Assume a grounding threshold setting meeting inequalities 7.1. It is possible to set grounding thresholds, so that normal epistemic relations* $CS \models^{\mathrm{E}} p \to Pos(q)$ *and* $CS \models^{\mathrm{E}} p \to Pos(\neg q)$ *can be both met. A necessary and sufficient condition is:*

$$\lambda_{minPos} < 0.5 \leqslant \lambda_{maxPos}$$

*Proof.* From definition 6.8 for $p \to Pos(q)$ we know that:

$$\lambda_{minPos} < \lambda^{p\to q} \leqslant \lambda_{maxPos}$$

and from the same definition for $p \to Pos(\neg q)$ we know that:

$$\lambda_{minPos} < \lambda^{p\to\neg q} \leqslant \lambda_{maxPos}$$

Meeting both conditions requires that their sums are also met:

$$2\lambda_{minPos} < 1 \leqslant 2\lambda_{maxPos}$$

dividing by 2 gives:

$$\lambda_{minPos} < 0.5 \leqslant \lambda_{maxPos}$$

this proves necessity.

To prove sufficiency we have to show that for every threshold setting, meeting condition in theorem, there exists at least one setting of $\lambda^{p\to q}$ and $\lambda^{p\to\neg q}$ that meets both epistemic relations. Assume $\lambda^{p\to q} = \lambda^{p\to\neg q} = 0.5$. From definitions of epistemic relations we get two exactly the same inequalities:

$$\lambda_{minPos} < 0.5 \leqslant \lambda_{maxPos}$$

that are the same as the assumed condition. This ends the proof. □

**Theorem 7.11.** *(Normal) Assume a grounding threshold setting meeting inequalities 7.1. Normal epistemic relations $CS \models^{\mathrm{E}} p \to \Pi(q)$ and $CS \models^{\mathrm{E}} q \to \Xi(\neg p)$ can be both met if and only if $\Xi \neq Know$.*

*Proof.* Let $c_1, c_2, c_3, c_4$ denote:

$$c_1 = \mathrm{card}(C^{p\wedge q}),\ c_2 = \mathrm{card}(C^{p\wedge\neg q}),\ c_3 = \mathrm{card}(C^{\neg p\wedge q}),\ c_4 = \mathrm{card}(C^{\neg p\wedge\neg q}).$$

In such case, grounding strengths are:

$$\lambda^{p\to q} = \frac{c_1}{c_1+c_2}, \quad \lambda^{q\to\neg p} = \frac{c_3}{c_3+c_1} \tag{7.4}$$

When $CS \models^{\mathrm{E}} p \to \Pi(q)$ is met, then $\lambda^{p\to q} > 0$ and $c_1 > 0$. This means that $\lambda^{q\to\neg p} < 1$. Both epistemic relations for $p \to \Pi(q)$ and $q \to Know(\neg p)$ can't be simultaneously met.

On the other hand one can choose $c_1, c_2, c_3, c_4$, so that grounding strengths 7.4 are independently any rational numbers such that: $\lambda^{p\to q} \in (0,1]$ and $\lambda^{q\to\neg p} \in (0,1)$. All combinations of modal operators, such that $\Xi \neq Know$, can be simultaneously grounded. □

### 7.1.1. Constraints for the grounding thresholds

**Theorem 7.12.** *(Normal) It is possible to set grounding thresholds so that all conditions from theorems 7.1-7.11 are met simultaneously. A necessary and sufficient conditions are:*

$$0 \leqslant \lambda_{minPos} < 1 - \lambda_{minBel} \leqslant 0.5 \leqslant \lambda_{maxPos} \leqslant \lambda_{minBel} < \lambda_{maxBel} \leqslant 1 \tag{7.5}$$

*Proof.* All theorems 7.1-7.11 provide necessary and sufficient conditions required for different properties of epistemic relation for conditional formulas. These conditions are:

1. $\lambda_{minPos} \leqslant \lambda_{minBel} \wedge \lambda_{maxPos} \leqslant \lambda_{maxBel}$ (from initial restriction 6.10),
2. $\lambda_{maxPos} \leqslant \lambda_{minBel} < \lambda_{maxBel} \leqslant 1$ (from 7.4)

3. $0 \leqslant \lambda_{minPos}$ (from 7.5),
4. $\lambda_{minBel} \geqslant 0.5 \ \vee \ 0.5 \leqslant \lambda_{maxBel}$ (from 7.8),
5. $\lambda_{minPos} + \lambda_{minBel} < 1 < \lambda_{maxPos} + \lambda_{maxBel}$ (from 7.9),
6. $\lambda_{minPos} < 0.5 \leqslant \lambda_{maxPos}$ (from 7.10).

Intersection of these constraints shall provide sufficient and necessary conditions for all properties to be simultaneously met.

From 1, 2 and 3 we obtain initial constraints for grounding thresholds already defined in equation 7.1:

$$0 \leqslant \lambda_{minPos} < \lambda_{maxPos} \leqslant \lambda_{minBel} < \lambda_{maxBel} \leqslant 1$$

Constraints 1 ($\lambda_{maxPos} \leqslant \lambda_{maxBel}$) and 6 ($0.5 \leqslant \lambda_{maxPos}$) imply that within constraint 4, inequality $0.5 \leqslant \lambda_{maxBel}$ is false, so $\lambda_{minBel} \geqslant 0.5$ must be true.

Inequalities $0.5 \leqslant \lambda_{maxBel}$ and $0.5 \leqslant \lambda_{maxPos}$ together imply $1 < \lambda_{maxPos} + \lambda_{maxBel}$ that is part of constraint 5.

Inequality $\lambda_{minPos} + \lambda_{minBel} < 1$ from 5 can be transformed to an equivalent form: $\lambda_{minPos} < 1 - \lambda_{minBel}$. From $\lambda_{minBel} \geqslant 0.5$ we obtain $1 - \lambda_{minBel} \leqslant 0.5$.

Finally, the intersection of all constraints can be written as:

$$0 \leqslant \lambda_{minPos} < 1 - \lambda_{minBel} \leqslant 0.5 \leqslant \lambda_{maxPos} \leqslant \lambda_{minBel} < \lambda_{maxBel} \leqslant 1$$

Any grounding threshold setting meeting constraints 1-6 also meets condition 7.5 and vice versa. □

## 7.2. Theorems for pragmatic epistemic relation

Following theorems apply to pragmatic epistemic relation as proposed by definition 6.14. Pragmatic epistemic relation is a stricter version of normal epistemic relation. It requires a dependence between the antecedent and the consequent in form of the conditonal relation (definition 6.6).

Some proofs use general lemmas provided later in section 7.2.2.

Final sufficient constraints on boundary functions that ensure proper grounding of conditionals, have been proposed in section 7.2.1.

**Theorem 7.13.** *(Normal, Pragmatic) If pragmatic epistemic relation* $CS \models^{\mathrm{PE}} p \to \Pi(q)$ *is met, then normal epistemic relation* $CS \models^{\mathrm{E}} p \to \Pi(q)$ *is also met.*

*Proof.* Normal pragmatic epistemic relation requires respective grounding thresholds to be met. Pragmatic epistemic relation requires meeting of the same grounding thresholds and additionally meeting of conditional relation. □

Theorem 7.13 states that meeting of pragmatic epistemic relation implies meeting of respective normal epistemic relation.

**Theorem 7.14.** *(Pragmatic) It is possible to set boundary functions, so that pragmatic epistemic relation $CS \models^{\mathrm{PE}} p \to Know(q)$ can be met a necessary and sufficient condition is: $\overline{f}(0) < 1$.*

*Proof.* Firstly I assume that $CS \models^{\mathrm{PE}} p \to Know(q)$ and wish to prove $\overline{f}(0) < 1$. The only grounding strength $\lambda^{p \to q}$ meeting epistemic relation for knowledge operator must be equal to 1. From conditional relation we obtain:

$$\lambda^{p\to q} > \overline{f}(\lambda^{\neg p \to q}) \quad \text{or} \quad \lambda^{p\to q} < \underline{f}(\lambda^{\neg p \to q})$$

Second inequality can't be met because: $\lambda^{p\to q} = 1 \geqslant \underline{f}(\lambda^{\neg p \to q})$. When the first inequality $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p \to q})$ is met, then:

$$1 = \lambda^{p\to q} > \overline{f}(\lambda^{\neg p \to q}) \geqslant \overline{f}(0)$$

We obtain the condition $\overline{f}(0) < 1$.

Secondly I assume $\overline{f}(0) < 1$ and wish to prove that there exists grounding strength setting meeting $CS \models^{\mathrm{PE}} p \to Know(q)$. Assume grounding strengths: $\lambda^{p\to q} = 1,\ \lambda^{\neg p \to q} = 0$. In such case:

$$\lambda^{p\to q} = 1 > \overline{f}(0) = \overline{f}(\lambda^{\neg p \to q})$$

This setting meets all criteria for pragmatic epistemic relation for knowledge operator. □

**Theorem 7.15.** *(Pragmatic) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set boundary functions and grounding thresholds so that pragmatic epistemic relation $CS \models^{\mathrm{PE}} p \to Bel(q)$ can be met. A necessary and sufficient condition is:*

$$\overline{f}(0) < \lambda_{maxBel} \quad \textit{or} \quad \underline{f}(1) > \lambda_{minBel}.$$

*Proof.* Firstly, I assume $CS \models^{\mathrm{PE}} p \to Bel(q)$ is met. In such case $\lambda_{minBel} < \lambda^{p\to q} < \lambda_{maxBel}$ and one of conditional relations: (case 1) $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p \to q})$ or (case 2) $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p \to q})$ holds. In case 1:

$$\lambda_{maxBel} > \lambda^{p\to q} > \overline{f}(\lambda^{\neg p \to q}) \geqslant \overline{f}(0)$$

and in case 2:

$$\lambda_{minBel} < \lambda^{p\to q} < \underline{f}(\lambda^{\neg p \to q}) \leqslant \underline{f}(1)$$

so indeed either $\overline{f}(0) < \lambda_{maxBel}$ or $\underline{f}(1) > \lambda_{minBel}$.

Secondly, I assume (case 1) $\overline{f}(0) < \lambda_{maxBel}$ or (case 2) $\underline{f}(1) > \lambda_{minBel}$ and prove that there exists a grounding strengths setting meeting pragmatic epistemic relation. Let's start with case 1 and a setting:

$$\lambda^{\neg p \to q} = 0, \quad \lambda^{p\to q} = \min \lambda_{maxBel} - \epsilon.$$

Where $\epsilon$ is any value from set $\epsilon \in (0, \epsilon_{max})$ and $\epsilon_{max} > 0$ is:

$$\epsilon_{max} = \min\{\lambda_{maxBel} - \lambda_{minBel},\ 1 - \overline{f}(0),\ \lambda_{maxBel} - \overline{f}(0)\}$$

Then grounding threshold requirement is met:

$$\lambda^{p \to q} = \min\{1, \lambda_{maxBel}\} - \epsilon < \lambda_{maxBel}$$
$$\lambda^{p \to q} = \lambda_{maxBel} - \epsilon > \lambda_{maxBel} - \lambda_{maxBel} + \lambda_{minBel} = \lambda_{minBel}$$

and conditional relation is met:

$$\lambda^{p \to q} = \lambda_{maxBel} - \epsilon > \lambda_{maxBel} - \lambda_{maxBel} + \overline{f}(0) = \overline{f}(\lambda^{\neg p \to q})$$

Now let's assume case 2 and a setting:

$$\lambda^{\neg p \to q} = 1, \quad \lambda^{p \to q} = \lambda_{minBel} + \epsilon.$$

Where $\epsilon$ is any value from set $\epsilon \in (0, \epsilon_{max})$ and $\epsilon_{max} > 0$ is:

$$\epsilon_{max} = \min\{\lambda_{maxBel} - \lambda_{minBel},\ \underline{f}(1) - \lambda_{minBel}\}$$

Then grounding threshold requirement is met:

$$\lambda^{p \to q} = \lambda_{minBel} + \epsilon > \lambda_{minBel}$$
$$\lambda^{p \to q} = \lambda_{minBel} + \epsilon < \lambda_{minBel} + \lambda_{maxBel} - \lambda_{minBel} = \lambda_{maxBel}$$

and conditional relation is met:

$$\lambda^{p \to q} = \lambda_{minBel} + \epsilon < \lambda_{minBel} + \underline{f}(1) - \lambda_{minBel} = \underline{f}(1) = \underline{f}(\lambda^{\neg p \to q})$$

This ends proof. □

**Theorem 7.16.** *(Pragmatic) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set grounding thresholds and boundary function so that pragmatic epistemic relation $CS \models^{\mathrm{PE}} p \to Pos(q)$ can be met. A necessary and sufficient condition is:*

$$\overline{f}(0) < \lambda_{maxPos} \quad or \quad \underline{f}(1) > \lambda_{minPos}.$$

*Proof.* Proof of theorem 7.16 is similar to the proof of theorem 7.15. One needs to change grounding thresholds respectively. □

Theorems 7.14-7.16 provide sufficient and necessary conditions that ensure all pragmatic epistemic relations can be met at all. They are similar to theorems 7.1-7.3 for normal epistemic relation.

Because of dependence between normal and pragmatic epistemic relations (theorem 7.13), theorem 7.4 provides sufficient conditions that ensure at most one of formulas

$p \to Know(q)$, $p \to Bel(q)$, $p \to Pos(q)$ can meet pragmatic epistemic relation concurrently.

Normal epistemic relation allows for simultaneous grounding of statements $p \to \Pi(q)$ and $\neg p \to \Xi(q)$ regardless of chosen modal operators. Pragmatic epistemic relation also allows it, if at least one of modal operators is not a knowledge operator. If one says both $p \to Know(q)$ and $\neg p \to Know(q)$, then he simply knows $q$. Using of conditional statement is not rational in this situation and should be denied by pragmatic epistemic relation (theorem 7.21). Theorems 7.17-7.22 provide conditions that ensure all remaining pairs are accepted.

**Theorem 7.17.** *(Pragmatic) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Bel(q)$ and $CS \models^{\mathrm{PE}} \neg p \to Bel(q)$ can be both met. A sufficient and necessary condition is:*

$$\lambda_{maxBel} > \overline{f}(\lambda_{minBel}) \quad and \quad \underline{f}(\lambda_{maxBel}) > \lambda_{minBel}$$

*Proof.* The proof of theorem 7.17 can be implied directly from lemmas 7.41 and 7.42. Assume $\lambda_{min} = \lambda_{minBel}$ and $\lambda_{max} = \lambda_{maxBel}$. If $\lambda_{max} > \overline{f}(\lambda_{min})$ and $\underline{f}(\lambda_{max}) > \lambda_{min}$, then there exist numbers $\underline{\lambda} = \lambda^{p \to q}$ and $\overline{\lambda} = \lambda^{\neg p \to q}$ that meet all requirements of pragmatic epistemic relation (lemma 7.42). Otherwise such numbers don't exist (lemma 7.41). □

**Theorem 7.18.** *(Pragmatic) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Pos(q)$ and $CS \models^{\mathrm{PE}} \neg p \to Bel(q)$ can be both met. A sufficient and necessary condition is:*

$$\lambda_{maxBel} > \overline{f}(\lambda_{minPos}) \quad and \quad \underline{f}(\lambda_{maxBel}) > \lambda_{minPos}$$

*Proof.* Assume $\lambda_{maxBel} > \overline{f}(\lambda_{minPos})$ and $\underline{f}(\lambda_{maxBel}) > \lambda_{minPos}$. From lemma 7.42 ($\lambda_{min} = \lambda_{minPos}$, $\lambda_{max} = \lambda_{maxBel}$) we already know that there exist numbers $\underline{\lambda} = \lambda^{p \to q}$ and $\overline{\lambda} = \lambda^{p \to q}$ meeting:

$$\lambda_{minPos} < \lambda^{p \to q} < \lambda_{maxBel}, \ \lambda_{minPos} < \lambda^{\neg p \to q} < \lambda_{maxBel},$$
$$\lambda^{p \to q} < \underline{f}(\lambda^{\neg p \to q}), \ \lambda^{\neg p \to q} > \overline{f}(\lambda^{p \to q})$$

I only need to prove that among all such numbers there always exist ones that also meet:

$$\lambda^{p \to q} \leqslant \lambda_{maxPos} \quad \text{and} \quad \lambda_{minBel} < \lambda^{\neg p \to q}$$

For that I shall use lemma 7.36. Numbers $\underline{\lambda}, \overline{\lambda}$ can be constructed as $\underline{\lambda} = \lambda_{minPos} + \epsilon$ and $\overline{\lambda} = \lambda_{maxBel} - \epsilon$. Where $\epsilon$ is any value between 0 and $\epsilon_{max}$. It is enough to choose $\epsilon$ small enough to fit $\lambda^{p \to q}, \lambda^{\neg p \to q}$ into the provided inequalities. Let:

$$0 < \epsilon < \min\{\epsilon_{max}, \ \lambda_{maxBel} - \lambda_{minBel}, \ \lambda_{maxPos} - \lambda_{minPos}\}$$

then:

$$\lambda^{p\to q} = \lambda_{minPos} + \epsilon < \lambda_{maxPos}$$
$$\lambda^{\neg p\to q} = \lambda_{maxBel} - \epsilon > \lambda_{maxBel} - \lambda_{maxBel} + \lambda_{minBel} = \lambda_{minBel}$$

This proves condition from theorem 7.18 is sufficient.

Now I shall prove necessity. The only possible setting of inequalities related to boundary functions is:

$$\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q}),\ \lambda^{\neg p\to q} > \overline{f}(\lambda^{p\to q}).$$

Other settings are impossible, because we quickly obtain contradictions:

$$\lambda_{maxPos} \geqslant \lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q}) \geqslant \lambda^{\neg p\to q} > \lambda_{minBel} \geqslant \lambda_{maxPos}$$
$$\lambda_{minBel} < \lambda^{\neg p\to q} < \underline{f}(\lambda^{p\to q}) \leqslant \lambda^{p\to q} \leqslant \lambda_{minPos} < \lambda_{minBel}$$

If $\lambda_{maxBel} > \overline{f}(\lambda_{minPos})$ and $\underline{f}(\lambda_{maxBel}) > \lambda_{minPos}$ is not met, then there is no setting $\underline{\lambda} = \lambda^{p\to q}$, $\overline{\lambda} = \lambda^{\neg p\to q}$ that would meet all conditions for pragmatic epistemic relation (from lemma 7.41). □

**Theorem 7.19.** *(Pragmatic) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Pos(q)$ and $CS \models^{\mathrm{PE}} \neg p \to Pos(q)$ can be both met. A sufficient and necessary condition is:*

$$\lambda_{maxPos} > \overline{f}(\lambda_{minPos}) \quad \textit{and} \quad \underline{f}(\lambda_{maxPos}) > \lambda_{minPos}$$

*Proof.* The proof of theorem 7.17 can be implied directly from lemmas 7.41 and 7.42. Assume $\lambda_{min} = \lambda_{minPos}$ and $\lambda_{max} = \lambda_{maxPos}$. If $\lambda_{max} > \overline{f}(\lambda_{min})$ and $\underline{f}(\lambda_{max}) > \lambda_{min}$, then there exist numbers $\underline{\lambda} = \lambda^{p\to q}$ and $\overline{\lambda} = \lambda^{\neg p\to q}$ that meet all requirements of pragmatic epistemic relation (lemma 7.42). Otherwise such numbers don't exist (lemma 7.41). □

**Lemma 7.20.** *(Pragmatic) Meeting of all conditions provided by theorems 7.14 - 7.19 implies that:*

$$\overline{f}(0) < \lambda_{maxPos} \quad \textit{and} \quad \underline{f}(1) > \lambda_{minBel} \tag{7.6}$$

Joining the conditions from theorems 7.14 - 7.19 leads to a more general requirements as stated by lemma 7.20. These requirements shall be asserted in most of the following theorems regarding pragmatic epistemic relation.

*Proof.* From theorem 7.19 we obtain:

$$\lambda_{maxPos} > \overline{f}(\lambda_{minPos}) \geqslant \overline{f}(0)$$

and from theorem 7.17 we obtain:

$$\lambda_{minBel} < \underline{f}(\lambda_{minBel}) \leqslant \underline{f}(1)$$

□

**Theorem 7.21.** *(Pragmatic) Meeting of pragmatic epistemic relation* $CS \models^{\mathrm{PE}} p \to Know(q)$ *excludes meeting of pragmatic epistemic relation* $CS \models^{\mathrm{PE}} \neg p \to Know(q)$.

*Proof.* Assume by contradiction that both pragmatic epistemic relations hold. In such case $\lambda^{p \to q} = 1$ and $\lambda^{\neg p \to q} = 1$. From conditional relation and an upper boundary function ($\overline{f}(x) \geqslant x$) we obtain a contradiction:

$$1 = \lambda^{p \to q} > \overline{f}(\lambda^{\neg p \to q}) \geqslant \lambda^{\neg p \to q} = 1$$

similarly from conditional relation and an lower boundary function ($\underline{f}(x) \leqslant x$) we obtain a contradiction:

$$1 = \lambda^{p \to q} < \underline{f}(\lambda^{\neg p \to q}) \leqslant \lambda^{\neg p \to q} = 1$$

This ends proof. □

**Theorem 7.22.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. Let* $\Pi \neq Know$. *Pragmatic epistemic relations* $CS \models^{\mathrm{PE}} p \to Know(q)$ *and* $CS \models^{\mathrm{PE}} \neg p \to \Pi(q)$ *can be both met.*

*Proof.* $CS \models^{\mathrm{PE}} p \to Know(q)$ implies $\lambda^{p \to q} = 1$. In order for $CS \models^{\mathrm{PE}} \neg p \to \Pi(q)$ to be met, one requires:

$$\lambda^{\neg p \to q} < \underline{f}(\lambda^{p \to q}) = \underline{f}(1)$$

From the proof of theorem 7.6 we already know $\lambda^{p \to q}$ and $\lambda^{\neg p \to q}$ can be fixed independently. When $\underline{f}(1) > \lambda_{minBel}$ (see equation 7.6) one can choose any $\lambda^{\neg p \to q}$ that meets respective grounding threshold setting for possibility ($\lambda_{minPos} < \lambda^{\neg p \to q} < \lambda_{maxPos} \leqslant \lambda_{minBel}$) or belief operator ($\lambda_{minBel} < \lambda^{\neg p \to q} < \underline{f}(1)$). □

Normal epistemic relation allows for simultaneous meeting of $p \to \Pi(q)$ and $\neg p \to \Xi(\neg q)$ (theorem 7.7). So does pragmatic epistemic relation, when constraints provided by theorems 7.23-7.28 are met.

**Theorem 7.23.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. Epistemic relations* $CS \models^{\mathrm{PE}} p \to Know(q)$ *and* $CS \models^{\mathrm{PE}} \neg p \to Know(\neg q)$ *can be both met.*

*Proof.* Let grounding strengths $\lambda^{p \to q} = 1$ and $\lambda^{\neg p \to \neg q} = 1$. Grounding threshold requirements are met. Conditional relations are also met, because:

$$\lambda^{p \to q} = 1 > \lambda_{maxPos} > \overline{f}(0) = \overline{f}(1 - \lambda^{\neg p \to \neg q}) = \overline{f}(\lambda^{\neg p \to q})$$
$$\lambda^{\neg p \to \neg q} = 1 > \lambda_{maxPos} > \overline{f}(0) = \overline{f}(1 - \lambda^{p \to q}) = \overline{f}(\lambda^{p \to \neg q})$$

Pragmatic epistemic relation is met for both formulas. □

**Theorem 7.24.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations* $CS \models^{\mathrm{PE}} p \to$

*Know(q) and CS* $\models^{\text{PE}} \neg p \rightarrow Bel(\neg q)$ *can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1-\lambda_{maxBel}) < 1$$

*Proof.* Assume both pragmatic epistemic relations are met, then $\lambda^{p\rightarrow q} = 1$, $\lambda^{p\rightarrow\neg q} = 0$ and $\lambda_{minBel} < \lambda^{\neg p\rightarrow\neg q} < \lambda_{maxBel}$. Conditional relation also must be met. The case where $\lambda^{p\rightarrow q} < \underline{f}(\lambda^{\neg p\rightarrow q})$ can't hold. So $\lambda^{p\rightarrow q} > \overline{f}(\lambda^{\neg p\rightarrow q})$ must be met and we obtain:

$$1 = \lambda^{p\rightarrow q} > \overline{f}(\lambda^{\neg p\rightarrow q}) = \overline{f}(1-\lambda^{\neg p\rightarrow\neg q}) \geqslant \overline{f}(1-\lambda_{maxBel}).$$

This proves necessity.

To prove condition's sufficiency I shall construct a grounding strengths setting meeting both pragmatic epistemic relations.

Let $\lambda_{min} = 1-\lambda_{maxBel}$ and $\lambda_{max} = 1$. From lemma 7.35 we know, there exists $\epsilon^{(1)}_{max} > 0$, such that for every $\epsilon \in (0, \epsilon^{(1)}_{max})$, $\lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$.

Similarly, let $\lambda_{min} = 0$ and $\lambda_{max} = \lambda_{maxBel}$, from the same lemma 7.35 we know there exists $\epsilon^{(2)}_{max} > 0$ such that for every $\epsilon \in (0, \epsilon^{(2)}_{max})$, $\lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$.

Let $\lambda^{p\rightarrow q} = 1$ and $\lambda^{\neg p\rightarrow\neg q} = \lambda_{maxBel} - \epsilon$, where $\epsilon = 0.5 \min\{\lambda_{maxBel} - \lambda_{minBel}, \epsilon^{(1)}_{max}, \epsilon^{(2)}_{max}\}$.

Grounding threshold requirement for belief operator is met because:

$$\lambda^{\neg p\rightarrow\neg q} = \lambda_{maxBel} - \epsilon < \lambda_{maxBel}$$

$$\lambda^{\neg p\rightarrow\neg q} = \lambda_{maxBel} - \epsilon > \lambda_{maxBel} - \lambda_{maxBel} + \lambda_{minBel} = \lambda_{minBel}$$

Conditional relation for both statements is also met, because of lemma 7.35:

$$\lambda^{p\rightarrow q} = 1 > 1-\epsilon > \overline{f}(1-\lambda_{maxBel}+\epsilon) = \overline{f}(1-\lambda^{\neg p\rightarrow\neg q}) = \overline{f}(\lambda^{\neg p\rightarrow q})$$

$$\lambda^{\neg p\rightarrow\neg q} = \lambda_{maxBel} - \epsilon > \overline{f}(0+\epsilon) \geqslant \overline{f}(0) = \overline{f}(\lambda^{p\rightarrow\neg q})$$

This proves sufficiency. □

**Theorem 7.25.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations CS* $\models^{\text{PE}} p \rightarrow Know(q)$ *and CS* $\models^{\text{PE}} \neg p \rightarrow Pos(\neg q)$ *can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1-\lambda_{maxPos}) < 1$$

*Proof.* Proof of theorem 7.25 can be constructed similarly to the proof of theorem 7.24. One needs to change grounding thresholds respectively. □

**Theorem 7.26.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \rightarrow Bel(q)$ and $CS \models^{\mathrm{PE}} \neg p \rightarrow Bel(\neg q)$ can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxBel}) < \lambda_{maxBel}$$

*Proof.* When both pragmatic epistemic relations are met, we have:

$$\lambda_{minBel} < \lambda^{p \rightarrow q} < \lambda_{maxBel} \quad \text{and} \quad \lambda_{minBel} < \lambda^{\neg p \rightarrow \neg q} < \lambda_{maxBel}$$

and

$$\lambda^{p \rightarrow q} > \overline{f}(\lambda^{\neg p \rightarrow q})$$

Case where $\lambda^{p \rightarrow q} < \underline{f}(\lambda^{\neg p \rightarrow q})$ can't be met, because (from $\lambda_{minBel} \geqslant 1 - \lambda_{minBel}$ and $\underline{f}(x) \leqslant x$):

$$\lambda^{p \rightarrow q} > \lambda_{minBel} \geqslant 1 - \lambda_{minBel} \geqslant \underline{f}(1 - \lambda_{minBel}) \geqslant \underline{f}(1 - \lambda^{\neg p \rightarrow \neg q}) = \underline{f}(\lambda^{\neg p \rightarrow q})$$

When $\lambda^{p \rightarrow q} > \overline{f}(\lambda^{\neg p \rightarrow q})$ is met, so must be:

$$\lambda_{maxBel} > \lambda^{p \rightarrow q} > \overline{f}(\lambda^{\neg p \rightarrow q}) = \overline{f}(1 - \lambda^{\neg p \rightarrow \neg q}) \geqslant \overline{f}(1 - \lambda_{maxBel})$$

This proves necessity.

To prove condition's sufficiency I shall construct a grounding strengths values that meet both pragmatic epistemic relations.

Let $\lambda_{min} = 1 - \lambda_{maxBel}$ and $\lambda_{max} = \lambda_{maxBel}$. According to lemma 7.35 there exists $\epsilon_{max}$, such that for every $\epsilon \in (0, \epsilon_{max})$, $\lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$. Let:

$$\epsilon = 0.5 \min\{\epsilon_{max}, \lambda_{maxBel} - \lambda_{minBel}\}$$

and let $\lambda^{p \rightarrow q} = \lambda^{\neg p \rightarrow \neg q} = \lambda_{maxBel} - \epsilon$. Grounding threshold requirement is met, because:

$$\begin{aligned}
\lambda^{p \rightarrow q} = \lambda^{\neg p \rightarrow \neg q} =& \lambda_{maxBel} - \epsilon < \lambda_{maxBel} \\
\lambda^{p \rightarrow q} = \lambda^{\neg p \rightarrow \neg q} =& \lambda_{maxBel} - \epsilon > \lambda_{maxBel} - \lambda_{maxBel} + \lambda_{minBel} = \lambda_{minBel}
\end{aligned}$$

and, according to lemma 7.35, conditional relation is met, because:

$$\begin{aligned}
\lambda^{p \rightarrow q} = \lambda^{\neg p \rightarrow \neg q} &= \lambda_{maxBel} - \epsilon \\
&> \overline{f}(1 - \lambda_{maxBel} + \epsilon) \\
&= \overline{f}(1 - \lambda^{p \rightarrow q}) = \overline{f}(1 - \lambda^{\neg p \rightarrow \neg q}) \\
&= \overline{f}(\lambda^{\neg p \rightarrow q}) = \overline{f}(\lambda^{p \rightarrow \neg q})
\end{aligned}$$

This proves sufficiency. □

**Theorem 7.27.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Bel(q)$ and $CS \models^{\mathrm{PE}} \neg p \to Pos(\neg q)$ can be both met. A sufficient and necessary condition is:*

$$(\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxBel} \;\wedge\; \overline{f}(1-\lambda_{maxBel}) < \lambda_{maxPos})$$
$$or$$
$$(\underline{f}(1-\lambda_{minPos}) > \lambda_{minBel} \;\wedge\; \underline{f}(1-\lambda_{minBel}) > \lambda_{minPos})$$

*Proof.* Meeting of pragmatic epistemic relations for both formulas means meeting of grounding threshold requirements:

$$\lambda_{minBel} < \lambda^{p\to q} < \lambda_{maxBel} \text{ and } \lambda_{minPos} < \lambda^{\neg p\to\neg q} \leqslant \lambda_{maxPos}$$

There are four possible cases for conditional relations:

1. $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q})$ and $\lambda^{\neg p\to\neg q} > \overline{f}(\lambda^{p\to\neg q})$
2. $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q})$ and $\lambda^{\neg p\to\neg q} < \underline{f}(\lambda^{p\to\neg q})$
3. $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q})$ and $\lambda^{\neg p\to\neg q} > \overline{f}(\lambda^{p\to\neg q})$
4. $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q})$ and $\lambda^{\neg p\to\neg q} < \underline{f}(\lambda^{p\to\neg q})$

Case 3. can't be met, because meeting it leads to a contradiction:

$$\begin{aligned}1-\lambda^{\neg p\to q} &= \lambda^{\neg p\to\neg q} > \overline{f}(\lambda^{p\to\neg q})\\ &= \overline{f}(1-\lambda^{p\to q}) \geqslant 1-\lambda^{p\to q}\\ &\geqslant 1-\underline{f}(\lambda^{\neg p\to q}) \geqslant 1-\lambda^{\neg p\to q}\end{aligned}$$

Similar contradiction can be shown for case 4, so it also can't be met.

It is either case 1. that implies:

$$\begin{aligned}\lambda_{maxBel} > \lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q}) = \overline{f}(1-\lambda^{\neg p\to\neg q}) \geqslant \overline{f}(1-\lambda_{maxPos}),\\ \lambda_{maxPos} \geqslant \lambda^{\neg p\to\neg q} > \overline{f}(\lambda^{p\to\neg q}) = \overline{f}(1-\lambda^{p\to q}) \geqslant \overline{f}(1-\lambda_{maxBel})\end{aligned}$$

or case 2. that implies:

$$\begin{aligned}\lambda_{minBel} < \lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q}) = \underline{f}(1-\lambda^{\neg p\to\neg q}) \leqslant \underline{f}(1-\lambda_{minPos}),\\ \lambda_{minPos} < \lambda^{\neg p\to\neg q} < \underline{f}(\lambda^{p\to\neg q}) = \overline{f}(1-\lambda^{p\to q}) \leqslant \underline{f}(1-\lambda_{minBel}).\end{aligned}$$

Cases 1. and 2. imply the condition from theorem 7.27. This proves necessity.

To prove sufficiency I shall construct grounding threshold setting meeting both pragmatic epistemic relations. Firstly assume (case 1):

$$\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxBel} \;\wedge\; \overline{f}(1-\lambda_{maxBel}) < \lambda_{maxPos}$$

Let $\lambda_{min} = 1-\lambda_{maxPos}$ and $\lambda_{max} = \lambda_{maxBel}$. According to lemma 7.35 there exists $\epsilon^{(1)}_{max}$, such that for every $\epsilon \in (0, \epsilon^{(1)}_{max})$, $\lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$.

Similarly let $\lambda_{min} = 1 - \lambda_{maxBel}$ and $\lambda_{max} = \lambda_{maxPos}$. According to lemma 7.35 there exists $\epsilon^{(2)}_{max}$, such that for every $\epsilon \in (0, \epsilon^{(2)}_{max})$, $\lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$.

Let grounding strengths be:

$$\begin{aligned}
\lambda^{p\to q} &= \lambda_{maxBel} - \epsilon \\
\lambda^{\neg p\to \neg q} &= \lambda_{maxPos} - \epsilon \\
\lambda^{p\to \neg q} &= 1 - \lambda_{maxBel} + \epsilon \\
\lambda^{\neg p\to q} &= 1 - \lambda_{maxPos} + \epsilon
\end{aligned}$$

where $\epsilon > 0$ is defined as:

$$\epsilon = 0.5\ \max\{\epsilon^{(1)}_{max},\ \epsilon^{(2)}_{max},\ \lambda_{maxBel} - \lambda_{minBel},\ \lambda_{maxPos} - \lambda_{minPos}\}$$

For such setting, according to already mentioned lemma 7.35, both conditional relations are met. Grounding thresholds are also met, because:

$$\begin{aligned}
\lambda^{p\to q} &= \lambda_{maxBel} - \epsilon < \lambda_{maxBel} \\
\lambda^{p\to q} &= \lambda_{maxBel} - \epsilon > \lambda_{maxBel} - \lambda_{maxBel} + \lambda_{minBel} = \lambda_{minBel} \\
\lambda^{\neg p\to \neg q} &= \lambda_{maxPos} - \epsilon < \lambda_{maxPos} \\
\lambda^{\neg p\to \neg q} &= \lambda_{maxPos} - \epsilon > \lambda_{maxPos} - \lambda_{maxPos} + \lambda_{minPos} = \lambda_{minPos}
\end{aligned}$$

Proof for (case 2), where:

$$\underline{f}(1 - \lambda_{minPos}) > \lambda_{minBel}\ \wedge\ \underline{f}(1 - \lambda_{minBel}) > \lambda_{minPos},$$

is performed similarly to the proof for case 1. One should use lemma 7.34 and a setting:

$$\begin{aligned}
\lambda^{p\to q} &= \lambda_{minBel} + \epsilon \\
\lambda^{\neg p\to \neg q} &= \lambda_{minPos} + \epsilon \\
\lambda^{p\to \neg q} &= 1 - \lambda_{minBel} - \epsilon \\
\lambda^{\neg p\to q} &= 1 - \lambda_{minPos} - \epsilon
\end{aligned}$$

Sufficiency is proved. □

**Theorem 7.28.** *(Pragmatic) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Pos(q)$ and $CS \models^{\mathrm{PE}} \neg p \to Pos(\neg q)$ can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxPos}) < \lambda_{maxPos}\quad \textit{or}\quad \underline{f}(1 - \lambda_{minPos}) > \lambda_{minPos}$$

*Proof.* Meeting of pragmatic epistemic relation for both formulas means grounding threshold requirements meet inequalities:

$$\lambda_{minPos} < \lambda^{p\to q} < \lambda_{maxPos} \;\text{ and }\; \lambda_{minPos} < \lambda^{\neg p\to\neg q} \leqslant \lambda_{maxPos}$$

Conditional relations can be met when either (case 1):

$$\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q}) \;\text{ and }\; \lambda^{\neg p\to\neg q} > \overline{f}(\lambda^{p\to\neg q})$$

or (case 2):

$$\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q}) \;\text{ and }\; \lambda^{\neg p\to\neg q} < \underline{f}(\lambda^{p\to\neg q})$$

other cases are impossible for the same reason as provided in the proof of theorem 7.27.

From case 1 we conclude:

$$\lambda_{maxPos} \geqslant \lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q}) = \overline{f}(1-\lambda^{\neg p\to\neg q}) \geqslant \overline{f}(1-\lambda_{maxPos})$$

and from case 2 we conclude:

$$\lambda_{minPos} < \lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q}) = \underline{f}(1-\lambda^{\neg p\to\neg q}) \leqslant \underline{f}(1-\lambda_{minPos})$$

Both cases lead to the condition from theorem 7.28. This proves necessity.

To prove sufficiency I shall construct a grounding threshold setting meeting both pragmatic epistemic relations. Firstly assume that (case 1) $\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxPos}$.

Let $\lambda_{min} = 1 - \lambda_{maxPos}$ and $\lambda_{max} = \lambda_{maxPos}$. According to lemma 7.35 there exists $\epsilon_{max}$, such that for every $\epsilon \in (0, \epsilon_{max})$, $\lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$.

Let grounding strengths be:

$$\lambda^{p\to q} = \lambda^{\neg p\to\neg q} = \lambda_{maxPos} - \epsilon$$
$$\lambda^{p\to\neg q} = \lambda^{\neg p\to q} = 1 - \lambda_{maxPos} + \epsilon$$

where $\epsilon = 0.5 \min\{\epsilon_{max},\ \lambda_{maxPos} - \lambda_{minPos}\}$.

Lemma 7.35 implies that conditional relations are met. Similarly grounding threshold requirements are met, because:

$$\lambda^{p\to q} = \lambda^{\neg p\to\neg q} = \lambda_{maxPos} - \epsilon < \lambda_{maxPos}$$
$$\begin{aligned}\lambda^{p\to\neg q} = \lambda^{\neg p\to q} &= 1 - \lambda_{maxPos} + \epsilon \\ &> 1 - \lambda_{maxPos} + \lambda_{maxPos} - \lambda_{minPos} = 1 - \lambda_{minPos} > \lambda_{minPos}\end{aligned}$$

Inequality $1 - \lambda_{minPos} > \lambda_{minPos}$ can be implied from lemma 7.33.

Proof for (case 2), where $\underline{f}(1-\lambda_{minPos}) > \lambda_{minPos}$, is performed similarly to the proof for case 1. One should use lemma 7.34 and a setting:

$$\lambda^{p\to q} = \lambda^{\neg p\to\neg q} = \lambda_{minPos} + \epsilon$$
$$\lambda^{p\to\neg q} = \lambda^{\neg p\to q} = 1 - \lambda_{minPos} - \epsilon$$

Sufficiency is proved. □

Because of dependence between normal and pragmatic epistemic relations (theorem 7.13), theorem 7.5 provides sufficient condition that disallows simultaneous meeting of pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Know(q)$ and $CS \models^{\mathrm{PE}} p \to \Pi(\neg q)$.

Similarly theorem 7.8 provides sufficient condition that disallows simultaneous meeting of pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Bel(q)$ and $CS \models^{\mathrm{PE}} p \to Bel(\neg q)$.

Other combinations of formulas $p \to \Pi(q)$ and $p \to \Xi(\neg q)$ can be simultaneously met by normal epistemic relation (theorems 7.9 and 7.10) and pragmatic epistemic relation. Theorems 7.29 and 7.30 provide required constraints in the case of pragmatic epistemic relation.

**Theorem 7.29.** *(Pragmatic) Assume a grounding threshold and boundary functions setting meeting inequalities 7.5 and 7.6. Pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to Pos(q)$ and $CS \models^{\mathrm{PE}} p \to Bel(\neg q)$ can be both met.*

*Proof.* I shall construct a grounding threshold setting that meets both pragmatic epistemic relations. Let:

$$\begin{aligned}
b_{max} &= \min\{\lambda_{maxBel},\ 1 - \lambda_{minPos}\},\\
b_{min} &= \max\{\lambda_{minBel},\ 1 - \lambda_{maxPos}\},\\
p_{max} &= 1 - b_{min} = \min\{1 - \lambda_{minBel},\ \lambda_{maxPos}\},\\
p_{min} &= 1 - b_{max} = \max\{1 - \lambda_{maxBel},\ \lambda_{minPos}\}
\end{aligned}$$

and let grounding strength setting be defined as:

$$\begin{aligned}
\lambda^{p\to q} &= p_{min} + \epsilon\\
\lambda^{p\to\neg q} &= 1 - \lambda^{p\to q} = b_{max} - \epsilon\\
\lambda^{\neg p\to q} &= 1\\
\lambda^{\neg p\to\neg q} &= 1 - \lambda^{\neg p\to q} = 0
\end{aligned}$$

where

$$\begin{aligned}
\epsilon =& 0.5\min\{b_{max} - b_{min},\ p_{max} - p_{min},\\
& \underline{f}(1) - \lambda_{minBel},\ \lambda_{maxPos} - \overline{f}(0)\}.
\end{aligned}$$

Values $\underline{f}(1) - \lambda_{minBel},\ \lambda_{maxPos} - \overline{f}(0)$ are both positive (lemma 7.33). Similarly $b_{max} - b_{min},\ p_{max} - p_{min}$ are positive (theorem 7.9 and inequality 7.5). In result $\epsilon$ is always positive.

Firstly I shall prove grounding threshold requirements are met:

$$\lambda^{p\to q} = p_{min} + \epsilon = \max\{1 - \lambda_{maxBel},\ \lambda_{minPos}\} + \epsilon > \lambda_{minPos}$$

$$\begin{aligned}
\lambda^{p\to q} &= p_{min} + \epsilon\\
&< p_{min} + p_{max} - p_{min} = p_{max}\\
&= \min\{1 - \lambda_{minBel},\ \lambda_{maxPos}\}\\
&\leqslant \lambda_{maxPos}
\end{aligned}$$

and

$$\lambda^{p\to\neg q} = b_{max} - \epsilon = \min\{\lambda_{maxBel},\ 1-\lambda_{minPos}\} - \epsilon < \lambda_{maxBel}$$

$$\begin{aligned}
\lambda^{p\to\neg q} &= b_{max} - \epsilon \\
&> b_{max} - b_{max} + b_{min} = b_{min} \\
&= \max\{\lambda_{minBel},\ 1-\lambda_{maxPos}\} \\
&\geqslant \lambda_{minBel}
\end{aligned}$$

Secondly I shall prove conditional relations $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q})$ and $\lambda^{p\to\neg q} > \overline{f}(\lambda^{\neg p\to\neg q})$ hold:

$$\begin{aligned}
\lambda^{p\to q} &= p_{min} + \epsilon \\
&< p_{min} + \underline{f}(1) - \lambda_{minBel} \\
&= \max\{1-\lambda_{maxBel},\ \lambda_{minPos}\} + \underline{f}(1) - \lambda_{minBel} \\
&\leqslant \underline{f}(1) = \underline{f}(\lambda^{\neg p\to q}) \\
\lambda^{p\to\neg q} &= b_{max} - \epsilon \\
&> b_{max} - \lambda_{maxPos} + \overline{f}(0) \\
&= \min\{\lambda_{maxBel},\ 1-\lambda_{minPos}\} - \lambda_{maxPos} + \overline{f}(0) \\
&\geqslant \overline{f}(0) = \overline{f}(\lambda^{\neg p\to\neg q})
\end{aligned}$$

In conclusion a threshold setting meeting both epistemic relations exists. This ends proof of sufficiency. □

**Theorem 7.30.** *(Pragmatic) Assume a grounding threshold and boundary functions setting meeting inequalities 7.5 and 7.6. Pragmatic epistemic relations* $CS \models^{\mathrm{PE}} p \to Pos(q)$ *and* $CS \models^{\mathrm{PE}} p \to Pos(\neg q)$ *can be both met.*

*Proof.* I shall construct a grounding strength setting that meets both pragmatic epistemic relations. Let grounding strength setting be defined as:

$$\begin{aligned}
\lambda^{p\to q} &= \lambda_{maxPos} - \epsilon, \\
\lambda^{p\to\neg q} &= 1 - \lambda^{p\to q} = 1 - \lambda_{maxPos} + \epsilon \\
\lambda^{\neg p\to q} &= 0 \\
\lambda^{\neg p\to\neg q} &= 1 - \lambda^{\neg p\to q} = 1
\end{aligned}$$

where $\epsilon$ is a positive number such that:

$$\epsilon = 0.5\ \min\{2\lambda_{maxPos} - 1,\ \lambda_{maxPos} - \overline{f}(0),\ \underline{f}(1) - \lambda_{minBel}\}$$

Such setting meets grounding thresholds requirement, because:

$$\lambda^{p\to q} = \lambda_{maxPos} - \epsilon < \lambda_{maxPos}$$

$$\begin{aligned}\lambda^{p\to q} &= \lambda_{maxPos} - \epsilon \\ &> \lambda_{maxPos} - 2\lambda_{maxPos} + 1 = 1 - \lambda_{maxPos} > \lambda_{minPos}\end{aligned}$$

$$\lambda^{p\to\neg q} = 1 - \lambda_{maxPos} + \epsilon > 1 - \lambda_{maxPos} > \lambda_{minPos}$$

$$\begin{aligned}\lambda^{p\to\neg q} &= 1 - \lambda_{maxPos} + \epsilon \\ &< 1 - \lambda_{maxPos} + 2\lambda_{maxPos} - 1 = \lambda_{maxPos}\end{aligned}$$

and it meets conditional relations:

$$\begin{aligned}\lambda^{p\to q} &= \lambda_{maxPos} - \epsilon \\ &> \lambda_{maxPos} - \lambda_{maxPos} + \overline{f}(0) \\ &= \overline{f}(0) = \overline{f}(\lambda^{\neg p\to q})\end{aligned}$$

$$\begin{aligned}\lambda^{p\to\neg q} &= 1 - \lambda_{maxPos} + \epsilon \\ &< 1 - \lambda_{maxPos} + \underline{f}(1) - \lambda_{minBel} \\ &\leqslant \underline{f}(1) + 1 - \lambda_{maxPos} - \lambda_{maxPos} \\ &= \underline{f}(1) + 1 - 2\lambda_{maxPos} \leqslant \underline{f}(1) + 1 - 2\cdot 0.5 \\ &= \underline{f}(1) = \underline{f}(\lambda^{\neg p\to\neg q})\end{aligned}$$

Proposed grounding strength setting meets pragmatic epistemic relations for both formulas. □

Theorem for normal epistemic relation 7.11 states $CS \models^{\mathrm{E}} p \to \Pi(q)$ and $CS \models^{\mathrm{E}} q \to \Xi(\neg p)$ can be both met iff $\Xi \neq Know$. Pragmatic epistemic relation has exactly the same property as stated by theorem 7.31.

**Theorem 7.31.** *(Pragmatic) Assume a grounding threshold and boundary functions setting meeting inequalities 7.5 and 7.6. Pragmatic epistemic relations $CS \models^{\mathrm{PE}} p \to \Pi(q)$ and $CS \models^{\mathrm{PE}} q \to \Xi(\neg p)$ can be both met if and only if $\Xi \neq Know$.*

*Proof.* Theorems 7.11 and 7.13 imply that when both epistemic relations are met, then $\Xi \neq Know$.

Let $c_1, c_2, c_3, c_4$ denote:

$$c_1 = \mathrm{card}(C^{p\wedge q}),\ c_2 = \mathrm{card}(C^{p\wedge\neg q}),\ c_3 = \mathrm{card}(C^{\neg p\wedge q}),\ c_4 = \mathrm{card}(C^{\neg p\wedge\neg q}).$$

Now assume $\Xi \neq Know$. From the proof of theorem 7.11 we know that one can choose $c_1, c_2, c_3, c_4$, so that grounding strengths become any rational numbers: $\lambda^{p\to q} \in (0, 1]$ and

$\lambda^{q\to\neg p} \in (0,1)$. We only need to prove that conditional relation can also be met. It can be met, when:

$$\lambda^{p\to q} = \frac{c_1}{c_1+c_2} > \overline{f}\left(\frac{c_3}{c_3+c_4}\right) = \overline{f}(\lambda^{p\to\neg q})$$
$$\lambda^{q\to\neg p} = \frac{c_3}{c_3+c_1} < \underline{f}\left(\frac{c_4}{c_4+c_2}\right) = \underline{f}(\lambda^{\neg q\to\neg p})$$

Left sides of inequalities do not depend on $c_4$. When $c_4$ increases:

$$\lim_{c_4\to\infty} \lambda^{p\to\neg q} = 0 \quad\text{and}\quad \lim_{c_4\to\infty} \lambda^{\neg q\to\neg p} = 1.$$

One can choose $c_4$ big enough to obtain $\lambda^{p\to\neg q}$ as close to 0 and $\lambda^{\neg q\to\neg p}$ as close to 1 as one wishes.

Boundary functions are continuous, non-decreasing and meet $\underline{f}(1) > \lambda_{minBel}$, $\overline{f}(0) < \lambda_{maxPos}$. This implies there exist values $\lambda^{\neg p\to q}$ and $\lambda^{\neg q\to\neg p}$, respectively small enough and big enough, to meet $\overline{f}(\lambda^{p\to\neg q}) > \lambda_{minBel}$ and $\underline{f}(\lambda^{\neg q\to\neg p}) < \lambda_{maxPos}$.

In result one can choose values $c_1, c_2, c_3, c_4$ that meet both grounding thresholds and conditional relations. Formal construction of grounding strengths can be performed similarly as in previous theorems. □

### 7.2.1. Constraints for the boundary functions

Some of theorems 7.14-7.31 provide constraints on grounding thresholds and boundary functions. Meeting of these constraints guarantees some formulas can or can not be grounded simultaneously and in result used together. It is possible to set grounding thresholds and boundary functions so that all of the constraints are met all together. Theorem 7.32 provides general conditions that ensure meeting of all the properties. This theorem states that all common-sense criteria described in chapter 5 can be met by the proposed grounding theory of conditional formulas.

**Theorem 7.32.** *(Pragmatic) Let grounding thresholds setting meet inequalities 7.5. It is possible to set lower and upper boundary functions so that all conditions from theorems 7.14-7.31 are simultaneously met by pragmatic epistemic relation. Sufficient conditions are:*

$$\overline{f}(\max\{\lambda_{minPos},\ 1-\lambda_{maxPos}\}) < \lambda_{maxPos} \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel} \tag{7.7}$$
$$\underline{f}(\min\{\lambda_{maxBel},\ 1-\lambda_{minPos}\}) > \lambda_{minBel} \geqslant \underline{f}(1-\lambda_{minBel}) > \lambda_{minPos} \tag{7.8}$$

*Proof.* When 7.5 is assumed, theorems 7.14-7.31 provide necessary and sufficient constraints for particular properties of pragmatic epistemic relation. To provide necessary and sufficient conditions for all properties to be met together, one needs to take intersection of these constraints.

Theorem 7.17 requires:

$$\lambda_{maxBel} > \overline{f}(\lambda_{minBel}) \text{ and } \underline{f}(\lambda_{maxBel}) > \lambda_{minBel}$$

and theorem 7.19 requires:

$$\lambda_{maxPos} > \overline{f}(\lambda_{minPos}) \text{ and } \underline{f}(\lambda_{maxPos}) > \lambda_{minPos}$$

The intersection of these constraints is simply:

$$\begin{aligned}\overline{f}(\lambda_{minPos}) < \lambda_{maxPos} \text{ and } \overline{f}(\lambda_{minBel}) < \lambda_{maxBel}\\ \underline{f}(\lambda_{maxBel}) > \lambda_{minBel} \text{ and } \underline{f}(\lambda_{maxPos}) > \lambda_{minPos}\end{aligned}$$

We can use $\lambda_{maxPos} \leqslant \lambda_{minBel} \leqslant \overline{f}(\lambda_{minBel})$ and $\lambda_{minBel} \geqslant \lambda_{maxPos} \geqslant \underline{f}(\lambda_{maxPos})$ to rewrite constraints in a neater form:

$$\overline{f}(\lambda_{minPos}) < \lambda_{maxPos} \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel} \tag{7.9}$$

$$\underline{f}(\lambda_{maxBel}) > \lambda_{minBel} \geqslant \underline{f}(\lambda_{maxPos}) > \lambda_{minPos} \tag{7.10}$$

One can notice that:

$$\begin{aligned}&\overline{f}(0) \leqslant \overline{f}(\lambda_{minPos}) < \lambda_{maxPos} \leqslant \lambda_{minBel} \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel} \leqslant 1,\\ &\underline{f}(1) \geqslant \underline{f}(\lambda_{maxBel}) > \lambda_{minBel} \geqslant \lambda_{maxPos} \geqslant \underline{f}(\lambda_{maxPos}) > \lambda_{minPos} \geqslant 0,\end{aligned}$$

so conditions for theorems: 7.14, 7.15, 7.16 and 7.18, as less restrictive, are already guaranteed by inequalities 7.9 and 7.10.

From lemma 7.33 we know $\lambda_{minBel} > 1 - \lambda_{maxBel}$ and $\lambda_{maxPos} < 1 - \lambda_{minPos}$, so $\lambda_{maxBel} > 1 - \lambda_{minBel}$ and $\lambda_{minPos} < 1 - \lambda_{maxPos}$. This, together with inequalities 7.9, 7.10, implies that:

$$\begin{aligned}&\overline{f}(1 - \lambda_{maxBel}) \leqslant \overline{f}(1 - \lambda_{maxPos}) \leqslant\\ &\quad \leqslant \overline{f}(\lambda_{maxPos}) \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel} \leqslant 1\end{aligned} \tag{7.11}$$

This means conditions from theorems 7.24, 7.25 and 7.26 are already guaranteed by 7.9 and 7.10.

Theorem 7.28 provides condition $\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxPos}$ or $\underline{f}(1-\lambda_{minPos}) > \lambda_{minPos}$. It is already met on $\underline{f}(1-\lambda_{minPos}) > \lambda_{minPos}$, as from lemma 7.33 and inequalities 7.10 we obtain:

$$\underline{f}(1 - \lambda_{minPos}) \geqslant \underline{f}(\lambda_{maxPos}) > \lambda_{minPos}$$

Since we are searching only for sufficient criteria, we can still add $\overline{f}(1 - \lambda_{maxPos}) < \lambda_{maxPos}$ to them. We obtain:

$$\overline{f}(\max\{\lambda_{minPos},\ 1 - \lambda_{maxPos}\}) < \lambda_{maxPos} \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel} \tag{7.12}$$

The only theorem, whose condition is not met yet, is 7.27 with a constraint:

$$(\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxBel} \ \wedge \ \overline{f}(1-\lambda_{maxBel}) < \lambda_{maxPos})$$
$$\text{or}$$
$$(\underline{f}(1-\lambda_{minPos}) > \lambda_{minBel} \ \wedge \ \underline{f}(1-\lambda_{minBel}) > \lambda_{minPos})$$

Inequality $\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxBel}$ is already guaranteed by 7.11. We have two possible situations (case 1) $\overline{f}(1-\lambda_{maxBel}) < \lambda_{maxPos}$ or (case 2) $\underline{f}(1-\lambda_{minPos}) > \lambda_{minBel} \ \wedge \ \underline{f}(1-\lambda_{minBel}) > \lambda_{minPos}$.

In case 1 inequality $\overline{f}(1-\lambda_{maxBel}) < \lambda_{maxPos}$ is not guaranteed by 7.11 but it is guaranteed by 7.12. This constraint has to be added to 7.9.

$$\overline{f}(\max\{\lambda_{minPos},\ 1-\lambda_{maxPos}\}) < \lambda_{maxPos} \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel}$$

that is the same as condition 7.7 from the theorem.

In case 2 neither of inequalities $\underline{f}(1-\lambda_{minPos}) > \lambda_{minBel}, \underline{f}(1-\lambda_{minBel}) > \lambda_{minPos}$ is guaranteed and they have to be added to 7.10. We obtain:

$$\underline{f}(\min\{\lambda_{maxBel},\ 1-\lambda_{minPos}\}) > \lambda_{minBel} \geqslant \underline{f}(1-\lambda_{minBel}) > \lambda_{minPos}$$

that is the same as condition 7.8 from the theorem.

Given constraints 7.7,7.8 are not necessary because only one of inequalities from theorem 7.27 must be met and second inequality from theorem 7.28 is already met by weaker criteria. □

### 7.2.2. Useful lemmas

**Lemma 7.33.** *Inequalities:* $\underline{f}(1) > \lambda_{minBel} > 1-\lambda_{maxBel}$ *and* $\overline{f}(0) < \lambda_{maxPos} < 1-\lambda_{minPos}$ *hold for all grounding threshold and boundary function settings meeting inequalities 7.5 and 7.6.*

*Proof.* From 7.5 we can imply:

$$\lambda_{minPos} < 1-\lambda_{minBel} \leqslant 1-\lambda_{maxPos}, \text{ hence } \lambda_{maxPos} < 1-\lambda_{minPos}$$
$$1-\lambda_{maxBel} < 1-\lambda_{minPos} \leqslant 0.5 \leqslant \lambda_{minBel}$$

From above and from 7.6:

$$1-\lambda_{minPos} > \lambda_{maxPos} > \overline{f}(0)$$
$$1-\lambda_{maxBel} < 0.5 \leqslant \lambda_{minBel} < \underline{f}(1)$$

□

Let $0 \leqslant \lambda_{min} < \lambda_{max} \leqslant 1$ be some real numbers and let $\underline{f}, \overline{f}$ be lower and upper boundary functions.

**Lemma 7.34.** *If $\underline{f}(\lambda_{max}) > \lambda_{min}$, then:*

$$\exists \epsilon_{max} > 0 : \forall \epsilon \in (0, \epsilon_{max}) \;\Rightarrow\; \lambda_{min} + \epsilon < \underline{f}(\lambda_{max} - \epsilon)$$

*Proof.* Assume $\underline{f}(\lambda_{max}) > \lambda_{min}$. We wish to prove $\epsilon_{max}$ exists.

Let $\underline{x}$ be a solution of equation:

$$\underline{f}(\underline{x}) = -\underline{x} + \lambda_{min} + \lambda_{max} \tag{7.13}$$

Boundary function $\underline{f}(x) \leqslant x$ is non-decreasing, continuous and defined on $[0, 1]$. $g(x) = -x + \lambda_{min} + \lambda_{max}$ is a line crossing points $(\lambda_{min}, \lambda_{max})$ and $(\lambda_{max}, \lambda_{min})$. Both points lie within a square area $[0, 1] \times [0, 1]$. If $\underline{f}(1) \geqslant \lambda_{min} + \lambda_{max} - 1$, then the crossing point $g(\underline{x}) = \underline{f}(\underline{x})$ exists and lies within $[0, 1]$.

The $\underline{f}(1) \geqslant \lambda_{min} + \lambda_{max} - 1$ is already known to hold, because:

$$\underline{f}(1) \geqslant \underline{f}(\lambda_{max}) > \lambda_{min} = \lambda_{min} + 1 - 1 \geqslant \lambda_{min} + \lambda_{max} - 1$$

Assume by contradiction solution $\underline{x} \geqslant \lambda_{max}$, then:

$$\begin{aligned} \lambda_{min} + \lambda_{max} &= \underline{f}(\underline{x}) + \underline{x} \\ &\geqslant \underline{f}(\lambda_{max}) + \lambda_{max} \\ &> \lambda_{min} + \lambda_{max} \end{aligned}$$

we obtain a contradiction, hence $\underline{x} < \lambda_{max}$. This in turn implies that $\epsilon_{max} = \lambda_{max} - \underline{x} > 0$. Now I will prove $\lambda_{min} + \epsilon < \underline{f}(\lambda_{max} - \epsilon)$ for every $\epsilon \in (0, \epsilon_{max})$.

$$\begin{aligned} \lambda_{min} + \epsilon &< \lambda_{min} + \epsilon_{max} \\ &= \lambda_{min} + \lambda_{max} - \underline{x} \\ &= \underline{f}(\underline{x}) + \underline{x} - \underline{x} = \underline{f}(\underline{x}) \\ &= \underline{f}(\lambda_{max} - \epsilon_{max}) \\ &\leqslant \underline{f}(\lambda_{max} - \epsilon) \end{aligned}$$

This ends proof. □

**Lemma 7.35.** *If $\overline{f}(\lambda_{min}) < \lambda_{max}$, then:*

$$\exists \epsilon_{max} > 0 : \forall \epsilon \in (0, \epsilon_{max}) \;\Rightarrow\; \lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$$

*Proof.* Proof construction method is the same as for lemma 7.34. □

**Lemma 7.36.** *If $\underline{f}(\lambda_{max}) > \lambda_{min}$ and $\overline{f}(\lambda_{min}) < \lambda_{max}$, then:*

$$\exists \epsilon_{max} > 0 : \forall \epsilon \in (0, \epsilon_{max}) \;\Rightarrow\; \lambda_{min} + \epsilon < \underline{f}(\lambda_{max} - \epsilon) \wedge \lambda_{max} - \epsilon > \overline{f}(\lambda_{min} + \epsilon)$$

*Proof.* Lemma 7.36 is a direct consequence of lemmas 7.34 and 7.35. Simply let $\epsilon_{max} = \min\{\epsilon_{max}^{(1)}, \epsilon_{max}^{(2)}\}$, where $\epsilon_{max}^{(1)}, \epsilon_{max}^{(2)}$ are found $\epsilon_{max}$'s from lemmas 7.34 and 7.35. □

**Lemma 7.37.** *If numbers $\underline{\lambda}, \overline{\lambda}$, such that:*

$$\lambda_{min} < \underline{\lambda} \leqslant \lambda_{max}, \ \lambda_{min} < \overline{\lambda} \leqslant \lambda_{max} \quad \textit{and} \quad \underline{\lambda} < \underline{f}(\overline{\lambda}),$$

*exist, then:*

$$\underline{f}(\lambda_{max}) > \lambda_{min}.$$

*Proof.* Let $\underline{\lambda}, \overline{\lambda}$ be some numbers meeting conditions given in the lemma. Then:

$$\underline{f}(\lambda_{max}) \geqslant \underline{f}(\overline{\lambda}) \quad \text{and} \quad \underline{f}(\overline{\lambda}) > \underline{\lambda} \geqslant \lambda_{min}$$

so $\underline{f}(\lambda_{max}) > \lambda_{min}$ must be met. □

**Lemma 7.38.** *If:*

$$\underline{f}(\lambda_{max}) > \lambda_{min},$$

*then there exist numbers $\underline{\lambda}, \overline{\lambda}$, such that:*

$$\lambda_{min} < \underline{\lambda} < \lambda_{max}, \ \lambda_{min} < \overline{\lambda} < \lambda_{max} \quad \textit{and} \quad \underline{\lambda} < \underline{f}(\overline{\lambda}).$$

*Proof.* Let: $\overline{\lambda} = \lambda_{max} - \epsilon$ and $\underline{\lambda} = \lambda_{min} + \epsilon$, where $\epsilon \in (0, \min\{\lambda_{max} - \lambda_{min}, \epsilon_{max}\})$ and $\epsilon_{max}$ is constructed according to lemma 7.34. The $\epsilon_{max}$ exists because $\underline{f}(\lambda_{max}) > \lambda_{min}$.

According to the same lemma 7.34: $\underline{\lambda} < \underline{f}(\overline{\lambda})$ holds.

Obviously $\overline{\lambda} = \lambda_{max} - \epsilon < \lambda_{max}$ and $\underline{\lambda} = \lambda_{min} + \epsilon > \lambda_{min}$. I only need to prove $\overline{\lambda} > \lambda_{min}$ and $\underline{\lambda} < \lambda_{max}$:

$$\underline{\lambda} = \lambda_{min} + \epsilon < \lambda_{min} + \lambda_{max} - \lambda_{min} = \lambda_{max}$$
$$\overline{\lambda} = \lambda_{max} - \epsilon > \lambda_{max} - \lambda_{max} + \lambda_{min} = \lambda_{min}$$

This ends proof. □

**Lemma 7.39.** *If numbers $\underline{\lambda}, \overline{\lambda}$, such that:*

$$\lambda_{min} < \underline{\lambda} \leqslant \lambda_{max}, \ \lambda_{min} < \overline{\lambda} \leqslant \lambda_{max} \quad \textit{and} \quad \overline{\lambda} > \overline{f}(\underline{\lambda}),$$

*exist, then:*

$$\overline{f}(\lambda_{min}) < \lambda_{max}.$$

*Proof.* Proof of lemma 7.39 can be constructed in the same manner as the proof of lemma 7.37. □

**Lemma 7.40.** *If:*

$$\overline{f}(\lambda_{min}) < \lambda_{max},$$

*then there exist numbers $\underline{\lambda}, \overline{\lambda}$, such that:*

$$\lambda_{min} < \underline{\lambda} < \lambda_{max}, \ \lambda_{min} < \overline{\lambda} < \lambda_{max} \quad \textit{and} \quad \overline{\lambda} > \overline{f}(\underline{\lambda}).$$

*Proof.* Proof of lemma 7.40 can be constructed in the same manner as the proof of lemma 7.38. □

**Lemma 7.41.** *If numbers* $\underline{\lambda}, \overline{\lambda}$*, such that:*

$$\lambda_{min} < \underline{\lambda} \leqslant \lambda_{max}, \ \lambda_{min} < \overline{\lambda} \leqslant \lambda_{max} \quad \text{and} \quad \underline{\lambda} < \underline{f}(\overline{\lambda}), \ \overline{\lambda} > \overline{f}(\underline{\lambda}),$$

*exist, then:*

$$\underline{f}(\lambda_{max}) > \lambda_{min} \quad \text{and} \quad \overline{f}(\lambda_{min}) < \lambda_{max}.$$

*Proof.* Lemma 7.41 is a direct consequence of lemmas 7.37 and 7.39. □

**Lemma 7.42.** *If:*

$$\underline{f}(\lambda_{max}) > \lambda_{min} \quad \text{and} \quad \overline{f}(\lambda_{min}) < \lambda_{max},$$

*then there exist numbers* $\underline{\lambda}, \overline{\lambda}$*, such that:*

$$\lambda_{min} < \underline{\lambda} < \lambda_{max}, \ \lambda_{min} < \overline{\lambda} < \lambda_{max} \quad \text{and} \quad \underline{\lambda} < \underline{f}(\overline{\lambda}), \ \overline{\lambda} > \overline{f}(\underline{\lambda}).$$

*Proof.* Lemma 7.42 is a direct consequence of lemmas 7.38 and 7.40. □

## 7.3. Theorems for strictly pragmatic epistemic relation

**Theorem 7.43.** *(Pragmatic, Strict) If strictly pragmatic epistemic relation* $CS \models^{\text{SPE}} p \to \Pi(q)$ *is met, then pragmatic epistemic relation* $CS \models^{\text{PE}} p \to \Pi(q)$ *is also met.*

*Proof.* Proof is straightforward. Pragmatic epistemic relation can be met when $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q})$ or $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q})$. Strictly pragmatic epistemic relation can be met only when $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q})$. □

Theorems 7.13 and 7.43 together imply that whenever strictly pragmatic epistemic relation is met, then also normal epistemic relation is met.

**Theorem 7.44.** *(Pragmatic, Strict) Strictly pragmatic epistemic relation is equivalent to pragmatic epistemic relation, if the lower boundary function is a constant function of 0 (* $\forall_x \underline{f}(x) = 0$ *)*

*Proof.* When $\underline{f}(x)$ is a constant function of 0, condition $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q})$ is never met and effectively we are imposing strict conditional relation requirements ($\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q})$). □

Theorem 7.44 implies that any pair of formulas prohibited to be simultaneously grounded by normal or pragmatic epistemic relation shall also be prohibited by strictly pragmatic epistemic relation. This feature applies to theorems: 7.5, 7.8, 7.21. When one substitutes normal or pragmatic epistemic relations with strictly pragmatic epistemic relation in these theorems, provided conditions shall be sufficient but not always necessary.

All pairs of formulas allowed to be simultaneously grounded by pragmatic epistemic relation are not always allowed by strictly pragmatic epistemic relation. Whenever one of formulas must be grounded using lower boundary function requirement $\lambda^{p\to q} < \underline{f}(\lambda^{\neg p\to q})$ the pair will not be allowed. This remark applies to theorems: 7.17, 7.18, 7.19.

**Theorem 7.45.** *(Strict) It is possible to set boundary functions so that strictly pragmatic epistemic relation $CS \models^{\mathrm{SPE}} p \to Know(q)$ can be met. A necessary and sufficient condition is: $\overline{f}(0) < 1$.*

**Theorem 7.46.** *(Strict) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set boundary functions and grounding thresholds so that strictly pragmatic epistemic relation $CS \models^{\mathrm{SPE}} p \to Bel(q)$ can be met. A necessary and sufficient condition is:*

$$\overline{f}(0) < \lambda_{maxBel}.$$

**Theorem 7.47.** *(Strict) Assume a grounding threshold setting meeting inequalities 7.5. It is possible to set grounding thresholds and boundary function so that strictly pragmatic epistemic relation $CS \models^{\mathrm{SPE}} p \to Pos(q)$ can be met. A necessary and sufficient condition is:*

$$\overline{f}(0) < \lambda_{maxPos}.$$

*Proof.* Theorems 7.45-7.47 can be directly implied from theorem 7.44 and theorems 7.14-7.16. □

Theorems 7.45-7.47 provide conditions that ensure all types of conditional formulas can be grounded according to strictly pragmatic epistemic relation.

**Lemma 7.48.** *(Strict) Meeting of all conditions provided by theorems 7.45 - 7.47 implies that:*

$$\overline{f}(0) < \lambda_{maxPos} \tag{7.14}$$

Joining the conditions from theorems 7.45 - 7.47 leads to a more general requirement, as stated by lemma 7.48. This requirement ensures strictly pragmatic epistemic relation can be met for all modal operators. Requirement 7.48 shall be assumed in most of the following theorems.

*Proof.* Because $\lambda_{maxPos} < \lambda_{maxBel} \leqslant 1$, theorem 7.47 provides most restrictive requirements, which are stated in the lemma as inequality 7.14. □

Normal epistemic relation allows for simultaneous grounding of all combinations of formulas $p \to \Pi(q)$ and $\neg p \to \Xi(q)$ (see theorem 7.6). Pragmatic epistemic relation allows all combinations where at least one of modal operators is not a knowledge operator. Required constraints have been provided in theorems 7.17-7.19, 7.21 and 7.22. Strictly pragmatic epistemic relation does not allow for grounding of any of two formulas of the forms $p \to \Pi(q)$ and $\neg p \to \Xi(q)$. The chance for consequent (assuming antecedent) can't incerase in both conditionals. Theorem 7.49 states this fact and is true even without constraints 7.5 or 7.5, 7.7, 7.8.

**Theorem 7.49.** *(Strict) Meeting of strictly pragmatic epistemic relation* $CS \models^{\text{SPE}} p \to \Pi(q)$ *excludes meeting of* $CS \models^{\text{SPE}} \neg p \to \Xi(q)$.

*Proof.* Assume both strictly pragmatic epistemic relations are met, then, from strict conditional relation:

$$\lambda^{p \to q} > \overline{f}(\lambda^{\neg p \to q}) \quad \text{and} \quad \lambda^{\neg p \to q} > \overline{f}(\lambda^{p \to q})$$

Using the requirement $\overline{f}(x) \geqslant x$ and joining two inequalities leads to:

$$\lambda^{p \to q} > \overline{f}(\lambda^{\neg p \to q}) \geqslant \lambda^{\neg p \to q} > \overline{f}(\lambda^{p \to q}) \geqslant \lambda^{p \to q}$$

We obtain a contradiction, so both formulas can't be simultaneously met. □

When considering pairs of the forms $p \to \Pi(q)$ and $\neg p \to \Xi(\neg q)$ strictly pragmatic epistemic relation behaves similarly to pragmatic epistemic relation. Theorems 7.50-7.55 provide sufficient conditions.

**Theorem 7.50.** *(Strict) Assume a grounding threshold and boundary functions setting meeting inequalities 7.5 and 7.14. Strictly pragmatic epistemic relations* $CS \models^{\text{SPE}} p \to Know(q)$ *and* $CS \models^{\text{SPE}} \neg p \to Know(\neg q)$ *can be both met.*

*Proof.* Proof is the same as the proof of theorem 7.23. □

**Theorem 7.51.** *(Strict) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that strictly pragmatic epistemic relations* $CS \models^{\text{SPE}} p \to Know(q)$ *and* $CS \models^{\text{SPE}} \neg p \to Bel(\neg q)$ *can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxBel}) < 1$$

*Proof.* Proof is the same as the proof of theorem 7.24. □

**Theorem 7.52.** *(Strict) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that strictly pragmatic epistemic relations* $CS \models^{\text{SPE}} p \to Know(q)$ *and* $CS \models^{\text{SPE}} \neg p \to Pos(\neg q)$ *can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxPos}) < 1$$

*Proof.* Proof of theorem 7.52 can be constructed similarly to the proof of theorem 7.24. One needs to change grounding thresholds respectively. □

**Theorem 7.53.** *(Strict) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that strictly pragmatic epistemic relations* $CS \models^{\text{SPE}} p \to Bel(q)$ *and* $CS \models^{\text{SPE}} \neg p \to Bel(\neg q)$ *can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxBel}) < \lambda_{maxBel}$$

*Proof.* Proof is the same as the proof of theorem 7.26. □

**Theorem 7.54.** *(Strict) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that strictly pragmatic epistemic relations $CS \models^{\mathrm{SPE}} p \rightarrow Bel(q)$ and $CS \models^{\mathrm{SPE}} \neg p \rightarrow Pos(\neg q)$ can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxPos}) < \lambda_{maxBel} \ \wedge \ \overline{f}(1 - \lambda_{maxBel}) < \lambda_{maxPos}$$

*Proof.* Proof of theorem 7.54 is a special case of the proof of theorem 7.27. Simply use case 1 situations, whenever different cases are considered. □

**Theorem 7.55.** *(Strict) Assume a grounding threshold and boundary function setting meeting inequalities 7.5 and 7.6. It is possible to set grounding thresholds and upper and lower boundary functions so that strictly pragmatic epistemic relations $CS \models^{\mathrm{SPE}} p \rightarrow Pos(q)$ and $CS \models^{\mathrm{SPE}} \neg p \rightarrow Pos(\neg q)$ can be both met. A sufficient and necessary condition is:*

$$\overline{f}(1 - \lambda_{maxPos}) < \lambda_{maxPos}$$

*Proof.* Proof of theorem 7.54 is a special case of the proof of theorem 7.28. Simply use case 1 situations, whenever different cases are considered. □

Let us consider pairs of formulas the form $p \rightarrow \Pi(q)$ and $p \rightarrow \Xi(\neg q)$. Normal epistemic relation allows for simultaneous grounding of two pairs of this form: $p \rightarrow Bel(q)$ with $p \rightarrow Pos(\neg q)$ and $p \rightarrow Pos(q)$ with $p \rightarrow Pos(\neg q)$ (theorems 7.9 and 7.10). Other pairs are disallowed by theorems 7.5 and 7.8. Pragmatic epistemic relation allows and disallows the same pairs as normal epistemic relation. Required constraints have been provided by theorems 7.29 and 7.30.

Strictly pragmatic epistemic relation does not allow for simultaneous grounding of any of both formulas of the form $p \rightarrow \Pi(q)$ and $p \rightarrow \Xi(\neg q)$ (theorem 7.56). This property is a result from strict conditional relation, where a raise in the chance of the consequent is required.

**Theorem 7.56.** *(Strict) Meeting of strictly pragmatic epistemic relation $CS \models^{\mathrm{SPE}} p \rightarrow \Pi(q)$ excludes meeting of $CS \models^{\mathrm{SPE}} p \rightarrow \Xi(\neg q)$.*

*Proof.* Assume by contradiction that epistemic relations for both formulas can be met. Strictly pragmatic epistemic relation for $p \rightarrow \Pi(q)$ requires:

$$\lambda^{p \rightarrow q} > \overline{f}(\lambda^{p \rightarrow \neg q}),$$

while strictly pragmatic epistemic relation for $p \rightarrow \Xi(\neg q)$ requires:

$$\lambda^{p \rightarrow \neg q} > \overline{f}(\lambda^{p \rightarrow q}).$$

From above and the definition for upper boundary function: $\overline{f}(x) > x$ we have:

$$\lambda^{p\to q} > \overline{f}(\lambda^{p\to\neg q}) > \lambda^{p\to\neg q} > \overline{f}(\lambda^{p\to q}) > \lambda^{p\to q}$$

what is a contradiction. □

Both normal and pragmatic epistemic relations allow for pairs of the form: $p \to \Pi(q)$ and $q \to \Xi(\neg p)$ if $\Xi \neq Know$ (theorems 7.11, 7.31). Strictly pragmatic epistemic relation does not allow for such pairs. This is stated in theorem 7.57.

**Theorem 7.57.** *(Strict) Meeting of strictly pragmatic epistemic relation $CS \models^{\mathrm{SPE}} p \to \Pi(q)$ excludes meeting of $CS \models^{\mathrm{SPE}} q \to \Xi(\neg p)$.*

*Proof.* Let $c_1, c_2, c_3, c_4$ denote:

$$c_1 = \mathrm{card}(C^{p\wedge q}),\ c_2 = \mathrm{card}(C^{p\wedge\neg q}),\ c_3 = \mathrm{card}(C^{\neg p\wedge q}),\ c_4 = \mathrm{card}(C^{\neg p\wedge\neg q}).$$

Suppose by contradiction that epistemic relations for both formulas can be met. From strict conditional relation for $CS \models^{\mathrm{SPE}} \Pi(p \to q)$ we get $\lambda^{p\to q} > \overline{f}(\lambda^{\neg p\to q}) \geqslant \lambda^{\neg p\to q}$, hence:

$$\frac{c_1}{c_1 + c_2} = \frac{\mathrm{card}(C^{p\wedge q})}{\mathrm{card}(C^{p\wedge q} \cup C^{p\wedge\neg q})} > \frac{\mathrm{card}(C^{\neg p\wedge q})}{\mathrm{card}(C^{\neg p\wedge q} \cup C^{\neg p\wedge\neg q})} = \frac{c_3}{c_3 + c_4}$$

and for $CS \models^{\mathrm{SPE}} \Xi(q \to \neg p)$ we get $\lambda^{q\to\neg p} > \overline{f}(\lambda^{\neg q\to\neg p}) \geqslant \lambda^{\neg q\to\neg p}$, hence:

$$\frac{c_3}{c_3 + c_1} = \frac{\mathrm{card}(C^{\neg p\wedge q})}{\mathrm{card}(C^{\neg p\wedge q} \cup C^{p\wedge q})} > \frac{\mathrm{card}(C^{\neg p\wedge\neg q})}{\mathrm{card}(C^{\neg p\wedge\neg q} \cup C^{p\wedge\neg q})} = \frac{c_4}{c_4 + c_2}$$

Multiplying by denominators leads to:

$$c_1c_3 + c_1c_4 > c_1c_3 + c_2c_3 \quad \text{and} \quad c_2c_3 + c_3c_4 > c_1c_4 + c_3c_4$$

Simplifying and joining two inequalities leads to:

$$c_1c_4 > c_2c_3 > c_1c_4$$

We obtain a contradiction. □

### 7.3.1. Constraints for strictly pragmatic epistemic relation

Similarly as for normal and pragmatic epistemic relations, all properties given by theorems 7.45-7.57 can be simultaneously met by strictly pragmatic epistemic relation. Theorem 7.58 provides sufficient constraints on grounding threshold setting and boundary functions. These constraints are not necessary.

**Theorem 7.58.** *(Strict) Let grounding thresholds meet inequalities 7.5. It is possible to set lower and upper boundary functions so that all conditions from theorems 7.45-7.57 can be simultaneously met by strictly pragmatic epistemic relation. A sufficient condition is:*

$$\overline{f}(\max\{\lambda_{minPos},\ 1-\lambda_{maxPos}\}) < \lambda_{maxPos} \leqslant \overline{f}(\lambda_{minBel}) < \lambda_{maxBel} \tag{7.15}$$

*Proof.* Theorem 7.58 is a consequence of theorem 7.32. Theorems 7.45-7.57 do not provide new constraints regarding upper boundary function. The setting that is suitable for pragmatic epistemic relation given in equation 7.7, is also suitable for strictly pragmatic epistemic relation. □

## 7.4. Theorems for epistemic relations for modal conditionals

**Theorem 7.59.** *(ALL) All theorems for epistemic relations for conditional modalities (theorems 7.2-7.58) also hold for respective epistemic relations for modal conditionals.*

*One can simply substitute $\phi \rightarrow \Pi(\psi)$ with $\Pi(\phi \rightarrow \psi)$ and $\lambda_{minPos}$, $\lambda_{maxPos}$, $\lambda_{minBel}$, $\lambda_{maxBel}$ with $\beta_{minPos}$, $\beta_{maxPos}$, $\beta_{minBel}$, $\beta_{maxBel}$ respectively.*

*Proof.* Epistemic relations for modal conditionals defined by 6.11-6.13, 6.15 and 6.17 additionally require $\overline{C}^{p\wedge\neg q} = \emptyset$. This is the only difference to epistemic relations for conditional modalities (6.8-6.10, 6.14 and 6.16).

Conscious grounding sets $\overline{C}^{p\wedge q}, \overline{C}^{p\wedge\neg q}, \overline{C}^{\neg p\wedge q}, \overline{C}^{\neg p\wedge\neg q}$ can be empty regardless of the contents of grounding sets $C^{p\wedge q}, C^{p\wedge\neg q}, C^{\neg p\wedge q}, C^{\neg p\wedge\neg q}$. This additional requirement does not influence any of previous theorems. □

**Theorem 7.60.** *(ALL) Normal/pragmatic/strictly pragmatic epistemic relation for $Know(p \rightarrow q)$ is equivalent to respective epistemic relation for $p \rightarrow Know(q)$.*

*Proof.* Condition 2 from definition 6.13 also implies $\overline{C}^{p\wedge\neg q} = \emptyset$. If $\overline{C}^{p\wedge\neg q}$ would not be empty, then:

$$1 = \lambda^{p\rightarrow q} = \frac{\mathrm{card}(C^{p\wedge q})}{\mathrm{card}(C^{p\wedge q}) + \mathrm{card}(C^{p\wedge\neg q})} < \frac{\mathrm{card}(C^{p\wedge q})}{\mathrm{card}(C^{p\wedge q})} = 1$$

This leads to a contradiction. Similarly for pragmatic and strictly pragmatic epistemic relations (additional conditions on conditional relation are the same). □

Theorem 7.60 states that statements “I know that if $p$, then $q$” and “If $p$, then I know that $q$” have equivalent grounding conditions. Hence either both statements can be simultaneously grounded or neither of them.

**Theorem 7.61.** *(ALL) It is possible to set grounding thresholds so that meeting of normal/pragmatic/strictly pragmatic epistemic relation $\Xi(p \to q)$ implies meeting of respective epistemic relation for $p \to \Xi(q)$. A sufficient condition is:*

$$\lambda_{minPos} = \beta_{minPos}, \quad \lambda_{maxPos} = \beta_{maxPos},$$
$$\lambda_{minBel} = \beta_{minBel}, \quad \lambda_{maxBel} = \beta_{maxBel}$$

*Proof.* When respective grounding thresholds are equal and normal epistemic relation for modal conditional is met, then all conditions provided by definitions 6.8-6.10 for conditional modalities are also met. Similarly for pragmatic and strictly pragmatic epistemic relations. □

## 7.5. Exemplary grounding threshold and boundary function setting

One can choose grounding threshold and boundary function setting meeting all inequalities 7.5, 7.7, 7.8. This setting is adequate for all types of proposed epistemic relations. An exemplary grounding threshold setting is:

$$\lambda_{minPos} = 0, \ \lambda_{maxPos} = \lambda_{minBel} = 0.6, \ \lambda_{maxBel} = 1 \tag{7.16}$$

It meets constraints 7.5, because:

$$0 = \lambda_{minPos} < 1 - \lambda_{minBel} = 0.4 \leqslant 0.5$$
$$\leqslant \lambda_{maxPos} = 0.6 = \lambda_{minBel} < \lambda_{maxBel} = 1$$

The key advantage of choosing $\lambda_{maxPos} = \lambda_{minBel}$ is that there is no gap where agent could obtain too big grounding strength to use possibility operator and too small to use belief modal operator. Choice of $\lambda_{minPos} = 0$, $\lambda_{maxBel} = 1$ ensures all values $\lambda^{p \to q} \in (0, 1]$ of the grounding strength are covered by some modal operator[2].

Exemplary upper and lower boundary functions can be constructed from $\overline{f}_c$ (see equation 6.7):

$$\overline{f}(x) = \sqrt{r^2 - (x - x_0)^2} - x_0 + 1, \quad r = 2, \ x_0 \simeq 1.823 \tag{7.17}$$
$$\underline{f}(x) = 1 - \overline{f}(1 - x)$$

[2] Except zero that denotes the conequent is impossible in case of the antecedent.

These functions and a threshold setting 7.16, meet inequalities 7.7 and 7.8:

$$\begin{aligned}
\overline{f}(\max\{\lambda_{minPos},\ 1-\lambda_{maxPos}\}) &\simeq 0.583 \\
&< \lambda_{maxPos} = 0.6 \\
&\leqslant \overline{f}(\lambda_{minBel}) \simeq 0.760 \\
&< \lambda_{maxBel} = 1 \\
\underline{f}(\min\{\lambda_{maxBel},\ 1-\lambda_{minPos}\}) &= 1 \\
&> \lambda_{minBel} = 0.6 \\
&\geqslant \underline{f}(1-\lambda_{minBel}) \simeq 0.240 \\
&> \lambda_{minPos} = 0
\end{aligned}$$

One has too choose a radius $r$ big enough to meet all inequalities. For example radius $r = 1$ does not meet them. The bigger radius one chooses, the less 'impact' of the consequent is required. When one chooses too small radius, inequality $\overline{f}(1-\lambda_{maxPos}) < \lambda_{maxPos}$ is not met. This inequality is a consequence of condition from theorem 7.55 that in turn allows for simultaneous usage of $CS \models^{\mathrm{SPE}} p \rightarrow Pos(q)$ and $CS \models^{\mathrm{SPE}} \neg p \rightarrow Pos(\neg q)$.

## 7.6. Comparison to the simultaneous usage constraints

Table 7.1 presents pairs of formulas that can or can not be simultaneously met by respective epistemic relations when constraints 7.5, 7.7, 7.8 are met. Intuitively formulas that can be met together can be uttered together.

For example both formulas $p \rightarrow Bel(q)$, $p \rightarrow Pos(\neg q)$ (S9) can be uttered according to normal and pragmatic epistemic relations, but not according to strictly pragmatic epistemic relation. Antecedent $p$ can influence consequent $q$. Assuming $p$, we can be rather certain of $q$ and think that $\neg q$ is possible. Yet it is impossible that whenever antecedent holds, consequent's chance of holding increases and concurrently decreases. That is why this pair can't be simultaneously uttered in the most restrictive meaning of both conditionals. Either $p$ causes $q$ or it causes $\neg q$.

Please compare table 7.1 with table 5.2 that in turn summaries what pairs should be accepted/denied according to common-sense constraints. Both tables match on all pairs. This exemplifies that the grounding theory fulfils assumed common-sense constraints.

Table 7.1. Some non-trivial pairs of modal conditionals and their acceptability according to proven theorems. Columns 'Normal', 'Pragmatic' and 'Strict' refer to epistemic relation, pragmatic epistemic relation and strictly pragmatic epistemic relation respectively. Symbol ✓ denotes that simultaneous usage of statements is possible. Symbol ✕ denotes that simultaneous usage is disallowed. Numbers refer to corresponding theorems for simultaneous acceptance or denial of given pairs.

| no. | pair | Normal | Pragmatic | Strict |
|---|---|---|---|---|
| S1 | $p \to Know(q)$, $\neg p \to Know(q)$ | ✓ 7.6 | ✕ 7.21 | ✕ 7.49 |
| S2 | $p \to Know(q)$, $\neg p \to Know(\neg q)$ | ✓ 7.7 | ✓ 7.23 | ✓ 7.50 |
| S3 | $p \to Know(q)$, $p \to Know(\neg q)$ | ✕ 7.5 | ✕ | ✕ |
| S4 | $p \to Know(q)$, $p \to Bel(q)$ | ✕ 7.4 | ✕ | ✕ |
| S5 | $p \to Know(q)$, $p \to Pos(q)$ | ✕ 7.4 | ✕ | ✕ |
| S6 | $p \to Bel(q)$, $p \to Pos(q)$ | ✕ 7.4 | ✕ | ✕ |
| S7 | $p \to Know(q)$, $p \to Pos(\neg q)$ | ✕ 7.5 | ✕ 7.5 | ✕ 7.56 |
| S8 | $p \to Bel(q)$, $p \to Bel(\neg q)$ | ✕ 7.8 | ✕ 7.8 | ✕ 7.56 |
| S9 | $p \to Bel(q)$, $p \to Pos(\neg q)$ | ✓ 7.9 | ✓ 7.29 | ✕ 7.56 |
| S10 | $p \to Know(q)$, $q \to Pos(\neg p)$ | ✓ 7.11 | ✓ 7.31 | ✕ 7.57 |
| S11 | $p \to Bel(q)$, $q \to Bel(\neg p)$ | ✓ 7.11 | ✓ 7.31 | ✕ 7.57 |
| S12 | $p \to Bel(q)$, $q \to Pos(\neg p)$ | ✓ 7.11 | ✓ 7.31 | ✕ 7.57 |

# 8. Comparison to other theories on conditionals

Proposed grounding theory covers the conventional meaning of indicative conditionals. It proposes the formal criteria in form of epistemic relations required for proper grounding of conditionals. Grounding is performed on empirical material accessible to the agent. Further it is proven that the theory meets a series of common-sense criteria on simultaneous usage of conditionals.

It is important to notice the grounding process is designed from speakers perspective to allow for a rational utterance of a conditional statement in a typical context. From the listeners perspective the grounding process works differently, as he has to compare his empirical material with the speakers knowledge and intentions. A cooperative listener modifies a meaning of a conditional message in order to fit it to the possessed knowledge. In result the listener conforms to the speaker. Many of other theories on conditionals do not specify whether statements are considered from the speaker's or the listener's perspective. But the analysis process and empirical experiments are usually performed from the listener's perspective. This has an important impact on obtained results and shall be the source of crucial differences between the grounding theory and other solutions.

Within this chapter proposed grounding theory is compared to the most known theories on indicative conditionals. Similarities and differences are outlined.

## 8.1. Shortly on the material implication

A few words on the material implication have been already mentioned in section 4.1. It has been said that truth table of material implication does not model all aspects of conditionals. The falsity of the antecedent or the truth of the consequent is enough for the truth of the material implication. Speakers subjective incomplete knowledge and mood are not considered. In result material implication doesn't model rational usage patterns as it allows for statements like:

— If the moon is a piece of cheese, then I can jump 100 meters high.
— If birds can fly, then Roosevelt was a president of United States.
— If I am a snail, then Earth is round.

Proposed grounding theory constraints usage of such statements. Statements 1 and 3 are disallowed due to $C^{p(o)} \neq \emptyset$ criterion. These statements can't be used as indicative conditionals because the antecedents are known not to hold. Second statement is allowed only by the normal epistemic relation 6.10 and 6.13 that models only the most broad meaning of a conditional. The only requirement for normal epistemic relation is that the antecedent can't be known not to hold. Pragmatic epistemic relation does not allow for the second statement, because there are no proofs for empirical dependence between the antecedent and the consequent. If the speaker doesn't see such dependence, rational usage of a conditional is forbidden. For pragmatic and strict epistemic relations neither the antecedent nor the consequent can be known.

## 8.2. A conditional probability or belief based theories

Between 1965 and 1987 emerged a broad group of theories (Adams 1975; Stalnaker 1980; Jackson 1987) based on Ramsey test (already presented in section 4.2) and conditional belief (already presented in section 4.4).

Theories were broadly discussed and differed in some aspects but one claim stayed constant:

> Our belief in a conditional 'If $A$, $B$' is equal to the conditional belief in $B$ assuming that $A$.

Such approach had an interesting advantages over material implication as it allowed for a partial belief in a conditional. Later this claim has been used to model subjunctives with the help of possible worlds (see (Gibbard 1980) for a review).

The normal epistemic relations from grounding theory are compatible with this claim. Relative grounding strength $\lambda^{p(o) \to q(o)}$ is similar to conditional belief. Depending on the strength's value one shall say $p(o) \to Pos(q(o))$ or $p(o) \to Bel(q(o))$ or $p(o) \to Know(q(o))$. In result normal epistemic relations form a theory similar to theories based on conditional belief.

The problem with this claim is that it is enough for belief in the consequent $B$ to be high when antecedent $A$ holds. When it comes to uttering a conditional this constraint is not strict enough. One may have high belief in $B$ regardless of $A$. For example: 'If you eat apples, you will die before 2080'. The speaker may be almost certain the listener will die before 2080, so his belief in conditional should be high. Unfortunatelly it isn't, as it does not depend on eating apples. Pragmatic and strictly pragmatic epistemic relations of the grounding theory are free of this fallacy.

## 8.3. Modal logic and Kripke semantics

When it comes to modal operators one cannot omit mentioning of modal logic with Kripke semantics (Garson 2013). Modal logic introduces two modal operators of possibility '$\Diamond$' and necessity '$\Box$'. Kripke introduced relational semantics based on frames.

A frame is a pair $\langle W, R\rangle$, where elements of $W$ are worlds and $R$ is a relation between the elements of $W$. $R$ is often called accessibility relation denoting which worlds are accessible from a given world $w \in W$. Semantics for non-modal formulas are taken directly from propositional logic. Kripke semantics allow for interpretation of modal formulas. Let $p, q$ be atomic formulas of propositional logic and $A, B$ be modal formulas. Further, let $w, x \in W$ and $\models$ be a satisfaction relation, then:

$$\begin{array}{ll}
w \models A & \text{iff } w \text{ satisfies } A \\
w \models \neg A & \text{iff } w \not\models A \\
w \models A \wedge B & \text{iff } w \models A \text{ and } w \models B \\
w \models A \Rightarrow B & \text{iff } w \not\models A \text{ or } w \models B \\
w \models \Box A & \text{iff } \forall_{x \in W} : (w\, R\, x) \Rightarrow (x \models A) \\
w \models \Diamond A & \text{iff } \exists_{x \in W} : (w\, R\, x) \text{ and } (x \models A)
\end{array}$$

In particular, satisfaction relation for modal implications is defined as:

$$\begin{array}{ll}
w \models p \Rightarrow \Box q & \text{iff } w \not\models p \text{ or } \forall_{x \in W}(w\, R\, x) \Rightarrow (x \models q) \\
w \models p \Rightarrow \Diamond q & \text{iff } w \not\models p \text{ or } \exists_{x \in W}(w\, R\, x) \text{ and } (x \models q) \\
w \models \Box(p \Rightarrow q) & \text{iff } \forall_{x \in W}(w\, R\, x) \Rightarrow (x \models p \Rightarrow q) \\
w \models \Diamond(p \Rightarrow q) & \text{iff } \exists_{x \in W}(w\, R\, x) \Rightarrow (x \models p \Rightarrow q)
\end{array}$$

Accessibility relation $R$ can be transitive, reflexive, symmetric, etc.. Possible interpretations of satisfaction relation vary greatly, depending on the shape of the accessibility relation. One can construct many types of modal logics by simply changing the properties of accessibility relation.

Modal logic with Kripke semantics uses definition of implication from Boolean logic. In result modal implication $\Box(p \Rightarrow q)$ is not free of shortcomings already mentioned for material implication. The falsity of the antecedent or the truth of consequent is still a sufficient condition for the truth of implication.

Kripke semantics define satisfaction relation with respect to some chosen world $w$ often denoted as a factual world. This world is used to evaluate whether some modal formula is satisfied or not. In the considered case, we would like the worlds to be interpreted as possible variants of some not thoroughly known environment. Unfortunately, in such case, there exists a major interpretative problem with the choice of the factual world $w$ (Katarzyniak 2007). If one knows which of the worlds is factual, he does not need to consider other worlds (unless he is thinking counter-factually). If one does not know which world is factual, he does not know how to evaluate $w \models \Box A$ or $w \models \Diamond A$, as $w$ is not fixed. It is also impossible not to know whether $A$ is satisfied in $w$, as it stands in contradiction with the principle of bivalence.

Let us focus on a case where the accessibility relation is an equivalence relation (it is transitive, reflexive and symmetric). In such case $[w]_R \subseteq W$ forms an equivalence class with respect to $R$. When $R$ is an equivalence relation, $[w]_R$ may be interpreted as a set of all worlds consistent with agent's knowledge about the environment. For modal formulas $\Diamond p$ and $\Box p$ the satisfaction relation either holds in all worlds from $[w]_R$ or in none of them. For modal formulas $\Diamond p$ and $\Box p$ it no longer matters which of the worlds from $[w]_R$ is the factual one.

Unfortunately this is not the case for formulas of propositional logic. In my opinion there are serious problems with semantics of modal implications $p \Rightarrow \Box q$ and $p \Rightarrow \Diamond q$. Problems arise due to the the choice of the factual world. Suppose $w_1, w_2 \in [w]_R$. If $w_1 \not\models p$, then $w_1 \models p \Rightarrow \Box q$, regardless of the chosen consequent $q$. At the same time, if $w_2 \models p$, then $w_2 \models p \Rightarrow \Box q$ may not hold. Depending on the chosen factual world the implication can be true or not. But I don't know which world is the factual one. How can I tell whether this modal implication is true or not? If I know which of the two worlds is the factual one I do not need to use a conditional. I simply know $p$ and $q$.

Even greater problems arise for possibility operator. In my opinion the semantics of $\Diamond(p \Rightarrow q)$ are wrongly modelled. Suppose $R$ is an equivalence relation and we have a total of 10 worlds in one equivalence class $[w]_R$. In 9 of these worlds there is $p$ and $\neg q$ and in the 10th world there is $\neg p$ and $\neg q$. For such a setting $w \models \Box(\neg q)$ and $w \models \Diamond(p \Rightarrow q)$ but nobody sane would utter this conditional. The only setting of worlds where $\Diamond(p \Rightarrow q)$ does not hold, is where every accessible world satisfies $p$ and $\neg q$. This means the only situation where one can't say $\Diamond(p \Rightarrow q)$ is when one knows that $p$ and $\neg q$. This is irrational.

Finally, for every $R$ such that $\forall_w \exists_x : wRx$, whenever $\Box A$ is true, $\Diamond A$ is also true. This is intuitive in the context the most broad meaning of possibility. On the other hand, uttering $\Diamond A$, when one knows $A$, is misleading.

As a result modal logic with Kripke semantics does not model uttering of conditionals well. There are interpretative problems related to the choice of factual world. It repeats shortcomings of material implication. It uses a very broad unintuitive understanding of the modal operator of possibility.

## 8.4. Mental models and possibilities

Johnson presented an interesting theory on conditionals (Johnson-Laird and Byrne 2002; Byrne and Johnson-Laird 2009) that tries to define the meaning of conditionals in terms of the theory of mental models and possibilities (Johnson-Laird and Savary 1999). Johnson focuses on conditionals of the forms 'If $p$ then $q$' and 'If $p$ then possibly $q$'. The world 'possibly' has a meta-meaning as it is a general substitute for phrases like: 'may be', 'is permissible', 'is allowed', 'can' etc.. The theory of conditionals and the theory of mental models are quite complex so it is difficult to shortly describe them. I will try to outline some of the key facts of Johnson's theory to compare it to the grounding theory of conditionals.

The theory of mental models (Johnson-Laird and Savary 1999) has been originally formulated to model human's elementary deductive reasoning. Theory takes into account the fact that humans often construct incomplete minimal mental models for statements.

Informally, the fundamental principle states that the mental model represents a setting of one or more assertions that are true (hypothetically). If such assertions can be all concurrently true, the mental model represents a possibility. Exemplary mental models for the statement 'The battery is dead or the circuit is not connected' are (Johnson-Laird and Byrne 2002):

| | | |
|---|---|---|
| factual possibilities: | dead | |
| | | ¬ connected |
| | dead | ¬ connected |

Each row represents a mental model. Row '¬ connected' denotes that battery is not connected and tells nothing about battery being dead. Third row 'dead' and '¬ connected' is associated with a mental model of battery being both dead and not connected. There is no mental model for '¬ dead' and 'connected' as it is not a possibility (it is false) for the considered statement. Mental models represent assertions supporting the sentence, not neglecting it[1].

Depending on background knowledge and the amount of time spent on reasoning about a statement mental model can be simpler or more complex. If one knows the battery can't be both dead and not connected he will not consider a mental model holding both 'dead' and '¬ connected'. If one thinks longer about a statement he may construct a fully explicit model:

| | | |
|---|---|---|
| factual possibilities: | dead | connected |
| | ¬ dead | ¬ connected |
| | dead | ¬ connected |

The background idea of this theory is that mental models are constructed in a predefined order. Humans construct models that are as simple as possible and concentrate on the models 'most strongly' supporting a statement, not neglecting it. The models that are minimal for understanding a statement are constructed as first. Proposed approach seems to explain typical mistakes made in humans' reasoning processes what has been confirmed by numerous experiments (Johnson-Laird and Savary 1999; Oberauer 2010).

Proposed theory of mental models (and conditionals) defines what mental models are constructed for simple and complex statements. The fact that some mental models are constructed prior to others is taken into account. The process of constructing mental models is considered from listeners perspective. Proper understanding of a heard statement is not the same as feeling the urge to utter a statement[2]. This is an important difference to the grounding theory, where this process is analysed from speakers perspective.

[1] According to this theory false assertions can be later inferred from true assertions

[2] For example we adjust the meaning of the statement in order to properly understand the speaker.

To model the meaning of conditionals authors extend the theory of mental models with 5 principles (Johnson-Laird and Byrne 2002). Principle 2 refers to subjunctive conditionals and will be skipped here.

First principle plays the key role in proposed theory and refers to something Johnson calls a core meaning. The core meaning is understood as a meaning assigned to basic conditionals with neutral content that is independent from context and background knowledge. The principle states that the core meaning of a conditional 'If $p$ then $q$' refers to three possibilities:

$$\begin{array}{rr} P & Q \\ \neg P & Q \\ \neg P & \neg Q \end{array}$$

And a core meaning of a conditional 'If $p$ then possibly $q$' is the tautological interpretation, which refers to all four possibilities:

$$\begin{array}{rr} P & Q \\ P & \neg Q \\ \neg P & Q \\ \neg P & \neg Q \end{array}$$

Principle 1 simply states that the listener of a 'true' conditional sentence 'If $p$ then $q$' can construct at most 3 possibilities. Mental model associated with $P$ and $\neg Q$ is a factual impossibility. For a conditional 'If $p$ then possibly $q$' all mental models can be possibilities. Some of the mental models are optional and may be missing. For example, for a statement 'If $p$ then possibly $q$', the pragmatics of the situation often rule out mental model $\neg P$ and $Q$. Listener often assumes that $Q$ is impossible in the absence of $P$.

The most interesting part of Johnson's theory comes with principle 3 called the principle of implicit models:

> 'Basic conditionals have mental models representing the possibilities in which their antecedents are satisfied, but only implicit mental models for the possibilities in which their antecedents are not satisfied. ...'

For a statement 'If $p$ then $q$' mental model $P$, $Q$ is explicitly present and mental models $\neg P$, $Q$ and $\neg P$, $\neg Q$ are only implicit. Term 'implicit' is a bit confusing. It means the models are not built or fully developed in reasoners mind. Reasoner is aware of their existence but does not explicitly construct and consider them.

Similarly for a statement 'If $p$ then possibly $q$' mental models $P$, $Q$ and $P$, $\neg Q$ are explicitly present and mental models $\neg P$, $Q$ and $\neg P$, $\neg Q$ are only implicit.

According to principle 3, when a human considers a conditional he constructs explicit models with satisfied antecedent. Other models are optional and depend on listeners background knowledge and time spent on the analysis of a statement. At first the listener constructs a mental model for $P$ and $Q$ if such model is not a factual possibility the conditional can't be accepted. Further the listener may try to construct a mental model for $P$ and $\neg Q$. In the end listener may, but doesn't have to analyse mental models for $\neg P$, $Q$ and $\neg P$, $\neg Q$.

Principle 4 refers to the listeners background knowledge on the antecedent and the consequent. It states that constructed mental models depend on the meaning of the antecedent and the consequent. For example, if the listener knows the consequent is impossible in the absence of the antecedent he will not construct a mental model for it.

Principle 5 tells that the context of a conditional depends on long-term memory and particular circumstances of the utterance. This context influences the process of construction of the mental models.

Principles 4 and 5 are rather general and do not formally define how long-term memory, background knowledge or context influence mental models construction process. Johnson considers many types of conditionals and proposes a total of 10 different settings of possibilities depending on the contents of a conditional statement. Johnson describes the construction of mental models with the help of many examples. Unfortunately he does not provide formal criteria on when which setting to use. When compared to the grounding theory, his theory is meant to cover more types of conditionals including subjunctives. Yet, he does not consider the belief modal operator.

Johnson's theory has much in common with the grounding theory proposed here. Mental models have similar interpretation to grounding sets. Explicit and implicit models are similar to conscious and unconscious areas of cognitive state. In the grounding theory the cognitive state is constructed from empirical experiences within some context. Cognitive state defines the contents of grounding sets. Johnson's theory does not explain how exactly the context and long-term memory influences the mental models.

Unlike in Johnoson's theory, the grounding theory focuses on providing criteria for uttering a statement. Johnson's theory focuses on possibilities constructed from the perspective of sentence recepient. It is important whether some mental model is a possibility or not. There is no measure quantifying the influence of particular possibilities. If Johnson's theory were used to decide whether a conditional can be uttered, it would not be free of some of the typical shortcomings mentioned earlier. For example the tautological setting of mental models for 'If $p$ then possibly $q$' allows for statements like: 'If he is tall, then it is possible that the apple is red'.

Principles 1 and 3 constrain mental models for two types of statements: 'If $p$ then $q$', 'If $p$ then possibly $q$'. These conditions are much aligned with normal epistemic relations proposed in definitions 6.8 and 6.10 respectively. For the formula $p \rightarrow Pos(q)$ it is required that $C^p \neq \emptyset$ and $\lambda_{minPos} < \lambda^{p \rightarrow q} \leqslant \lambda_{maxPos}$. This implies that neither $C^{p \wedge q}$ nor $C^{p \wedge \neg q}$ can be empty. For Johnson's mental models, this means both $P$, $Q$ and $P$, $\neg Q$ have to be factual possibilities.

For the formula $p \rightarrow Know(q)$ it is required that $C^p \neq \emptyset$ and $\lambda^{p \rightarrow q} = 1$. This in turn implies that $C^{p \wedge \neg q} = \emptyset$. For Johnson's mental models, $P$, $Q$ is a possibility and $P$, $\neg Q$ is an impossibility.

Johnson's work focuses on mental models constructed for conditional sentences but does not provide explicit criteria of pragmatic and rational statement usage. His theory aligns in key aspects with the grounding theory proposed here. Merging of both theories is certainty an interesting aim worth pursuing. The predicates of Johnson's theory may be helpful in defining and implementing the construction process of the cognitive state.

# 9. Usage examples

## 9.1. Computational example

Following section presents a simple computational example constructed upon agent's knowledge state of 8 base profiles. The example is meant to explain the grounding process. Three cases of the grounding process and its results are presented and discussed.

### 9.1.1. Knowledge state

Let agent's perceptive abilities be limited to recognition of three objects $O = \{o_1, o_2, o_3\}$ and four properties $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$. Further, let grounding threshold and boundary functions be set as in equations 7.16 and 7.17.

Agent has gathered observations for 8 time moments, where $t = 8$ is the current time moment. Internal reflections are gathered in 8 base profiles held in agent's knowledge state $KS = \{BP(1), BP(2), ..., BP(8)\}$.

Contents of base profiles are presented in table 9.1. Rows and columns represent base profiles and properties respectively. Cells contain perceptions of objects (not) exhibiting particular properties. For example, at time moment $\hat{t} = 5$ agent has observed that $o_1$ and $o_2$ exhibit property $P_3^+$ ($o_1 \in P_3^+(5)$ and $o_2 \in P_3^+(5)$), while $o_3$ does not exhibit it ($o_3 \in P_3^-(5)$). At the same time moment $t = 5$ property $P_2$ was not covered by perception for object $o_3$ ($o_3 \notin P_2^+(5)$ and $o_3 \notin P_2^-(5)$). Agent has not observed whether $o_3$ exhibits $P_2$ or not.

Agent can use empirical knowledge presented in table 9.1 to reason about environment, draw conclusions and eventually utter statements. Agent processes empirical knowledge using its mental capabilities in accordance with its intentions. This results in a particular cognitive state. This cognitive state can be later used to ground conditional formulas. The formulas properly grounded may be eventually uttered. The following three sections present different cases of grounding process and its results depending on the formed cognitive state.

Table 9.1. Agent's knowledge state up to current time moment $t = 8$

| $\hat{t}$ | $P_1^+$ | $P_1^-$ | $P_2^+$ | $P_2^-$ | $P_3^+$ | $P_3^-$ | $P_4^+$ | $P_4^-$ |
|---|---|---|---|---|---|---|---|---|
| 8 | $o_1$ | $o_3$ | $o_1, o_2$ | | $o_1$ | $o_3$ | | $o_1$ |
| 7 | $o_3$ | $o_1$ | $o_1$ | | $o_1$ | $o_3$ | | $o_1, o_2$ |
| 6 | $o_1$ | | $o_1, o_2$ | | $o_2$ | | $o_2$ | |
| 5 | $o_3$ | | $o_2$ | $o_1$ | $o_1, o_2$ | $o_3$ | $o_2$ | $o_1$ |
| 4 | $o_1$ | | $o_2$ | | $o_3$ | $o_2$ | | $o_2$ |
| 3 | | $o_3$ | | $o_1, o_2$ | $o_1$ | $o_2, o_3$ | $o_2$ | |
| 2 | | $o_1, o_3$ | | $o_2$ | $o_2, o_3$ | | $o_2$ | $o_1$ |
| 1 | $o_3$ | $o_1$ | | $o_1, o_2$ | $o_2, o_3$ | | | $o_2$ |

### 9.1.2. Case 1: Context free example

Assume all empirical knowledge resides in conscious area of cognitive state model. Such state can be used to utter a context free (general) statement of whole empirical knowledge.

$$\overline{CS}(t) = KS(t), \quad \underline{CS}(t) = \emptyset \tag{9.1}$$

**Case 1.1:** Assume agent is focused on object $o_1$ and its properties $P_1$ and $P_2$, for which grounding sets are:

$$\begin{aligned} C^{p_1(o_1)\wedge p_2(o_1)} &= \{BP(6), BP(8)\}, \\ C^{p_1(o_1)\wedge \neg p_2(o_1)} &= \emptyset, \\ C^{\neg p_1(o_1)\wedge p_2(o_1)} &= \{BP(7)\}, \\ C^{\neg p_1(o_1)\wedge \neg p_2(o_1)} &= \{BP(1)\} \end{aligned}$$

The grounding strengths are calculated from the grounding sets:

$$\begin{aligned} &\lambda^{p_1(o_1)\to p_2(o_1)} = 1, \quad \lambda^{p_1(o_1)\to \neg p_2(o_1)} = 0, \\ &\lambda^{\neg p_1(o_1)\to p_2(o_1)} = 0.5, \quad \lambda^{\neg p_1(o_1)\to \neg p_2(o_1)} = 0.5, \end{aligned}$$

One can notice that whenever $o_1$ exhibited $P_1$, it also exhibited $P_2$. There are observations where $o_1$ exhibited $P_1$ and where it did not. Similarly, there are observations where $o_1$ exhibited $P_2$ and where it did not. Neither the antecedent, nor the consequent is known to (not) hold.

Upper boundary function meets $\lambda^{p_1(o_1)\to p_2(o_1)} > \overline{f}(\lambda^{\neg p_1(o_1)\to p_2(o_1)})$.

For such a cognitive state model the following epistemic relations (definitions 6.8-6.17) are met:

— $CS \models^{\text{SPE}} Know(p_1(o_1) \to p_2(o_1))$,
— $CS \models^{\text{SPE}} p_1(o_1) \to Know(p_2(o_1))$,
— $CS \models^{\text{E}} \neg p_1(o_1) \to Pos(p_2(o_1))$,

— $CS \models^{\mathrm{E}} \neg p_1(o_1) \to Pos(\neg p_2(o_1))$,

and the following exemplary epistemic relations are NOT met:

— $CS \not\models^{\mathrm{E}} p_1(o_1) \to Pos(\neg p_2(o_1))$
— $CS \not\models^{\mathrm{PE}} \neg p_1(o_1) \to Pos(p_2(o_1))$
— $CS \not\models^{\mathrm{PE}} \neg p_1(o_1) \to Pos(\neg p_2(o_1))$

Formulas $Know(p_1(o_1) \to p_2(o_1))$ and $p_1(o_1) \to Know(p_2(o_1))$ meet all types of epistemic relations (see theorems 7.13 and 7.43). These two formulas can be used as conditionals in their conventional meaning having all typical implicatures.

Formula $p_1(o_1) \to Pos(\neg p_2(o_1))$ can't be grounded, because there was never $\neg p_2(o_1)$, in case of $p_1(o_1)$.

Formulas $\neg p_1(o_1) \to Pos(p_2(o_1))$ and $\neg p_1(o_1) \to Pos(\neg p_2(o_1))$ meet only normal epistemic relation. Pragmatic epistemic relation is not met, because conditional relation is not met. These formulas imply only that $p_2(o_1)$ is possible when $\neg p_1(o_1)$ (degraded meaning).

**Case 1.2:** Assume the cognitive state given as in equation 9.1. Let now the agent focus on object $o_3$ and its properties $P_1$ and $P_3$. Respective grounding sets are:

$$\begin{aligned} C^{p_1(o_3)\wedge p_3(o_3)} &= \{BP(1)\}, \\ C^{p_1(o_3)\wedge \neg p_3(o_3)} &= \{BP(5), BP(7)\}, \\ C^{\neg p_1(o_3)\wedge p_3(o_3)} &= \{BP(2)\}, \\ C^{\neg p_1(o_3)\wedge \neg p_3(o_3)} &= \{BP(8), BP(3)\} \end{aligned}$$

and the calculated grounding strengths are:

$$\begin{aligned} \lambda^{p_1(o_3)\to p_3(o_3)} &= \frac{1}{3},\ \lambda^{p_1(o_3)\to\neg p_3(o_3)} = \frac{2}{3}, \\ \lambda^{\neg p_1(o_3)\to p_3(o_3)} &= \frac{1}{3},\ \lambda^{\neg p_1(o_3)\to\neg p_3(o_3)} = \frac{2}{3} \end{aligned}$$

Because $\lambda^{p_1(o_3)\to p_3(o_3)} = \lambda^{\neg p_1(o_3)\to p_3(o_3)}$, $p_3(o_3)$ is not conditionally related to $p_1(o_3)$.

For such a cognitive state the following epistemic relations (definitions 6.8-6.17) are met:

— $CS \models^{\mathrm{E}} p_1(o_3) \to Pos(p_3(o_3))$,
— $CS \models^{\mathrm{E}} p_1(o_3) \to Bel(\neg p_3(o_3))$

and the following exemplary epistemic relations are NOT met:

— $CS \not\models^{\mathrm{PE}} p_1(o_3) \to Pos(p_3(o_3))$,
— $CS \not\models^{\mathrm{PE}} p_1(o_3) \to Bel(\neg p_3(o_3))$.

Pragmatic epistemic relations are not met, because conditional relation is not met. Agent can utter $p_1(o_3) \to Pos(p_3(o_3))$ or $p_1(o_3) \to Bel(\neg p_3(o_3))$ only in their degraded meaning, where dependence between the antecedent and the consequent is not required. It is possible to simultaneously claim that $p_3(o_3)$ is possible and we believe $\neg p_3(o_3)$ (assuming $p_1(o_3)$).

### 9.1.3. Case 2: Known antecedent or consequent

Let the cognitive state, similarly as the in previous case, consist of all empirical knowledge accessible to the agent:

$$\overline{CS}(t) = KS(t), \quad \underline{CS}(t) = \emptyset \tag{9.2}$$

Let agent focus on $o_1$ and properties $P_2$ and $P_3$, for which grounding sets are:

$$\begin{aligned} C^{p_2(o_1) \wedge p_3(o_1)} &= \{BP(7), BP(8)\}, \\ C^{p_2(o_1) \wedge \neg p_3(o_1)} &= \emptyset, \\ C^{\neg p_2(o_1) \wedge p_3(o_1)} &= \{BP(3), BP(5)\}, \\ C^{\neg p_2(o_1) \wedge \neg p_3(o_1)} &= \emptyset \end{aligned}$$

**Case 2.1:** Let agent consider $p_2(o_1)$ as an antecedent. Grounding strengths are:

$$\begin{aligned} &\lambda^{p_2(o_1) \to p_3(o_1)} = 1, \ \lambda^{p_2(o_1) \to \neg p_3(o_1)} = 0, \\ &\lambda^{\neg p_2(o_1) \to p_3(o_1)} = 1, \ \lambda^{\neg p_2(o_1) \to \neg p_3(o_1)} = 0 \end{aligned}$$

Feature $p_2(o_1)$ is not conditionally related to $p_3(o_1)$, because $\lambda^{p_2(o_1) \to p_3(o_1)} = \lambda^{\neg p_2(o_1) \to p_3(o_1)}$.

For such a cognitive state model the following exemplary normal epistemic relations are met:

— $CS \models^{\mathrm{E}} Know(p_2(o_1) \to p_3(o_1))$,
— $CS \models^{\mathrm{E}} Know(\neg p_2(o_1) \to p_3(o_1))$

and respective pragmatic epistemic relations are NOT met:

— $CS \not\models^{\mathrm{PE}} Know(p_2(o_1) \to p_3(o_1))$,
— $CS \not\models^{\mathrm{PE}} Know(\neg p_2(o_1) \to p_3(o_1))$.

When the consequent $p_3(o_1)$ is known to hold, the conditional statement $p_2(o_1) \to p_3(o_1)$ can be uttered only in its degraded meaning (one simply knows $p_3(o_1)$ assuming $p_2(o_1)$). Pragmatic epistemic relations are not met, there is no point in uttering a conditional, when the consequent holds regardless of the antecedent.

**Case 2.2:** Now let agent consider $p_3(o_1)$ as an antecedent. Respective grounding strengths are:

$$\lambda^{p_3(o_1) \to p_2(o_1)} = 0.5, \ \lambda^{p_3(o_1) \to \neg p_2(o_1)} = 0.5$$

Antecedent is already known to hold because set $C^{\neg p_3(o_1)} = \emptyset$. Grounding strengths $\lambda^{\neg p_3(o_1) \to p_2(o_1)}$, $\lambda^{\neg p_3(o_1) \to \neg p_2(o_1)}$ can't be calculated.

The normal epistemic relations are met:

— $CS \models^{\mathrm{E}} p_3(o_1) \to Pos(p_2(o_1))$,
— $CS \models^{\mathrm{E}} p_3(o_1) \to Pos(\neg p_2(o_1))$

and the follwoing normal epistemic relations are NOT met:

— $CS \not\models^{\mathrm{E}} \neg p_3(o_1) \rightarrow Pos(p_2(o_1))$,
— $CS \not\models^{\mathrm{E}} \neg p_3(o_1) \rightarrow Pos(\neg p_2(o_1))$

because of $C^{\neg p_3(o_1)} = \emptyset$.

When antecedent is known not to hold, one deals with subjunctive conditionals so considered *indicative* conditionals can't be grounded.

When the antecedent or the consequent is known (not) to hold in advance, one can't use a conditional in its full conventional meaning. Pragmatic epistemic relations are never met.

### 9.1.4. Case 3: Different contexts

The contents of cognitive state are context driven and depend on current observations, agent's intentions and desires. The choice of empirical material included in the cognitive state model influences grounding process. Different statements can be grounded depending on the contents of conscious and unconscious areas of cognitive state. Following three cases exemplify this phenomenon.

Let agent focus on object $o_2$ and its properties $P_3$ and $P_4$.

**Case 3.1:** In this case the agent considers only the observations where $o_2$ exhibited $P_2$. In result, conscious area contains all situations where $o_2 \in P_2^+$ and unconscious area is empty ($\underline{C} = \emptyset$).

$$\overline{CS}(t) = \{BP(4), BP(5), BP(6), BP(7), BP(8)\}, \quad \underline{CS}(t) = \emptyset$$

The grounding sets are:

$$\begin{array}{llll}
\overline{C}^{p_3(o_2)\wedge p_4(o_2)} & = \{BP(5), BP(6)\}, & \underline{C}^{p_3(o_2)\wedge p_4(o_2)} & = \emptyset \\
\overline{C}^{p_3(o_2)\wedge \neg p_4(o_2)} & = \emptyset, & \underline{C}^{p_3(o_2)\wedge \neg p_4(o_2)} & = \emptyset \\
\overline{C}^{\neg p_3(o_2)\wedge p_4(o_2)} & = \emptyset, & \underline{C}^{\neg p_3(o_2)\wedge p_4(o_2)} & = \emptyset \\
\overline{C}^{\neg p_3(o_2)\wedge \neg p_4(o_2)} & = \{BP(4)\}, & \underline{C}^{\neg p_3(o_2)\wedge \neg p_4(o_2)} & = \emptyset
\end{array}$$

Such conscious state can be suitable for example when speaking in the context of current moment $t = 8$. In this moment agent has observed $o_2 \in P_2^+(8)$ (see table 9.1). Agent may conclude property $P_2$ is important and influences properties $P_3$ and $P_4$. For that reason agent considers only empirical knowledge where $o_2 \in P_2^+$.

In such case epistemic relation:

— $CS \models^{\mathrm{SPE}} Know(p_3(o_2) \rightarrow p_4(o_2))$

is met.

Formula $CS \models^{\mathrm{SPE}} Know(p_3(o_2) \rightarrow p_4(o_2))$ can be properly grounded and eventually uttered. This time agent does not generally summarize its whole knowledge. The meaning of conditional is constrained to a particular context (where $o_2 \in P_2^+$). The speaking agent and a recipient must have previously aligned contexts on $P_2$. This is required, so that the speaking agent shall not mislead the recipient.

**Case 3.2:** Assume conscious area contains all observations where $o_2 \in P_2^+$ and unconscious area contains observations where $o_2 \in P_2^-$.

$$\overline{CS}(t) = \{BP(4), BP(5), BP(6), BP(7), BP(8)\},$$
$$\underline{CS}(t) = \{BP(1), BP(2), BP(3)\}$$

This situation is similar to previous one, but observations $o_2 \in P_2^-$ are not excluded from mental state and influence agent's reasoning. Yet situations where $o_2 \in P_2^+$ are more important.

Grounding sets are:

$$\begin{array}{llll}
\overline{C}^{p_3(o_2)\wedge p_4(o_2)} & = \{BP(5), BP(6)\}, & \underline{C}^{p_3(o_2)\wedge p_4(o_2)} & = \{BP(2)\} \\
\overline{C}^{p_3(o_2)\wedge \neg p_4(o_2)} & = \emptyset, & \underline{C}^{p_3(o_2)\wedge \neg p_4(o_2)} & = \{BP(1)\} \\
\overline{C}^{\neg p_3(o_2)\wedge p_4(o_2)} & = \emptyset, & \underline{C}^{\neg p_3(o_2)\wedge p_4(o_2)} & = \{BP(3)\} \\
\overline{C}^{\neg p_3(o_2)\wedge \neg p_4(o_2)} & = \{BP(4)\}, & \underline{C}^{\neg p_3(o_2)\wedge \neg p_4(o_2)} & = \emptyset
\end{array}$$

and grounding strengths are:

$$\lambda^{p_3(o_2)\to p_4(o_2)} = 0.75,\ \lambda^{p_3(o_2)\to \neg p_4(o_2)} = 0.25,$$
$$\lambda^{\neg p_3(o_2)\to p_4(o_2)} = 0.5,\ \lambda^{\neg p_3(o_2)\to \neg p_4(o_2)} = 0.5.$$

The following epistemic relations are met:

— $CS \models^{\text{SPE}} Bel(p_3(o_2) \to p_4(o_2))$
— $CS \models^{\text{SPE}} Pos(\neg p_3(o_2) \to \neg p_4(o_2))$
— $CS \models^{\text{PE}} p_3(o_2) \to Pos(\neg p_4(o_2))$
— $CS \models^{\text{PE}} \neg p_3(o_2) \to Pos(p_4(o_2))$

and epistemic relations are NOT met:

— $CS \not\models^{\text{SPE}} p_3(o_2) \to Pos(\neg p_4(o_2))$
— $CS \not\models^{\text{SPE}} \neg p_3(o_2) \to Pos(p_4(o_2))$

In comparison to case 3.1, agent no longer claims $Know(p_3(o_2) \to p_4(o_2))$, as the current context is broader. There is some uncertainty in the agent, disallowing it to exclude situations where $o_2 \in P_2^-$. This uncertainty may result from lack of time to fully process and analyse possessed knowledge.

**Case 3.3:** Now assume conscious area contains all situations where $o_2 \in P_2^-$ and unconscious area contains situations where $o_2 \in P_2^+$. This is a reversed situation in comparison to case 3.2. Conscious grounding sets switch places with unconscious ones. Grounding strengths stay the same as in case 3.2.

Now the following epistemic relation:

— $CS \models^{\text{PE}} Pos(\neg p_3(o_2) \to p_4(o_2))$

is met, because $\overline{C}^{\neg p_3(o_2) \wedge \neg p_4(o_2)} = \emptyset$.

Yet the following epistemic relations are no longer met:

— $CS \models^{\mathrm{SPE}} Bel(p_3(o_2) \rightarrow p_4(o_2))$
— $CS \models^{\mathrm{SPE}} Pos(\neg p_3(o_2) \rightarrow \neg p_4(o_2))$

because $\overline{C}^{p_3(o_2) \wedge \neg p_4(o_2)} = \{BP(1)\} \neq \emptyset$ and $\overline{C}^{\neg p_3(o_2) \wedge p_4(o_2)} = \{BP(3)\} \neq \emptyset$.

Cases 3.1-3.3 exemplify how the contents of the cognitive state influence grounding process. Depending on circumstances different statements can be properly grounded from the same empirical material.

## 9.2. Summarizing transaction base

Proposed grounding theory has been utilized to summarize simple transaction base (Skorupa and Katarzyniak 2012) with modal conditional formulas. The transaction base has been filled with random data, so that one has through understanding of provided data characteristics. Obtained results can be compared to known probability distributions.

### 9.2.1. Transaction base

Let $\mathcal{P} = \{p_1, p_2, ..., p_K\}$ be a set of attributes. Further let database $D$ be defined as a multiset: $D = \{d^{(1)}, d^{(2)}, ..., d^{(t)}\}$ where each transaction $d_{\hat{t}} \in D$, $\hat{t} = 1, 2, ..., t$ is a vector of attributes' values $d^{(\hat{t})} = (p_1^{(\hat{t})}, p_2^{(\hat{t})}, ..., p_K^{(\hat{t})})$. Each attribute can take one of three values $p_k^{(\hat{t})} \in \{-1, 0, 1\}$. A following interpretation is assumed:

— $p_k^{(\hat{t})} = 1$ - transaction $d_{\hat{t}}$ has attribute $p_k$.
— $p_k^{(\hat{t})} = -1$ - transaction $d_{\hat{t}}$ does not have attribute $p_k$.
— $p_k^{(\hat{t})} = 0$ - attribute $p_k$ is unknown for transaction $d_{\hat{t}}$.

### 9.2.2. Method for Choosing Conditional Statements

Proposed grounding theory, presented in chapter 6, has been used to describe the transaction base with conditional sentences. Let $p, q \in \mathcal{P}$. A user may ask questions regarding the antecedent or the consequent or both. Questions may take the one of three forms:

— $p \rightarrow ?$ - What does $p$ imply? (find conditionals with $p$ as an antecedent)
— $? \rightarrow q$ - What does $q$ depend on? (find conditionals with $q$ as a consequent)
— $p \rightarrow q$ - Does $q$ depend on $p$? (find conditionals with $p$ as an antecedent and $q$ as a consequent)

Agent analyses transaction base to find suitable conditionals. In the analysis process agent utilizes proposed theory to constructs grounding sets and calculate grounding strengths.

Depending on the type of the input question, method searches for proper conditionals with the fixed antecedent or the consequent. Method checks all required combinations of pairs of attributes' values against strictly pragmatic epistemic relation 6.16. Each statement satisfying respective relation, is returned as an output.

### 9.2.3. Program simulation

Proposed method has been implemented to check its work-flow and present exemplary results.

A grounding thresholds setting given by equation 7.16 was assumed. If not stated otherwise, $\overline{f}_c$ (see eq. 6.7) was taken as an upper boundary function. Radius $r$ was set to 1.5.

### 9.2.4. Used data

Transaction base used within program consisted of 5000 transactions. There were 10 attributes $p_1, p_2, ..., p_{10}$. Values of each attribute were generated randomly. Each attribute could hold, not hold or be unknown. To simulate dependencies between attributes some conditional distributions were used. A random database has been chosen, to allow for through understanding of delivered data. This way one can discuss results against input distributions. Whole setting of uniform distributions is given below:

| | |
|---|---|
| $P(p_1) = 0.6$ | $P(p_2 \vert \neg p_1) = 0.3, P(p_2 \vert p_1) = 0.8$ |
| $P(p_3 \vert \neg p_2) = 0.5, P(p_3 \vert p_2) = 1$ | $P(p_4) = 0.3$ |
| $P(p_5 \vert \neg p_1) = 0.5, P(p_5 \vert p_1) = 0.1$ | $P(p_6 \vert \neg p_4) = 0.05, P(p_6 \vert p_4) = 0.2$ |
| $P(p_7) = 0.5$ | $P(p_8 \vert \neg p_5) = 0.8, P(p_8 \vert p_5) = 0.9$ |
| $P(p_9) = 0.2$ | $P(p_{10} \vert \neg p_9) = 0, P(p_{10} \vert p_9) = 1$ |

Attributes $p_1, p_4, p_7$ and $p_9$ are independent. All other attributes directly or indirectly depend on values of independent attributes. For an example: $p_2$ depends on $p_1$. If $p_1$ holds, probability of $p_2$ is equal to 0.8. When $p_1$ doesn't hold, probability of $p_2$ is only 0.3. For a second example: $p_{10}$ is equivalent to $p_9$, either both attributes hold or none of them. For a third example: $p_6$ has an overall low probability. It is slightly higher when $p_4$ holds.

Finally some of values have been randomly masked, so that they are unknown to the program. About 20% of values were unknown.

### 9.2.5. Exemplary Results

Table 9.2 presents exemplary questions asked (column 2) and answers given by the agent (column 3).

Question 1 asks about attributes influencing $p_{10}$. Program correctly recognizes that $p_9$ and $p_{10}$ are equivalent by stating two conditional formulas with knowledge operator (see input data description). No other conditionals are returned as $p_{10}$ does not depend on any other attributes.

There is no answer to question 5, as attribute $p_7$ does not influence other attributes. No conditional is suitable as an answer. Again system behaves correctly.

Question 6 on $p_8$ has no answers, because dependence between $p_5$ and $p_8$ is very weak. According to input data, probabilities are $P(p_8|\neg p_5) = 0.8, P(p_8|p_5) = 0.9$. The 0.1 difference between probabilities is not enough to claim that they are conditionally related. Statement testing fails on conditional relation that requires a more significant dependence.

Table 9.2. Exemplary questions and answers made ($\overline{f} = \overline{f}_c, r = 1.5$)

| no. | question | answers |
|---|---|---|
| 1. | $? \rightarrow p_{10}$ | $p_9 \rightarrow Know(p_{10}), \neg p_9 \rightarrow Know(\neg p_{10})$ |
| 2. | $? \rightarrow p_2$ | $p_1 \rightarrow Bel(p_2), \neg p_1 \rightarrow Bel(\neg p_2),$ <br> $p_3 \rightarrow Bel(p_2), \neg p_3 \rightarrow Know(\neg p_2)$ |
| 3. | $? \rightarrow p_6$ | $p_4 \rightarrow Pos(p_6), \neg p_4 \rightarrow Bel(\neg p_6)$ |
| 4. | $p_2 \rightarrow p_3$ | $p_2 \rightarrow Know(p_3), \neg p_2 \rightarrow Pos(\neg p_3)$ |
| 5. | $p_7 \rightarrow ?$ | *no answers* |
| 6. | $? \rightarrow p_8$ | *no answers* |

For attribute $p_6$ (question 3), there is also a sight probability difference of 0.15, depending on $p_4$. Here we receive conditionals as answers. This happens, because in case of $p_4$, $p_6$ is four times as probable. Slight change in probability greatly increases chance for $p_6$. Agent notices that and utters statements forming a desired result. Small probability of a consequent doesn't necessarily mean that there is no conditional dependence.

Table 9.3. Program answers for different upper boundary functions $\overline{f}$

| question | function $\overline{f}$ | answers |
|---|---|---|
| $? \rightarrow p_6$ | $\overline{f}_c, r = 1$ | *no answers* |
| | $\overline{f}_c, r = 3$ | $p_4 \rightarrow Pos(p_6), \neg p_4 \rightarrow Bel(\neg p_6)$ |
| | $\overline{f}_c, r = 4$ | $p_4 \rightarrow Pos(p_6), \neg p_4 \rightarrow Bel(\neg p_6),$ <br> $p_5 \rightarrow Pos(p_6), \neg p_5 \rightarrow Bel(\neg p_6)$ |
| | $\overline{f}_s, n = 2$ | $\neg p_4 \rightarrow Bel(\neg p_6)$ |
| | $\overline{f}_q, n = 2$ | $p_4 \rightarrow Pos(p_6)$ |
| $? \rightarrow p_8$ | $\overline{f}_c, r = 1$ | *no answers* |
| | $\overline{f}_c, r = 3$ | $p_5 \rightarrow Bel(p_8), \neg p_5 \rightarrow Pos(\neg p_8)$ |
| | $\overline{f}_c, r = 4$ | $p_1 \rightarrow Pos(\neg p_8), \neg p_1 \rightarrow Bel(p_8),$ <br> $p_5 \rightarrow Bel(p_8), \neg p_5 \rightarrow Pos(\neg p_8)$ |

The choice of upper boundary function $\overline{f}$ has a crucial impact on the answer. Table 9.3 presents program answers for two exemplary questions and different functions. Attribute $p_6$ has low probability that is slightly higher when $p_4$ holds. Attribute $p_8$ has high probability that is slightly higher when $p_5$ holds (see data specification in section 9.2.4). One can assume that proper common-sense answers are $p_4 \rightarrow Pos(p_6), \neg p_4 \rightarrow Bel(\neg p_6)$ for $p_6$ and $p_5 \rightarrow Bel(p_8), \neg p_5 \rightarrow Pos(\neg p_8)$ for $p_8$. Choosing $\overline{f}_c$ with $r = 3$, gave such answers for both questions. Choosing too big radius $r$ additionally allowed sentences about unrelated attributes ($p_5$ and $p_1$ respectively). Setting to small radius $r = 1$ gave no answers.

For other upper boundary functions $\overline{f}_s$ and $\overline{f}_q$ (see equations 6.8 and 6.9), there were different answers. Function $\overline{f}_s$ prefers consequents with high conditional probability ($\neg p_6$), $\overline{f}_q$ prefers ones with low conditional probability ($p_6$).

## 9.3. The application example: Mushroom adviser

Proposed grounding theory may be utilized to describe dataset with discrete attributes. Dataset can be transformed to agent's knowledge state and instances held within dataset may be converted to base profiles. Knowledge state may later be used to check epistemic relations of conditional formulas. Because grounding theory meets a series of common-sense criteria, returned formulas should possess conventional and intuitive meaning of natural language sentences.

Mushrooms dataset (Bache and Lichman 2013) has been chosen for an exemplary implementation of description of data with conditional formulas. This dataset is a well known reference set often used for testing classification algorithms. Dataset consists of hypothetical samples corresponding to 23 species of gilled mushrooms in the Agaricus and Lepiota Family. Each species is described by 22 discrete attributes and a classification attribute. Classification attribute tells whether mushroom is edible or poisonous. There are total of 8124 instances in the data set and about half of them are poisonous. Some values for one attribute (stalk-root) are missing.

Let $p$ and $e$ denote 'poisonous' and 'edible' respectively. Suppose $a$ is some attribute describing mushroom instances and $v$ is some value of this attribte. Let $a = v$ denote 'Attribute $a$ is $v$' or 'Attribute $a$ has property $v$'. Because grounding theory meets a series of common-sense criteria, a properly grounded formula[1] of the form $(a = v) \rightarrow Pos(p)$ has a conventional meaning of a conditional: 'If $a$ is $v$, then it is possible that mushroom is poisonous'. Formula possesses the following conventional implicatures of a conditional statement:

— Attribute $a$ may, but doesn't have to have value of $v$.
— Mushroom may, but doesn't have to be poisonous.
— When $a = v$, it is possible that the mushroom is poisonous.
— When $a \neq v$, mushroom rather isn't or can't be poisonous.

The grounding theory has been defined on binary properties, whereas mushrooms are described with discrete and usually non-binary attributes. To allow application of grounding theory, 23 discrete attributes have been converted to 126 binary properties $\mathcal{P} = \{p_1, p_2, ..., p_{126}\}$. Missing values stayed as unknown. In this form the dataset can be interpreted as a knowledge state $KS(t)$ where each mushroom sample is treated as a base profile $BP(\hat{t})$. For a general description of the dataset, cognitive state simply contains all knowledge ($\overline{CS}(t) = KS(t)$).

[1] In this example, it is assumed, formula is properly grounded, when strictly pragmatic epistemic relation is met.

### 9.3.1. General knowledge

In an exemplary implementation I was searching for conditionals of the forms: $(a = v) \to \Pi(e)$ and $(a \neq v) \to \Pi(e)$, where $\Pi \in \{Pos, Bel, Know\}$, $a$ is an attribute name, $v$ is its value and $e$ denotes 'edible'.

Data set contains 126 binary properties, so there were total of 252 possible antecedents. For each antecedent there were 3 possible modal operators. In result there were a total of 756 candidates for conditional sentences.

Every valid attribute and value combination was checked against strictly pragmatic epistemic relation (definition 6.16). Grounding thresholds and boundary function were chosen as in equations 7.16 and 7.17. Every time relation held, respective formula was added to the result.

A total of 87 formulas were properly grounded: 24 with knowledge operator, 36 with belief operator and 27 with possibility operator. These formulas were later translated to natural language sentences using simple substitution table. *Exemplary* returned sentences were:

1. If cap-shape is sunken, then I know that mushroom is edible.
2. If cap-color is green, then I know that mushroom is edible.
3. If odor is almond, then I know that mushroom is edible.
4. If odor is anise, then I know that mushroom is edible.
5. If habitat is waste, then I know that mushroom is edible.
6. If odor is none, then I believe that mushroom is edible.
7. If habitat is meadows, then I believe that mushroom is edible.
8. If odor is not creosote, then it is possible that mushroom is edible.[2]
9. If cap-shape is not knobbed, then it is possible that mushroom is edible.[3]

Running the method for the rules of the form $(a = v) \to \Pi(p)$ ($p$ denotes 'poisonous') lead to similar results. Exemplary statements were:

1. If odor is fishy, then I know that mushroom is poisonous.
2. If gill-color is green, then I know that mushroom is poisonous.
3. If odor is creosote, then I know that mushroom is poisonous.
4. If odor is not none, then I believe that mushroom is poisonous.
5. If habitat is not meadows, then it is possible that mushroom is poisonous.

All returned formulas generally summarize mushrooms dataset and have intuitive natural language meaning.

[2] When the odor is creosote, the mushroom is certainly poisonous. There were no instances of edible mushrooms with creosote odor.

[3] When cap-shape is knobbed, mushroom may also be edible. About 27% of mushrooms with knobbed cap shape are edible and about 54% of all other mushrooms instances are edible.

### 9.3.2. Context specific utterances

The grounding theory can be also utilized in a context specific circumstances to advise a mushroom picker on the edibility of a mushroom.

Suppose the picker sees a mushroom, but does not know if it is edible. The picker takes a photo of the mushroom using his smart-phone. The photo is passed to a software that analyses it and extracts as many mushroom's features as possible. This way features such as the cap shape or the colour can be determined, yet it is impossible to obtain the odour. Later the features are passed to a specialized software agent that compares them against mushroom database. Firstly the agent tries to determine on mushroom edibility. If it is not possible to determine the edibility, the agent uses a grounding theory to support the picker with helpful tips in the form of conditionals. For example the program may suggest: 'If the odour is almond, then I know the mushroom is edible'. The picker may consider the advise and smell the mushroom to check its odour.

In order not to complicate the example too much I introduce only a sketch of the solution. The algorithm works more or less as follows:

1. Construct current base profile $BP(t)$ and fill it according to the features determined from the mushroom's photo.
2. Within the mushroom dataset find instances that are consistent with all known attributes from $BP(t)$ (where respective attributes have the same values as the known attributes in $BP(t)$). Put all found instances into conscious area of mental state $\overline{CS}(t)$.
3. If cardinality of $\overline{CS}(t)$ is very small, find instances that are most similar to $BP(t)$ and put them into unconscious area of mental state $\underline{CS}$.
4. If all instances of mushrooms in mental state $CS(t) = \overline{CS}(t) \cup \underline{CS}(t)$ are edible (or all of them are poisonous), simply return 'I know the mushroom is edible (poisonous)'.
5. Similarly as in section 9.3.1 apply grounding theory to find all suitable conditional statements of the forms $(a = v) \rightarrow \Pi(e)$ and $(a = v) \rightarrow \Pi(p)$. Instead of the whole database use mental state $CS(t) = \overline{CS}(t) \cup \underline{CS}(t)$ obtained in steps 2 and 3.
6. Rank obtained conditional statements (for example on their usability to the picker). Return highest ranked statement(s).

In step 3 a predefined cardinality should be picked. Too small cardinality may lead to unpredictable results. One should choose a similarity measure that depends only on the known attributes from $BP(t)$.

In step 6 the ranking measure may prefer the statements with antecedents that the picker can easily check. Additionally the ranking measure may prefer the statements with antecedents that are likely to hold.

The grounding theory was used to extract suitable conditional statements. Because of that, agent returns context specific conditional utterances that possess conventional meaning. A simple ranking mechanism should ensure that the returned statements are useful for the picker.

# 10. Summary

Work presented a new approach to conditionals. The approach starts with a shift from paradigm of conditionals truth conditions to subjective and rational usage conditions. Indicative conditionals usage patterns are analysed from the speaker's perspective. Work provides the grounding theory in form of formal conditions that allow for the choice of conditionals with accordance to speakers empirical incomplete knowledge. It has been proven that proposed theory meets a common-sense constraints derived from conditionals conventional usage patterns. This novel approach allows for the choice of utterances that sustains intuitive natural language meaning of conditionals. This choice can be performed autonomously by a properly designed cognitive agent.

Proposed theory is an extension of existing grounding theory (Katarzyniak 2007) of modal formulas where modality is reperesented by three modal operators of possiblity, belief and knowledge. Provided theory extends existing grounding theory by adding modal conditionals to considered formal language. Formal conditions of proper grounding of conditional formulas in form of epistemic relations have been presented. Three types of epistemic relations have been proposed. Normal epistemic relation models the most broad and general meaning of conditionals. Strictly pragmatic epistemic relation models very narrow and intuitive understanding of a conditional statement.

Two possible applications of presented theory have been proposed. The grounding theory has been utilized to offer a summary of an artificial transaction base and a benchmark dataset with discreet attributes. Promising results have been obtained in a series of simulations. Returned utterances seem to sustain conventional and intuitive meaning of conditionals.

**Acknowledgements:** Some parts of this work have been supported by Grant no. N N519 444939 funded by Polish Ministry of Science and Higher Education (2010-2013).

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