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Contents

TL-MOMENTS: ANALOGY OF CLASSICAL L-MOMENTS

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Abstract
Moments and cumulants are commonly used to characterize the probability distribution or observed data set. The use of the moment method of parameter estimation is also common in the construction of an appropriate parametric distribution for a certain data set. The moment method does not always produce satisfactory results. It is difficult to determine exactly what information concerning the shape of the distribution is expressed by its moments of the third and higher order. In the case of small samples in particular, numerical values of sample moments can be very different from the corresponding values of theoretical moments of the relevant probability distribution from which the random sample comes. Parameter estimations of the probability distribution made by the moment method are often considerably less accurate than those obtained using other methods, particularly in the case of small samples. The present paper deals with an alternative approach to the construction of an appropriate parametric distribution for the considered data set using order statistics.

Key words: *L-moments and TL-moments of probability distribution, sample L-moments and TL-moments, order statistics.*

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1. Introduction

L-moments form the basis for a general theory which includes the summarization and description of theoretical probability distributions and obtained sample data sets, parameter estimation of theoretical probability distributions and hypothesis testing of parameter values for theoretical probability distributions. The theory of L-moments includes the established methods such as the use of order statistics and the Gini middle difference. It leads to some auspicious innovations in the area of measuring skewness and kurtosis of the distribution and provides relatively new methods of parameter estimation for an individual distribution. L-moments can be defined for any random variable whose expected value exists. The main advantage of L-moments over conventional moments is that they can be estimated by linear functions of sample values and are more resistant to the influence of sample variability. L-moments are more robust than conventional moments to the existence of outliers in the data, facilitating better conclusions made on the basis of small samples of the basic probability distribution. L-moments sometimes bring even more efficient parameter estimations of the parametric distribution than those acquired by the maximum likelihood method for small samples in particular.

L-moments have certain theoretical advantages over conventional moments consisting in the ability to characterize a wider range of the distribution. They are also more resistant and less prone to estimation bias, approximation by the asymptotic normal distribution being more accurate in finite samples.

Let X be a random variable being distributed with the distribution function $F(x)$ and quantile function $x(F)$ and let $X_1, X_2, \ldots, X_n$ be a random sample of the sample size n from this distribution. Then $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ are order statistics of the random sample of the sample size n which comes from the distribution of the random variable X.

L-moments are analogous to conventional moments. They can be estimated on the basis of linear combinations of sample order statistics, i.e. L-statistics. L-moments are an alternative system describing the shape of the probability distribution.

2. L-Moments of Probability Distributions

The issue of L-moments is discussed, for example, in (Adamowski, 2000) or (Ulrych, 2000). Let X be a continuous random variable being distributed with the distribution function $F(x)$ and quantile function $x(F)$. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be order statistics of a random sample of the sample size n which comes from the distribution of the random variable X. L-moment of the r-th order of the random variable X is defined as

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r-j:r}), \quad r = 1, 2, \ldots . \tag{1}$$

An expected value of the r-th order statistic of the random sample of the sample size n has the form

$$E(X_{r:n}) = \frac{n!}{(r-1)! \cdot (n-r)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-1} \cdot [1-F(x)]^{n-r} \mathrm{d}\, F(x). \tag{2}$$

If we substitute equation (2) into equation (1), after adjustments we obtain

$$\lambda_r = \int_0^1 x(F) \cdot P_{r-1}^*[F(x)]\, \mathrm{d}\, F(x), \quad r = 1, 2, \ldots , \tag{3}$$

where

$$P_r^*[F(x)] = \sum_{j=0}^{r} p_{r,j}^* \cdot [F(x)]^j \quad \text{a} \quad p_{r,j}^* = (-1)^{r-j} \cdot \binom{r}{j} \cdot \binom{r+j}{j}, \tag{4}$$

$P_r^*[F(x)]$ being the r-th shifted Legendre polynomial. Having substituted expression (2) into expression (1), we also obtained

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{r!}{(r-j-1)! \cdot j!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-j-1} \cdot [1-F(x)]^j \,\mathrm{d}\, F(x), \quad r = 1, 2, \ldots . \tag{5}$$

The letter "L" in "L-moments" indicates that the r-th L-moment λ_r is a linear function of the expected value of a certain linear combination of order statistics. The very estimation of the r-th L-moment λ_r, based on the obtained data sample, is thus the linear combination of order data values, i.e. L-statistics. The first four L-moments of the probability distribution are now defined as

$$\lambda_1 = E(X_{1:1}) = \int_0^1 x(F)\,\mathrm{d}\,F(x), \tag{6}$$

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^1 x(F)\cdot[2F(x)-1]\,\mathrm{d}\,F(x), \tag{7}$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 x(F)\cdot\{6[F(x)]^2 - 6F(x) + 1\}\,\mathrm{d}\,F(x), \tag{8}$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) = \int_0^1 x(F)\cdot\{20[F(x)]^3 - 30[F(x)]^2 + 12[F(x)] - 1\}\,\mathrm{d}\,F(x). \tag{9}$$

The probability distribution can be specified by its L-moments even if some of its conventional moments do not exist, the opposite, however, is not true. It can be proved that the first L-moment λ_1 is a location characteristic, the second L-moment λ_2 being a variability characteristic. It is often desirable to standardize higher L-moments λ_r, $r \geq 3$, so that they can be independent of specific units of the random variable X. The ratio of L-moments of the r-th order of the random variable X is defined as

$$\tau_r = \frac{\lambda_r}{\lambda_2}, \quad r = 3, 4, \ldots. \tag{10}$$

We can also define the function of L-moments which is analogous to the classical coefficient of variation, i.e. the so called L-coefficient of variation

$$\tau = \frac{\lambda_2}{\lambda_1}. \tag{11}$$

The ratio of L-moments τ_3 is a skewness characteristic, the ratio of L-moments τ_4 being a kurtosis characteristic of the corresponding probability distribution. Main properties of the probability distribution are very well summarized by the following four characteristics: L-location λ_1, L-variability λ_2, L-skewness τ_3 and L-kurtosis τ_4. L-moments λ_1 and λ_2, the L-coefficient of variation τ and ratios of L-moments τ_3 and τ_4 are the most useful characteristics for the summarization of the probability distribution. Their main properties are existence (if the expected value of the distribution exists, then all its L-moments exist) and uniqueness (if the expected value of the distribution exists, then L-moments define the only distribution, i.e. no two distributions have the same L-moments). More for example, see (Hosking, 1990) or (Hosking, 1997).

3. Sample L-Moments

L-moments are usually estimated by a random sample obtained from an unknown distribution. Since the r-th L-moment λ_r is the function of the expected values of order statistics of a random sample of the sample size r, it is natural to estimate it using the so-called U-statistic, i.e. the corresponding function of sample order statistics (averaged over all subsets of the sample size r, which may be formed from the obtained random sample of the sample size n).

Let $x_1, x_2, \ldots, x_n$ be the sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ the order sample. Then the r-th sample L-moment can be written as

$$l_r = \binom{n}{r}^{-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \ldots \sum \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot x_{i_{r-j}:n}, \quad r = 1, 2, \ldots, n. \tag{12}$$

Hence the first four sample L-moments have the form

$$l_1 = \frac{1}{n} \cdot \sum_i x_i, \tag{13}$$

$$l_2 = \frac{1}{2} \cdot \binom{n}{2}^{-1} \cdot \sum_{i>j} \sum (x_{i:n} - x_{j:n}), \tag{14}$$

$$l_3 = \frac{1}{3} \cdot \binom{n}{3}^{-1} \cdot \sum_{i>j>k} \sum \sum (x_{i:n} - 2x_{j:n} + x_{k:n}), \tag{15}$$

$$l_4 = \frac{1}{4} \cdot \binom{n}{4}^{-1} \cdot \sum_{i>j>k>l} \sum \sum \sum (x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{l:n}). \tag{16}$$

U-statistics are widely used especially in nonparametric statistics. Their positive properties are the absence of bias, asymptotic normality and a slight resistance due to the influence of outliers.

When calculating the r-th sample L-moment, it is not necessary to repeat the process over all sub-sets of the sample size r, since this statistic can be expressed directly as a linear combination of order statistics of a random sample of the sample size n.

If we assume an estimate of $E(X_{r:r})$ obtained with the use of U-statistics, it can be written as $r \cdot b_{r-1}$, where

$$b_r = \frac{1}{n} \cdot \binom{n-1}{r}^{-1} \cdot \sum_{j=r+1}^{n} \binom{j-1}{r} \cdot x_{j:n}, \tag{17}$$

namely

$$b_0 = \frac{1}{n} \cdot \sum_{j=1}^{n} x_{j:n}, \tag{18}$$

$$b_1 = \frac{1}{n} \cdot \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} \cdot x_{j:n}, \tag{19}$$

$$b_2 = \frac{1}{n} \cdot \sum_{j=3}^{n} \frac{(j-1)\cdot(j-2)}{(n-1)\cdot(n-2)} \cdot x_{j:n}, \tag{20}$$

and so generally

$$b_r = \frac{1}{n} \cdot \sum_{j=r+1}^{n} \frac{(j-1)\cdot(j-2)\cdot\ldots\cdot(j-r)}{(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r)} \cdot x_{j:n}. \tag{21}$$

Thus the first sample L-moments can be written as

$$l_1 = b_0, \tag{22}$$

$$l_2 = 2b_1 - b_0, \tag{23}$$

$$l_3 = 6b_2 - 6b_1 + b_0, \tag{24}$$

$$l_4 = 20b_3 - 30b_2 + 12b_1 - b_0. \tag{25}$$

We can therefore write generally

$$l_{r+1} = \sum_{k=0}^{r} p_{r,k}^{*} \cdot b_k, \quad r = 0, 1, \ldots, n-1, \tag{26}$$

where

$$p_{r,k}^{*} = (-1)^{r-k} \cdot \binom{r}{k} \cdot \binom{r+k}{k} = \frac{(-1)^{r-k} \cdot (r+k)!}{(k!)^2 \cdot (r-k)!}. \tag{27}$$

Sample L-moments are used in a similar way as sample conventional L-moments, summarizing the basic properties of the sample distribution, which are the location (level), variability, skewness and kurtosis. Thus, sample L-moments allow an estimation the corresponding properties of the probability distribution from which the sample originates and can be used in estimating the parameters of the relevant probability distribution. We often prefer L-moments to conventional moments within such applications, since sample L-moments – as the linear functions of sample values – are less sensitive to sample variability or measurement errors in extreme observations than conventional moments. L-moments therefore lead to more accurate and robust estimates of characteristics or parameters of the basic probability distribution.

Sample L-moments have been used previously in statistics, but not as part of a unified theory. The first sample L-moment l_1 is a sample L-location (sample average), the second sample L-moment l_2 being a sample L-variability. The natural estimation of L-moments (10) ratio is the sample ratio of L-moments

$$t_r = \frac{l_r}{l_2}, \quad r = 3, 4, \ldots. \tag{28}$$

Hence t_3 is a sample L-skewness and t_4 is a sample L-kurtosis. Sample ratios of L-moments t_3 and t_4 may be used as the characteristics of skewness and kurtosis of a sample data set.

The Gini middle difference relates both to sample L-moments, having the form of

$$G = \binom{n}{2}^{-1} \cdot \sum\sum_{i>j} (x_{i:n} - x_{j:n}), \tag{29}$$

and the Gini coefficient which depends only on a single parameter σ in the case of the two-parametric lognormal distribution, depending, however, on the values of all three parameters in the case of the three-parametric lognormal distribution. More for example in (Elamir, 2003).

4. TL-Moments of Probability Distributions

An alternative robust version of L-moments is introduced in this subchapter. The modification is called "trimmed L-moments" and it is termed TL-moments. The expected values of order statistics of a random sample in the definition of L-moments of probability distributions are replaced with those of a larger random sample, its size growing correspondingly to the extent of the modification, as shown below.

Certain advantages of TL-moments outweigh those of conventional L-moments and central moments. TL-moment of the probability distribution may exist despite the non-existence of the corresponding L-moment or central moment of this probability distribution, as it is the case of the Cauchy distribution. Sample TL-moments are more resistant to outliers in the data. The method of TL-moments is not intended to replace the existing robust methods but rather supplement them, particularly in situations when we have outliers in the data.

In this alternative robust modification of L-moments, the expected value $E(X_{r-j:r})$ is replaced with the expected value $E(X_{r+t_1-j\,:\,r+t_1+t_2})$. Thus, for each r, we increase the sample size of a random sample from the original r to $r + t_1 + t_2$, working only with the expected values of these r modified order statistics $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, \ldots, X_{t_1+r:r+t_1+t_2}$ by trimming the smallest t_1 and largest t_2 from the conceptual random sample. This modification is called the r-th trimmed L-moment (TL-moment) and marked as $\lambda_r^{(t_1,t_2)}$. Thus, TL-moment of the r-th order of the random variable X is defined as

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r+t_1-j:r+t_1+t_2}), \quad r = 1, 2, \ldots. \tag{30}$$

It is evident from the expressions (30) and (1) that TL-moments are reduced to L-moments, where $t_1 = t_2 = 0$. Although we can also consider applications where the adjustment values are not equal, i.e. $t_1 \neq t_2$, we will focus here only on the symmetric case $t_1 = t_2 = t$. Then the expression (30) can be rewritten

$$\lambda_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r+t-j:r+2t}), \quad r = 1, 2, \ldots. \tag{31}$$

Thus, for example, $\lambda_1^{(t)} = E(X_{1+t:1+2t})$ is the expected value of the median of the conceptual random sample of 1 + 2*t* size. It is necessary to note that $\lambda_1^{(t)}$ is equal to zero for distributions that are symmetrical around zero.

For *t* = 1, the first four TL-moments have the form

$$\lambda_1^{(1)} = E(X_{2:3}), \tag{32}$$

$$\lambda_2^{(1)} = \frac{1}{2} E(X_{3:4} - X_{2:4}), \tag{33}$$

$$\lambda_3^{(1)} = \frac{1}{3} E(X_{4:5} - 2X_{3:5} + X_{2:5}), \tag{34}$$

$$\lambda_4^{(1)} = \frac{1}{4} E(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}). \tag{35}$$

The measurements of location, variability, skewness and kurtosis of the probability distribution analogous to conventional L-moments (6)-(9) are based on $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$ a $\lambda_4^{(1)}$.

The expected value $E(X_{r:n})$ can be written using the formula (2). With the use of the equation (2), we can express the right side of the equation (31) again as

$$\lambda_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{(r+2t)!}{(r+t-j-1)! \cdot (t+j)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r+t-j-1} \cdot [1-F(x)]^{t+j} \, \mathrm{d}F(x), \; r = 1, 2, \ldots. \tag{36}$$

It is necessary to point out that $\lambda_r^{(0)} = \lambda_r$ represents a normal *r*-th L-moment with no respective adjustments.

Expressions (32)–(35) for the first four TL-moments (*t* = 1) may be written in an alternative way as

$$\lambda_1^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1-F(x)] \, \mathrm{d}F(x), \tag{37}$$

$$\lambda_2^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1-F(x)] \cdot [2F(x)-1] \, \mathrm{d}F(x), \tag{38}$$

$$\lambda_3^{(1)} = \frac{20}{3} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1-F(x)] \cdot \{5[F(x)]^2 - 5F(x) + 1\} \, \mathrm{d}F(x), \tag{39}$$

$$\lambda_4^{(1)} = \frac{15}{2} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1-F(x)] \cdot \{14[F(x)]^3 - 21[F(x)]^2 + 9[F(x)] - 1] \, \mathrm{d}F(x). \tag{40}$$

The distribution can be determined by its TL-moments, even though some of its L-moments or conventional moments do not exist. For example, $\lambda_1^{(1)}$ (the expected value of the median of a conceptual random sample of sample size three) exists for the Cauchy distribution, despite the non-existence of the first L-moment λ_1.

TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ can be defined analogously as L-skewness τ_3 and L-kurtosis τ_4

$$\tau_3^{(t)} = \frac{\lambda_3^{(t)}}{\lambda_2^{(t)}}, \tag{41}$$

$$\tau_4^{(t)} = \frac{\lambda_4^{(t)}}{\lambda_2^{(t)}}. \tag{42}$$

5. Sample TL-Moments

Let $x_1, x_2, \ldots, x_n$ be a sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ an order sample. The expression

$$\hat{E}(X_{j+1:j+l+1}) = \frac{1}{\binom{n}{j+l+1}} \cdot \sum_{i=1}^{n} \binom{i-1}{j} \cdot \binom{n-i}{l} \cdot x_{i:n} \tag{43}$$

is considered to be an unbiased estimate of the expected value of the $(j+1)$-th order statistic $X_{j+1:j+l+1}$ in the conceptual random sample of sample size $(j+l+1)$. Now we will assume that in the definition of TL-moment $\lambda_r^{(t)}$ in (31), the expression $E(X_{r+t-j:r+2t})$ is replaced by its unbiased estimate

$$\hat{E}(X_{r+t-j:r+2t}) = \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \tag{44}$$

which is obtained by assigning $j \to r+t-j-1$ a $l \to t+j$ in (43). Now we get the r-th sample TL-moment

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \hat{E}(X_{r+t-j:r+2t}), \quad r = 1, 2, \ldots, n-2t, \tag{45}$$

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \quad r = 1, 2, \ldots, n-2t, \tag{46}$$

which is an unbiased estimate of the r-th TL-moment $\lambda_r^{(t)}$. Let us note that for each $j = 0, 1, \ldots, r-1$, the values $x_{i:n}$ in (46) are not equal to zero only for $r+t-j \le i \le n-t-j$, taking combination numbers into account. A simple adjustment of equation (46) provides an alternative linear form

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{i=r+t}^{n-t} \left[\frac{\sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j}}{\binom{n}{r+2t}} \right] \cdot x_{i:n}. \tag{47}$$

For $r = 1$, for example, we obtain for the first sample TL-moment

$$l_1^{(t)} = \sum_{i=t+1}^{n-t} w_{i:n}^{(t)} \cdot x_{i:n}, \tag{48}$$

where the weights are given by

$$w_{i:n}^{(t)} = \frac{\binom{i-1}{t} \cdot \binom{n-i}{t}}{\binom{n}{2t+1}}. \tag{49}$$

The above results can be used for the estimation of TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ by simple ratios

$$t_3^{(t)} = \frac{l_3^{(t)}}{l_2^{(t)}}, \tag{50}$$

$$t_4^{(t)} = \frac{l_4^{(t)}}{l_2^{(t)}}. \tag{51}$$

We can choose $t = n\alpha$, representing the size of the adjustment from each end of the sample, where α is a certain ratio, where $0 \leq \alpha < 0{,}5$.

6. Results

L-moments method used to be employed in hydrology, climatology and meteorology in the research of extreme precipitation, having mostly used smaller data sets. This study presents applications of L-moments and TL-moments to large sets of economic data. The research variable is the net annual household income per capita (in CZK) in the Czech Republic (nominal income). The data collected by the Czech Statistical Office come from the Mikrocenzus statistical investigation covering the years 1992, 1996 and 2002 and EU-SILC survey spanning the period 2004-2007. In total, 168 income distributions were analyzed – for all households in the Czech Republic as well as with the use of particular criteria: gender, region (Bohemia and Moravia), social group, municipality size, age and the highest educational attainment. With only minor exceptions, the TL-moments method produced the most accurate results. L-moments was the second most effective method in more than half of the cases, the differences between this method and that of maximum likelihood not being significant enough as far as the number of cases, when the former gave better results than the latter. Table 1 represents distinctive outcomes for all 168 income distributions, showing the results for the total household sets in the Czech Republic. Apart from the estimated parameter values of the three-parametric lognormal distribution, which were obtained having simultaneously employed TL-moments, L-moments and maximum likelihood methods, Table 1 contains the values of the test criterion χ^2, indicating that the L-moments method produced – in four out of seven cases – more accurate results than the maximum likelihood method, the most accurate outcomes in all seven cases being produced by the TL-moments method.

Table 1. Parameter estimations of three-parametric lognormal curves obtained using three various robust methods of point parameter estimation and the value of χ^2 criterion

Year	Method of TL-moments			Method of L-moments			Maximum likelihood method		
	μ	σ^2	θ	μ	σ^2	θ	μ	σ^2	θ
1992	9.722	0.521	14,881	9.696	0.700	14,491	10.384	0.390	-325
1996	10.334	0.573	25,981	10.343	0.545	25,362	10.995	0.424	52.231
2002	10.818	0.675	40,183	10.819	0.773	37,685	11.438	0.459	73.545
2004	10.961	0.552	39,899	11.028	0.675	33,738	11.503	0.665	7.675
2005	11.006	0.521	40,956	11.040	0.677	36,606	11.542	0.446	-8.826
2006	11.074	0.508	44,941	11.112	0.440	40,327	11.623	0.435	-42.331
2007	11.156	0.472	48,529	11.163	0.654	45,634	11.703	0.421	-171.292
Year	Criterion χ^2			Criterion χ^2			Criterion χ^2		
1992	739.512			811.007			1,227.325		
1996	1,503.878			1,742.631			2,197.251		
2002	998.325			1,535.557			1,060.891		
2004	494.441			866.279			524.478		
2005	731.225			899.245			995.855		
2006	831.667			959.902			1,067.789		
2007	1,050.105			1,220.478			1,199.035		

Source: Own research

7. Conclusion

A relatively new class of moment characteristics of probability distributions has been introduced in the present paper. They are the characteristics of the location (level), variability, skewness and kurtosis of probability distributions constructed with the use of L-moments and TL-moments that represent a robust extension of L-moments. The very L-moments were implemented as a robust alternative to classical moments of probability distributions. L-moments and their estimates, however, are lacking in some robust features that are associated with TL-moments.

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