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MULTIPLE LIFE INSURANCE – PENSION CALCULATION*

STANISŁAW HEILPERN
University of Economics, Department of Statistics,
Komandorska 118-120, 54-345 Wrocław, Poland
email: stanislaw.heilpern@ue.wroc.pl

Abstract
The contribution is devoted to the dependent multiple life insurance of married couple. A more realistic assumption of dependent lifetime of married couple is investigated as distinct from classical approach, which assumes the independent lives. Two models: first is based on the Markov chain and second uses the copulas, mainly Archimedean are studied. The actuarial values of three cases of pensions: widow's, n-year joint-life and n-year last survival annuities are calculated in these models. The differences between the values of pensions in the independent model and model based on the dependences are investigated using the empirical data from Poland. The results will be compare with the results obtained in the author's earlier investigations.

Key words: multiple life insurance, copula, Markov model, pension.

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1. Introduction

We will study the multiple life insurance concerns with the married couple in order to evaluate the premiums of contracts. Classical actuarial theory connected with the multiple life insurance assumes the independence for the remaining lifetimes (Bowers et al., 1986; Frees, Carriere, Valdez, 1996). But it is not realistic assumption. In the real life, the spouses may be exposed to the same risks and the their lifetimes are often little dependent, but dependent. We also may observe so-called the "broken heart syndrome".

In the paper we investigate two models allowing the dependence of lifetime of spouses. First is based on the Markov chain and second uses the copulas. We derive the values of three annuities: the widow's, the n-year joint-life and n-year last survival annuities. We study the impact of a dependences on the values of these annuities.

The paper is based on the Denuit's et al. (2001) paper. The authors studied in it the situation in Belgium. Heilpern (2011) tried to apply the methods from this paper in Polish case based on the data from 2002. Now, we continue this work and use the new data from Polish Central Statistical Office to 2011. The aim of this paper is study the impact dependences on the value of above three pensions. The differences of the values of these annuities between the variants based on independence and dependence are calculated.

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2. General assumptions and notations

Now, we introduce the general notation and assumption connected with this subject. Let T_x^M and T_y^W be the remaining lifetimes of a x-year-old man and y-year-old woman taking values in [0, w_x^M] and [0, w_y^W]. The w_x^M (resp. w_y^W) denotes the difference between the border age of the man (resp. woman) and x (resp. y).

The distribution function ${}_tq_x^M$ and survival function ${}_tp_x^M$ of T_x^M are given by formulas:

$$ {}_tp_x^M = P(T_x^M > t) = 1 - {}_tq_x^M . $$

We can also derive ${}_tp_x^M$ using a force of mortality μ_x^M :

$$ {}_tp_x^M = \exp\left(-\int_0^t \mu_{x+s}^M ds \right). \tag{1} $$

The survival probability ${}_tp_y^W$ and the death probability ${}_tq_y^W$ of wife are obtained in the similar way.

If we want to study the dependence of random variables T_x^M , T_y^W we must know their joint distribution. The probability of a joint-life status surviving to time t is given by formula:

$$ {}_tp_{xy} = P(T_x^M > t, T_y^W > t) $$

and a last-survival status

$$ {}_tp_{\overline{xy}} = P(\max\{T_x^M, T_y^W\} > t) = {}_tp_x^M + {}_tp_y^W - {}_tp_{xy}. $$

The random variables T_x^M , T_y^W are positive quadrant dependence (PQD) when (Lehmann, 1966; Dhaene, Goovaerts, 1997)

$$ P(T_x^M > t, T_y^W > s) \geq P(T_x^M > t) P(T_y^W > s) . $$

We can see, that if the lifetimes T_x^M, T_y^W are PQD, then we obtain ${}_tp_{xy} \geq {}_tp_x^M \, {}_tp_y^W$.

3. Pensions

Now we present three pensions connected with the multiple life insurance of spouses. First we study the widow's pension:

$$ a_{x|y} = a_y - a_{xy}, $$

where $a_y = \sum_{k=1}^{w_y^W} v^k \, {}_kp_y^W$, $a_{xy} = \sum_{k=1}^{w_x^M \wedge w_y^W} v^k \, {}_kp_{xy}$ and $v = (1 + \xi)^{-1}$ be the discount factor connected with the annual effective rate ξ. The payments starts with the husband's death and terminating with the death of his wife in this case.

Second pensions are the n-year joint-life survival annuities described by formula:

$$ a_{xy;\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_kp_{xy}. $$

Third, n-year last survival annuities is equal

$$ a_{\overline{xy};\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_kp_{\overline{xy}}. $$

In last two annuities they pays \$1 at the end of the years as long both or either spouse survives.

When the lifetimes are independent we denote these pension by symbols: $a^{\perp}_{x|y}$, $a^{\perp}_{x|y;\overline{n}|}$ and $a^{\perp}_{\overline{x|y};\overline{n}|}$. When the lifetimes T_x^M, T_y^W are PQD then we obtain the following relation between these pension with respect the independent case:

$$a_{x|y} \le a^{\perp}_{x|y}, \quad a_{\overline{xy};\overline{n}|} \le a^{\perp}_{\overline{x|y};\overline{n}|} \quad \text{and} \quad a_{xy;\overline{n}|} \ge a^{\perp}_{x|y;\overline{n}|}.$$

We see, that when we assume the independence, we can overestimate or underestimate the value of the annuity. The second case occurs when we want to compute the value of the n-year joint-life survival annuities.

4. Markov model

In this section we investigate the Markov model based on stationary Markov chain. It is an appreciated tool for the calculation of life contingencies functions and pensions (see Wolthuis, Van Hoeck, 1986; Norberg, 1989). This Markov chain have four states and the forces of mortalities μ_{ij}, $i,j = 0, 1, 2, 3$, in this case (see fig. 1).

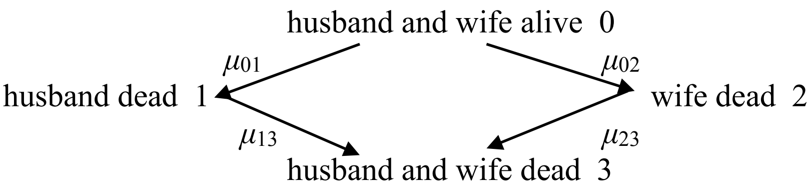

Figure 1. The space of states of Markov model
Source: Denuit et al., 2001.

We denote by symbol $p_{ij}(t, s)$ the transition probabilities. This is the conditional probability that the couple is in state j at time s, given that it was in state i at time t. The forces of mortality $\mu_{ij}(t)$ from the state i to state j at time t is done by formula

$$\mu_{ij}(t) = \lim_{\Delta t \to 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t}.$$

The transition probabilities $p_{ij}(t, s)$ can be represented by the forces of mortalities in the following way (Denuit et al. 2001):

$$p_{00}(t,s) = \exp\left(-\int_t^s (\mu_{01}(u) + \mu_{02}(u))du \right),$$

$$p_{ii}(t,s) = \exp\left(-\int_t^s \mu_{i3}(u)du \right),$$

$$p_{0i}(t,s) = \int_t^s p_{00}(t,u)\mu_{0i}(u)\, p_{ii}(u,s)du,$$

where $i = 1, 2$.

If we know the probabilities of staying at state 0, we can compute the joint and marginal survival functions of random variables T_x^M and T_y^W using the formula (Denuit et al. 2001):

$$P(T_x^M > t, T_y^W > s) = \begin{cases} p_{00}(0,s) + p_{00}(0,t)\,p_{01}(t,s) & 0 \le t \le s \\ p_{00}(0,t) + p_{00}(0,s)\,p_{02}(s,t) & 0 \le s \le t \end{cases}, \tag{2}$$

$$P(T_x^M > t) = p_{00}(0,t) + p_{02}(0,t),$$

$$P(T_y^W > t) = p_{00}(0,t) + p_{01}(0,t).$$

The lifetimes T_x^M and T_y^W are independent iff $\mu_{01}(t) = \mu_{23}(t)$, $\mu_{02}(t) = \mu_{13}(t)$ and if $\mu_{01}(t) < \mu_{23}(t)$, $\mu_{02}(t) < \mu_{13}(t)$, then they are PQD (Norberg, 1989). In our paper we use for fixed ages of husband x and wife y the following simplifying assumption done by Denuit et al. (2001):

$$\mu_{01}(t) = (1 - \alpha_{01})\,\mu_{x+t}^M \qquad \mu_{23}(t) = (1 + \alpha_{23})\,\mu_{x+t}^M \tag{3}$$

$$\mu_{02}(t) = (1 - \alpha_{02})\,\mu_{y+t}^W \qquad \mu_{23}(t) = (1 + \alpha_{13})\,\mu_{y+t}^W.$$

These formulas link the Markov forces of mortality $\mu_{ij}(t)$ and the marginal lifetime forces of mortality μ_{x+t}^M and μ_{y+t}^W using the constants α_{ij}. So, we can compute the joint survival function:

$$_t p_{xy} = p_{00}(0,t) = \exp\left(-\int_0^t \left(\mu_{01}(u) + \mu_{02}(u)\right) du \right) = \left(_t p_x^M \right)^{1-\alpha_{01}} \left(_t p_y^W \right)^{1-\alpha_{02}}.$$

We see, that if we want to use this model in practice, we must estimate the coefficients α_{01}, α_{02} and we obtain the marginal survival functions $_t p_x^M$ and $_t p_y^W$ from the survival life tables (Heilpern, 2011). We estimate the parameters α_{01} and α_{02} using the Nelson-Aalen estimator based on the cumulative function (Jones, 1997; Denuit et al. 2001):

$$\Omega_{ij}(t) = \int_0^t \mu_{ij}(s)\,ds.$$

The Nelson-Aalen estimator minimizes the sum of squared differences between the increments $\Delta\Omega_{ij}$ and their estimator $\Delta\widehat{\Omega}_{ij}$, i.e. the statement:

$$\sum_{k=t_1}^{t_2} \left(\Delta\widehat{\Omega}_{ij}(k) - \int_0^1 \mu_{ij}(k+t)\,dt \right)^2.$$

Using (3) we obtain

$$\int_0^1 \mu_{01}(k+t)\,dt = \int_0^1 (1-\alpha_{01})\mu_{x+k+t}^M\,dt = (1-\alpha_{01}) \ln {}_1p_{x+k}^M,$$

$$\int_0^1 \mu_{02}(k+t)\,dt = \int_0^1 (1-\alpha_{02})\mu_{y+k+t}^W\,dt = (1-\alpha_{02}) \ln {}_1p_{y+k}^W.$$

The estimators of the coefficients α_{01}, α_{02} are solutions of the above optimization problem:

$$\widehat{\alpha}_{01} = 1 + \frac{\sum_{k=t_1}^{t_2} \Delta\widehat{\Omega}_{01}(k) \ln {}_1p_{x+k}^M}{\sum_{k=t_1}^{t_2} (\ln {}_1p_{x+k}^M)^2}, \qquad \widehat{\alpha}_{02} = 1 + \frac{\sum_{k=t_1}^{t_2} \Delta\widehat{\Omega}_{02}(k) \ln {}_1p_{y+k}^W}{\sum_{k=t_1}^{t_2} (\ln {}_1p_{y+k}^W)^2}, \tag{4}$$

where the estimator $\Delta\widehat{\Omega}_{0i}$ (Denuit et al. 2001) is equal to

$$\Delta\widehat{\Omega}_{0i}(k) = \frac{\Delta L_{0i}(k)}{L_i(k+1) - L_i(k)} (\ln L_i(k+1) - \ln L_i(k)). \tag{5}$$

The symbol $\Delta L_{01}(k)$ (resp. $\Delta L_{02}(k)$) means a number of k-year-old husbands (resp. wives) dying during fixed year, e.g. 2011, $L_1(k)$ (resp. $L_2(k)$) is a number of k-year-old husbands (resp. wives) at 2011 and $L_1(k+1)$ (resp. $L_2(k+1)$) is a number of $(k+1)$-year-old husbands (resp. wives) at 2012.

5. Copula model

5.1. Basic definition and properties

We can describe the dependent structure of joint lifetimes using copula. Copula is a link between the joint and marginal distributions (Genest, MacKay, 1986; Nelsen, 1999):

$$P(T_x^M \le t, T_y^W \le s) = C(P(T_x^M \le t), P(T_y^W \le s)).$$

But in our analysis we need more the joint and marginal survival functions than cumulative distribution functions. We use the survival copula C^* to this end:

$$P(T_x^M > t, T_y^W > s) = C^*(P(T_x^M > t), P(T_y^W > s)).$$

The function C^* is a copula too and it satisfies the following relation (Nelsen, 1999):

$$C^*(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

The probabilities ${}_tp_{xy}$ can be computed using the survival copula C^* in the following way:

$${}_tp_{xy} = P(T_x^M > t, T_y^W > t) = C^*({}_tp_x^M, {}_tp_y^W).$$

For the independent random variables, the corresponding copula takes the simple form:

$$C_I(u, v) = uv$$

and for the strict positive C_W and the strict negative C_M dependence we have:

$$C_W(u, v) = \min\{u, v\}, \qquad C_M(u, v) = \max\{u + v - 1, 0\}.$$

These extreme copulas satisfy the following relation:

$$C_M(u, v) \le C(u, v) \le C_W(u, v) \tag{6}$$

for every copula C (Nelsen 1999).

Using the relation (6) we obtain the following inequalities:

$$\max\{\overline{F}_1(x_1) + \overline{F}_2(x_2) - 1, 0\} \le \overline{F}(x_1, x_2) \le \min\{\overline{F}_1(x_1), \overline{F}_2(x_2)\}$$

for every joint and marginal survival functions. The left and right sides of these inequalities are called the Frechet bounds. So, we can estimate the a joint-life status surviving to time t:

$$\max\{{}_tp_x^M + {}_tp_y^W - 1, 0\} \le {}_tp_{xy} \le \min\{{}_tp_x^M, {}_tp_y^W\},$$

and a last-survival status

$$1 - \min\{{}_tq_x^M, {}_tq_y^W\} \le {}_tp_{\overline{xy}} \le 1 - \max\{{}_tq_x^M + {}_tq_y^W - 1, 0\}.$$

The above relations let us estimate the pensions and we obtain (Denuit et al. 2001):

$$a_{x|y}^{\min} \le a_{x|y} \le a_{x|y}^{\max}, \qquad a_{xy;\overline{n}|}^{\min} \le a_{xy;\overline{n}|} \le a_{xy;\overline{n}|}^{\max}, \quad a_{\overline{xy};\overline{n}|}^{\min} \le a_{\overline{xy};\overline{n}|} \le a_{\overline{xy};\overline{n}|}^{\max},$$

where

$$a_{x|y}^{\min} = \sum_{k=1}^{w_y^W} v^k {}_kp_y^W - \sum_{k=1}^{w_x^M \wedge w_y^W} v^k \min\{{}_kp_x^M, {}_kp_y^W\},$$

$$a_{x|y}^{\max} = \sum_{k=1}^{w_y^W} v^k {}_kp_y^W - \sum_{k=1}^{w_x^M \wedge w_y^W} v^k \max\{{}_kp_x^M + {}_kp_y^W - 1, 0\},$$

$$a_{\overline{xy};n|}^{\min} = \sum_{k=1}^{n} v^k \max\{{}_k p_x^M + {}_k p_y^W, 0\}, \qquad a_{\overline{xy};n|}^{\max} = \sum_{k=1}^{n} v^k \min\{{}_k p_x^M, {}_k p_y^W\},$$

$$a_{\overline{xy};n|}^{\min} = \sum_{k=1}^{n} v^k (1 - \min\{{}_k q_x^M + {}_k q_y^W\}, \qquad a_{\overline{xy};n|}^{\max} = \sum_{k=1}^{n} v^k (1 - \max\{{}_k q_x^M + {}_k q_y^W, 0\}.$$

In Heilpern's paper (Heilpern, 2011) the dependent structure of joint lifetimes was described by the Archimedean copula. It is simple copula done by formula (Nelsen 1999):

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)),$$

where φ: $[0, 1] \rightarrow R_+$ is decreasing function, called generator, satisfying condition $\varphi(1) = 0$. Archimedean copulas form the families of copulas characterized by some parameter. This parameter described the degree of dependence. The Kendall's coefficient of correlation τ is done by formula

$$\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt.$$

5.2. Copula selection

Now, we present the method of selection of copula best fit to the data (Genest, Rivest, 1993; Heilpern, 2007). We restricted ourselves to Archimedean copulas only. This methods proceeds in four steps:

i) set the families of Archimedean copulas,
ii) estimate Kendall's τ coefficient of correlation based on the empiric data,
iii) select the copula connected with this Kendall's τ from every family,
iv) choose optimal copula using some criterion.

We can use the criterion based on the on the Kendall function (Genest and Rivest, 1993):

$$K_C(t) = P(F(T_x^M, T_y^W) \le t) = t - \frac{\varphi(t)}{\varphi'(t)}.$$

Then, we choose the copula minimized the distance between empirical $K_n(t)$ and theoretical $K_C(t)$ Kendall's functions:

$$S_n = \int_0^1 |K_n(t) - K_C(t)|^2 dK_C(t).$$

Denuit et al. (2001) collected the ages at death of 533 couples buried in two cemeteries in Brussels. They used this data to select the Archimedean copula describing the dependent structure of the joint lifetimes (T_0^M, T_0^W) of spouses and used the criterion based on the Kendall's function. They select Gumbel copula $C_\alpha(u, v) = \exp(-((-\ln u)^\alpha + (-\ln v)^\alpha)^{1/\alpha})$, $\alpha \ge 1$ with parameter $\alpha = 1.1015$ using this data. Heilpern (2011) used the date $n = 360$ from two cemeteries in Wrocław and the criterion based on the Kendall's function, too. He took into account the Clayton, Gumbel, Frank and AMH families of copulas. The AMH copula $C_\alpha(u, v) = uv/(1 - \alpha(1 - u)(1 - v))$ with the parameter $\alpha = 0.5879$ proved to be the best copula in this case. They computed the values of pensions and they compared them with the values obtained under independent assumption.

The copulas C_α obtained in these papers are connected with the survival probability of joint lifetimes (T_0^M, T_0^W). If we want to obtain probability ${}_tp_{xy}$ we must compute the following conditional survival probability

$$ {}_i p_{xy} = P(T_x^M > t, T_y^W > t) = \frac{C_\alpha({}_{x+t}p_0^M,\ {}_{y+t}p_0^W)}{C_\alpha({}_xp_0^M,\ {}_yp_0^W)}. $$

6. Example

In this section we present the results of the investigation of the spouses in Poland. We use the Markov model and the data from Polish Central Statistical Office from 2011. There was the Polish General Census in 2011 and the data are more detail in this year. So, we can obtain the values $L_i(k)$, but $L_i(k + 1)$, as the number of $(k + 1)$-year-old husbands (resp. wives) at 2012, is unattainable. We can estimate the statement $\Delta L = L_0(k + 1) - L_0(k)$ as the difference between the number of k-year-old men getting married during 2011 and the sum of the number of k-year-old married men dying during 2011, k-year-old married men whose wife died during 2011 and k-year-old married men getting divorced during 2011. Then $L_0(k + 1) = L_0(k) + \Delta L$. We obtain the value of $L_1(k + 1)$ in the similar way.

These data were grouped in the 5-year classes. So, they were evenly distributed over the one year periods. The effective rate $\xi = 0.03$. Using (5) and (4) we obtain the following values of the parameters:

$$\alpha_{01} = 0{,}1257, \qquad \alpha_{02} = 0{,}2009.$$

Heilpern (2011) conducted the similar study based on the data from 2002 and obtain the values $\alpha_{01} = 0{,}0706$ and $\alpha_{02} = 0{,}1155$. The parameters obtained in investigation using the data from Belgium in 1991 (Denuit et al. 2001) are equal to $\alpha_{01} = 0{,}0929$ and $\alpha_{02} = 0{,}1217$.

The relative values of the widow's pension $a_{x|y}$ when the spouse are in the same age, i.e $x = y$, for minimum, independent and maximum cases received toward Markov model (the pension for Markov model is equal to 1) are given in Table 1. We see, that if the Markov model is truth, then the window's pension when we assume independent lifetimes is overestimate. This overestimate is equal average 20% and it increases with age x. We obtain the similar results for Frechet bounds, but the errors are bigger, particularly for upper bound.

Table 1. The relative values of widow pension toward Markov model

x	40	50	60	70	80	90
Markov	3,463	3,959	3,998	3,450	2,452	1,372
min	0,768	0,752	0,722	0,653	0,498	0,304
indpendent	1,188	1,187	1,187	1,192	1,212	1,240
max	1,449	1,470	1,500	1,552	1,642	1,744

Source: own elaboration.

Table 2 contains the values of the pensions $a_{xy;\overline{n}|}$ when $x = y = 50$ for different values of n. We see, that the independent case underestimates the truth pension in this case. But, the errors are smaller than in the case of widows pension.

Table 2. The relative values of pension $a_{xy;\overline{n}|}$ toward Markov model

n	10	20	30	40	50
Markov	8,003	12,789	15,053	15,629	15,665
independet	0,989	0,976	0,962	0,954	0,953
max	1,010	1,024	1,041	1,058	1,063
min	0,988	0,964	0,917	0,883	0,881

Source: own elaboration.

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