DIDACTICS

OF

MATHEMATICS

9(13)

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OPINIONS

OPTIMUM CLASS SIZE.
TESTING LAZEAR'S THEOREM WITH
INTERMEDIATE MATHEMATICS SCORES
IN POLISH SECONDARY SCHOOLS

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Abstract. The paper is directed at testing Lazear's proposition which argues that educational outcome depends on the non-uniformity of distribution of knowledge and skills of students and on class size. Lazear asserts that students' performance achieves its maximum when students are segregated by skills and behaviour. Using the 2010 data (mathematics scores in *gimnazjum* and maturity exams), we corroborate this theorem. We also demonstrate that improved teaching conditions in Polish primary schools and *gimnazjums*, i.e. lower class sizes, better salaries of teachers and higher unit costs did not result in an increased educational outcome as measured by the final examination scores in primary and secondary schools between 2006 and 2011.

Keywords: efficiency of teaching mathematics, class size, non-uniformity of knowledge distribution in class.

1. Introduction

The modern economy is founded on knowledge, or more precisely, on high quality human capital. Barro and Lee (2001), emphasize that scores in exact science are the educational factor resulting in highest economic growth. The efficiency of teaching intermediate mathematics between 2003 and 2009 in Poland compared to other countries, measured by the PISA examination, has been relatively stable and close to the average score attained by students in the OECD countries, with the number of weakest students (those who do not pass level 1 of PISA difficulty in OECD surveys) close to 20 per cent, while the number of best students (those solving problems at levels 5 and 6), tends consistently to be 10 per cent (Biernacki, Czesak, 2012).

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To become innovative, Poland's economy needs excellent resear-chers, engineers, economists, physicists, etc. On the other hand, the weakest students, who mostly come from poor families, have no chance to rise above the poverty level (Sparkers, 1999). Hence, a natural ques-tion emerges: what should be done in order to reduce the percentage of weakest students and to increase the percentage of best students.

Teaching mathematics consists mainly in instructing primary, secondary and tertiary students about new structures and systems found-ed on known ones. Therefore, students are required to work systematical-ly and diligently. Before going on to the next part of the material, teach-ers must be sure that their students have effectively comprehended the current part. Students are able to properly master the material included in Analysis 2 if they have adequately mastered Analysis 1; likewise, a student at any level of education in mathematics from primary to sec-ondary school. After concluding one area of mathematics with their students, teachers mostly face a dilemma: either to revise the past materi-al during one or two lessons or to start a new area. The first option is usually a must in classes with a considerable non-uniformity of knowledge and skills distribution. The problem of large non-uniformity of distribution of knowledge in mathematics involves all students at primary, secondary and tertiary schools alike.

2. Educational production function

The educational production function (Bowles, 1969), assumes a dependence of educational outcome on a number of factors such as: expenditure per student, class size, education and commitment of teachers, characteristics of students and their families, etc. Next, some facts con-cerning this function will be explained.

A report by Coleman (1966), argues that educational outcome de-pends equally on the school and on the characteristics of students and their families. Hanushek (1986), derives similar conclusions following his research on children and youngsters at public schools in the 1980s, i.e. that the amount of expenditure per student exerts a lesser impact on students' achievements than their background. Card and Krueger (1996), examined the school performance of students at public primary and secondary schools in the US, and mostly found a significant positive correlation between their achievements and their parents' wages. Hanushek (2003) analysed the schooling effects of U.S. public schools

between 1960 and 2000 to find out that the basic factors of the educational production function, such as the amount of expenditure per student, number of students per teacher, teacher's experience, education and wages are statistically insignificant.

Lazear (2001) proved several interesting theorems regarding the educational function:

1. The optimal class size rises in line with the teacher's salary, falls in line with the value of a unit of education (in the labour market) and increases in line with the probability that students learn well and behave well.

2. Educational output is higher in larger classes with good learners and well-behaved students than in smaller classes with poor learners and less well-behaved students.

3. Educational output is maximized when students are segregated by their skills and behaviour.

Lazear assumes that the educational outcome of the class (group of students) achieved during a lesson (lecture, tutorial) is determined by the skills, capabilities and behaviour of students in the classroom. To examine this relation, he introduces a parameter regarding student behaviour in the class (that depends on the level of their knowledge and capabilities). In his analysis, the following notation is used:

p – the probability that any given student is not impeding his own or other students' learning, i.e. that he is a good learner and behaves well,

V – the value of a unit of knowledge capital, as determined by the market,

W – the unit cost of student education,

Z – the number of students at school,

m – the number of classes.

The function of the school's profit is represented in the following way:

$$\pi\left(p, Z, V, W, m\right) = ZVp^{Z/m} - Wm. \tag{1}$$

The model (1) includes several important items of information that are interlinked by a certain interaction. One example is given by lowering the class size, which naturally increases school operating costs. The objective of a school, i.e. that of local self-government or a state, is the maximization of profit given by formula (1). While V, W and Z are assumed to be independent, the optimal class size $n = Z / m$ can be found, given m and the equal size for all classes. In order to determine m that

maximizes model (1), we differentiate it with respect to m to obtain the first-order condition:

$$\frac{\partial \pi}{\partial m} = -V \frac{Z^2}{m^2} p^{Z/m} \ln(p) - W = 0 \tag{2}$$

and solve it for m assuming that p, Z, V and W are constant.

The above Lazear's theorem was generalized by Ejsmont (2009). Assume there are two partitions of a set with cardinality Z of all students at school into k equinumerous classes with regard to the probability of behaviour p. In the first partition students are segregated, i.e. classes are behaviour–homogenous. Let $p_1, \ldots, p_k$ denote respective probabilities of behaviour in classes denoted by $A_1,\ldots, A_k$. In the second partition students are not segregated. It is assumed that in each class there is at least one student from the set A_i, and $\alpha_1, \ldots, \alpha_k$ denote the numbers of students from $A_1, \ldots, A_k$, respectively. Hence, $\alpha_1 + \ldots + \alpha_k = 1$ and $\alpha_1, \ldots, \alpha_k > 0$, therefore $\alpha_i Z$ students belong to the group A_i. Then, the total output of a school segregating students by the probability p can be presented with the above notation as:

$$\pi' = ZV\left(\alpha_1 p_1^n + \alpha_k p_k^n\right) - Wm, \tag{3}$$

whereas the output of a school not segregating students by the behaviour probability p is:

$$\pi'' = ZV\left(p_1^{\alpha_1 n} \cdot \ldots \cdot p_k^{\alpha_k n}\right) - Wm. \tag{4}$$

It follows directly from the Jensen's inequality that $\pi' \geq \pi''$. Thus, it implies that students segregation by their skills, knowledge and behaviour raises school's output and social wealth. The above reasoning also confirms the soundness of introducing a non-uniformity parameter to calculate a school's optimal profit. This means that the larger the inequality of knowledge in a class, the harder the optimization of educational output. Therefore, one should consider the level of non-uniformity (e.g. measured with the Gini coefficient G) when calculating educational effectiveness. Figure 1 shows a simulation that explains the positive effect of student segregation by skills and capabilities ($V = 1$, $Z = 999$, $W = 5$, $30 \leq m \leq 89$, $p_1 = 0{,}98$, $p_2 = 0.95$, $p_3 = 0.9$, $\alpha_1 = 0.167$ $\alpha_2 = 0.33$ $\alpha_3 = 0.5$).

$n = Z/m$

No segregation - Maximum class size is 13
Segregation - Maximum class size is 16
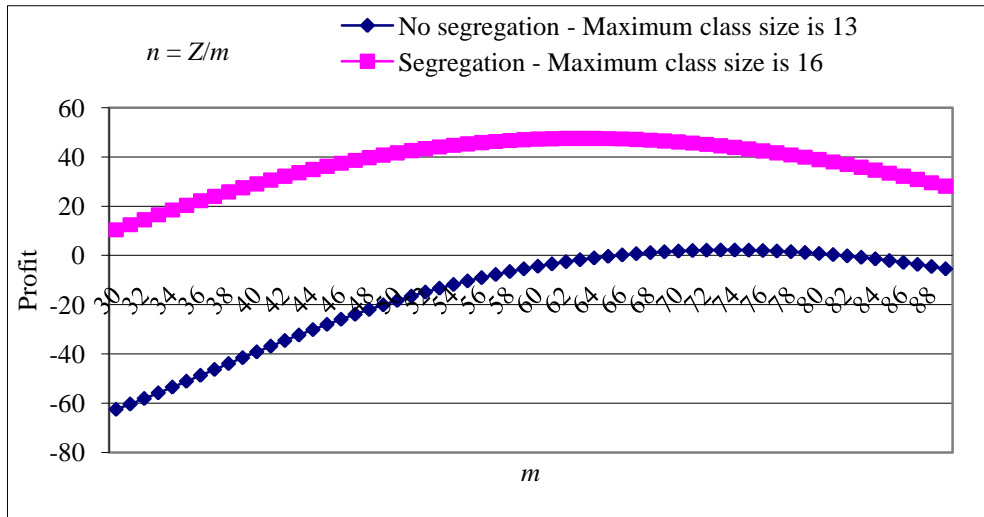
Fig. 1. Simulation of educational effectiveness
with regard to student segregation in classes

Source: Ejsmont (2011).

Similar findings have been reported by Gary-Bobo and Mahjoub (2006) who examined the above problem by means of Markov processes.

3. Testing Lazear's theorem with data
from Polish secondary schools

Dobbelsteen, Levin and Oosterbeek (2002) analyzed empirical data and showed that under some conditions a larger class size results in higher educational output. A similar result was obtained with data from 844 secondary schools in Poland and 44,621 graduates. The best educational output measured by the educational value added (EVA, cf. Biernacki, Ejsmont, 2011) was **achieved in large classes of 30-32** students. Exceeding this threshold resulted in a lower educational output. Next, we present the results of the analysis by location of schools and class size at Polish secondary schools. It should be mentioned that the best secondary schools admit applicants with high scores, i.e. classes are relatively homogenous with respect to knowledge distribution. Accordingly, students apply to secondary schools where the previous year's admission threshold was close to their individual examination scores.

The LH vertical axis shows EVA, the RH axis – non-uniformity of knowledge distribution, while both axes have different scales. The diffe-

rences between educational value added and location become remarkably blurred. The differences between smaller and bigger towns are not as distinct as in the case of Polish language (Ejsmont, 2009). One may notice that the obtained non-uniformities of knowledge distribution evidently are negatively correlated with students' performance measured by EVA (see Table 1). If we allow non-uniformity of knowledge distribution as parameter p in model (1), then the assumption of Lazear's theorem is satisfied.

Table 1. Correlation between EVA and non-uniformity of knowledge distribution[1]

	Rural areas	Small towns	Medium towns	Big cities
Coefficient of correlation	−0.727	−0.983	−0.973	−0.984

Source: own calculation.

Fig. 2. Educational value added and non-uniformity of knowledge distribution in Polish secondary schools by location and class size obtained for mathematics in 2010[2]

Source: own elaboration based on data from the Central Examination Board.

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[1] The *p*-value is above the 0.05 level of significance.
[2] Four types of school location were adopted: village – V; city up to 20 thousand residents – C20; city between 20 and 100 thousand residents – C20-100; city over 100 thousand residents – C100. The class size values have been assumed for every three students, starting from 10 and ending at 38 students.

A small non-uniformity of student knowledge distribution in class promises "better" behaviour of students. At the same time, as demonstrated by Figure 2, the EVA rises, therefore the assumptions of Lazear's theorem are met. However, the conducted analysis does not respond to a natural question of whether lowering the size of a class with a small non-uniformity of knowledge and skills distribution will result in a significant improvement of student performance. So far, under Polish conditions, small class sizes occur at private schools, not necessarily implying the low non-uniformity of knowledge distribution.

4. Class size and average educational output at primary schools and *gimnazjum* (secondary schools) by rural and urban locations

Tables 2 and 3 and Figure 2 present the average graduation examination scores obtained at primary schools and *gimnazjum* (secondary schools). Certainly, the picture provided by the analysis of student performance is not complete, as schools are obliged to prepare students to live in the community and for the community, not just to pass the final examinations. Data regarding average class sizes and average performance at primary schools and *gimnazjum* in Poland during the past six years are presented in Tables 2 and 3 and in Figure 2. The average educational output at primary schools remained constant during the analyzed period, despite the significant increase of educational inputs (increased teachers' salaries in 2010, lower class sizes, increased unit expenses) at primary schools and *gimnazjum*, whereas the average educational output at the latter follows a downward trend.

Table 2. Class sizes at primary schools and *gimnazjums* in 2006-2011

Year	Primary school			*Gimnazjum*		
	Rural	Urban	U–R	Rural	Urban	U–R
2006	16.90	23.18	6.28	23.40	25.44	2.04
2007	16.39	22.85	6.46	22.99	25.12	2.13
2008	15.86	22.21	6.35	22.42	24.60	2.18
2009	15.59	21.89	6.3	21.82	24.15	2.33
2010	15.27	21.88	6.61	21.36	23.84	2.48
2011	14.91	21.65	6.74	20.88	23.55	2.67

Source: based on Jeżowski (2012) with GUS data.

Where U-R is the difference between the average number of students in class in the urban and rural area. A maximum score to attain in the final test at primary school equals 40 points, and that at *gimnazjum* – 50 points, therefore in order to compare the differences, the difference in performance at primary schools was multiplied by 1.25. The difference between the average scores achieved at urban and rural primary schools is decreasing, while the difference between *gimnazjum* remains constant and equal to one point.

Table 3. Average graduation examination scores in 2006-2011

Year	Primary school graduation tests				*Gimnazjum* graduation exams		
	Rural	Urban	U–R	5/4(U–R)	Rural	Urban	U–R
2006	24.43	25.94	1.51	**1.89**	30.27	31.44	1.17
2007	25.60	27.22	1.62	**2.03**	30.25	31.46	1.21
2008	24.90	26.35	1.45	**1.81**	29.69	30.74	1.05
2009	22.64	23.40	0.76	0.95	30.88	32.10	**1.22**
2010	24.56	25.33	0.77	0.96	30.26	30.40	**0.14**
2011	25.27	25.85	0.58	0.73	24.65	25.67	**1.02**

Source: based on Jeżowski (2012) with data from the Central Examination Board.

In Table 3 the same color is used for showing the difference of the points gained during the final exams between the same students in the urban and rural area during the following stages of their education.

At rural primary schools, the coefficient of correlation between average scores and average class sizes equals 0.069, while at urban primary schools 0.509. At *gimnazjums* the coefficients of correlation between average scores and average class sizes are 0.599 and 0.639 for rural and urban schools, respectively. Regrettably, the correlations are significant and positive (except for rural primary schools), i.e. lowering average class sizes are accompanied by lower average educational output.

The analysis of differences over time between the average scores by rural and urban same groups of students (with a 3-year shift) implies the lack of correlation. Unfortunately, such results evidence problems in Polish education.
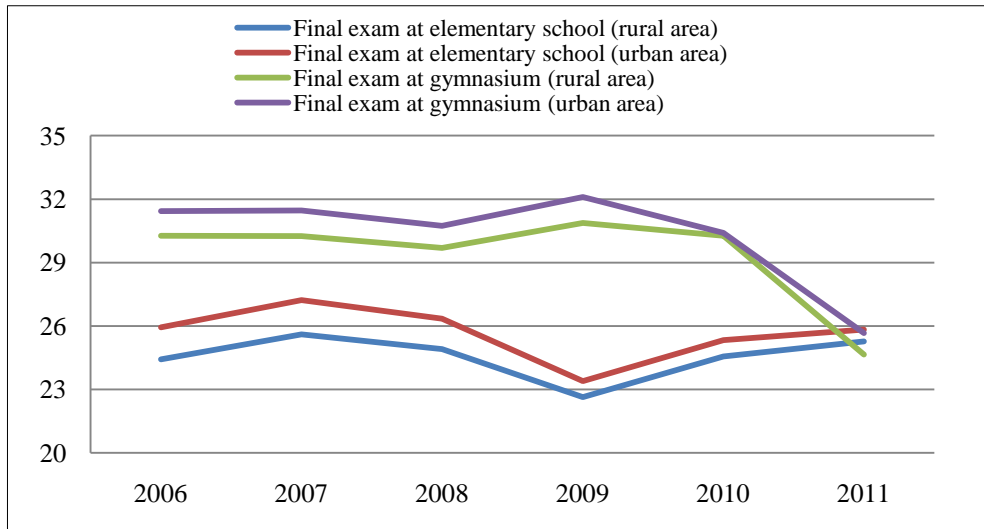
Fig. 2. Examination scores after graduating from primary school and *gimnazjum*
by rural and urban schools

Source: Jeżowski (2012) with GUS data.

In 2010, the salaries of teachers at primary and secondary schools increased by approximately 33 per cent, however, even after this rise, the salary of a Polish teacher is still lower than that in the European Union on average. Maybe the effect of the rise will be noticeable in 2013 (after the 3-year education cycle). Rural primary schools, typically with just one class at each level, cannot afford the segregation of students. Rural *gimnazjums*, on the other hand, are usually large community schools where the manipulation of class sizes and non-uniformity of knowledge distribution is attainable.

5. Conclusion

The segregation of *gimnazjum* and university students based on their knowledge and skills in training groups or in classes should cause an increase of educational efficiency (represented by an increase of points gained in final exams) and should give the opportunity for more dynamic progress for talented students (e.g. through problem teaching). The freshmen at the Wrocław University of Economics are grouped for classes in mathematics based on their choice of specializations of study, which has nothing to do with their segregation by the level of knowledge and skills in mathematics.

Large class sizes and the considerable non-uniformity of knowledge distribution in student groups, adversely influence the optimization of educational output measured by examination scores (Biernacki, Czesak, 2012).

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