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EXTENDED SUMMARIES OF SELECTED PAPERS

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PAGE'S CUSUM IN SEQUENTIAL CHANGE-POINT ANALYSIS – ASYMPTOTIC DISTRIBUTION OF THE STOPPING TIME

Stefan Fremdt (University of Cologne)

1. Introduction

The two worldwide economic crises of the last decade have revealed the consequences of mispricing of risks and assets due to a misspecification of the valuation models applied. The importance of monitoring procedures to guarantee the validity of the underlying model is therefore beyond all question. As a consequence sequential monitoring procedures have evoked more and more interest in recent years and are required to cope with the dependence structures inherent in many economic data sets. One of the most common approaches for the construction of such a monitoring procedure is based on so-called cumulative sum (CUSUM) detectors, bringing with them the property that they work best for very early change-points. We will present a CUSUM-based procedure going back to an idea of Page [1954] that offers a higher stability concerning the time of change and the underlying model is chosen in such a way that it is applicable to a variety of time series models from economy and econometrics. After the introduction of this procedure in the context of linear models a result on the distribution of the corresponding delay time in the special case of the so-called location model is presented that shows the desired properties for our procedure.

2. Model description

Consider the linear model:

$$y_i = \mathbf{x}_i^T \beta_i + \varepsilon_i, \qquad 1 \le i < \infty$$

where $\mathbf{x}_i$ is a $p \times 1$ random vector and β_i is a real vector of the same dimension.

We assume that for the first m observations the so-called "noncontamination assumption" (cf. [Chu et al. 1996]) holds, i.e.,

$$\beta_i = \beta_0, \quad 1 \le i \le m \tag{1}$$

We would like to test the constancy of the regression parameters β_i in time which leads to the null hypothesis

$$H_0: \quad \beta_i = \beta_0, \quad i = m+1, m+2; \ldots ;$$

and we choose to test alternatives given by one abrupt change in the regression parameter at an unknown change-point, i.e.,

H_A: there is $k^* \ge 1$ such that $\beta_i = \beta_0$, $m < i < m + k^*$,

and $\beta_i = \beta_*$, $i = m + k^*$, $m + k^* + 1, \ldots$, with $\Delta_m = \beta_* - \beta_0 \ne 0$: k – k.

The detection procedure will consist of a stopping time τ chosen in such a way that under the null hypothesis

$$\lim_{m\to\infty} P(\tau(m) < \infty) = \alpha \tag{2}$$

for some prescribed level α, $0 < \alpha < 1$, and under the alternative

$$\lim_{m\to\infty} P(\tau(m) < \infty) = 1 \tag{3}$$

We assume the following conditions on the regressors and the error terms:

(**A.1**) $E\varepsilon_i = 0$, $E\varepsilon_i\varepsilon_j = 0$ $(i \ne j)$ and $E\varepsilon_i^2 \le D$ with some D > 0,

(**A.2**) $\mathbf{x}_i^T = (1, x_{2i}, \ldots, x_{pi})$, $1 \le i < \infty$,

(**A.3**) $\{\varepsilon_i,\ 1 \le i < \infty\}$ and $\{\mathbf{x}_i,\ 1 \le i < \infty\}$ are independent,

there are a positive definite matrix **C** and a constant $\xi > 0$ such that

$$\left|\frac{1}{n}\sum_{1\le i<n} \mathbf{x}_i\mathbf{x}_i^T - \mathbf{C}\right| = O\left(n^{-\xi}\right) \quad \text{a. s.} \tag{4}$$

(**A.4**) For every m there are a constant $\sigma > 0$ and independent Wiener processes

$\{W_{1,m}(t): \ t \geq 0\}$ and $\{W_{0,m}(t): \ t \geq 0\}$ such that

$$\sup_{1 \leq k < \infty} \frac{1}{k^{\kappa}} \left| \sum_{i=m+1}^{m+k} \varepsilon_i - \sigma W_{1,m}(k) \right| = O_P(1) \qquad (m \to \infty) \qquad (\ )$$

and

$$\sum_{l=1}^{m} \varepsilon_l - \sigma W_{2,m}(m) = O_P(m^{\kappa}) \qquad (m \to \infty) \qquad (6)$$

with some $\kappa < 1/2$.

The residuals of the model are defined via

$$\varepsilon_i = y_i - \mathbf{x}_i^T \hat{\beta}_m, \quad i = 1, 2, \ldots,$$

where $\hat{\beta}_m$ denotes the OLSE for β from the "training period" $(y_1, \mathbf{x}_1)$, $\ldots$, $(y_m; \mathbf{x}_m)$, i.e.,

$$\hat{\beta}_m = \left(\sum_{1 \leq i \leq m} \mathbf{x}_i \mathbf{x}_i^T \right)^{-1} \sum_{1 \leq j \leq m} \mathbf{x}_j y_j.$$

3. Page's CUSUM procedure and its asymptotic properties

Many sequential detection procedures in the literature are constructed as first-passage time of a so-called detector over a certain boundary function. E.g., Horváth et al. [2004] proposed as a detector the (ordinary) CUSUM of the residuals, i.e.,

$$\hat{Q}(m,k) = \sum_{m < i \leq m+k} \hat{\varepsilon}_i \qquad k = 1,2, \ldots,$$

and as a boundary function

$$g(m,k) = cm^{\frac{1}{2}} \left(1 + \frac{k}{m}\right) \left(\frac{k}{k+m}\right)^{\gamma} =: c\tilde{g}(m,k),$$

with $c = c(\alpha, \gamma)$ and

$$0 \le \gamma < \min(\xi, \tfrac{1}{2}). \tag{7}$$

The approach of Horváth et al. [2004] for i.i.d. error terms was extended by Aue et al. [2006] to allow certain dependence structures by introducing Assumptions (A.1) and (A.4). We refer to them for example time series satisfying the assumptions of the model. Now we want to introduce a monitoring procedure that, as was already mentioned, goes back to an idea of Page [1954] and we define the Page CUSUM detector by:

$$\hat{Q}_P(m,k) = \max_{0\le i\le k} \left|\hat{Q}(m,k) - \hat{Q}(m,i)\right|.$$

The corresponding stopping time is then given by

$$\tau^{\text{Page}}(m) = \inf\{k \ge 1:\ \hat{Q}_P(m,k) > g(m,k)\},$$

where inf Ø = ∞ and the constant c in the definition of g can be derived from Theorem 1 below which implies that (2) holds for $\tau^{\text{Page}}(m)$.

Theorem 1 [Fremdt 2012b]
Assume that (1), (**A.1**) - (**A.4**) and (7) hold. Then under the null hypothesis we have

$$\lim_{m\to\infty} P\left(\frac{1}{\hat{\sigma}_m}\sup_{1\le k<\infty}\frac{\hat{Q}_P(m,k)}{g(m,k)} \le c\right) = P\left(\sup_{0<t<1}\sup_{0\le s\le t}\frac{1}{t^\gamma}\left|W(t) - \frac{1-t}{1-s}W(s)\right| \le c\right)$$

for all real c, where $\hat{\sigma}_m^2 = \frac{1}{m-p}\sum_{i=1}^{m}\left(\hat{\varepsilon}_i - \frac{1}{m}\sum_{l=1}^{m}\hat{\varepsilon}_l\right)^2$.
Under the alternative hypotheses the detector diverges and hence $\tau^{\text{Page}}(m)$ also satisfies (3) as the following theorem shows.

Theorem 2 [Fremdt 2012b]
Let c_1 denote the first column of **C** and assume that (1), (**A.1**) – (**A.4**) and (7) hold. Then under the alternative H_A, if $\boldsymbol{c}_1^T\boldsymbol{\Delta}_m \neq 0$ and $\sqrt{m}|\boldsymbol{c}_1\boldsymbol{\Delta}_m| \to \infty$ as $m \to \infty$ we have

$$\frac{1}{\hat{\sigma}_m}\sup_{1\le k<\infty}\frac{\hat{Q}_P(m,k)}{g(m,k)} \xrightarrow{P} \infty \quad \text{as } m \to \infty.$$

4. Asymptotic distribution of the stopping time

Aue and Horváth [2004] investigated in a sequential setup the asymptotic normality of the CUSUM-stopping-time in the case of the so-called location model with the alternative hypothesis of a change in the mean for relatively early changes (i.e. the change-point k^* is located relatively close to the end of the training period in terms of m). In this section we will present the asymptotic distribution of the stopping times based on Page's CUSUM detector in this location model and show hereby that the Page CUSUM procedure from Section 3 offers more stability concerning the time of change. The location model as a special case of the model (1) is given via

$$X_i = \begin{cases} \mu + \varepsilon_i & i = 1, \dots, m + k^* - 1, \\ \mu + \varepsilon_i + \Delta_m & i = m + k^* - 1, \dots, \end{cases} \tag{8}$$

where μ and Δ_m are real numbers and $1 \leq k^* < \infty$ denotes the unknown time of change. The corresponding hypotheses are then

$$H_0\colon\ \mu = 0 \text{ and } \quad H_A\colon\ |\Delta_m| > 0$$

We need the following assumptions on Δ_m and k^*:

(**A.5**) there exists a $\theta > 0$ such that $k^* = \lfloor \theta m^\beta \rfloor$ with $0 \leq \beta < 1$,

(**A.6**) $\sqrt{m}|\Delta_m| \xrightarrow{(m\to\infty)} \infty$,

(**A.7**) $|\Delta_m| = O(1)$:

The limit behaviour of the stopping time is determined by the behaviour of $m^{\beta(1-\gamma)-1/2+\gamma}|\Delta_m|$ for which we distinguish the following three cases:

(I) $m^{\beta(1-\gamma)-1/2+\gamma}|\Delta_m| \xrightarrow{(m\to\infty)} 0$,

(II) $m^{\beta(1-\gamma)-1/2+\gamma}|\Delta_m| \xrightarrow{(m\to\infty)} \tilde{c}_1 \in (0, \infty)$ and

(III) $m^{\beta(1-\gamma)-1/2+\gamma}|\Delta_m| \xrightarrow{(m\to\infty)} \infty$.

To state our main result on the asymptotic distribution of the stopping time we first introduce the distribution function Ψ depending on the set of assumptions, i.e. the given case (I), (II) or (III). Under (II) denote by d_1 the unique solution of

$$d_1 = 1 - \frac{c}{c_1} d_1^{1-\gamma}.$$

For all real x let

$$\Psi(x) = \begin{cases} \Phi(x) & \text{under (I)} \\ P\left(\sup\limits_{d_1<t<1} W(t) \le x\right) & \text{under (II) and} \\ P\left(\sup\limits_{d_1<t<1} W(t) \le x\right) = \begin{cases} 0 & x < 0 \\ 2\Phi(x) - 1 & x \ge 0 \end{cases} & \text{under (III)} \end{cases}$$

where $\Phi(x)$ denotes the standard normal distribution function.

Theorem 3 [Fremdt 2012a]
Let $\{X_n\}_{n=1,2,\ldots}$ be a sequence of random variables according to (8) such that (**A.1**), (**A.4**) and (**A.5**) – (**A.7**) are satisfied and let $\gamma \in [0, 1/2)$. Then for all real x under H_A

$$\lim_{m\to\infty} P\left(\frac{\tau^{\text{Page}}(m) - a_m}{b_m} \le x\right) = 1 - \Psi(-x) = \bar{\Psi}(x),$$

where a_m is the unique solution of

$$a_m = \left(\frac{cm^{1/2-\gamma}}{|\Delta_m|} + \frac{k^*}{a_m^{\gamma}}\right)^{1/(1-\gamma)}$$

and

$$b_m = \sigma\sqrt{a_m}|\Delta_m|^{-1}\left(1 - \gamma\left(1 - \frac{k^*}{a_m}\right)\right)^{-1}.$$

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A BRIEF GLIMPSE AT BLOCKWISE ALGEBRAIC MULTIGRID

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1. Introduction

"Algebraic Multigrid" (AMG) is a highly efficient iterative method for solving large (1-100 millions of variables) sparse linear systems of equations $Ax = b$. It became a major research topic in numerical analysis in the last 15 years and is applied in industry, mainly to solve systems arising from discretization of PDEs. Stochastic applications are for example simulation of stochastic PDEs and computation of the stationary probability vector for Markov transition matrices [Rosseel et al. 2008; De Sterck et al. 2010].

Nowadays effort is put into applying AMG to "strongly" coupled PDE-systems e.g. by blockwise approaches. Clees [2008] for example showed a convergence result for blockwise AMG under the assumption of blockwise diagonally dominance. The main aim of the present note is to derive a generalization of this result.

Blockwise diagonal dominance is defined for block-structured matrices

$$C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,N} \\ \vdots & \ddots & \vdots \\ C_{N,1} & \dots & C_{N,N} \end{bmatrix}$$

with square matrices $C_{i,i}$ as follows.

Definition. A matrix $C = [C_{i,j}]$ is blockwise diagonally dominant in a p-norm if all $C_{i,i}$ are invertible and

$$\max_{1\le i\le N}\sum_{j\neq i}\left\|C_{i,i}^{-1}C_{i,j}\right\|_p \le 1$$

with $1 \le p < \infty$.

Remark. For the trivial partitioning into 1×1-blocks a blockwise diagonally dominant matrix is simply (weakly) diagonal dominant. Furthermore it is easy to see that for a fixed partition blockwise diagonally dominance in a p_1-norm is generally not implied by blockwise diagonally dominance in a p_2-norm as well as it is generally not implied by weak diagonal dominance.

2. Algebraic Multigrid

Classical *one-level* iteration-schemes for solving $Ax = b$ like Gauss-Seidel iteration and Jacobi iteration (as well as their weighted and/or blockwise versions) are cheap and fast for many low dimensional problems but typically far too slow for high-dimensional problems [cf. Trottenberg et al. 2001]. AMG combines those classical iteration-schemes (which *smooth* the *error*[1] in the *algebraic sense* for the considered system, [Brandt 1986; Stüben 2001] with a *coarse-level correction* in a hierarchical way to obtain a cheap and fast method also for high-dimensional problems. The idea is that an error, which cannot be diminished by a classical scheme is then efficiently reduced by a coarse-level correction[2] and vice versa. For a review of the fundamental principles of "smoothing" and "coarse-grid correction" we refer to Trottenberg et al. [2001].

The algorithm of (blockwise) AMG can be introduced on the basis of a *two-level cycle*. (Note that the terms "level" and "grid" are used

[1] Exact solution of $Ax = b$ minus the approximation.
[2] Because for properly chosen classical iteration-schemes such an error is smooth.

as synonyms.) For solving the original block-structured *fine-level system*

$$Ax = \begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix} = b \qquad (1)$$

with $A \epsilon R^{n_1 \times n_1}$ and $B \epsilon R^{n_1}$ a two-level method constructs *coarse-level systems* of much lower dimension

$$A^H x^H = \begin{bmatrix} A^H_{i_1,i_1} & \cdots & A^H_{i_1,i_n} \\ \vdots & \ddots & \vdots \\ A^H_{i_n,i_1} & \dots & A^H_{i_n,i_n} \end{bmatrix} \begin{bmatrix} X^H_{i_1} \\ \vdots \\ X^H_{i_n} \end{bmatrix} = \begin{bmatrix} B^H_{i_1} \\ \vdots \\ B^H_{i_n} \end{bmatrix} = b^H \qquad (2)$$

with $A^H \epsilon R^{n_2 \times n_2}$ and $B^H \epsilon R^{n_2}$ and $1 \le i_1 < \ldots \le i_n \le N$. We assume that system (2) is partitioned according to (1), i.e. block $A^H_{i_k,i_k}$ has the same size as A_{i_k,i_k}. To map fine-level vectors into the coarse-level and vice-versa we need an *interpolation* $I^h_H \epsilon R^{n_1 \times n_2}$ and a *restriction* $I^H_h \epsilon R^{n_2 \times n_1}$ as transfer-operators. We will consider a symmetric positive-definite matrix A and in that case the restriction-mapping is (typically) defined as the transpose of the interpolation, i.e. $I^H_h = (I^h_H)^T$ and the coarse-level Matrix A^H is defined purely algebraically according to the *Galerkin-Principle*

$$A^H = I^h_H A I^H_h, \qquad I^H_h = (I^h_H)^T.$$

As interpolation-mapping we consider the *blockwise interpolation* which is defined by

$$(I^h_H e^H)_i = \begin{cases} E^H_i & i \epsilon C \\ \sum_{i \epsilon C} W_{ij} E^H_j & i \epsilon F' \end{cases} \qquad e^H = \begin{bmatrix} E^H_{i_1} \\ \vdots \\ E^H_{i_n} \end{bmatrix}$$

with matrix-valued "weights" W_{ij} and set $C = \{i_1 < \ldots < i_n\}$ representing the *coarse-level* and the complementary set $F = \{1, \ldots, N\} - C$. We implicitly assume that $I^h_H e^H$ is partitioned according to (1), i.e. $(I^h_H e^H)_i$ has the same size as block X_i and that for each $i \in \{1, \ldots, N\}$ there is at least one $j \in C$ with $A_{i,j} \neq 0$.

Finally we need a (blockwise) *smoother*, i.e. a classical iteration-scheme (with *iteration matrix S*, certain vector *d* and an initial approximation $u^{(0)}$ to the exact solution u^* of (1))

$$u^{(k+1)} := Su^{(k)} + d, \quad k = 1, 2, \ldots$$

that smoothes the error in the algebraic sense for the considered system. Appropriate choices are e.g. block-Gauss-Seidel and suitable weighted block-Jacobi iteration (cf. [Clees 2005]).

Suppose that $u^{(0)}$ is an approximation to u^* and we want to obtain a better approximation $\bar{u}^{(0)}$. The smoother and the coarse-level correction are therefore combined in the following way:

1. **Pre-Smoothing**: Do $\nu_1 \geq 1$ iteration steps with a (consistent) blockwise smoother $u^{(0)} \to u^{(\nu_1)}$.
2. **Coarse-Level Correction:**
   - Build the residual of the last approximation r := $B - Au^{(\nu_1)}$.
   - Restrict this residual on the coarse level $r^H := I_h^H(r)$.
   - Solve the coarse-level equation
   $$A^H = e^H r^H. \tag{3}$$
   - Prolongate the exact solution $e_*^H = (A^H)^{-1} r^H$ onto the fine level, i.e. compute $I_H^h(e_*^H)$ and correct the last approximation as following $\bar{u}^{(0)} := u^{(\nu_1)} + I_H^h(e_*^H)$.
3. **Post-Smoothing:** Do $\nu_2 \geq 1$ iteration steps with a (consistent) blockwise smoother $\bar{u}^{(0)} \to \bar{u}^{(\nu_2)}$.

Remark. To obtain a multilevel (multigrid) procedure we apply the same two-level procedure recursively to solve the coarse-level system (3).

For further discussion we restate the procedure in terms of operators. The iteration step of the coarse-level correction $u^{(\nu_1)} \to \bar{u}^{(0)}$ is given by

$$\bar{u}^{(0)} = Ku^{(\nu_1)} + d$$

with $d = I_H^h (A^H)^{-1} I_h^H b$ and the iteration matrix $K = I - I_H^h (A^H)^{-1} I_h^H A$. One iteration of the whole two-level procedure $u^{(\nu_1)} \to \bar{u}^{(\nu_2)}$ then can be written as

$$\bar{u}^{(\nu_2)} = M(\nu_1, \nu_2) u^{(\nu_1)} + w$$

with iteration matrix $M(\nu_1, \nu_2) = S^{\nu_2} K S^{\nu_1}$ and a suitable w.

3. Convergence result

The following theorem was shown by Clees [2008] for the special case $p = 2$. We show that the assertion holds true for the general case $1 \leq p < \infty$. Note that the *spectral radius* $\rho\big(M(v_1, v_2)\big)$ is the *asymptotic convergence* rate of a two-level method, i.e. approximately the average error-reduction in each step.

Theorem. Suppose $A = [A_{i,j}]$ is symmetric positive-definite and blockwise diagonally dominant in p-norm with $1 \leq p < \infty$. For I_H^h defined by

$$W_{ij} = -\alpha_i A_{i,i}^{-1} A_{i,j}, \qquad \alpha_i = \frac{\sum_{j \neq i} \left\| A_{i,i}^{-1} A_{i,j} \right\|_p}{\sum_{j \in C} \left\| A_{i,i}^{-1} A_{i,j} \right\|_p}$$

for $i \in F$ and $j \in C$ we have

$$\rho\big(M(v_1, v_2)\big) \leq \sqrt{1 - \sigma / \max\{\alpha_i\}}$$

and $\sigma = 1/4$ with blockwise-Gauss-Seidel as smoother.

This Theorem extends the classical convergence results (cf. Theorem 4.1 Brandt [1986] and Theorem 1 in Stüben [2001]) to the blockwise case proving an upper bound on convergence of the same type. The same result with a comparable σ can be obtained for a suitable weighted block-Jacobi smoother (cf. [Torgovitski 2011]). This bound shows that it is possible to control the convergence rate by a clever choice of the coarse level, i.e. the set C and (as long as the blocks $A_{i,i}$ are not "too big") easy computable weights. Note that only the set C and the weights W_{ij} have to be specified for the two-level method. The proof of the above theorem is elaborated in detail in Torgovitski [2011] and reduces to showing that certain matrices with a rather complicated structure are positive semi-definite. This can be done using the following characterization which was stated in Clees [2008] for the special case $p = 2$.

Lemma. Suppose C is symmetric. If $C = [C_{i,j}]$ is blockwise diagonally dominant in a p-norm ($1 \leq p < \infty$) and $C_{i,i}$ are positive-definite for all i then C is positive-semi-definite.

For the special case of a trivial partitioning this is a well-known result which is a direct consequence of the Gershgorin Theorem. The general case $1 \leq p < \infty$ was proven in Torgovitski [2011] by use of blockwise p-norms. Here is a short alternative proof.

Proof. From Theorem 2 in Feingold and Varga [1962] we conclude that all eigenvalues of blockdiag$(A)^{-1}A$ lie inside

$$\left\{ z \in \mathbb{C}: \ |1-z| \leq \max_{1 \leq i \leq N} \sum_{i \neq j} \left\| C_{i,i}^{-1} C_{i,j} \right\|_p \right\}$$

and since C is blockwise diagonally dominant their real-part is non-negative. The assertion follows now from Theorem 7.6.3 in Horn and Johnson [1990].

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