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Financial Investment and Insurance – Global Trends and the Polish Market
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ON A CONCEPT OF A HOUSEHOLD FINANCIAL PLAN OPTIMIZATION MODEL

Summary: In the paper a model of life-long financial plan for households is presented. This is a cashflow-based, discrete time, two-person household model. Its main concept is expressing risk aversion of a household as the maximum range of survival scenarios for which the plan should guarantee stable financing of household needs. This approach allows at the same time for reducing the number of survival scenarios for which the plan is optimized. The aim of this research is to present assumptions and construction of the model and show its performance under some chosen scenarios. The choice of scenarios is made in the way that the different riskiness of different plans is explicitly shown without a need of performing a thorough sensitivity analysis nor applying any kind of risk measure.

Keywords: Personal finance, household, retirement, financial planning, intertemporal choice, life-length risk aversion.

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1. Introduction

In this paper an approach to life-long financial planning of a household is presented. There is also proposed an original way of adjusting the optimization procedure to life-length risk aversion. The underlying concept is internal transfer of financial resources between household members (and resulting internal transfer of risk), which makes the approach different to a vast extent from analogous single-individual models.

Some basic notions and the motivations of this model were announced by Feldman, Pietrzyk and Rokita [2014a] and Rokita, Pietrzyk, Feldman [2014]. This article is in part based on the same set of assumptions as a sketch of model that was introduced there. It is a cashflow-based, two-person household model. The main type of outcome, distinguishing different financial plans from one another, is the term structure of cumulated net cashflow (referred further to as *cumulated surplus*). At the current stage of the research the model is focused on retirement goal. Other financial goals may, however, be worked into the plan even in this version, since their completion may be treated as a part of consumption.

The aim of the optimization is to fill the retirement gap of a household under a set of constraints, budget constraint being the most important one. The decision variables are: a) initial proportion of consumption and investments, b) initial proportion of investment contributions assigned to particular persons. The financial plan should guarantee fulfillment of the retirement goal and the constraints under all considered scenarios. The scenarios are driven by two stochastic factors, namely: dates of death of the two persons. The choice of scenarios to be taken into account is adjusted to life-length risk aversion of the household.

The proposed model is an attempt to generalize personal financial planning to a household case. There exists a multitude of other models that, in some other respects, are more general and more advanced (the scope of possible financial goals, sources of financing, types or risk, underlying survival models, etc.). There are only few models exploiting the fact that finance of, for example, two persons running together their common household differs significantly from the case of two separate individuals (age, sex, education and labor income unchanged). Kotlikoff and Spivak [1981] analyzed annuity choice decisions by couples. They indicated the role of internal longevity risk sharing between members of a couple. This research was, however, focused on the decision what to do with private retirement capital once it has already been accumulated, rather than life-long financial planning. A life-long financial plan model of a married couple was, in turn, proposed by Hurd [1999]. This was indeed a consumption optimization model with a bivariate underlying survival process, but it was not a model answering such practical questions as what the household should actually do (in terms of investment decisions, for example) to obtain the optimal consumption path or whether the retirement gap is filled under the optimal solution. Another significant contribution in the area of personal finance of couples was proposed by Brown and Poterba [2000]. Their research was concentrated on the possibility of constructing life annuity products dedicated to married couples. They also made a review of existing products of this type. This was, however, not a model of a whole-life financial plan for couples.

Generally speaking, the main difference between a single individual case and a multi-person household is in the possibility of sharing cashflows and transferring cumulated financial wealth between household members, which, as a result, gives an internal risk transfer effect. This property is not always correctly used in practice of personal finance. On the contrary, it is often recommended that each individual plans her or his retirement so that the retirement income covers the whole amount of household fixed (common) costs and the full amount of individual, variable, costs of a given person. This is an expensive solution, entailing high contributions to private pension plans in the accumulation phase of the life cycle. From the perspective of the household approach this is the safest but not the only possible approach to preserve pre-retirement standard of life in retirement. It is not even a recommended one, because of unnecessary overlapping coverage of some part of household costs.

With the reservation discussed above, the proposition presented in this article is built, to a vast extent, on the existing contributions in the area of personal finance.

Personal finance, and, particularly, financial planning for individuals, has developed considerably since the half of the 20th century, and its origins reach back to Ramsey [1928], Fisher [1930] and Samuelson [1937]. Drawing on their intertemporal choice concepts, life cycle consumption models were developed by Modigliani and Brumberg [1954], Ando and Modigliani [1957], and Yaari [1965]. Together with dynamic asset allocation, investigated by Merton [1969, 1971] and Richard [1975], they formed the basis for further development of modern financial planning models for personal finance. Amongst their concepts that stood the test of time is expected discounted utility, introduced by Modigliani and Brumberg [1954].

Yaari's [1965] model assumed maximization of discounted expected utility of consumption (expressed in the form of consumption rate). Utility of consumption in subsequent periods was weighted with conditional probability of survival. The only argument of the utility functions was consumption. The utility function of a decision maker did not change over time. Assumption of time separable preferences was held. Bodie, Merton and Samuelson [1992] proposed to optimize both consumption and investment decisions. Another seminal proposition was taking consumption of leisure time into account. They also started the discussion on the role of human capital and its risk in making decisions regarding consumption and investment. The classical models by Yaari [1965] and Merton [1969, 1971] were further developed and generalized in different directions. Amongst the augmentations there are, for instance, such as: allowing for multiple risky assets [Bodie et al. 2004], optimization of the date of retirement [Sundaresan, Zapatero 1997], taking into account the maximum psychological planning horizon [Carbone, Infante 2012], modeling some kinds of behavioral biases in decision making by means of hyperbolic discounting [Ainslie 1975, 1991; Kirby, Herrnstein 1995]. Geyer, Hanke and Weissensteiner [2009] proposed a model with stochastic labor income and investment opportunities. Scholz and Seshadri [2012] started to treat health as one of assets and "building" one's health as investment. There are also propositions of generalizing the underlying survival model by introducing stochastic hazard rate function [Huang, Milevsky, Salisbury 2012]. There is also a discussion on optimal retirement capital distribution, represented by: Huang and Milevsky [2011], Milevsky and Huang [2011], Gong and Webb [2008], Dus, Maurer and Mitchell [2004].

Here, building on the basic concepts of classical financial planning models for individuals, a two-person household retirement planning model is proposed. The hypothesis is that the model allows for reducing life-length risk to the extent that is imposed by a pre-defined risk aversion of the household. The risk reduction effect is here observed as stable positive cumulated surplus throughout the whole life-cycle of the household for such survival scenarios that fit given risk aversion level (the higher risk aversion declared, the more adverse scenarios the financial plan should be immune against). The method used to show this property is based on stylized case studies illustrating how the financial plan performs under the scenarios fitting the declared risk aversion level and falling beyond it.

2. Assumptions and definitions

As it has already been announced, the model is constructed for a two-person household (Def. 1). Its definition builds on a more general one by Zalega [2007].

Definition 1 (*Household*)

An autonomous economic entity distinguished according to the criterion of individual property, whose members are two natural persons, further on referred to as the *main members of the household*, fulfilling the following conditions:

1) jointly set and solidary realize goals as regards majority of the most important needs of all members of this entity,

2) are supposed to intend to remain members of this entity, to the extent possible, throughout the whole period of its existence,

3) together with other natural persons, who, in some phases of the life cycle of this entity, fulfill the condition 1 or are (at least economically) dependent on any of the two main members of the household (compare also [Feldman, Pietrzyk, Rokita 2014a]).

At the current stage of the research it is assumed that:

1. The main household members show the intention to remain members of the household until their death.
2. Household members agree on a common attitude towards life-length risk.
3. The household has a value function (goal function) composed of:

- utility of consumption,
- utility of bequest.

4. In pre-retirement period income of the household increases at a constant rate in real terms.
5. Rate of return on private pension plan is constant in real terms.
6. Income from private retirement is constant in real terms (inflation indexed).
7. Replacement rate of public pension system is fixed (it may be, however, different for women and men).
8. Financial goals other than retirement are not yet considered explicitly (but the assumed level of consumption may include additional expenditures that may be interpreted as costs of other goal completion).

9. Household invests into systematic investment programs with a purpose of amassing capital for future retirement (private pension plans). And this is the only aim of investing in the model.

10. Each person has one's own investment program. The programs are separated and they do not depend on each other, however, if a person dies before retirement age, the capital is transferred to the other one.

11. The capital accumulated as a result of investments in the pre-retirement phase is fully spent on purchasing life annuities.

12. The only not-annuitized capital that the household may possess in the retirement phase of the life cycle comes from cumulated surplus.

13. Consumption growths at a constant rate in real terms – this refers to the assumed consumption which may turn out to be feasible or not (the only situation when the consumption may be different from its assumed level is when household loses financial liquidity).

14. Household members do not have to (but may) contribute such amounts into their private retirement programs that their individual retirement incomes fully cover the sum of their individual costs and common fixed costs of the household – put differently, *partial retirement* schemes, as defined by Feldman, Pietrzyk and Rokita [2014a], are allowed for.

15. There are two different rates of return considered: rate of return on investment and interest rate of a cumulated surplus account; the surplus over consumption and investments is kept in a low-interest liquid account (here, the second rate is assumed to be zero in real terms).

16. The rate of return for investment is adjusted to general attitude towards market risk of the household, which is not discussed in this paper; it is assumed that the asset allocation decisions have already been made, no information supporting investment policy formulation is utilized in this model; the rate or return on investment is set to be fixed and equal to a long-run expected return on the chosen class of assets.

17. There are two sources of financing consumption:

- income (in pre-retirement phase – from job, in retirement phase – from public and private retirement, and, if the other person has not retired yet – from labor income of that other person),
- cumulated surplus (the only part of household financial wealth in this model that is liquid enough to be treated as a source of consumption financing, in the periods of insufficient income).

There are four particular dates distinguished in the household life cycle, called here *critical dates*. There are six such dates, four deterministic and two random.

To the deterministic critical dates belong:

- retirement dates ($R1$ and $R2$),
- expected life lengths ($E(D1)$, $E(D2)$).

Stochastic critical dates are, in turn, dates of death ($D1$ and $D2$).

The position of realized values of random variables $D1$ and $D2$ between the four constants ($R1$, $R2$, $E(D1)$ and $E(D2)$) is crucial to household financial situation.

3. The risk

In the model at its current stage only life-length risk is taken into consideration. Unlike financial plans for single individuals, where only longevity risk needs to be analyzed, both longevity and premature death of household members matter. For a single individual, premature death may affect only a bequest. In a household, premature death of one person, particularly if it is the one who earns better, may threat financial liquidity of the other one.

Since only life-length risk is taken into account, the risk factors to be considered are dates of death of the two persons – $D1$ and $D2$. Risk is not measured here, but it is managed by constructing a plan which for a set of ($D1$, $D2$) pairs, referred further to as *survival scenarios*, guarantees financial liquidity throughout the whole planning period. The planning period is usually longer than the maximum of unconditional expected times of death of the two persons, determined at the moment of the plan start. How much longer it is, it depends, amongst others, on risk aversion of the household.

The risk consists in the fact that deviations of $D1$ and $D2$ from their expected values, $E(D1)$ and $E(D2)$, may cause serious deterioration of household financial situation, as compared to the expected scenario. It can not be, however, stated that the higher deviation the worse. There is no such straightforward rule, because of interconnections between persons, as well as a step (switching) nature of the relationship between time of death and total amount of household financial wealth.

For example, if one person dies before her or his retirement, (vast part of) the cumulated private retirement capital is inherited by the other one. If it is shortly before retirement (say, one day), the whole capital has already been cumulated and it does not vanish. If the same person dies very shortly after the retirement date, but having already purchased a life annuity, the capital is lost to the household. Whether the person dies shortly before or shortly after the retirement date, it is before expected date of death, and the interval between the realized and expected time of death is in both cases almost the same. But financial consequences are dramatically different.

4. Household cashflows and cumulated surplus

The main characteristic feature describing household financial situation is available level of sustainable consumption throughout the whole planning period. In the model, consumption is composed of three parts:

– common consumption (not attributed to any particular person),
– consumption of Person 1,
– consumption of Person 2.

At start, household cashflow is divided so that the part of period 0 not consumed income is fully allocated into private pension programs. In subsequent periods the initial proportion of consumption and investment remains unchanged, and, if income growths faster than consumption, some surplus remains. If no unexpected events occur, at least until the first retirement, there should be a surplus generated. This surplus does not need to be spent on consumption and may be cumulated. After retirement the relationship between income and consumption may change, dependent on the amount of retirement income. The last depends, in turn, on the choice of retirement scheme (2 × *Full*, *Full-Partial*, 2 × *Partial*) as defined by Feldman, Pietrzyk and Rokita [2014a], that is – amounts of private pension plan contributions of the pre-retirement period, as well as their division between the persons. In retirement, (some part of) consumption may be financed from cumulated surplus. Also unexpected events in the area of life time, like premature death or longevity, may lead to the situation that cumulated surplus needs to be used to finance consumption. As it has already been explained in Section 3, no other sources of risk are taken into consideration in this version of the model (e.g., medical condition, loss of the job, damage to property).

The relations between consumption, investment and surplus of the household are provided below (compare also [Feldman, Pietrzyk, Rokita 2014b]).

- Assumed consumption:

$$C_{a_t} \equiv VC_t^{(1)} + VC_t^{(2)} + FC, \tag{1}$$

where: C_{a_t} – assumed consumption; $VC_t^{(i)}$ – variable costs assigned to i-th person; FC – common costs of the household (not assigned to any person).

- Savings:

$$S_t = Ic_t - C_{a_t} = Ic_t^{(1)} + Ic_t^{(2)} + Ic_t^{(c)} - VC_t^{(1)} - VC_t^{(2)} - FC, \tag{2}$$

where: Ic_t – joint income at the moment t; $Ic_t^{(1)}$ – income of the first person; $Ic_t^{(2)}$ – income of the second person; $Ic_t^{(c)}$ – income to the household that is not assigned to any person (e.g.: an income from renting out a real estate being a part of conjugal community).

- Surplus (the part of savings that remains uninvested):

$$\begin{aligned}\mathrm{NCF}_t &= S_t - Iv_t = Ic_t - C_{a_t} - Iv_t = \\ &= Ic_t^{(1)} + Ic_t^{(2)} + Ic_t^{(c)} - VC_t^{(1)} - VC_t^{(2)} - FC - Iv_t^{(1)} - Iv_t^{(2)} - Iv_t^{(c)}\end{aligned} \quad (3)$$

$$\begin{aligned}&(Ic_t = Ic_t^{(1)} + Ic_t^{(2)} + Ic_t^{(c)}; \\ &Iv_t = Iv_t^{(1)} + Iv_t^{(2)} + Iv_t^{(c)}; \\ &\text{if } t > Ri, \text{ then } Ic_t^{(i)} = Icb_t^{(i)} + Icc_t^{(i)}),\end{aligned}$$

where: Iv_t – investments of the household in the period t; $Iv_t^{(1)}$ – investments of the first person in the period t; $Iv_t^{(2)}$ – investments of the second person in the period t; $Iv_t^{(c)}$ – investments of the household that are not assigned to any person in the period t; moreover: $Icb_t^{(i)}$ – i-th person retirement income from public pension system (all pillars included); $Icc_t^{(i)}$ – i-th person retirement income from private pension plan(s); Ri – retirement date of person i.

- Maximum surplus that does not affect surplus for given level of investment:

$$C_{ft}^{*} = Ic_t - Iv_t \quad (4)$$

(no cumulated surplus generated).

- Cumulated surplus – cumulated net cashflow:

$$\mathrm{CNCF}_t = \sum_{\tau=0}^{t-1} NCF_\tau. \quad (5)$$

- Maximum consumption that can be realized indeed at a given moment t, assuming that until the moment only the assumed consumption was realized:

$$C_{ft}^{*} = Ic_t + \mathrm{CNCF}_t - Iv_t. \quad (6)$$

- Consumption realized by the household in this model (i.e., assumed consumption, but up to the available amount):

$$C_t = \min\{C_{a_t}, Ic_t + \mathrm{CNCF}_t - Iv_t\} = \min\{C_{a_t}, C_{ft}^{*}\} \quad (7)$$

or equivalently (eq. 8):

$$C_t = C_{a_t} + \min\{0, \mathrm{CNCF}_t\}. \quad (8)$$

The formulas (6) and (7) do not reflect detailed decomposition of costs, incomes and investments. It is, however, important to distinguish individual contribution of each person to the total net cashflow of the household. This allows to model the impact of stochastic elements of the model (namely, $D1$ and $D2$).

The level of consumption set at the start of the plan determines whether the household will be able to generate surplus. The higher consumption, the lower surplus of each period, and thus lower cumulated surplus. Also the higher investments, other things unchanged, the lower surplus. But contribution to systematic investment programs on the date of plan start (Iv_0) is here strictly determined by the assumed consumption (C_{a_0}). And there is no surplus at the starting point of the plan. The whole income is then divided into consumption and investment. In the next periods surplus accumulation depends on the difference between income growth on the side of cash inflows and consumption and investment growths on the side of cash outflows.

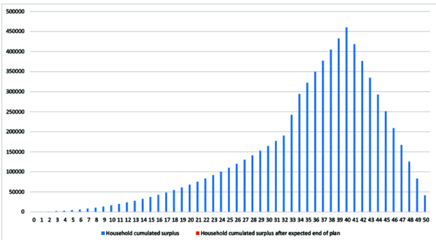
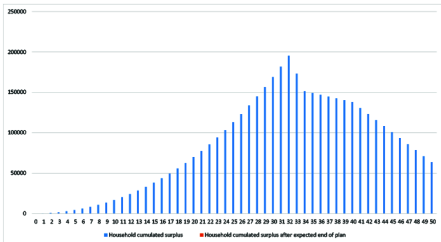
Plan A	Plan B
	
Optimized for: (0,0); realized: (0,0)	Optimized for: (5,5); realized: (0,0)

Figure 1. Cumulated surplus trajectories of two financial plans for the expected survival scenario

Source: own study.

In the model where consumption and investments are determined (and only the initial level of assumed desired consumption is set by the decision maker), the feature that best describes financial situation of the household is cumulated surplus. For instance, under the expected scenario of $D1 = E(D1)$ and $D2 = E(D2)$, consumption trajectories throughout the whole life cycle will be identical for all financial plans in which the cumulated surplus trajectory is non-negative at each time. But financial plans with expected non-negative cumulated surplus trajectory may differ a lot. The difference is in risk. For example, in Figure 1 there are presented trajectories of cumulated surplus for two financial plans that guarantee financial liquidity for the expected scenario. Unless another scenario than the expected one is realized, it may be not so obvious that the Plan B is less vulnerable to premature death or longevity than the Plan A. However, if household members die earlier or later than expected, the difference in the plans becomes evident, which is presented in Figure 2.

The plans are different in a very significant respect – premature death and longevity risk. The difference in plan sensitivity is reflected in different dynamics

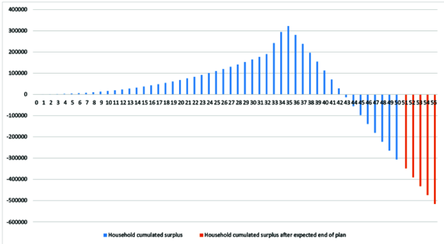
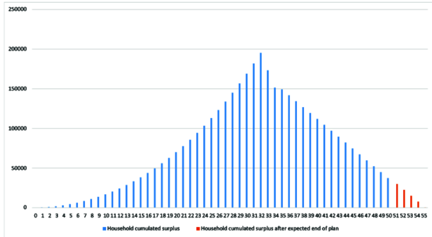
Plan A	Plan B
	
Optimized for: (0,0); realized: (-5,5)	Optimized for (5,5); realized: (-5,5)

Figure 2. Cumulated surplus trajectories of two financial plans for an unexpected scenario

Source: own study.

of cumulated net cash flow under a bunch of survival scenarios. For an experienced analyst, it might be even readable from the shape of cumulated surplus trajectory for the expected scenario.

5. Value function and the role of risk aversion

Because of path dependence, step nature of the relationship between financial wealth and risk factors, where abrupt switches in financial regime may be encountered, and the cumulative character of the main quantity to be analyzed, namely – cumulated surplus, the classical model of expected discounted utility is not fully applicable here. Yaari's model [Yaari 1965] and relative ones are based on conditional survival probability. In a discrete version this might be formulated like in the equation 9.

$$EDU = \sum_{t=0}^{\infty} \left[\frac{1}{(1+\rho)^t} \left({}_1p_{t-1}\right) u(c_t)\right], \tag{9}$$

where: ${}_1p_{t-1}$ – conditional probability of surviving at least one more period under the condition of surviving until the end of the previous period; $u(c_t)$ – utility of consumption and ρ is a rate used for discounting.

The probability may be based on a survival model with deterministic force of mortality, like Gompertz [1825] or Makeham [1860], or stochastic, like the one proposed by Huang, Milevsky and Salisbury [2012]. But in any case models of this type are best suited to a single individual who may be either alive or dead. There are, however, four possible survival states of a two-person household. Both persons may be alive at a moment, Person 1 may be alive and Person 2 dead, Person 1 may be dead and Person 2 alive and both persons may be dead. This fact, combined with the aforementioned path dependence and switches, makes it problematic to use conditional probability of household survival state at a given moment under the condition of the state at the preceding moment. Instead, it is much more convenient to analyze a whole trajectory of two-person survival process, from the starting point until the end of the planning period. There is, yet, a technical issue in this approach, namely the number of possible survival trajectories. For a discrete model with n sub-periods (e.g., years) spanned by the period of the plan, the number of survival scenarios is n^2, and allowing for scenarios in which any person lives longer than the horizon of the plan – even higher. One might reduce the number of scenarios by taking longer sub-periods. But this would also mean longer plan revision periods, which is definitely not recommended. Another way of simplifying the problem is adjusting the number of scenarios to be taken into consideration to risk aversion of the household.

The concept of selecting some particular scenarios in a way that the selection best suits risk aversion of the household finds its application in the form of what is called here a *range of concern*. The range of concern is defined in terms of a number of years before and after expected time of death of each person. For one person it may be expressed by the following formula:

$$G_i^* \in [E(Di) - \gamma^*; E(Di) + \delta^*], \tag{10}$$

where: G_i^* – range of concern for Person i; γ^* – premature-death risk aversion parameter (number of years that household takes into consideration); δ^* – longevity risk aversion parameter (number of years that household takes into consideration); $E(D_i)$ – unconditional expected time of death of Person i (i.e., $E(Di) \equiv E(Di|Di > t_0)$).

Since the model is constructed for two persons, the range of concern is a square of the form:

$$G_H^* = [E(D1) - \gamma^*; E(D1) + \delta^*] \times [E(D2) - \gamma^*; E(D2) + \delta^*]. \tag{11}$$

The motivation of this approach is that probabilities of some less typical scenarios are small, and some scenarios are practically unlikely because of reasons that lie beyond probabilistic model. For example, if two persons of the same age run their household, and one of them dies at a very young age, the second is unlikely to stay until an old age in the same household. Much more typical situation is that a young person who remains enters after some time into another relationship with somebody and launches a new household. The new household prepares a new financial plan, having nothing in common with the old one. Another reason for which reduction of the problem dimension is rational consists in avoiding overestimation of risk and, consistently, avoiding plans with absurdly low consumption. If a plan was optimized to guarantee financial liquidity at any point of time for even extremely adverse scenarios, the household would have to reduce its consumption aspiration to very low amounts. Each of these "extremely adverse" scenarios may be characterized by a low probability, but there are many of them, thus probability that any of them realizes may be sufficiently large to have significant impact on the optimization result. The third reason of introducing the range of concern is that some scenarios that are pretty distant from expected one may be not harmful in financial sense at all and there simply is no use "protecting" household finance against them. An example may be the case discussed in Section 2, in which the person who invests more dies a day before her or his retirement. However unpleasant might it be from the human point of view, no deterioration of financial situation is triggered by such event; just the opposite: rather an improvement.

On the basis of the range of concern the value function is constructed. It is a function of two kinds of utility: utility of consumption and utility of bequest. The value function is an aggregate constructed using the concept of expected discounted utility. Characteristic feature of this model is that two financial plans guaranteeing the same level of consumption for a given range of concern may end up with different level of cumulated surplus. This difference between plans should be taken into consideration. At the present stage of development of the model, where no other investments than pension plans are taken into consideration, the only financial capital that remains after the last person dies is unused cumulated surplus (unless the last person dies before retirement – then there may be both cumulated surplus and cumulated investment left). Taking bequest into account in the analysis of household utility allows for distinguishing between plans that end with different unused cumulated surplus.

$$V(c_0, v) = \sum_{D_2^*=E(D2)-\gamma^*}^{E(D2)+\delta^*} \sum_{D_1^*=E(D1)-\gamma^*}^{E(D1)+\delta^*} p_{D_1,D_2} \left[\begin{array}{l} \alpha \left(\displaystyle\sum_{t=0}^{\max\{D_1^*,D_2^*\}} \frac{1}{(1+r_c)^t} u\big(C(t; D_1^*, D_2^*)\big)\big(\gamma(t)+\delta(t)\big) \right) + \\ \beta \dfrac{1}{(1+r_B)^{\max\{D_1^*;D_1^*\}}} u\big(B(\max\{D_1^*, D_2^*\}; D_1^*, D_2^*)\big) \end{array} \right] \to \max, \quad (12)$$

where: $u(.)$ – utility function (the same in all segments of the formula); c_0 – consumption rate at the moment 0; v – proportion of Person 1 investment in joint one-period contribution of the household ($v \equiv v_1, v_1 = 1 - v_2$); $\gamma(t)$ – premature death risk aversion measure (depends on γ^*); p_{D_1,D_2} – (unconditional) probability of such scenario that ($D1 = D_1^*, D2 = D_2^*$); α – consumption preference parameter; β – bequest preference parameter; $\max\{D_1^*, D_2^*\}$ – time of household end under a scenario of ($D1 = D_1^*, D2 = D_2^*$); $C(t; D_1^*, D_2^*)$ – consumption at the moment t in the scenario where $D1 = D_1^*$, $D2 = D_2^*$; $B(t; D_1^*, D_2^*)$ – cumulated surplus at the moment t in the scenario where $D1 = D_1^*$ and $D2 = D_2^*$; for $t = \max\{D_1^*, D_2^*\}$ this is just amount of available bequest; r_C – discount rate of consumption; r_B – discount rate of bequest.

Multipliers $\gamma(t)$ and $\delta(t)$ are functions of time and are dependent on parameters γ^* and δ^*. It is proposed that they are defined in such a way that $\gamma(t)$ decreases until $t = \max\{E(D1), E(D2)\}$ and reaches the value 1 at this point. The function $\delta(t)$, in turn starts at $t = \max\{E(D1), E(D2)\}$ from the value 1 and then increases. Both are convex. It is also recommended that the slope of $\delta(t)$ is higher than that of $\gamma(t)$. This is because the financial consequences of scenarios when a household lasts longer than expected are usually more severe than consequences of premature death. Moreover, it is much more difficult to recover from financial shortfall in an old age.

Feldman, Pietrzyk and Rokita [2014b] give some hints on how to optimize the financial plan, taking into account also the properties discussed in Section 3. There is a minor technical issue that needs to be solved. The value function shows angularities and sudden jumps (because it is a discrete model, it is hard to talk about this function as of a non-differentiable, but in a continuous generalization of the model it would be probably not differentiable with sharp peaks, steps and jumps).

6. Plan optimization by an example

Let there be given a household with a 30-year-old man and 26-year-old woman. Retirement age of both is 67 years. Thus, time remaining to retirement is 37 for the man and 41 for the woman. Further life time at the moment of plan preparation is 44 and 56 years, respectively. Labor income of the man is 48 000 and 32 000 for the woman. Common part of household consumption is 45 000, individual consumption is subject to optimization. It is assumed that income growth rate in long run is 1% for the man and 0.5% for the woman in real terms (the difference may be, for instance, caused by career stoppage during maternity). Consumption growth rate is set to 0.2% in real terms for the household common consumption and the growth rates of individual consumption are the same as corresponding factors for income (i.e., 1% and 0.5% in real terms, respectively). It may be assumed that other parameters used in the model are independent of the household and are obtained from external sources of data (macroeconomic parameters, expected rates of return of capital assets, etc.).

The decision variables of the optimization procedure are:

c_0 – consumption rate at the moment 0,

v – proportion of Person 1 investment in joint one-period contribution of the household.

For the sake of simplicity it is assumed that retirement capital accumulated in private pension plan of a given person is used to buy life annuity of this particular person. This is certainly only one of possible solutions and in a more general model the decision which annuity should be financed with which capital might be also subject to optimization.

The direct outcome is maximum value of the goal function and the pair of corresponding optimal $\left(c_0^{(\max)}, v^{(\max)}\right)$.

The indirect output is a financial plan, whose main characteristics are:

- initial consumption $c_0^{(\max)}$,
- expected trajectory of subsequent amounts of consumption until plan horizon,
- expected trajectory of cumulated surplus,
- a bunch of possible trajectories of cumulated surplus, as well as consumption, for all survival scenarios belonging to the range of concern (compare eq. (11)).

If the financial plan is optimized for a given range of concern, financial liquidity of the household is usually preserved for all scenarios belonging to the range of concern. This is why all trajectories of consumption are positive and identical to the expected one. Trajectories of cumulated surplus differ, however, and the differences in their shapes may be substantial, with the restriction that for the optimized plan neither should show any unrecoverable shortfall.

There are analyzed three financial plans. Performance of each is observed under scenarios fitting corresponding range of concern and falling beyond it. The scenarios that belong to the range of concern are extreme in this sense that they embrace the bounds of the range. If the financial plan guarantees stable financing of household needs for the boundary scenarios, this is the more so for the scenarios from within the range of concern. Thus, analyzing more scenarios belonging to the range of concern is unnecessary.

Let the household optimize its financial plan only for the expected scenario. For the interpretation of risk aversion adopted in this research it means that the household shows indifference towards risk and its range of concern boils down to a point $\big(E(D1), E(D2)\big)$. In Figure 3a) there is presented cumulated surplus trajectory.

The solution proposed in Figure 3a) is, however, very sensitive to any deviations from the expected scenarios. In Figure 3b) the result of realization of another scenario for the same plan is shown. In this scenario Person 1 (the man) dies 5 years before and Person 2 (the woman) 5 years after their expected times of death.

As it is shown in Figure 3b), the household loses its financial liquidity (the shortfall is unrecoverable).

What would happen if the household had had optimized its plan for a broader range of concern (which would also mean higher risk aversion)? Figure 4 shows cumulated surplus behavior for a plan optimized with the range of concern of: $\gamma = 5$ and $\delta = 5$. In Figure 4a) cumulated surplus trajectory of this plan is presented for the scenario when Person 1 (the man) dies 5 years before and Person 2 (the woman) 5 years after their expected times of death, respectively. Financial liquidity of the household is still preserved. If, however, the realized

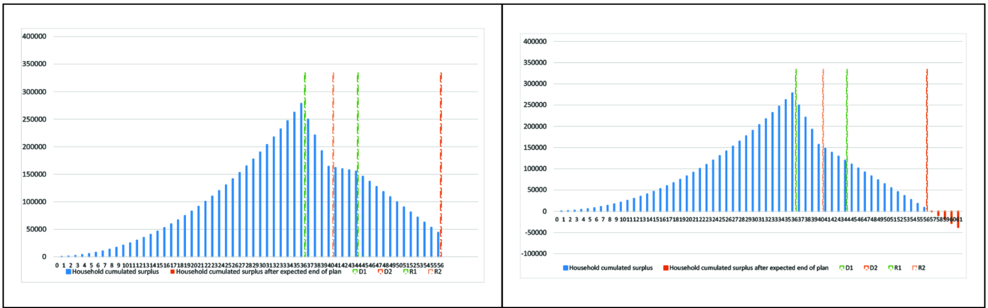


Fig. 3 a) Optimized for: (0,0); realized: (0,0) | Fig. 3 b) Optimized for: (0,0); realized: (-5,5)

Figure 3. Cumulated surplus trajectories of financial plan without risk aversion ($\gamma = 0, \delta = 0$) under: a) expected survival times – left vs. b) scenario with longevity and premature-death risk realized (–5,5) – right[1]

Source: own study.

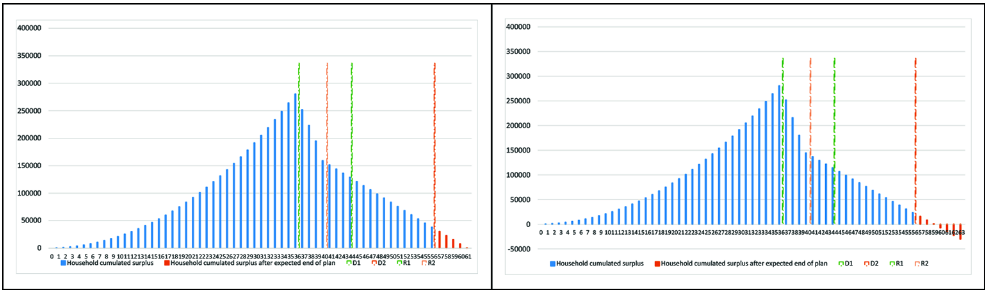


Fig. 4 a) Optimized for: (5,5); realized: (–5,5) | Fig. 4 b) Optimized for: (5,5); realized: (–7,7)

Figure 4. Cumulated surplus trajectories of financial plan with risk aversion ($\gamma = 5, \delta = 5$) under scenarios when longevity and premature-death risk realized at a) (–5,5) – left and b) (–7,7) – right

Source: own study.

scenario fell out of the range of concern, the household might incur an unrecoverable shortfall after all. This situation is shown in Figure 4b). It this scenario the man dies 7 years before his expected time of death and the woman dies 7 years after her expected time of death. The woman faces then an unrecoverable shortfall in the old age.

[1] Notation: "Optimized for: (v, x); realized: (z, y)" means that the plan had been optimized for a household with risk aversion of: $\gamma = v$ and $\delta = x$, whereas the realized scenario was: $D1 = E(D1) + z, D2 = E(D2) + y$.

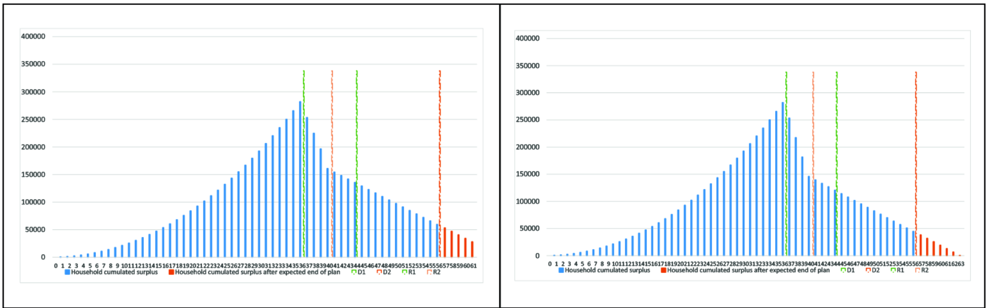


Fig. 5 a) Optimized for: (7,7); realized: (-5,5)	Fig. 5 b) Optimized for: (7,7); realized: (-7,7)

Figure 5. Cumulated surplus trajectories of financial plan with risk aversion ($\gamma = 7, \delta = 7$) under scenarios when longevity and premature-death risk realized at a) (–5,5) – left and b) (–7,7) – right

Source: own study.

The financial plan would have been immune even against the (–7, 7) scenario, that is – the one on Figure 5b), if the household had optimized it for the range of concern of: $\gamma = 7$ and $\delta = 7$. The effect of this more conservative approach is shown in Figure 5.The example presented above shows range of concern at work. The connection between attitude towards risk and cautiousness (or recklessness) in the choice of scenarios to be taken into account for the optimization procedure was the main idea behind defining life-length risk aversion in terms of the range of concern.

7. Conclusions

There are five characteristic features of the proposition discussed here.

The first is a household approach, allowing for internal transfer of financial resources between persons. This, amongst others, makes it possible for household members to construct a financial plan that sustains their life standard in retirement without bearing costs of a *full retirement* – in the sense of Feldman, Pietrzyk and Rokita [2014a] – for all. The scope of decisions in financial planning broadens then, as compared to the models for single individuals. For instance, in this particular model of retirement planning, the financial situation of a household depends not only on consumption-investment proportion, but also on division of investments between household members. In a more general version there might be also added another decision option, namely – whether a life annuity for a given person should be bought with the capital accumulated in a systematic investment program dedicated for this particular person until her or his retirement, or with the capital accumulated by that other household member. Or should the retirement capital be spent in yet another way? In the research discussed here this was automatically set by the division of investments between household members, since it was assumed that each household member bought a life annuity for the capital accumulated in her or his own private pension plan. There are, however, no conceptual obstacles against adding the nest degree of freedom to the model. The only problem would then be the increase of the number of dimensions in the optimization procedure. In addition to consumption-investment proportion and division of investment between household members there would appear another decision variable, responsible for the way of spending retirement capital.

The second feature of the model is the role of cumulated surplus. Two financial plans guaranteeing the same level of consumption may end at a given point of time with different levels of unused cumulated surplus. This cumulated surplus left may be treated as bequest available to descendants. Moreover, financial plans with the same level of consumption for a set of most probable scenarios may differ a lot as far as their sensitivity to scenarios from outside this set is concerned. The very shape of cumulated surplus trajectory for the expected scenario may, to some extent, inform about vulnerability of the plan to adverse scenarios.

The third characteristic is that the household model is based on a two-person household definition. This, however, does not limit significantly the generality of the model, because existence of any other household members may be reflected in the model just as part of financial situation of the main two household members. A single individual may be modeled as a two-person household in which the second person is just nonexistent (it is as if the second person died at time 0).

The fourth feature is that it is a discrete model. All calculations and optimization were performed numerically. One can, thus, list and analyze all possible scenarios. The survival process is, however, bidimensional. This is why the number of possible scenarios increases fast with the number of financial plan revision sub-periods. Taking into account the whole bunch of scenarios would be technically very problematic. For some reasons explained in the text it would be also in many cases irrational.

The problem of a large number of survival scenarios was overcome by an original approach to risk aversion. The concept of the range of concern allows for combining an easy-to-use and intuitive way of understanding premature-death risk aversion and longevity risk aversion with reduction of the set of survival scenarios.

The model may be used as a basis for developing a more generalized household financial planning model. The directions of further augmenting the model might be explicit treatment of other financial goals than retirement (currently "hidden" as part of consumption), adding analysis of other types of risk than risk of premature death and longevity (here, particularly market risk connected with investments is worth taking account of), considering different sources of financing (for other goals that retirement also post-financing with credits should be allowed for), decisions concerning managing risk of insurance events by means of insurance. Other directions of research, of a more technical nature, may be using other two-person survival models than just two independent survival processes. Also in the area of description of household preferences there are a number of possible improvements to be considered, like for instance the way of discounting. The model itself should be also tested for stability of outcome to changes of variables and parameter, as well as its stability to changes of basic parameters of optimization procedure, like the starting point of optimization. This is strictly connected with another important area of research, that is – analytical properties of the value function.

The main contribution of the model in its current version is that it may be used as a tool facilitating financial planning for a household, not only a single individual, that it takes into account both longevity and premature-death risk, also the fact that it allows for expressing household attitude towards life-length risk in a simple and intuitive way and, finally, that this risk aversion interpretations is in compliance with scenario bunch reduction, making the model practically tractable.

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References

Ainslie G., 1975, *Specious Reward: A Behavioral Theory of Impulsiveness and Impulse Control*, Psychological Bulletin, vol. 82, no. 4, p. 463–496.

Ainslie G., 1991, *Derivation of Rational Economic Behavior from Hyperbolic Discount Curves*, The American Economic Review, vol. 81, no. 2, p. 334–340.

Ando A., Modigliani F., 1957, *Tests of the Life Cycle Hypothesis of Saving: Comments and Suggestions*, Oxford Institute of Statistics Bulletin, vol. 19, p. 99–124.

Bodie Z., Detemple J., Otruba S., Walter S., 2004, *Optimal Consumption-Portfolio Choices and Retirement Planning*, Journal of Economic Dynamics and Control, vol. 28, p. 1115–1148.

Bodie Z., Merton R.C., Samuelson W.F., 1992, *Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model*, Journal of Economic Dynamics and Control, vol. 16, no. 3-4,p. 427–249.

Brown J.R., Poterba J.M., 2000, *Joint Life Annuities and Annuity Demand by Married Coupl*es, The Journal of Risk and Insurance, Vol. 67, no. 4, p. 527–554.

Carbone E., Infante G., 2012, *The Effect of a Short Planning Horizon on Intertemporal Consumption Choices*, LABSI: Research Paper, http://papers.ssrn.com/sol3/papers.cfm?abstract id=2187911 (retrieved: 28.03.2013).

Dus I., Maurer R., Mitchell O.S., 2004, *Betting on Death and Capital Markets in Retirement: A Shortfall Risk Analysis of Life Annuities versus Phased Withdrawal Plans*, Goethe-Universität, Frankfurt am Main, and The Wharton School, University of Pennsylvania: Working Paper http://www.actuaries.org/AFIR/Colloquia/Boston/Dus Maurer Mitchell.pdf (retrieved: 27.03.2013).

Feldman L., Pietrzyk R., Rokita P., 2014a, *A Practical Method of Determining Longevity and Premature-death Risk Aversion in Households and Some Proposals of Its Application*, [in:]

M. Spiliopoulou, L. Schmidt-Thieme, R. Janning (eds.), *Data Analysis, Machine Learning and Knowledge Discover*y, Springer, Berlin–Heidelberg, p. 255–264.

Feldman L., Pietrzyk R., Rokita P., 2014b, *Multiobjective Optimization of Financing Household Goals with Multiple Investment Programs*, Statistics in Transition new series, 15, no. 2, p. 243–268.

Fisher I., 1930, *The Theory of Interest*, Macmillan, New York.

Geyer A., Hanke M., Weissensteiner A., 2009, *Life-Cycle Asset Allocation and Consumption Using Stochastic Linear Programming*, The Journal of Computational Finance, vol. 12, no. 4, p. 29–50.

Gompertz B., 1825, *On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies*, Philosophical Transactions of the Royal Society of London, vol. 115, p. 513–585, 1825.

Gong G., Webb A., 2008, *Mortality Heterogeneity and the Distributional Consequences of Mandatory Annuitization*, The Journal of Risk and Insurance, vol. 75, no. 4, p. 1055–1079.

Hurd M., 1999, *Mortality Risk and Consumption by Couples*, NBER Working Paper no, 7048, http://www.nber.org/papers/w7048.pdf (retrieved: 27.09.2014).

Huang H., Milevsky M.A., 2011, *Longevity Risk Aversion and Tax-Efficient Withdrawals*, SSRN, http://ssrn.com/abstract=1961698 (retrieved: 22.03.2012).

Huang H., Milevsky A., Salisbury T.S., 2012, *Optimal Retirement Consumption with a Stochastic Force of Mortality*, Insurance: Mathematics and Economics, vol. 51, no. 2, p. 282–291.

Kirby K.N., Herrnstein R.J., 1995, *Preference Reversals Due to Myopic Discounting of Delayed Reward*, Psychological Science, vol. 6, no. 2, p. 83–89.

Kotlikoff L.J., Spivak A., 1981, *The Family as an Incomplete Annuities Market*, Journal of Political Economy, vol. 89, no. 2, p. 372–391.

Makeham W.M., 1860, *On the Law of Mortality and the Construction of Annuity Tables*, Journal of the Institute of Actuaries and Assurance Magazine, vol.8, p. 301–310.

Merton R.C., 1969, *Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case*, The Review of Economics and Statistics, vol. 51, no. 3, p. 247–257.

Merton R.C., 1971, *Optimum Consumption and Portfolio Rules in a Continuous-time Model*, Journal of Economic Theory, vol. 3, no. 4, p. 373–413.

Milevsky M.A., Huang H., 2011, *Spending Retirement on Planet Vulcan: The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates*, Financial Analysts Journal, vol. 67, no. 2, p. 45–58.

Modigliani F., Brumberg R.H., 1954, *Utility Analysis and the Consumption Function: An Interpretation of Cross-section Data*, [in:] K.K. Kurihara (ed.), *Post-Keynesian Economics*, Rutgers University Press, New Brunswick, NJ, p. 388–436.

Ramsey F.P., 1928, *A Mathematical Theory of Saving*, The Economic Journal, vol. 38, no. 152, p. 543–559.

Richard S.F., 1975, *Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in a Continuous Time Model*, Journal of Financial Economics, vol. 2, p. 187–203.

Rokita P., Pietrzyk R., Feldman L., 2014, *Cumulated Surplus Approach and a New Proposal of Life-Length Risk Aversion Interpretation in Retirement Planning for a Household with Two Decision Makers*, http://ssrn.com/abstract=2473156 (retrieved: 30.08.2013).

Samuelson P.A., 1937, *A Note on Measurement of Utility*, The Review of Economic Studies, vol. 4, p. 155–161.

Scholz K., Seshadri A., 2012, *Health and Wealth in a Lifecycle Model*, University of Wisconsin-Madison and NBER and University of Wisconsin-Madison: Working Paper, http://www.ssc.wisc.edu/~/Research/Health and Wealth v16.pdf (retrieved: 23.11.2013).

Sundaresan S., Zapatero F., 1997, *Valuation, Optimal Asset Allocation and Retirement Incentives of Pension Plans*, Review of Financial Studies, vol. 10, p. 631–660.

Yaari M.E., 1965, *Uncertain Lifetime, Life Insurance and Theory of the Consumer*, The Review of Economic Studies, vol. 32, no. 2, p. 137–150.

Zalega T., 2007, *Gospodarstwo domowe jako podmiot konsumpcji*, Materials and Studies, Faculty of Management, University of Warsaw.

KONCEPCJA MODELU OPTYMALIZACJI PLANU FINANSOWEGO GOSPODARSTWA DOMOWEGO

Streszczenie: W artykule zaprezentowana została koncepcja modelu długoterminowego planu finansowego gospodarstwa domowego. Jest to model dla dwuosobowego gospodarstwa domowego, oparty na przepływach pieniężnych i skonstruowany w czasie dyskretnym. Zastosowano w nim oryginalne rozwiązanie, polegające na tym, że awersja do ryzyka jest wyrażona jako maksymalny przedział scenariuszy przeżycia, dla których plan powinien gwarantować stabilne finansowanie potrzeb gospodarstwa domowego. Takie podejście stwarza zarazem możliwość ograniczenia liczby scenariuszy, dla których przeprowadzana jest optymalizacja planu. Celem tej pracy jest zaprezentowanie założeń i budowy tego modelu oraz przedstawienie jego działania dla przykładowych scenariuszy. Scenariusze te zostały tak dobrane, aby różnice między planami pod względem ich ryzyka były wprost widoczne na przykładach, bez konieczności przeprowadzenia szerszej analizy wrażliwości, ani też bez stosowania jakiejkolwiek miary ryzyka.

Słowa kluczowe: finanse osobiste, gospodarstwo domowe, emerytura, planowanie finansowe, wybór międzyokresowy, ryzyko związane z czasem życia.