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OPTIMIZATION OF CONSUMER PREFERENCES – AN EXAMPLE

Barbara Fura, Marek Fura

Abstract. In the article we discuss a standard example of an optimization problem. In our problem we are optimizing the objective function, i.e. a consumer utility function with two variables representing quantities of two commodities denoted by x_1 and x_2. We consider the standard optimization problem in which we maximize the defined utility function subject to a budget constraint. More precisely, the problem is to choose quantities of two commodities 1st and 2nd, in order to maximize $u(x_1, x_2)$ function subject to the budget constraint. The aim of the article is to present how we can exemplify and solve this kind of problems in a classroom. In the paper we suggest four methods of finding the solution. The first one is based on the graphical interpretation of the problem. Based on this we can get the approximate solution of the defined optimization problem. Then, we present an algebraic approach to find the optimum solution of the given problem. In the first method we use the budget constraint to transform the utility function of two variables into the function of one variable. The second algebraic method of achieving the solution is based on the second Goosen's law, which is known as the law of marginal utility theory. In the third algebraic method applied to find the maximum of the utility function, we use the Lagrange multipliers. The text emphasizes the educational aspect of the theory of consumer choice.

Keywords: consumer preferences, theory of consumer choice, Gossen's laws, Lagrange multipliers.

JEL Classification: C02.

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1. Introduction

In this paper we use standard optimization methods that economists use to model the behavior of consumers. By a consumer we mean a person who has the opportunity to buy various commodities at fixed market prices. In such cases the main question is: how do consumers choose what to buy given their income and prices in the economy?

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First, we make the assumption that the consumer has a precisely defined set of desires or preferences which are represented by a numerical utility function. Furthermore, we assume that the consumer chooses optimally, i.e. that they choose the option with the highest utility of those available to them. This means that the consumer is solving an optimization problem.

Each optimization problem has its three components [Dean 2009]:

- Object of choice – represented by different bundles of goods that the consumer can purchase.
- Objective function – defined by the consumer's utility function.
- Constraints – expressed by a set of commodities that the consumer can afford.

We can find the solution of optimization problems using several ways. Among them are both graphical as well as numerical methods. In the next section we give an example of an elementary optimization problem which in the third section we solve graphically as well as numerically. The first solution of the defined task is based on the theory of consumer choice, the second is based on Goosen's laws, and the basis for the third solution is the Lagrange multiplier method.

2. Example of solutions of the optimization problem

Let us consider an example of the optimization problem. The preferences of a consumer over two goods (1^{st}) and (2^{nd}) are given by the utility function:

$$u(x_1,\ x_2) = (x_1+1)(x_2+1) = x_1x_2 + x_1 + x_2 + 1,$$

where x_1 is the number of units of 1^{st} good, and x_2 is the number of units of 2^{nd} good.

Utility functions represent general and stable patterns of consumers' decision-making [Clavien, Chapuisat 2015]. The prices of commodities are $p_1 = 1$ and $p_2 = 2$ respectively. The consumer income I is given by 30. The question is: *What bundle of goods will the consumer choose?*

3. Example of solutions of the optimization problem

3.1. Solution I – graphical

The theory of consumer choice is a branch of microeconomics that relates preferences to consumption expenditures and to consumer demand curves. It analyzes how consumers maximize the desirability of their consumption as measured by their preferences subject to limitations on their expenditures, by maximizing the utility subject to a consumer budget constraint [Binger, Hoffman 1998; Dean 2009].

The first assumption is that

$$x_1 \geq 0,\ x_2 \geq 0.$$

The price system assigns a price to each type of good, such that the cost of the bundle is $p_1x_1 + p_2x_2$. The consumer who seeks to distribute his/her income (I) across the purchase of the two goods he/she consumes cannot spend more than his/her total income.

The assumption of a fixed income along with the prices of goods implies a budget constraint:

$$p_1x_1 + p_2x_2 \leq I.$$

Thus in our example we get

$$x_1 + 2x_2 \leq 30$$

and therefore

$$x_2 \leq -\frac{1}{2}x_1 + 15.$$

Let us now describe the demand curve by

$$c := x_1x_2 + x_1 + x_2 + 1,\ c > 0,$$

where c is the certain level of utility.

$$x_2 = -1 + \frac{c}{x_1 + 1}, \qquad c > 0.$$

For each level of utility we get a different indifference curve (Figure 1). An indifference curve presents a collection of all the commodity bundles which provide the consumer with the same level of utility. The consumer wants to end up on the highest possible indifference curve. All of its bundles provide them with a higher utility than the bundles from a lower indifference curve. However the consumer choice is limited by his or her income. Thus the consumer to be the most satisfied with the consumption and at the same time stay within the budget choses x_1^* units of the first and x_2^* units of the second good. This bundle of goods is represented on Figure 1 by the contact point of the budget constraint and the indifference curve.

The consumer will choose the indifference curve with the highest utility that is attainable within his/her budget constraint. From Figure 1 we see that the utility maximization over the budget constraint occurs at the point of tangency between the indifference curve and the budget line. Thus, the consumer will purchase x_1^* units of the first good and x_2^* units of the second good.

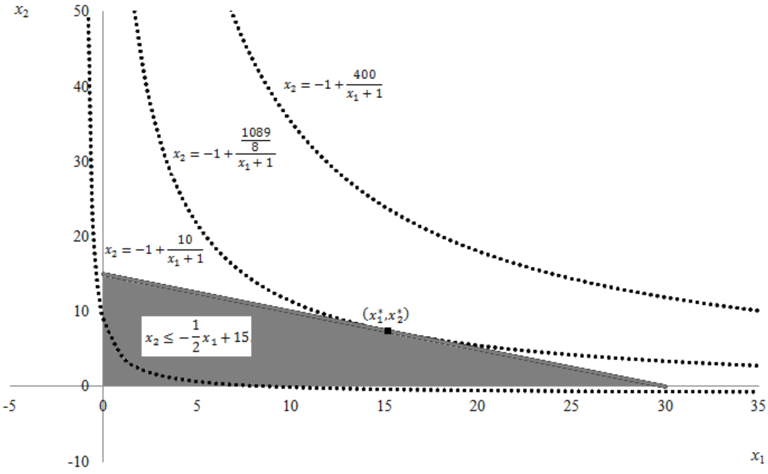

Fig. 1. Graphical solution of the example

Source: own elaboration.

3.2. Solution II – optimization of the function of one variable

The consumer is maximizing the utility function given by the formula:

$$u(x_1, x_2) = x_1x_2 + x_1 + x_2 + 1.$$

From the budget constraint we have:

$$x_1 = 30 - 2x_2$$

hence

$$u(x_2) = (30 - 2x_2)x_2 + 30 - 2x_2 + x_2 + 1 = -2x_2^2 + 29x_2 + 31$$

is a function of one variable (x_2). The first order derivative of $u(x_2)$ function is:

$$u'(x_2) = -4x_2 + 29.$$

From the necessary condition for local extremum we have:

$$u'(x_2) = 0 \quad -4x_2 + 29 = 0 \quad x_2 = \frac{29}{4}.$$

Since

$$u'(x_2) > 0 \quad x_2 \in \left(-\infty, \frac{29}{4}\right) \quad \text{and} \quad u'(x_2) < 0 \quad x_2 \in \left(\frac{29}{4}, +\infty\right),$$

then $u(x_2)$ function has in $x_2 = \frac{29}{4}$ a local maximum (Figure 2).

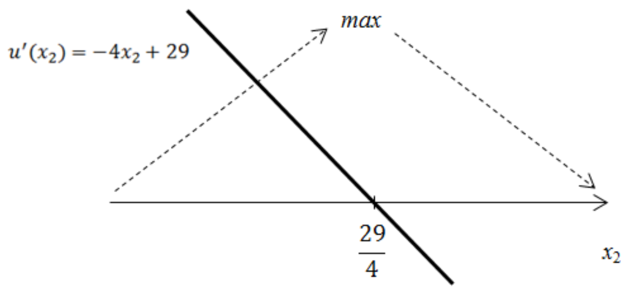

Fig. 2. Extremum of $u(x_2)$ function

Source: own elaboration.

$$u(x_2) \to max\ x_2 = \frac{29}{4}\ x_2^* = \frac{29}{4}.$$

Therefore

$$x_1^* = 30 - 2 \cdot \frac{29}{4} = \frac{31}{2}$$

and

$$\left(x_1^*, x_2^*\right) = \left(\frac{31}{2}, \frac{29}{4}\right).$$

3.3. Solution II – focus on Gossen's Law

Gossen's First Law: the magnitude (intensity) of pleasure decreases continuously if we continue to satisfy one the same enjoyment without interruption until satiety is ultimately reached [Negishi 2014]. In other words, the marginal utilities ($\partial u / \partial x_i$ for all i) of each homogenous unit decrease as the supply of units increases (and vice versa), or the marginal utilities are diminishing across the ranges relevant to decision-making.

The fundamental law of the theory of pleasure, is known as *Gossen's Second Law*: in order to maximize his life pleasure Man must distribute his time and energy among the preparation of various pleasures in such a way that the value of the last unity yielding each pleasure shall be equal to the magnitude of the discomfort experienced by him if this unity had been created in the very last moment of the employment of force. In other words, utility u is at least weakly quantified, that in equilibrium a consumer will allocate expenditures so that the ratio of marginal utility to price (marginal cost of acquisition) is equal across all goods and services [Gossen 1983; Rima (ed.) 2001; Hagendorf 2010; van Daal 2012].

From the second Gossen's law we have

$$\begin{cases} \dfrac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \dfrac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 = I \end{cases}$$

and hence

$$\begin{cases} \dfrac{x_2 + 1}{x_1 + 1} = \dfrac{1}{2} \\ x_1 + 2x_2 = 30 \end{cases}.$$

After solving this equation system we get

$$\left(x_1^*, x_2^*\right) = \left(\frac{31}{2}, \frac{29}{4}\right).$$

Thus, in order to maximize the utility the consumer will choose x_1^* units of first and x_2^* units of the second good.

3.4. Solution III – focus on the Lagrange multiplier method

The Lagrange multipliers (λ) method can be used to find extrema of a multivariate function e.g. $u(x_1, x_2, \ldots)$ subject to the constraint $g(x_1, x_2, \ldots) = 0$ where u and g are functions with continuous first partial derivatives on the open set containing the curve $g(x_1, x_2, \ldots) = 0$, and the gradient of function g is not 0 at any point on the curve. For an extremum of u to exist on g, the gradient of u must line up with the gradient of g [Arfken 1985]. In mathematical optimization, the *method of Lagrange multiplier* is a strategy for finding the local maxima and minima of a function subject to equality constraints [Chiang 1984].

Let us define the constraint condition by the formula

$$g\left(x_1,\ x_2\right) = I - p_1 x_1 - p_2 x_2 .$$

To solve our optimization problem we create a Lagrangeian function

$$L\left(x_1,\ x_2,\ \lambda\right) = u\left(x_1,\ x_2\right) + \lambda g\left(x_1, x_2\right),\ \lambda \neq 0 ,$$

which is our objective function we are going to maximize.

This function in our case takes the form of

$$L\left(x_1,\ x_2,\ \lambda\right) = x_1 x_2 + x_1 + x_2 + 1 + \lambda\left(30 - x_1 - 2x_2\right),\ \lambda \neq 0 .$$

First order sufficient condition for a local maximum is

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 0 \\ \dfrac{\partial L}{\partial x_2} = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \end{cases}.$$

In the defined problem it is given by the system of equations

$$\begin{cases} x_2 + 1 - \lambda = 0 \\ x_1 + 1 - 2\lambda = 0 \\ 30 - x_1 - 2x_2 = 0 \end{cases}.$$

As its result we have

$$\begin{cases} x_1 = \dfrac{31}{2} \\ x_2 = \dfrac{29}{4} \\ \lambda = \dfrac{33}{4} \neq 0 \end{cases}.$$

Let us apply the second order sufficient condition (bounded Hessian) which takes the following form

$$\det \bar{H} = \begin{vmatrix} \dfrac{\partial^2 L}{\partial \lambda^2} & \dfrac{\partial^2 L}{\partial \lambda \partial x_1} & \dfrac{\partial^2 L}{\partial \lambda \partial x_2} \\ \dfrac{\partial^2 L}{\partial x_1 \partial \lambda} & \dfrac{\partial^2 L}{\partial x_1^2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 L}{\partial x_2 \partial \lambda} & \dfrac{\partial^2 L}{\partial x_2 \partial x_1} & \dfrac{\partial^2 L}{\partial x_2^2} \end{vmatrix}.$$

Since

$$\det \bar{H} = \begin{vmatrix} 0 & -1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} > 0$$

the bundle (31/2, 29/4) maximizes the consumer's utility function. In other words, the consumer to maximize his utility will choose the bundle of

$$\left(x_1^*, x_2^*\right) = \left(\frac{31}{2}, \frac{29}{4}\right).$$

4. Conclusion

The problem of the optimization of consumer preferences is discussed at the first stage of a microeconomics course. In order to explain how consumers are making their decisions, apart from theoretical examples it is useful to apply mathematical methods of optimization. Such methods give us the option to find the solution $(x_1^*, x_2^*) = (31/2, 29/4)$ in several ways. This may be helpful to students in better understanding the given problem. The presented example might be useful from the didactic point of view. Suggesting the possibilities of finding the solution of the defined problem could constitute an area for the further exploration of more complex optimization problems [Fura 2012]. It can also encourage students to be more active in the classroom since they understand the discussed problem better.

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