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TWO PROOFS OF STOKES' THEOREM IN NEW CLOTHES

Arkadiusz Maciuk, Antoni Smoluk

Abstract. The paper presents two proofs of Stokes' theorem that are intuitively simple and clear. A manifold, on which a differential form is defined, is reduced to a three-dimensional cube, as extending to other dimensions is straightforward. The first proof reduces the integral over a manifold to the integral over a boundary, while the second proof extends the integral over a boundary to the integral over a manifold. A new idea consists in the definition of Sacała's line that inspired the authors to taking a different look at the proof of Stokes' theorem.

Keywords: Stokes' theorem, Sacała's column, additivity of integration.

JEL Classification: C02.

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1. Introduction

Stokes' theorem and Taylor's theorem on a local approximation of a smooth function by a polynomial are two fundamental theorems in mathematical analysis. It is a key part of each course in analysis, even if not included explicitly, because the Leibniz–Newton theorem, always in a core curriculum, is its special case. Stokes' theorem is explicitly included in most classical courses in mathematical analysis such as Cartan [1967], Fichtenholz [1949], Rudin [1976] and in many other books. Time and again, it has been motivating mathematicians. There are reported dozens of its proofs, in many variants. Petrello [1998] and Markvorsen [2008] are worth mentioning among the rich literature; the former is a M.Sc. in Mathematics thesis reviewing current proofs. The proofs presented in this paper are extremely short and straightforward, as they put forward an intuitively transparent and comprehensible idea, thus they are most valuable didactically.

This paper was directly motivated by working on a project of a tombstone to commemorate our late colleague, Dr Jerzy Sacała. The base of the monument is a cube, on which a similar cube is positioned, divided into eight cubes whose edges are half the length of the edges of the first cube.

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There is another cube on top, divided into sixty-four cubes with edges whose length is one fourth of the first length, and so on. The length of the edges of subsequent cubes is always halved, while the number of cubes is eightfold greater. A similar column is located in Wrocław, in Norwida Street, in front of the main building of the University of Technology, to commemorate the heroes of Solidarity. However, the order of cubes is reversed: a whole cube is on top, it is then divided to form a lower layer, with the third cube divided into sixty-four smaller cubes standing on the ground. Dr Jerzy Sacała was fond of studying fractals empirically. A line formed by the edges of the above-defined infinite column and compactified by the addition of one point is called Sacała's curve. A line here denotes a one-dimensional, connected topological space. Sacała's line is a compact continuum contained in $\mathbb{R}^3$ – or a kind of a quasi-fractal (see Figure 1).

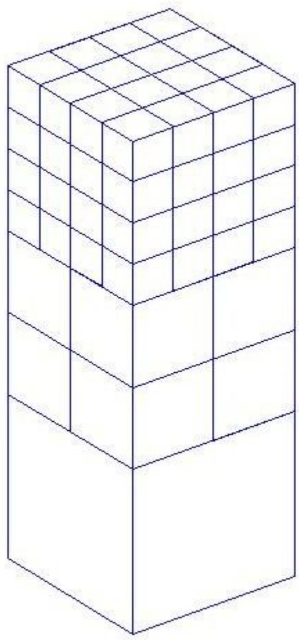

Fig. 1. Sacała's column

Source: own elaboration.

A similar line is exemplified by a Tatar trail. Tatars used to part and leave into the four cardinal directions so as to deceive the pursuers and to obstruct their localization. The line that is vertically formed, similarly to Sacała's line, and compactified by the addition of one point, is by definition a Tatar trail. To simplify the description, we consider the z-axis as time, the x-axis as an east-west circle of latitude, and the y-axis as a north-south meridian. The lowest level, corresponding to the initial moment, is the interval between the point (0,0,0) and the point (0,0,1). On the second level, i.e. with the next unit of time, we attach a cross of north-south and east-west arms, each with the length of one, to this interval. The endpoints of this cross are (–1, 0, 1), (0, –1, 1), (1, 0, 1) and (0, 1, 1). Each of them is an endpoint of the next interval with the length of one – another unit of time passes – closed by a cross with arms equal to half the length of the arms on a lower level. In this way one obtains sixteen endpoints on the third level, with the initial point on the first level. Next, one proceeds similarly, halving the length of the arms of subsequent crosses. Applying these steps infinitely yields a line that is called a Tatar trail when compactified by the addition of one point (Figure 2).

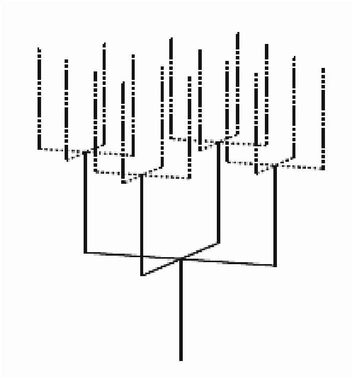

Fig. 2. A Tatar trail

Source: own elaboration.

Sacała's column embraces the main idea of the proof of Stokes' theorem that the integral of a differential k-form ω over the boundary of a $k + 1$-dimensional manifold M is equal to the integral of its exterior derivative $d\omega$ over the whole of M, i.e.

$$\int_{\partial M} \omega = \int_{M} d\omega .$$

We deal with the Euclidean space $\mathbb{R}^n$, where $k + 1 \leq n$. For the sake of simplicity we shall consider $\mathbb{R}^3$. Let K denote a cube in $\mathbb{R}^3$, and K_n – the same cube divided into 8^n smaller cubes denoted by K_{ni}, where $i \in \{1, 2, ..., 8^n\}$. The idea of the proof below consists in seeing that the oriented integral over the boundary of K is equal to the sum of oriented integrals along the boundaries of K_{ni}, whose orientation is induced by the orientation of the cube K (Figure 3).

Hence, we have

$$\int_{\partial K} \omega = \sum_{i=1}^{8^n} \int_{\partial K_{ni}} \omega,$$

where n is a natural number, including zero, i.e. $K = K_{01}$. The integrals over common faces of different cubes vanish, because one face is positively oriented and the other negatively.

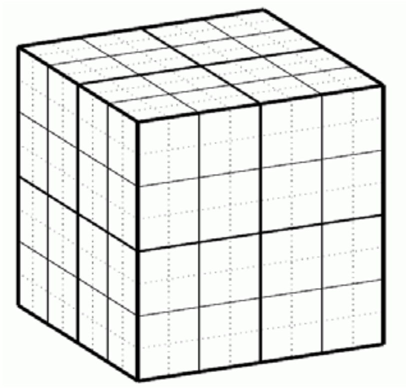

Fig. 3. Additivity of integration

Source: own elaboration.

Since each manifold may be approximated by cubes with any required accuracy, we will unfold the proof only for multidimensional cubes. Interchanging the variables yields the extension of the theorem from a cube into any manifold.

It is enough to prove the theorem for a (k+1)-dimensional cube K in $\mathbb{R}^{k+1}$. For simplicity, we assume that $k = 2$ and $n = 3$, then

$$\omega(x,y,z) = f(x,y,z)dxdy + g(x,y,z)dxdz + h(x,y,z)dydz,$$

where $(x,y,z)\in\mathbb{R}^3$. We assume that the functions f, g, h are not only continuous, but also smooth – they have continuous derivatives everywhere in $\mathbb{R}^3$. The cube K has vertices A,B,C,D,A',B',C',D', with the bottom face given by:

$$A = (x,y,z),\ B = (x+2r,y,z),\ C = (x+2r,y+2r,z),\ D = (x,y+2r,z),$$

and the top face given by:

$$A' = (x, y, z + 2r),\ B' = (x + 2r, y, z + 2r),$$
$$C' = (x + 2r, y + 2r, z + 2r),\ D' = (x, y + 2r, z + 2r)\text{'}$$

where $r > 0$ (Figure 4).

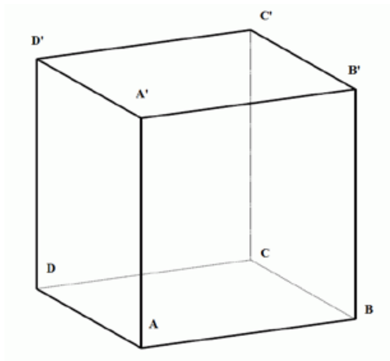

Fig. 4. Orientation of a cube's faces

Source: own elaboration.

Let K_1 denote a solid obtained by dividing the cube K into eight smaller cubes, then by properties of an integral, the integral of ω over the boundary of K is equal to the sum of the integrals over the boundaries of smaller cubes that form K_1. The integrals over the common faces of two smaller cubes vanish, because if one face is positively oriented, then the other is negatively oriented. This is true with smaller and smaller divisions, hence one may assume that the number r is suitably small.

For the sake of formality, let us recall the definition of the exterior derivative of a differential form. We have that:

$$d\omega(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z)\,dx + \frac{\partial f}{\partial y}(x,y,z)\,dy + \frac{\partial f}{\partial z}(x,y,z)\,dz\right)dxdy +$$
$$\left(\frac{\partial g}{\partial x}(x,y,z)\,dx + \frac{\partial g}{\partial y}(x,y,z)\,dy + \frac{\partial g}{\partial z}(x,y,z)\,dz\right)dxdz +$$
$$\left(\frac{\partial h}{\partial x}(x,y,z)\,dx + \frac{\partial h}{\partial y}(x,y,z)\,dy + \frac{\partial h}{\partial z}(x,y,z)\,dz\right)dydz.$$

Considering skew symmetry of the exterior product of vectors, we have

$$dxdy = -dydx \text{ and } dxdx = dydy = dzdz = 0,$$

hence,

$$d\omega(x,y,z)=\left(\frac{\partial f}{\partial z}(x,y,z)-\frac{\partial g}{\partial y}(x,y,z)+\frac{\partial h}{\partial x}(x,y,z)\right)dxdydz\,.$$

2. The first proof

The exterior product represents the surface area and is a bivector, while the scalar product is a number and if the vector is not a null vector, then the scalar product is a number greater than zero. The proof of Stokes' theorem may begin with the left-hand of equality and yield the right-hand side, or from right to left. The proof beginning with the right-hand side, i.e. with $\int_K d\omega$, is easier. One has to calculate three integrals. Let F_1 denote the sum of faces with vertices *A*, *B*, *C*, *D* and A', B', C', D', where one face is positively oriented and the other is negatively oriented. Let F_2 denote the sum of faces $ADD'A'$ and $BCC'B'$, with one face negatively oriented and the other positively oriented. Let F_3 denote the sum of faces $ABB'A'$ (negatively oriented) and $CDD'C'$ (positively oriented). Since:

$$\int_z^{z+2r}\frac{\partial f}{\partial z}z(x,y,\zeta)d\zeta = f(x,y,z+2r)-f(x,y,z),$$

$$\int_y^{y+2r}\frac{\partial g}{\partial y}(x,\epsilon,z)d\epsilon = g(x,y,z)-g(x,y+2r,z)$$

and

$$\int_x^{x+2r}\frac{\partial h}{\partial z}(\chi,y,z)d\chi = h(x,y,z)-h(x+2r,y,z)$$

hence,

$$\int_K d\omega=\int_{F_1}(\frac{\partial f}{\partial z}(x,y,z)dz)dxdy-\int_{F_2}(\frac{\partial g}{\partial y}(x,y,z)dy)dxdz+\int_{F_3}(\frac{\partial h}{\partial x}(x,y,z)dx)dydz,$$

thus showing that the theorem holds, because the sum of these three integrals is equal to the integral over the boundary of the cube *M*, which finishes the proof.

3. The second proof

The second proof begins with the integral of the $\int_{\partial K}\omega$ over the boundary of the manifold. We have $\int_{\partial K}\omega = \int_{\partial K} fdxdy + \int_{\partial K} gdxdz + \int_{\partial K} hdydz$. The integral over the faces, where differentials dx, dy and dz are equal to zero, is obviously also zero. Next, one gets $\int_{\partial K} fdxdy = \int_{F_1} fdxdy$, where F_1 is a union of faces $ABCD$ and $A'B'C'D'$ (Figure 4). Analogously,

$$\int_{\partial K} gdxdz = \int_{F_2} gdxdz \quad \text{and} \quad \int_{\partial K} hdxdz = \int_{F_3} hdydz,$$

where F_2 is a union of faces $ABB'A'$ and $DCC'D'$, also F_3 is a union of faces $ADD'A'$ and $BCC'B'$. One may assume that for suitably small values of r an average value of the function f on the face is equal to its value in the centre of the face. This holds analogously for the functions g and h. Such an assumption generates a bias, but this error decreases to zero along with the decrease of r.

Let $P = (x+r, y+r, z)$ be the centre of the face $ABCD$, and $P' = (x+r, y+r, z+2r)$ the centre of the face $A'B'C'D'$. Using this notation we have

$$\int_{F_1} fdxdy = \left(f\left(P'\right) - f\left(P\right)\right)dxdy + o(r).$$

It follows from Lagrange's theorem that $\int_{F_1} fdxdy = \left(\frac{\partial f}{\partial z}\left(W\right)dz\right)dxdy + o(r)$, where W is the centre of the cube K, i.e., $W = (x+r, y+r, z+r)$. Similarly,

$$\int_{F_2} gdxdz = \left(\frac{\partial g}{\partial y}\left(W\right)dy\right)dxdz + o(r) \quad \text{and} \quad \int_{F_3} hdydz = \left(\frac{\partial h}{\partial z}\left(W\right)dx\right)dydz + o(r).$$

The above equalities yield $\int_{\partial K}\omega = d\omega\left(W\right) + o(r)$, or $\int_{\partial K}\omega = \int_K d\omega$, with a negligible error. It is significant to see that $\int_K d\omega = \lim_{n\to\infty}(\sum_{i=1}^{8^n} d(P_{ni})dv)$, where P_{ni} is an arbitrary point from the cube K_{ni}, and dv is a non-oriented element of volume, i.e. with the usual notation of differential forms we have $dv = dxdydz$. This equality is identical with the definition of the usual Riemann integral.

Passing to any manifold M takes place in two steps. First, a manifold is approximated by a set of curved, deformed cubes, then the variables are interchanged, and the cube positioned on the manifold is converted into an ordinary Euclidean cube. The proof for a Euclidean cube is given above. Interchanging variables extends the proof for any manifold. The proof can be regarded as inductive, because it applies the Leibniz–Newton theorem that is relevant in the classical analysis of the functions of a single variable.

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