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PARTITIONS AND BRANCHING PROCESSES

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Abstract. A partition, i.e. a division of a finite set into nonempty subsets, is a simple and essential concept of quantitatively understanding the reality. A partition of a number *n* is a decreasing sequence of natural numbers whose sum equals *n*. Greater numbers are seen only in terms of the union of partitions. The most important processes such as stochastic processes of branching processes can be expressed most simply using the language of partitions. By means of partitions any Sacała's line defines a wide class of related quasi-branching processes which are more general than Markov processes. Didactically such an approach is extremely useful.

Keywords: partition, branching process, Sacała's line, tree (dendrite).

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1. Introduction

Mathematics is everywhere, it is an abstract of entire science and the peak of a scientific pyramid. We meet mathematical concepts in all places. We learn quantitative discrete phenomena by means of partitions. What is a partition? Formally it is a decreasing sequence of natural numbers, with a finite number of nonzero terms, i.e. a sequence of the finite type. A partition of the number seven is the sequence (5, 1, 1, 0,…), since $7 = 5 + 1 + 1$. If we withdraw PLN 200 from a cash machine, we might get one 200-zł note, or two 100-zł notes, or three notes: one 100-zł note and two 50-zł notes, and so on. These are precisely partitions of the number two hundred defined by the current legal tender [Narkiewicz 1977 quoted by Smoluk 2000]. Partitions form a structure called a semigroup. A union of two partitions is a partition with an identity element being a partition of zero. A semigroup is a simple concept and yet mathematically significant. It is a structure with one associative operation and identity element. If we combine the partition of seven (4, 2, 1, 0,…)with the partition of three

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(1, 1, 1, 0,…), we get the partition of ten (5, 3, 2, 0,…). We make use of partitions repeatedly each day, even if we are not aware of applying this concept. We see seven people in a photograph, because they are in three groupings of three, two and two people. Cognitive psychology fails to explain why most people perceive at first sight, without actually counting, either zero objects or one, two or at the most three objects; applying partitions in the case of more objects.

One might think that a concept of an injective function is irrelevant in the everyday life of most people, yet in many fields it is a fundamental notion. This can be demonstrated by the following story. In one of the major banks in Poland a customer wished to withdraw two hundred zloty. The cashier offered one 200-zloty note, but the customer demanded three banknotes: a one-hundred-zloty note and two fifty-zloty notes. The cashier accommodated the request, cancelled the earlier receipt and printed a new one, despite the fact that both printouts were identical. Both included the amount paid, that was the same, and the date of the transaction, without its time. The customer was surprised why a new receipt had to be printed and the cashier explained that the previous receipt was about a different partition, that is why it had to be cancelled. Information about numbers of specific denominations of cash is significant for a bank, therefore both transactions, even if involving the same total of currency, were not equivalent. The cashier matched an ordered pair composed in its first part of a partition, and in the second part, of the total, with an underlying receipt, so a new transaction required a new proof of payment. A function assigning a total to a partition is not naturally injective; one number may have many different partitions, although the same total is assigned to all of them.

2. Graph of a partition

Partitions seem to be the simplest and most fundamental concept of learning the reality quantitatively; they penetrate the whole science. A partition is essentially a discrimination, i.e. a division of a finite set into nonempty subsets. Since any discrimination is the same as equivalence, hence partitions are also equivalences. One partition is related to its class of discriminations and also to its equivalence class. In spite of being discrete, partitions adequately describe continuous phenomena, too. Partitions are essentially convex linear combinations or in other words, probabilities. They can be identified in stochastic branching processes, in the theory of the integral, in ergonomics and in price theory. A price is a linear functional – a special integral. Hence continuity can be derived from discrete partitions. We perceive branching processes more generally than at other times; they do not have to be Markov processes; one may call them quasi-branching processes. Any process of this type generates its dendrite, and even it is Sacała's line (Figure 1). The union of an increasing sequence of lines (S_n) compactified by one point is called Sacała's line when the union is not compact. Obviously, it holds $S_n \subset S_{n+1}$, each subsequent term contains branching points which do not belong to the precedent term, and naturally, it is a line, i.e. a connected and compact one-dimensional topological space; a branching point is defined as an initial or a final point, and points whose neighborhoods are equivalent to a tripod, a quadruped, etc. It is assumed further that each line S_n has a finite number of branching points. Except for the point of compactification, Sacała's line S is divided into floors; the n^{th} floor is defined by those branching points which do not belong to S_n and are not located on other floors.

If there are, say, $n!$ points on the n^{th} floor of the dendrite and each point is assigned a number from the set $\{1,\ldots,f_n\}$, where f_n is the n^{th} Fibonacci number, then arrangements of such numbers determine partitions of the number of branching points on the n^{th} floor. Certainly, instead of numbers they can be letters from a given alphabet which represent a word on each floor, generating a proper distribution of letters in this word. In addition, each number from the given set can occur more than once. The frequencies of occurrences define the distribution.

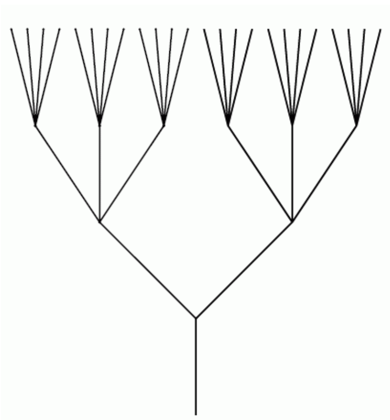

Fig. 1. Dendrite – a branching process

Source: own elaboration.

Each floor is therefore a probability distribution determined by a partition. If there is an arrangement of numbers $(1,2,3,2,2,3)$ on the fourth floor, a partition $(3,2,1,0,\ldots)$ of number 6 is assigned to this numerical word. It indicates that the number 2 occurs three times, the number 3 twice, and the number 1 once. We deal then with a probability distribution $\left\{\left(1,\frac{1}{6}\right),\left(2,\frac{3}{6}\right),\left(3,\frac{2}{6}\right)\right\}$, implying that a random variable takes just three values: 1 with probability $\frac{1}{6}$, 2 with probability $\frac{1}{2}$ and 3 with probability $\frac{1}{3}$. At this point it is worthwhile underlining that this simple distribution is connected with the 2/3 law of J. Łyko [Łyko, Smoluk 2000]. The remaining floors of the dendrite are analogous.

3. The Ulam sequence

Branching processes are exemplified by Fibonacci-like stochastic processes. One of best known processes of this type is the Ulam sequence. While the Fibonacci sequence describes rabbit breeding, the Ulam sequence models a chain reaction; the term was put forward by Ulam when he was working at Los Alamos [Drabik 2007; Florek et al. 2009]. If the Fibonacci sequence is defined by the recurrence equation $f_{n+2} = f_{n+1} + f_n$, $n \in \mathbb{N}$ with seed values $f_0 = 0, f_1 = 1$, an analog of the Ulam sequence is defined as follows: (a_n) is defined by the equation $a_{n+2} = a_{n+1} + b_n$, where b_n is a random variable taking values a_k for $k \in \{0,\ldots,n\}$ with equal probability $\frac{1}{n+1}$. The Fibonacci sequence determines the maximum potential value of this sequence. The values from 1 to f_n occur on the n^{th} floor. The distribution a_4 of this sequence is the above given distribution related to the 2/3 rule (Figure 2). This distribution explains the division of inheritance among a mother and her twins: a son and a daughter. A testator made a will allocating $\frac{1}{3}$ of his property to his wife if she gives birth to a son, and $\frac{2}{3}$ if a daughter is born, the remainder of the property being allocated to a descendant. Twins of different gender are born. How is the property to be divided? Lawyers in ancient Rome answered: $\frac{4}{7}$ to a son, $\frac{2}{7}$ to a wife, and $\frac{1}{7}$ to a daughter. They assumed a condition that a son's share should be twofold the mother's share, and the mother's share twofold the daughter's share. However, this division is not fair. In either case the mother was to receive at least $\frac{1}{3}$ of the property. The two equally probable events occurred. A son was born $\left(\frac{2}{3},\frac{1}{3},0\right)$ and a daughter was born $\left(0,\frac{2}{3},\frac{1}{3}\right)$. As the respective probabilities are equal, one should take their average, leading to a distribution $\left(\frac{2}{6},\frac{3}{6},\frac{1}{6}\right)$ with the property allocated as follows: $\frac{1}{3}$ to the son, half to the mother and $\frac{1}{6}$ to the daughter.

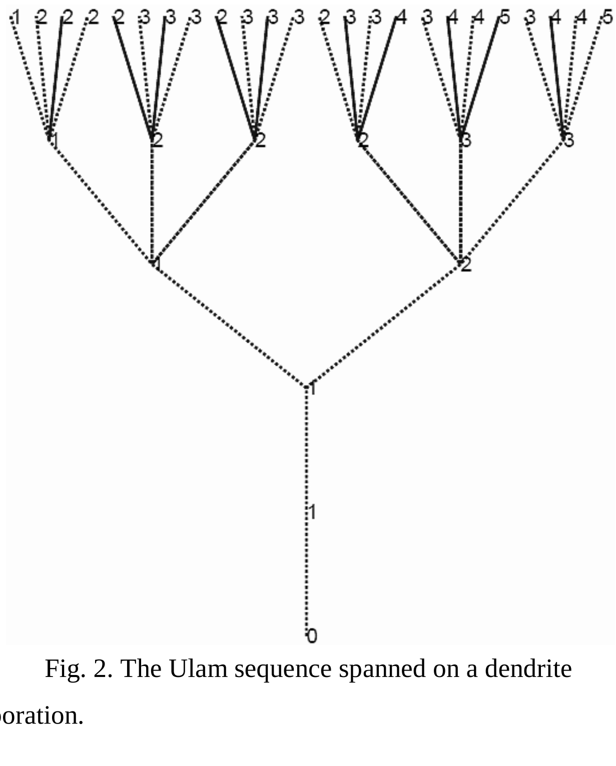

Fig. 2. The Ulam sequence spanned on a dendrite

Source: own elaboration.

4. Sacała's line

Lines of a square pyramid intersected by a plane perpendicular to its axis at equal distances (Figure 3) represent another Sacała's line that is interesting as regards the branching processes. Each square is dissected into smaller squares. The first square is dissected into four smaller squares, the next one into sixteen smaller squares, and each successive one into fourfold more squares than the preceding one. The edges of these dissections and the edges of the pyramid represent a Sacała's line. The dissection points located on the lower floor are also connected by a segment with their respective points on the upper floor. If there are r_n branching points on the n^{th} floor, then it holds $r_n = b_n^2$, where the auxiliary sequence (b_n) is defined by the condition $b_0 = 1, b_1 = 2, b_{n+2} = 2b_{n+1} - 1, n \in \mathbb{N}$. There are only three kinds of branching points on this line: fourfold, fivefold and sixfold points. If M_n denotes a matrix of branching points of the n^{th} level of this curve, then it obviously holds

$$M_0 = (4), \quad M_1 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 5 & 4 \\ 4 & 4 & 4 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 4 & 4 & 5 & 4 & 4 \\ 4 & 5 & 5 & 5 & 4 \\ 5 & 5 & 6 & 5 & 5 \\ 4 & 5 & 5 & 5 & 4 \\ 4 & 4 & 5 & 4 & 4 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 4 & 4 & 5 & 4 & 5 & 4 & 5 & 4 & 4 \\ 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 6 & 5 & 6 & 5 & 6 & 5 & 5 \\ 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 6 & 5 & 6 & 5 & 6 & 5 & 5 \\ 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 6 & 5 & 6 & 5 & 6 & 5 & 5 \\ 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 4 \\ 4 & 4 & 5 & 4 & 5 & 4 & 5 & 4 & 4 \end{pmatrix}$$

and so forth.

Each matrix is a variant of a two-dimensional word and generates a respective distribution d_n. A sequence of distributions (d_n) stabilizes, perhaps with an asymptotic distribution d of the form

$\left\{(4,0),\left(5,\frac{3}{4}\right),\left(6,\frac{1}{4}\right)\right\}$. As a consequence, when walking along this curve on high floors we encounter branching points of order 4 quite rarely, branching points of order 5 most frequently, with a probability close to $\frac{3}{4}$, and branching points of order 6 with a threefold smaller probability.

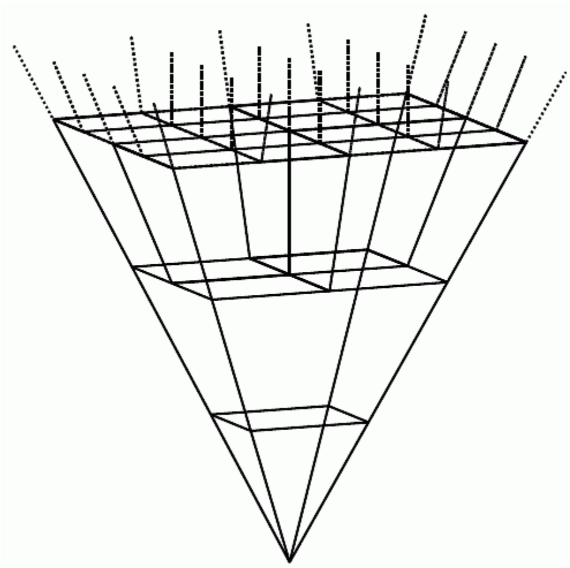

Fig. 3. Lines of a square pyramid – a variant of Sacała's line

Source: own elaboration.

5. Summary

The paper combined some matters which seem distant, however they are actually matching and complementary. Partitions express objects quantitatively as they are recognized on a daily basis, while branching processes facilitate a transition from discrete states to continuous asymptotic distributions. Our knowledge begins with a discrete perception of small collections and concludes with a continuous perception of the laws of nature. A network effect is evident here, exemplified by the radioactive decay of elements and the chain reaction, also a marketing and advertising trick that can be summarized as trying to emulate your neighbors and worthy people.

References

Drabik E. (2007). *Wykorzystanie ciągu Ulama do analizy fal giełdowych* [*An application of the Ulam sequence to analysis of the stock exchange waves*]. Ekonomika i Organizacja Gospodarki Żywnościowej 69. Pp. 77-87.

Florek J., Juzwiszyn J., Misztal A., Sacała J. (2009). *O ciągu Ulama, równaniu Pella i rotacjach rynku finansowego* [*On the Ulam sequence, Pell equation and financial market rotations*]. Didactics of Mathematics 5-6 (9-10). Pp. 5-18.

Łyko J., Smoluk A. (2000). *Problem of group choice and the 2/3 rule*. Ekonomia Matematyczna 4. Wydawnictwo AE we Wrocławiu.

Narkiewicz W. (1977). *Teoria liczb* [*Number theory*]. PWN.

Smoluk A. (2000). *Standardy, normy i algebra wielkości mian*owanych [*Standards, norms and algebra of quantities with dimensions*]. In: A. Smoluk [ed.]. *Elementy metrologii ekonomicznej* [*Elements of economic metrology*]. Wydawnictwo AE we Wrocławiu. Pp. 17-42.