STOCHASTIC SIMULATION OF STORAGE AND INVENTORY SYSTEMS

Wojciech Bombała, Zbigniew Michna

Abstract. This article aims to present the applications of Lévy processes for the stochastic modeling of storage resources. Two cases were considered. In the first one, the volume of supplies to the storehouse is described by a random process (Lévy process), while issuing the products is described by a deterministic and linear function. The second case is reversed: the delivery to the storehouse is described by a linear function (variable: time), while issuing the goods is described by a Lévy process. For both cases the form of the stock level process and examples of its trajectories, when the net supply is a Lévy process, are given. We investigated the following net supply processes: gamma process, α-stable Lévy process with $\alpha = 0.5$, Cauchy process, Wiener process.

Keywords: inventory system, storage system, stochastic process, Lévy process, probability of an overflow.

JEL Classification: L90, C02, C15.

1. Introduction

The inventory is a very important part in logistics systems (see e.g. Ghiani, Laporte, Musmanno, 2004; Krawczyk, 2011). Inventory management deals with the control and planning of stored resources. The aim of inventory management is to determine the stock level to minimize the total operating cost and to ensure satisfactory supplies for customers' demands. A good inventory policy should take into account several issues and one of them is the control and operating of the stock level. Thus, in practice we need to describe supplies (inflow) and demands (outflow) of a storehouse, then apply a certain storage model to determine its characteristics, such as the probability and mean time of an overflow of the storehouse and many others.

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In this article we present and discuss stochastic models of storage resources based on Lévy processes. The Lévy processes appear in many theoretical and practical fields, where they serve as a basic skeleton for a description of certain phenomena. They are applied in physics, economics, finance, insurance, queueing systems and other branches of knowledge. Their features, like independence and stationarity of increments or self-similarity, in certain cases permit to apply them to model for instance returns of stock prices, claims to insurance companies or a flow (outflow) to the buffer in queueing (telecommunications) systems. Moreover, the Lévy processes serve as a starting point for more complicated models, e.g. based on stochastic differential equations. Let us recall the notion of a Lévy process (see e.g. Sato, 1999).

Definition 1. A stochastic process $\{X(t) : 0 \leq t < \infty\}$ with values in $\mathfrak{R}^d$ is a Lévy process if:

- $X(0) = 0$ a.s.,
- X has independent increments,
- $X(t) - X(s)$ has the same distributions as $X(t-s)$, that is, X has stationary increments,
- X is stochastically continuous.

The sample paths of a Lévy process are cadlag functions, that is, right continuous with left limits. The structure of Lévy processes is not very complicated. The Lévy-Itô representation shows their stochastic construction, which is the following:

$$X(t) = B(t) + \int\limits_{|x|<1} x \big(N_t(dx) - tQ(dx) \big) + \int\limits_{|x|\geq 1} x N_t(dx) + at, \qquad (1)$$

where $B(t)$ is Wiener process, N is a point process generated by the jumps of X that is $N = \sum\limits_{\{t:\Delta X(t)\neq 0\}} \delta_{(t,\Delta X(t))}$. N is a random Poisson measure on $[0,\infty) \times \{\mathfrak{R}^d \setminus 0\}$ with the mean $ds \times Q(dx)$, where $Q(dx)$ is the so-called Lévy measure on $\mathfrak{R}^d \setminus 0$, and $a \in \mathfrak{R}^d$. The abbreviation $N_t(dx)$ means $N([0,t],dx)$.

2. The storage resources models based on Lévy processes

In this section we investigate a stochastic process describing a store level in an inventory system. We will distinguish two cases. The first one will assume that the inflow of goods or products (we do not distinguish products, i.e. we assume the we store one kind of the product) to a store-house is random due to a Lévy process and the outflow is deterministic and linear. Such a situation can happen when the products peel from a conveyor belt or natural resources are mined and after that are stored. We can have a similar situation in terminals, dams or telecommunication systems, where the flow is random and the outflow is deterministic. This case is particularly interesting as we know from the review by Silver (2008) and it appears commonly in the research. In the second case, we assume reversely that the inflow is deterministic and the outflow is random. Such a situation happens when clients buy a certain product, then a storehouse can guarantee deterministic linear supplies and the demands of clients are random. From Silver (2008) we know that these assumptions are also quite common in research.

Let us analyze the first case, that is, we assume that the supplies to a storehouse are random and due to a Lévy process X. From the practical point of view, it is more natural to assume that the process X is non-negative with non-decreasing sample paths, but we consider several examples where this assumption will be dropped. Then we can regard a certain uncertainty (a noise) is added to the outflow from the storehouse.

For simplicity, let us assume that the net supply is the following:

$$Y(t) = X(t) - t, \tag{2}$$

where X is a Lévy process with values in $[0, \infty)$, then its trajectories are non-decreasing. Thus, it is easy to notice that the process $Z(t)$ describing the stock level of products in the storehouse at time t satisfies the following equation:

$$Z(t) = Z(0) + Y(t) + \int_0^t \chi\{Z(s) = 0\}\, ds, \tag{3}$$

where $Z(0)$ is the stock level at time $t = 0$ (it can be a random or deterministic quantity) and:

$$\chi\{Z(s) = 0\} = \begin{cases} 1 & \text{if } Z(s) = 0 \\ 0 & \text{if } Z(s) > 0 \end{cases}, \tag{4}$$

which means that the stock level cannot be less than zero. One can find a unique solution of the integral equation (3) (see e.g. Prabhu, 1998).

Proposition 1. The integral equation (3) has a unique solution in the following form:

$$Z(t) = Z(0) + Y(t) + I(t),\qquad(5)$$

where:

$$I(t) = [\inf_{s \le t} Y(s) + Z(0)]^-,\qquad(6)$$

and:

$$X^- = -\min(0, X).\qquad(7)$$

Moreover if $Z(0) = 0$, then:

$$Z(t) = \sup_{s \le t}[Y(t) - Y(s)] \overset{d}{=} \sup_{s \le t} Y(s).\qquad(8)$$

Proof of Proposition 1. Subtracting the sides of the equation (3) for times t and τ ($\tau \le t$) we get:

$$Z(t) = Z(\tau-) + Y(t) - Y(\tau-) + \int_{\tau}^{t} \chi\{Z(s) = 0\}\, ds \ge Y(t) - Y(\tau-),$$

where $Z(\tau-)$ is the left-sided limit of Z at τ and the last inequality follows from the fact that the integral in the equation (3) is non-negative. First, let us assume that the storehouse is empty at some moments until time t (precisely on the time interval $(0, t]$). Then we can define the last moment when the storehouse is empty (until the moment t) that is $t_0 = \max\{\tau : \tau \le t, Z(\tau-) = 0\}$. Thus t_0 is the last moment when the store-house was empty on the time interval $(0, t]$ and:

$$Z(t) = Y(t) - Y(t_0-),$$

using the last equation and the fact that $\int_{t_0}^{t} \chi\{Z(s) = 0\}\, ds = 0$. Hence by the last inequality we obtain:

$$Z(t) = \sup_{\tau \le t}[Y(t) - Y(\tau-)]\qquad(9)$$

(we put $Y(0-) = 0$). Now let us assume that the storehouse is never empty on the time interval $(0, t]$. Then from the equation (3) it follows that

$Z(0) + Y(\tau-) > 0$ for all $0 < \tau \leq t$, hence $Z(t) = Z(0) + Y(t) > Y(t) - Y(\tau-)$. Thus by (9) we get:

$$Z(t) = \max\left\{\sup_{\tau \leq t}[Y(t) - Y(\tau-)], Z(0) + Y(t)\right\}$$

$$= \max\left\{Y(t) - \inf_{\tau \leq t} Y(\tau), Z(0) + Y(t)\right\},$$

where in the last equality we use the stochastic continuity of the process Y (check that in the first case the first term of max is greater than the second one). Thus using the last equality and (3) we get:

$$\int_0^t \chi\{Z(s) = 0\}ds = Z(t) - Z(0) - Y(t) = \max\left\{-\inf_{\tau \leq t} Y(\tau) - Z(0), 0\right\}$$

$$= [\inf_{\tau \leq t} Y(\tau) + Z(0)]^-.$$

Now let us assume $Z(0) = 0$ and notice that $Y_1(\tau) \overset{d}{=} Y(\tau)$ for $\tau \leq t$ (in the sense of finite dimensional distributions), where $Y_1(\tau) = Y(t) - Y(t-\tau)$. Then for a given t we have:

$$Z(t) = Y(t) - \inf_{s \leq t} Y(s) = \sup_{s \leq t}[Y(t) - Y(s)] = \sup_{s \leq t} Y_1(t-s) \overset{d}{=} \sup_{s \leq t} Y(s),$$

where the last equality is the sense of one dimensional distributions. The proof is completed.

Remark 1.1. The equality in (8) is not in the sense of finite dimensional distributions but in the sense of one dimensional distributions.

Remark 1.2. Except the equation (8), the statements of the above theorem are valid for any process X which starts from zero and has non-decreasing trajectories and is stochastically continuous, i.e. we can drop the assumption that X is a Lévy process.

Remark 1.3. The process Z in (5) can be used as a storage level process in storage systems even though the process X has values in $\Re$ and its sample paths are not non-decreasing. Then we add a certain uncertainty to the outflow from the storehouse and the process Z is a solution of a little different equation than (3).

Remark 1.4. The complexity of the model lies in the structure of the process X (see the equation (2)). In this article we consider X to be a Lévy process. In this natural and quite simple setting the theoretical results are very deep and complicated (see Michna, Bombała, Nielsen, 2013, and

references therein). Moreover, simulation methods in other cases than Lévy processes can be very sophisticated.

As we mentioned earlier, we will also investigate a model where the supplies are described by a linear function and the clients' demands (an outflow) is random. More precisely, we assume that the net supplies are the following:

$$Y(t) = t - X(t), \tag{10}$$

where X is a Lévy process with values in $[0, \infty)$ and with non-decreasing sample paths. Then, similarly as in the first case, the process $Z(t)$ describing the stock level at a moment t satisfies the following equation:

$$Z(t) = Z(0) + Y(t) + \int_0^t \chi\{Z(s) = 0\} dX(s), \tag{11}$$

where $Z(0)$ is the stock level at time $t = 0$ (it can be a random or deterministic quantity). The assumptions on the process X are the same as in the first case.

Proposition 2. The integral equation (11) has a unique solution in the following form:

$$Z(t) = Z(0) + Y(t) + I(t), \tag{12}$$

where:

$$I(t) = [\inf_{s \le t} Y(s) + Z(0)]^-, \tag{13}$$

and:

$$X^- = -\min(0, X). \tag{14}$$

Moreover if $Z(0) = 0$, then:

$$Z(t) = \sup_{s \le t}[Y(t) - Y(s)] \stackrel{d}{=} \sup_{s \le t} Y(s). \tag{15}$$

Proof of Proposition 2. The proof follows a similar way as the proof of Proposition 1.

Remark 1.4. The equality in (15) is not in the sense of finite dimensional distributions but in the sense of one dimensional distributions.

Remark 1.5. Except the equation (15), the statements of the above theorem are valid for any process X which starts from zero and has non--decreasing trajectories and is stochastically continuous, i.e. we can drop the assumption that X is a Lévy process.

Remark 1.6. The process Z in (12) can be used as a storage level process in inventory systems even though the process X has values in $\Re$ and its sample paths are not non-decreasing. Then we add a certain uncertainty to the inflow to the storehouse and the process Z is a solution of a little different equation than (11).

Remark 1.7. The complexity of the model lies in the structure of the process X (see the equation (10)). In this article we consider X to be a Lévy process. In this natural and quite simple setting the theoretical results are very deep and complicated (see Bombała, Michna, Nielsen, 2013, and references therein). Moreover, simulation methods in other cases than the Lévy processes can be very sophisticated.

3. Simulation of storage resources processes

In the previous section the stochastic stock level process was described by an integral equation and its solution. We can simulate this process using available pseudorandom simulators. The simulation of the storage resources allows to notice the difference between the two approaches in both models: the random outflow and the random inflow. Furthermore, simulations allow to see the behavior of these models and they can be used to find the probability of an overflow of the storehouse above the level u until the time t.

The family of Lévy processes is very rich. It includes gamma process, α-stable Lévy processes, inverse Gaussian Lévy process, normal inverse Gaussian Lévy process and many others (see e.g. Sato, 1999). We can generate random variables with α-stable distributions (see Janicki, Weron, 1994 or Samorodntisky, Taqqu, 1994, for the definition of stable distributions and Chambers, Mallows, Stuck, 1976; Weron, 1996, for the methods of their simulation), gamma distribution (see e.g. Ahrens, Dieter, 1982), inverse Gaussian distribution (see e.g. Michael, Schucany, Haas, 1976). Then it is easily to simulate the sample paths of Lévy processes and by the equations (5) and (12) the trajectories of the stock level Z. We show trajectories of the process Z when the net input Y is gamma process see Fig. 1. and Fig. 2., 0.5-stable Lévy process based on the so-called Lévy distribution ($\alpha = 0.5$, the skewness parameter $\beta = 1$) see Fig. 3. and Fig. 4., Cauchy process ($\alpha = 1, \beta = 1$) see Fig. 5 and Fig. 6 and Wiener process ($\alpha = 2$). Each sample path is generated in the time range: $[0,10]$ and the initial volume is: $Z(0) = 0$ (it is easily to simulate the stock level process with any positive initial value).
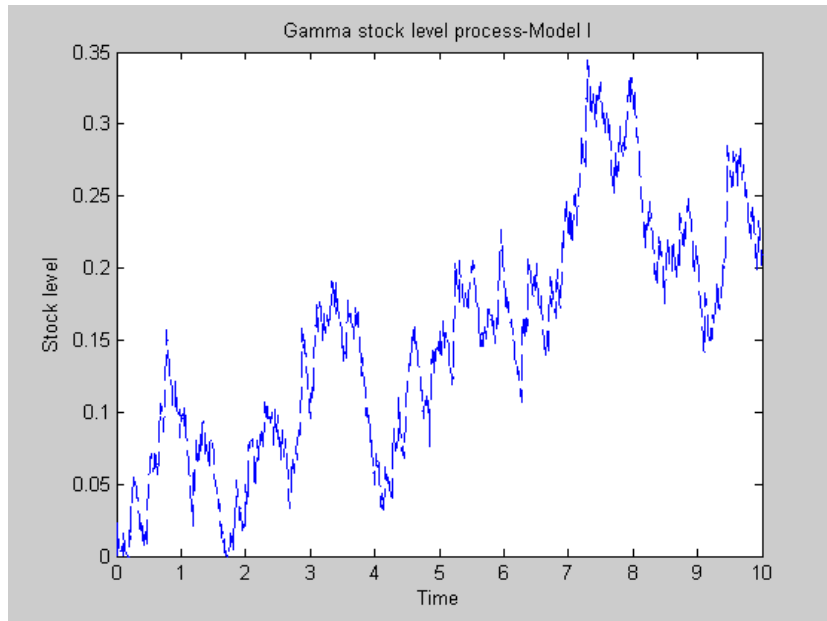
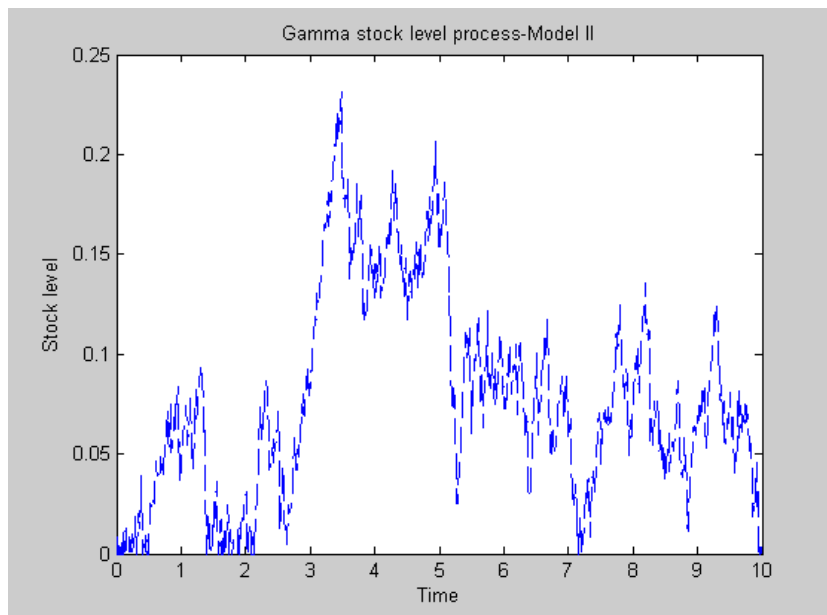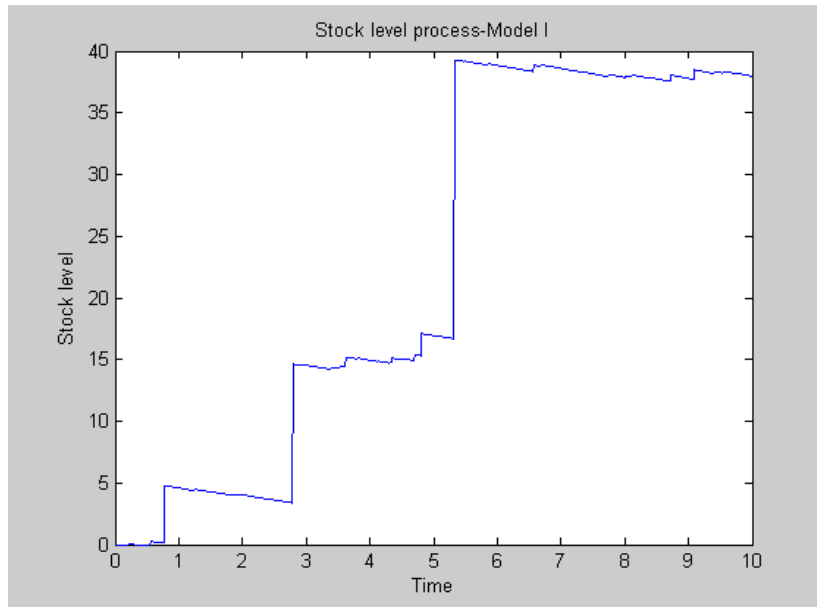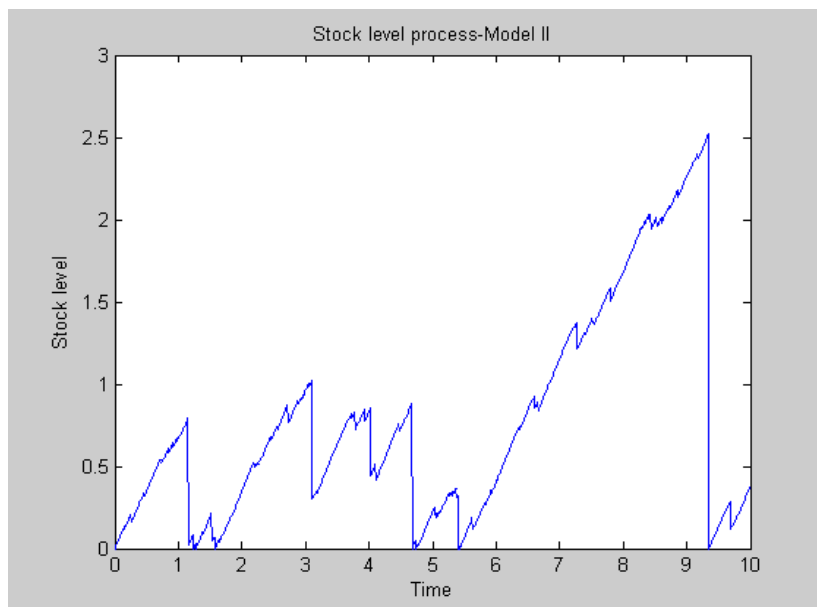Fig. 1. Gamma process (model II)

Source: own elaboration.

Fig. 2. Gamma process (model II)

Source: own elaboration.

Fig. 3. Lévy process with $\alpha = 0.5$ (model I)

Source: own elaboration.

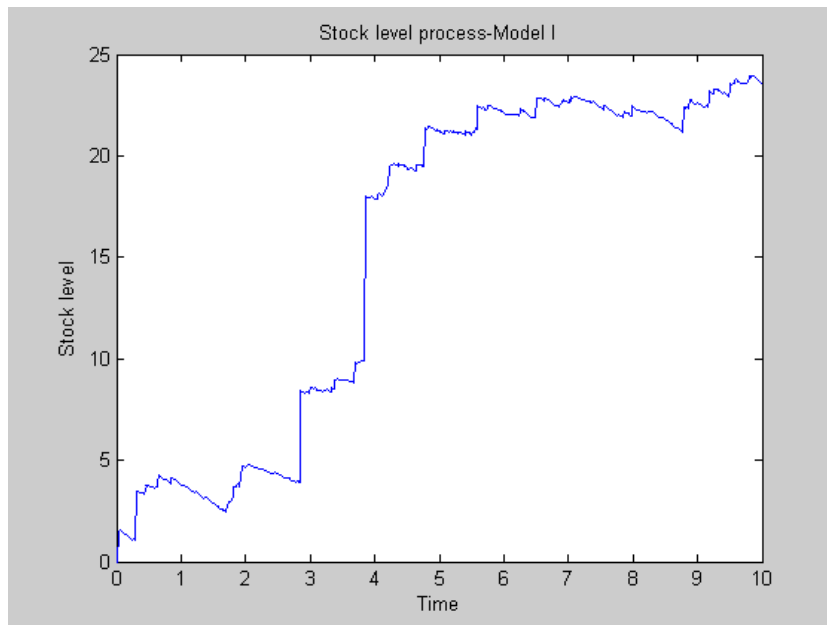Fig. 4. Lévy process with $\alpha = 0.5$ (model II)

Source: own elaboration.

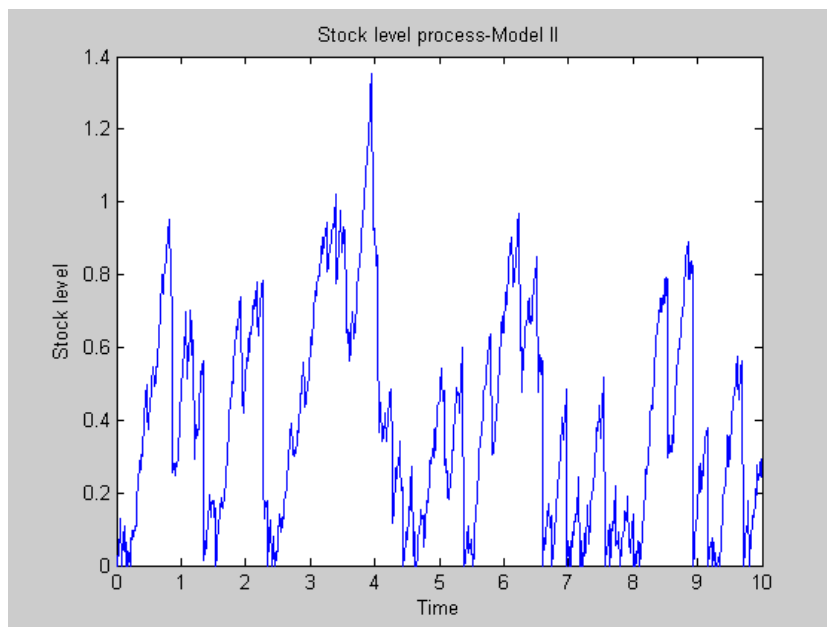Fig. 5. Cauchy process (model I)

Source: own elaboration.

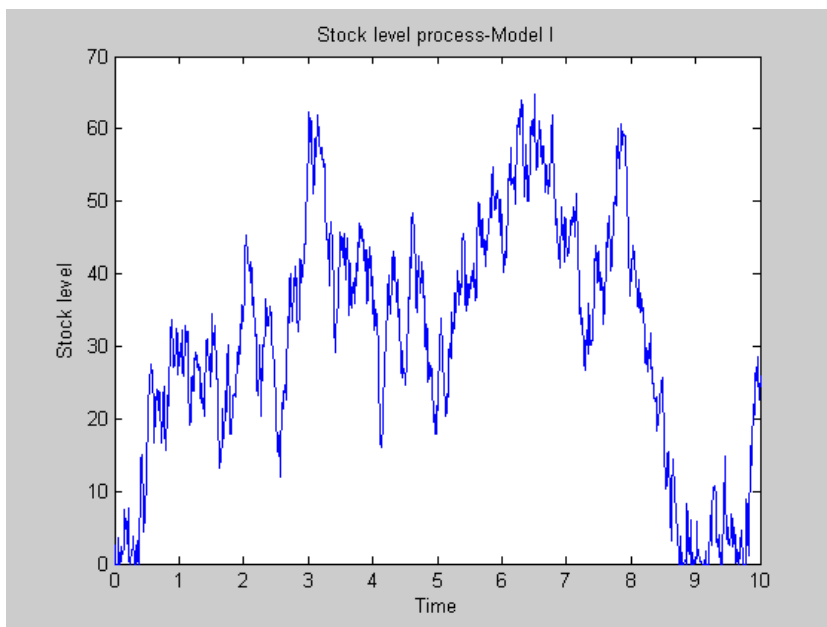Fig. 6. Cauchy process (model II)

Source: own elaboration.

Fig. 7. Wiener process (model I)
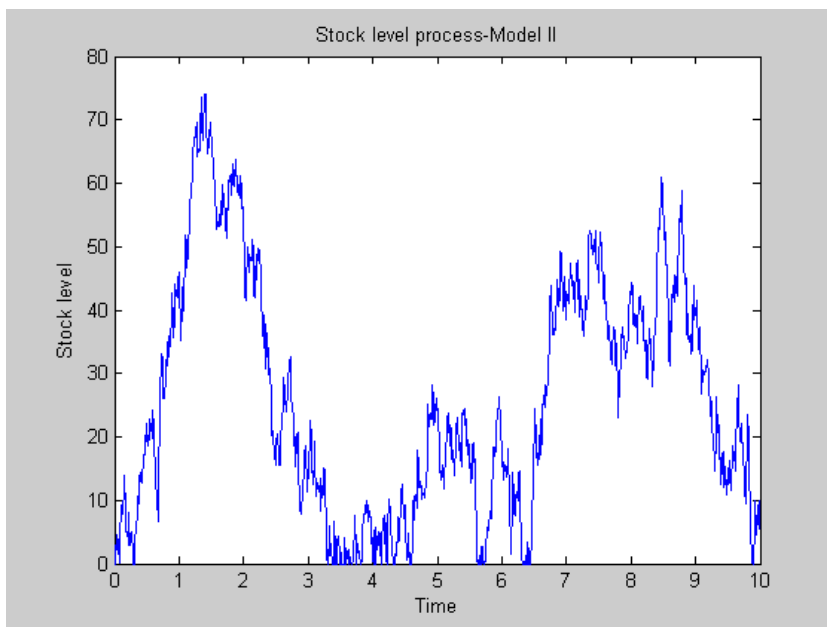
Source: own elaboration.

Fig. 8. Wiener process (model II)

Source: own elaboration.

Literature

Ahrens J., Dieter U. (1982). *Generating gamma variates by a modified rejection technique*. Communications of the ACM. Vol. 25. Pp. 47-54.

Chambers J.M., Mallows C.L. Stuck B.W. (1976). *A method for simulating stable random variables*. Journal of the American Statistical Association. Vol. 71. Pp. 340-344.

Dickson D.C.M., Waters H.R. (1993). *Gamma process and finite time survival probabilities*. ASTIN Bulletin. Vol. 23. Pp. 259-272.

Dufresne F., Gerber H.U., Shiu E.S.W. (1991). *Risk theory with the gamma Process*. ASTIN Bulletin. Vol. 21. Pp. 177-192.

Ghiani G., Laporte G., Musmanno R. (2004). *Introduction to Logistics Systems Planning and Control*. John Wiley and Sons. Chichester.

Janicki A., Weron A. (1994). *Simulation and Chaotic Behavior of α -Stable Stochastic Processes*. Marcel Dekker. New York.

Krawczyk S. (2011). *Logistyka. Teoria i praktyka*. Tom 1-2. Difin.

Michael J., Schucany W., Haas R. (1976). *Generating random variates using transformations with multiple roots.* American Statistician. Vol. 30. No. 2. Pp. 88-90.

Michna Z., Bombała, W., Nielsen P. (2013). *Lévy processes in storage and inventory models*. arXiv preprint. arXiv:1303.5964, 2013 – arxiv.org.

Nolan J.P. (1997). *Numerical calculation of stable densities and distribution functions.* Stochastic Models. Vol. 13. Pp. 759-774.

Prabhu N.U. (1998). *Stochastic Storage Processes*. Springer, New York.

Samorodnitsky G., Taqqu M. (1994). *Non-Gaussian Stable Processes: StochasticModels with Infinite Variance.* Chapman and Hall. London.

Sato K. (1999). *Lévy processes and infinitely divisible distributions*. Cambridge University Press. Cambridge.

Silver E.A. (2008). *Inventory Management: An Overview, Canadian Publications, Practical Applications and Suggestions for Future Research*. INFOR 46. Pp. 15-28.

Silver E.A., Pyke D.F., Peterson R. (1998). *Inventory Management and Production Planning and Scheduling*. John Wiley & Sons.

Weron R. (1996). *On the Chambers-Mallows-Stuck method for simulating skewed stable random variables*. Statistics and Probability Letters. Vol. 28. Pp. 165-171.