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ON THE WAYS OF FORMALIZATION AND INTERPRETATION OF THE NOTION OF "EFFICIENCY" – INTRODUCTORY REMARKS AND SOME EXAMPLES

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Abstract. In the paper we consider selected formal models, coming from the field of "pure" mathematics, as well as from some related areas, in which the notion "efficiency" appears. The presented essay may be seen as a continuation, development and (at the same time) specification of same ideas discussed in the previous article of the author: *On many-sideness, relativity and complexity of the "efficiency" (as a category)* (in Polish: *O wielostronności, relatywizmie i złożoności kategorii efektywności*). In addition to the proposals formulated in the above cited paper (concerning the classification and explanation of various "kinds" of efficiency) we introduce some new ways of meaning of this term, which we suggest to call: (a) basis-type efficiency, (b) sup (inf)-type efficiency, we also define and shortly discuss the following three types of efficiency, related to partial (pre)orders and formal logics, (c) informative capacity (reflecting the "richness" of an information contained in formulas defining given order), (d) linear similarity – efficiency (expressing a "distance of the (pre)order from the linear part" of the order in mind), (e) logical efficiency. In the final part we put together (and compare) "official" terms denoting "efficiency" and related notions presently functioning in economics, management and praxeology. The further forms of the meaning of notion "efficiency" are discussed in the "twin" paper submitted for publication in the present issue of Mathematical Economics.

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JEL Classification: C02, C60, C70.

1. Introduction

"Many a name efficiency has" – this paraphrase (of the Polish version) of the title of a renowned biographical book on the famous composer Felix B. Mendelsohn (La Mure, 1955) may be quickly decoded by all peo-

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ple of economic profession (and not only by them). This subject was relatively extensively analysed (from the semantical and praxeological points of view) by the present author in the earlier article *On many-sidedness, relativity and complexity of the "efficiency" (as a Category)* (Rybicki, 2005b). In the cited article various ways of the interpretation of this notion were proposed. The above mentioned "many-sidedness" and relativity (depending on the aspects and contexts they were used in) are observed and stressed. It is worth noting that this term functions – without any misunderstandings – in today's popular speech. So, the significant part of considered models is based on the field of economics and technical sciences, the other part – in optimization and theory of games and the remaining one – in econometric models and stochastic models of financial and insurance dynamics

The introduced notions are (sometimes "automatically") illustrated by properly chosen examples: some formal objects which appear in a linear algebra functional analysis, elementary topology, stochastic finance, prediction theory (Wold decomposition), stochastic orders and decision under risk as well as elements of welfare economics (Laplace order, Lorenz order). Some of them play "multiple roles", they provide valuable information concerning additional features of described items (attitudes toward wealth and risk of agents characterized by this function; i.e. Laplace order reflects so called mixed risk aversion of subjects, Lorenz order ranks inequality of incomes within societies). The majority of the presented examples are almost trivial from the mathematical point of view. All of them belong to classics of related fields but some notions require a bit more advanced mathematical apparatus and the knowledge of problems appearing in contemporary stochastic economics (i.e. ess sup $f(x)$, Wold decom-position, stochastic dominances, Hilbert space, martingale). The two last mentioned "kinds of efficiency" were treated in a slightly informal way. We only signal some loose ideas and thoughts.

Here we recall some metaphorical examples which may reflect the significant features of the discussed objects. Consider the case of the sport, bridge. The criterion of the assessment of the pairs (of players) is the skilful bidding and acquiring targeted result adequate to the distribution of their cards. This is in accordance with the common feeling of the term of efficiency. Effectiveness and skills of the team lie in its competence to deal with the optimization of complex problems. In general: the relations between the potential of the system (materialized in the physical and financial assets, human intelligence, as well as the quantity and quality of information) and the ways and degree of its usage, stand for the core of the formal and intuitive description of a praxeological category "efficiency".

The presentation of the article is intended to have the form of an essay. So, the considerations will be carried out in the "mixed" convention: fragments of a descriptive nature will be interwoven with formal, mathematical notations and formulas.

Before passing to the main themes of the article, we count the crucial elements of classification proposed in the cited paper. We there defined (and characterized, by examples) several approaches to the notion of efficiency, which we called (respectively): biologically-physical meaning of efficiency, mathematically economic interpretation of this notion and its cooperatively-constructive explanation. The present essay may be seen as a continuation, development and – at the same time – specification of some ideas discussed in the previous paper.

However, the primary aim of the article is the introduction of some new formal methods for "grasping the essence" of the contents of the notion "efficiency". We propose to call them in the following manner:

(a) basis-type efficiency,
(b) sup (inf)-type efficiency,
(c) linear-order efficiency,
(d) informative-order efficiency,
(e) logical efficiency.

The first type of efficiency may be interpreted as a generalization of the sup-type efficiency, but actually it constitutes the special case of this mathematical being.

The introduced proposals are accompanied by explanation, intuitive interpretations and examples. Roughly speaking: part of them refers to determining "small" sets sufficient for complete generating some "large" sets, part of them relates to determining "most economic" points limiting some sets, the others reflect some optimal properties concerning "capacity" of studied objects. Some of the proposed notions function in "pure" mathematics, some others appear in mathematical economics (in a wide sense, including so called stochastic economics). Some quoted examples are of quite an elementary level, others come from more sophisticated contexts.

The proposals formulated in points (c) and (e) were conducted in a less rigorous (more fictional) convention. At the end of the article the official terminology for the notions "efficiency", "effectiveness" and "economy" (of acting, functioning, etc.) is given.

2. The basis-type efficiency

In our opinion the proper way of approach to the formalization of the notion of efficiency would be: joining its mathematically-economic with cooperatively-constructive approaches (see (Rybicki, 2005b)). At the first approximation one may suggest to regard as "efficient" some points or subsets (of reasonably adjusted abstract spaces). The word "reasonably" should be understood as

(a) fulfilling appropriate postulates, and, at the same time,

(b) the "minimal" (in the well defined, mathematical sense) object for which these properties hold.

The above description expresses (in a little abstract formulation) the praxeological principle of attaining desirable aims with the use of minimal means. The classic phrase (somewhat incorrect from the formal point of view) states: "the maximal output at the cost of the minimal input". Passing to the standard mathematical terminology, we are going to distinguish two different formal beings which we call basis-type and sup-type efficiency, respectively. At the beginning we will discuss the first of the above approaches.

We can find elementary examples of the basis-type efficiency in the domain of linear algebra. However, one can easily notice its benchmarking character. In this case there are the bases (*sensu stricto*) which are efficient objects. Let us remember that by the basis in the usual sense we mean a maximal system of linearly independent vectors which can be found in a given linear space. At the same time this is a minimal such set spanning the whole space (in the classical case, finite-dimensional Euclidean spaces, the bases consist of finite number of elements – *ex definitione*). In standard notation we have

$$E = \operatorname{lin} B, \tag{1}$$

where:

E – linear space in mind,

$\mathrm{B} = \{b_1, ..., b_n\}$ – the set of linearly independent vectors in E. The symbol $\operatorname{lin} B$ denotes (as usually) the linear hull of B.

In the close "neighbourhood" of the above discussed bases one can find so called fundamental solutions systems of the system of linear equations (this is, of course, the special case of finite bases for model).

An analogous role play complete systems in linear normed spaces and linearly-dense subsets of topological linear spaces. Their fundamental property may be briefly indicated by the following equation

$$E = \overline{\text{lin } B} \tag{2}$$

(the bar over the right-hand term denotes topological closure).

The next class of examples (also-classical ones) are provided by bases in general topological spaces (as well as the bases of neighbourhoods). Let us remember that by the basis in the topological space (T, τ) we mean any minimal (in the sense of inclusion) family of open sets such that each open set in $T\,(G \in \tau)$ is a sum of sets belonging to this family.

An important, particular case make up countable bases in separable topological spaces – or spaces fulfilling the second countability axiom (each open set in such a space can be built as a sum derived from the countable family of open sets). In the case of metric spaces, the equivalent condition for separability is an existence of countable set of points (say D) which is dense in the space S

$$D \subset S, \quad D = \{d_i, i \in N\}, \quad S = \overline{D} \tag{3}$$

(then D itself may by meant as a basis-type efficient set).

A particularly important role is played here by so called "canonical" spaces (originated from functional spaces): separable Hilbert spaces and separable Banach spaces (including so called Polish spaces – metric, complete, separable spaces).

It seems to be an appropriate time to mention an important application of the basis-sense efficiency in contemporary stochastic finance. If every marketed financial asset is obtainable (or-replicable) as some combination of certain fundamental "primary" instruments (i.e. shares, bonds or suitable derivatives), then the market is called complete. So the minimal subset of such instruments, sufficient and necessary for replication of any evaluated asset, may be treated as a base (for the considered market). Thus, consequently, the above set may be regarded as efficient.

Some remarks should be made on the occasion. The questions concerning the efficiency of the financial markets (in dynamic, stochastic setting) à la E. Fama (1970); (Lipman, McCall, 1981) or efficient market hypothesis (EMH) as well as rational expectation hypothesis (REH) of J. Muth, R. Lucas and others (Muth, 1961; Lucas, 1972) was mentioned by the author in the article initiating the current series of papers; see also (Rybicki, 1998, 2008). We stressed there the strict connection of the martingale structure of the stochastic dynamics of finance phenomena with EMH (it is a commonly known fact). The general problem of market efficiency meant its ability to achieving the equilibrium when properly determined prices and quantities of commodities, will not be discussed here (in spite of its fundamental role in micro-, meso- and macroeconomics). These are the questions of a strictly economic character with some politics behind them: the "power of invisible hand", the role of a state as a specific agent, superior to the other participants of market, admissibility of the existence of a central planner. These themes go beyond the scope of our consideration, as we focus on searching for the common formal "denominators" to various types of efficiency. Anyway we may notice that in idealistic, competitive market assumptions, Pareto-type conclusions prevail.

We can meet an important example of basis-type efficiency while discussing the famous Wold decomposition of a stationary process (or – in a slightly general setting – the separable Hilbert space; see, i.e. (Urbanik, 1967; Mlak, 1970). Recall that the above decomposition plays a crucial role (as an important building block) in the prediction theory of stationary (wide sense) random sequences. The essence of this theoretical construction is contained in the observation that any such process admits representation as a sum of some "uninteresting" random sequence (anyway – significant for the description of the process), and the second sequence – "strictly random" one. In the case when we enumerate random variables, starting from the moment "zero" (or another finite point of time) this first part of the above decomposition becomes "trivial", being in fact generated by the finite collection of random variables (the so called determined part of the process). The remaining part of the process is called completely indetermined and it has been proved that it has a form of a sequence of cumulative sums built of white noise (discrete time) process (in general it is not a Gaussian process). This results in a specific form of mean-square optimal predictors of future variables: given "short" segment plus respective "moving average".

Following Urbanik (1967) we will present the above ideas in a more formalized way. Let $(X_n, n \in Z)$ (Z – integers) be a stationary sequence of random variables defined on a probability space $(\Omega, \boldsymbol{B}, P)$, and let $\mathbf{X}_k \subset \mathbf{X}$ be the subspace of space $\mathbf{X}$, spanned by $(X_n, n \leq k)$. $\boldsymbol{X}_k$ may be interpreted as the "linear past" of the sequence $(X_n, n \in Z)$ up to the time K (recall that $\boldsymbol{X}$ itself is also a linear subspace of Hilbert space $L^2(\Omega, \boldsymbol{B}, P)$ generated by all random variables of considered process).

Denoting $X_n^*(k) = \mathrm{Proj}_\mathrm{k}(X_n)$ (orthogonal projection of function X_n on subspace $\boldsymbol{X}_k$) we say that $X_n^*(k)$ is the predictor of X_n at time k (based on the history of process up to time k). In the cited paper the following definition was formulated.

Definition (Urbanik, 1967, p. 18)

(a) *The sequence* $(X_n, n \in Z)$ *is called determined if* $X_1^*(0) = X_1$,

(b) *The sequence* $(X_n, n \in Z)$ *is said to be completely indetermined if*

$$\lim_{k \to -\infty} X_1^*(k) = 0 . \tag{4}$$

We need to recall two important properties of introduced processes, before presenting the theorem of Wold (according to the version given in (Urbanik, 1967)

(a) A stationary sequence $(X_n, n \in Z)$ is determined if $\boldsymbol{X}_r = \boldsymbol{X}$ $(r \in Z)$,

(b) A (nontrivial) stationary sequence $(X_n, n \in Z)$ is completely indetermined if there exists a representation

$$X_n = \sum_{k=-\infty}^{0} a_k V_{k+n}, \quad (n \in Z), \tag{5}$$

where $(V_k,\ k \in Z)$ is an orthonormal sequence in $\boldsymbol{X}$.

Theorem (Wold Decomposition). *Every stationary sequence* $(X_n, n \in Z)$ *is a sum of two stationary sequences* $(X_n^{'}, n \in Z)$ *and* $(X_n^{"}, n \in Z)$ $(X_n = X_n^{'} + X_n^{"},\ n \in Z)$ *such that the following statements are true:*

(i) *for the spaces* $\boldsymbol{X}'$ *and* $\boldsymbol{X}''$ *spanned by* $(X_n^{'}, n \in Z)$ *and* $(X_n^{"}, n \in Z)$ *respectively, we have*

$$\boldsymbol{X}' \oplus \boldsymbol{X}'' = \boldsymbol{X} , \tag{6}$$

where the symbol $\oplus$ *denotes, as usual, the orthogonal sum,*

(ii) $(X_n^{'}, n \in Z)$ *is determined,*

(iii) $(X_n^{"}, n \in Z)$ *is completely indetermined.*

Remarks.

(i) If we drop the "determined" part, the role of basis (which enables efficient description of process) is taken over by an orthonormal sequence $(V_k,\ k \in Z)$.

(ii) One may easily observe the analogy with stationary autoregressive sequences admitting (infinite) moving average representation; see, i.e. (Box, Jenkins, 1970). In the latter case "all machinery governing" the random movement is "hidden" in generalized linear combinations of uncorrelated, identically distributed variables. After adding some restrictive conditions on spectral density we obtain "reasonable" expression of the present variable depending only on the "past" of the "efficient basis" $(V_n, n \in Z)$.

The subsequent example of a basis-type efficiency is provided by the family of intervals (open rays) on the reals. Immediately from the definition it follows that the family of sets

$$I = \{(-\infty, c);\ \ c \in R\} \tag{7}$$

makes the "efficient basis" for the family ***B*** of Borel sets in the space of real numbers R (let us note that ***B*** is defined as the smallest σ-algebra – in the sense of inclusion – of subsets of R, containing all open sets in R). This construction (and reasoning) automatically generalizes to higher dimensions with the obvious modification – substituting intervals by unbounded open rectangles

$$I^n = \{(-\infty, c_1)\times(-\infty, c_2)\times \ldots \times(\infty, c_n);\ \ (c_1, c_2, \ldots, c_n) \in R^n\}. \tag{8}$$

The last (but not least) mathematical example of basis-type efficiency comes from algebra. However, we will also demonstrate its importance at the field of comparisons of inequality in allocation of bundles of commodities, as well as the comparison of the riskiness of distributions (and attitudes for risky projects).

Consider the class of so called (square) permutation matrices. Such matrices consist of zeros and ones, fulfilling the condition: in each row and in each column only (and exactly) one element "1" appears. Note, by the way, that from the above assumption follows the summing of elements belonging to each row (as well as to each column) to unity. So the considered matrices make the subset of so called doubly stochastic (bi-stochastic) matrices. The latter is a significantly larger class than the former one. The famous theorem of Birkhoff (Marshall, Olkin, 1979; Arnold, 1986; Kolm, 1976; Nermuth,

1993) claims that the class of all doubly stochastic matrices can be in fact obtained as a convex hull of a class of permutation matrices. Let us write them in symbols. The set of all permutation matrices $(n \times n)$ will be denoted by $\boldsymbol{P}_n$. So

$$\mathbf{P}_n = \left\{ P_n = \left(p_{ij} \right); \quad i,\, j = 1, 2, ..., n \right\} \qquad p_{ij} \in \{0, 1\}, \quad \sum_{j=1}^{n} p_{ij} = \sum_{i=1}^{n} p_{ij} = 1. \tag{9}$$

To illustrate a "mechanics of operating" P_n consider the linear operator $p_3 : R^3 \to R^3$, the matrix of which P_3 has the form

$$P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Then

$$p_3\left((a,b,c)\right) = P_3 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}.$$

The thesis of the Birkhoff theorem may be written shortly in a form of equality

$$\boldsymbol{P}_{nn} = conv\left(\boldsymbol{P}_n\right), \tag{11}$$

where P_{nn} means here the (convex) set of all doubly stochastic matrices $n \times n$

$$\begin{gathered} \mathbf{P}_{nn} = \left\{ P_n = \left(p_{ij} \right); \quad i,\, j = 1, 2, ..., n \right\} \\ p_{ij} \geq 0; \quad i,\, j = 1, ..., n; \quad \sum_{i=1}^{n} p_{ij} = \sum_{j=1}^{n} p_{ij} = 1 \end{gathered} \tag{12}$$

and symbol $conv\left(\boldsymbol{P}_n\right)$ denotes, as usual, a convex hull of a set $\boldsymbol{P}_n$.

Concluding, we can observe that the relatively „thin" set of permutation matrices spans quite "large" set of doubly stochastic matrices – so the former may be regarded as a basis-type efficient generator of the latter.

Let us pass to presenting an important application of this theoretical fact. We will refer to a fragment of the article (Nermuth, 1993, p. 272). To this aim, consider two finite sets of objects (points) in a linear space X:

$x = \{x_1, ..., x_n\}$, $y = \{y_1, ..., y_n\}$, $x, y \in X$ and two probability vectors $\mu = \{\mu_1, ..., \mu_n\}$, $\upsilon = (\upsilon_1, ..., \upsilon_n)$ such that $\mu_i = \text{prob}(\{x_i\})$, $\upsilon_i = \text{prob}(\{y_i\})$; $i = 1, ..., n.$ So (μ, x) and (υ, y) denote the two probability distributions (supported by x and y, respectively). A distribution (μ, x) is called less dispersed than distribution (υ, y), when the following condition holds

$$\sum_{i=1}^{n} \mu_i f(x_i) \le \sum_{j=1}^{n} \upsilon_j f(y_j) \quad \text{for all convex functions} \quad f : X \to R. \qquad (13)$$

Take X to be R^k interpreting it as a set of possible quantities of commodity bundles of dimension k, and let $x_i = (x_{i1}, ..., x_{ik}) \in X$ denote the commodity bundle allocated to person *i*, in a population with n members. In this case $\mu_i = \upsilon_i = \frac{1}{n}$ for all i and x_{ih} is the amount of commodity h allocated to a person i, x itself is an $n \times k$ -matrix of allocation. Putting $u = -f$ we realize that u is a concave function. So we may adapt the previous definition.

An allocation x is more equal than an allocation y if the following condition holds

$$\sum_{i=n}^{n} u(x_i) \ge \sum_{j=1}^{n} u(y_i) \quad \text{for all concave functions } R^k \to R. \qquad (14)$$

The following crucial theorem, coming from the (Blackwell (1951)), and the appearing in the stochastic dominance as well inequality comparisons context, i.e. (Mosler, Scarsini, 1991; Kolm, 1976) gives the very important equivalent condition

Theorem. *An allocation* x *is more equal than an allocation* y *if*

$$x = P_{nn} y \qquad (15)$$

for some doubly stochastic matrix P_{nn}.

So, the definition contained in inequality (14) named more equal such allocations, which were preferred according to the utilitarian principle for (identical) utility functions with diminishing marginal utility (or – preferred by all risk averters as distributions of risky projects). Meanwhile the condition (15) says that the commodity bundle x_i of person i is a weighted mean of the bundles $y_1, ..., y_n$, with weights $p_{i1}, ..., p_{in}$. In the light of the cited theorem of Birkhoff we realize that the more equal allocation of x may be obtained from the original allocation y by a process of permuting and then mixing the original commodity bundles.

It may be seen that in each case mentioned above, the "efficient object" is a set. Generally speaking, such sets play the role of the minimal systems of generators. On the one hand, the set is sufficiently large to determine the "genuinely bigger" object. However, "depriving" it even of one element results in the vanishing of this property. It would be instructive to pay attention to some similarity of the discussed ideas with the introduced earlier (Rybicki, 2005b) cooperatively-constructive meaning of efficiency. "The case of table" can elicit its essence: three table legs constitute "a basis" for stabilizing it in a given place (on the plane). Two of them (any two) are not sufficient to this aim. Adding the fourth one creates three potential possibilities: if this additional table leg is properly adjusted, then it does not disturb the equilibrium, but it is not needed; similar effects may be observed if it is too short (in this case its "dummy" leverage); if it is too long, then it proves to be totally pointless and impossible to install.

3. Sup(inf)-type efficiency

Let us pass to sup-type efficiency. The classical "point-wise" efficient objects is "ordinary" supremum of the subset of partially ordered space (as a benchmark may serve the "most classical" $\sup A$, where A is a subset of reals: $A \subset R$) and, also, related notion: $ess \sup f(x)$, where $f(x)$ is a real function (both of them will be explained in the subsequent part of the article). If such an element belongs to the considered set, we may call it efficient in the proper sense. In the opposite case it can be named efficient in appearance. In this point we concentrate on the case of a sup-type efficiency (the inf-type efficiency could be elaborated in a perfectly analogous manner).

Without the loss of generality we may illustrate this notion in terms of subsets of reals. Let us begin by introducing the necessary assumptions and notation: $A \subset R$, A – bounded above. This means that the set of all points of R which are greater or equal to all elements of A – Upper Bound of A ($UB(A)$) is not empty.

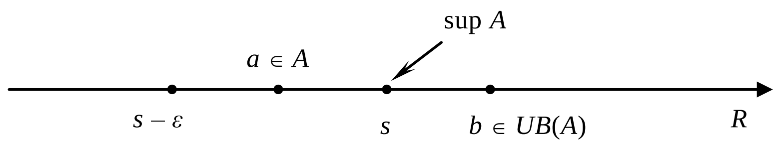

Fig. 1

The number s has two properties

(α) $\forall x \in A \quad s \geq x \quad \left(s \in UB(A)\right)$,

(β) $\forall \varepsilon > 0 \, \exists a \in A \quad a > s - \varepsilon$.

Equivalently: (β') s is the least element of $UB(A)$ or if any point (say: b) of the real line fulfils the condition (α), then $s \leq b$. In the latter formulation the definition may be immediately generalized to any ordered space $(X, \preceq)$ – it suffices to substitute the word "number" by the word "element" and "ordinary" weak inequality in a set of real numbers by the universally applied symbol $\preceq$. The idea is old and simple: s should bound A from above but any decreasing of it is impossible: the "new candidate" loses this essential property. So it is the least element bounding A from above. If we give up the assumption of boundedness of a set A, it is a natural way to "enreach" reals of element $\infty \, (-\infty)$, which leads to the so called extended real line. Then the role of "an efficient boundary" (for a set, which is not bounded above) plays an element ∞.

The related (however, slightly more subtle) notion is the so called essential supremum (essential infimum) of a measurable real function, defined on some set: $\sup_{x \in A} \operatorname{ess} f(x) \, (\inf_{x \in A} \operatorname{ess} f(x)$, (Sikorski, 1968, p. 13). Let $(X, \boldsymbol{B}, \mu)$ be the measure space. For any given $\boldsymbol{B}$-measurable real function $f : X \to R$ and the set $A \in \boldsymbol{B}$ ($\mu(A) > 0$) we define $\sup_{x \in A} \operatorname{ess} f(x)$ as the infimum of the set A^* where

$$A^* = \left\{ a \in R : \mu\left[x : (x \in A) \wedge \left(f(x) > a \right) \right] = 0 \right\}. \tag{16}$$

Equivalently we can say that $\sup_{x \in A} \operatorname{ess} f(x)$ is equal to $\inf_{g \in F} \left(\sup_{x \in A} g(x) \right)$, where F denotes the set of all functions μ-equivalent to f. It seems that this mathematical object perfectly reflects the essence of the notion of sup-efficiency (as a point-wise efficiency).

Consider the subsequent pair of (twin) mathematical examples which may prove to reveal in quite simple and elegant form of point-wise boundaries (despite the fact that one has to accept their somewhat abstract perspective). The topological closure $\bar{A}$ of the set A in topological space (X,τ), is defined as the least – in a sense of ordering by inclusion – closed set in this space, such that A is his subset (in short – the least closed superset of A). So $\bar{A}$ may be treated as a "point" (in the power set 2^x), which is inf-type effective for the family $\boldsymbol{F}_{X,A}$ closed supersets of A in (X,τ). We can write

$$\bar{A} = \inf\left\{F : F \in \boldsymbol{F}_{X,A}\right\} \tag{17}$$

Analogically, the interior of $A(\mathring{A})$ is a sup-type efficient element for the family $\boldsymbol{G}_{X,A}$ – open subsets of A and we can write – in short

$$\mathring{A} = \sup \boldsymbol{G}_{X,A} \tag{18}$$

In the next article we will discuss the subsequent approach to efficient bounding of the sets – but, unlike sup-efficiency, the introduced objects will be of a set-wise character. Anticipating this theme at the moment we will show the last example "in spirit" of sup-efficiency – but their "points" are of a complicated, "set-wise" nature.

The last several remarks of the current point will concern the problems appearing in the theory of stochastic ordering. In economics more popular is the other term: stochastic dominance. Although fields corresponding to both of the formulations almost coincide in economics' contexts, the range of the notion "stochastic (pre)orders" obeys in fact a slightly different (somewhat wider) family of relations (Mosler, Scarsini, 1991, 1991a; Szekli, 1995; Shaked, Shanthikumar, 1993).

Consider the measure space (m.s.) (Ω, S) and let $\boldsymbol{P}$ denote the set of all probability measures on (Ω, S). The preorders on $\boldsymbol{P}$ may be introduced in various ways. One of the possible definitions is to refer it to some family (say $\boldsymbol{A}$) of sets. This leads to the so called set dominance (defined) on $\boldsymbol{P}$ (Mosler, Scarsini, 1991). Formally: let $\boldsymbol{A}$ be the family of measurable subsets in m.s. (Ω, S) or $\boldsymbol{A}$.

The space $\boldsymbol{P}$ can be endowed with a partial preorder $\preceq_A$ in the following manner

(i) $P \preceq_A Q \overset{DEF}{\Leftrightarrow} P(A) \le Q(A)$ for all $A \in \boldsymbol{A}$,

or

(ii) $P \preceq_A Q \overset{DEF}{\Leftrightarrow} \int I_A(x)P(dx) \le \int I_A(x)Q(dx)$ for all $A \in \boldsymbol{A}$,

where the symbol I_A acts as an indicator function of the set A. The standard procedure (considering, step by step, simple measurable, positive functions, and next, passing monotonically increasing – to limit) leads, in the light of the Lebesgue monotone convergence theorem, to preserving generalized inequality (ii) "limit" relation

$$P \prec_{\boldsymbol{F},\boldsymbol{A}} Q \Leftrightarrow \int f(x)P(dx) \le \int f(x)Q(dx) \tag{19}$$

for all functions of the closed, convex cone, generated by indicators of sets $A \in \boldsymbol{A}$.

One may converse the procedure: starting from a family $\boldsymbol{F}$ (of real functions on Ω) to seek for family of sets generating $\boldsymbol{F}$. On the other hand, the "set stage" may be omitted (in a sense – ignored) and one can start defining the order from the family of integral kernels ($\boldsymbol{F}$). In this case we deal with the so called integral stochastic orders. It is possible to found a unified theory for such orders (Müller, 1998). It is worth noting that the "economic" aspects of such orderings (reflecting the postures of decision makers towards variable wealth or losses, including various kinds of attitudes to risk). They are done in "expected utility wine" and belong in fact to the class of stochastic dominances derived from the above integral conditions. The same is true for stochastic orders exploited in insurance mathematics, queuing and reliability theories, and some stochastic finance (Rolski, 1976; Stoyan, 1983; Alzaid, Kim, Proshan, 1991).

When we deal with real-valued variables and probability distributions on the real line, then the commonly used preorder in (such) $\boldsymbol{P}$ is an univariate stochastic order called: first order stochastic dominance, usual stochastic order or strong stochastic order (Mosler, Scarsini, 1991; Shaked, Shanthikumar, 1993; Szekli, 1995). The generating family of sets ($\boldsymbol{A}$) is then the set of all half-lines (rays) of the form (x, ∞), $x \in R$, and the corresponding set-dominance condition leads to comparisons of survival functions

$\bar{F}(x) = P\big((x,\infty)\big) = 1 - F(x)$, where F denotes the distribution function of measure P. In this case we may denote

$$P \preceq_1 Q \Leftrightarrow \bar{F}_P(x) \le \bar{F}_Q(x) \quad \forall x \in R. \tag{20}$$

The generalisation of this approach goes to considering the so called upper sets in partially ordered Polish spaces which are defined in analogous manner as "usual real half-lines" (of course, on a higher level of generality (Mosler, Scarsini, 1991; O'Brian, 1987).

There appear two dual questions when considering integral orders:

a) what is a minimal set of functions ($\boldsymbol{F}_m$) generating (as integral kernels) the order. Strictly related to this question is a problem of characterizing the smallest family of sets (***A***), which indicators can be "extended" to "whole" or "essentially true" family of kernels defining the order;

b) what is the maximal set of functions ($\boldsymbol{F}_M$), for a given (integral) stochastic order $\preceq$, compatible with this order in a sense of the relation (19):

$$P \preceq Q \;\Leftrightarrow\; \int f(x)P(dx) \le \int f(x)Q(dx) \quad \forall f \in \boldsymbol{F}_M. \tag{19a}$$

In other words, we are asking for the above mentioned "essentially true" ("necessary" and sufficient) family of integrands (integral kernels).

In the case (a) we are seeking for a "bases" for given stochastic orders. It is known that such basis for a usual stochastic order make indicators of upper rays (on line as well as in more general situations (Mosler, Scarsini, 1991). So such systems of the functions may be well regarded as basis-type efficient (for the stochastic order in mind). In the opposite extreme attempts are made for the identification of a maximal set of integral kernels, which, applied to an integral-wise definition of pre(order), provide the order which coincides with a given order. These collections play a role of sup-type efficient "objects" – in the sets consisting of generators of the order. Note that "points" of these, bounded above by the efficient element $\boldsymbol{F}_M$, have a somewhat strange and abstract nature: they are themselves families of sets of functions!

At the end we will present examples of maximal sets (or sup-type efficient elements) of functions compatible with some partial ordering for probability distribution (or, equivalently, for distribution functions if we restrict analyses to the univariate orders). The strong stochastic order is determined by strictly increasing integral kernels. This condition may be quite well accepted as a definition of usual order. In economics we say about cardinal utilities which reflect the rationale of preferring more rather than less – this is the idea of classical stochastic dominance. The so called second dominance corresponds to preferences of agents who prefer more than less and are risk averters or prefer the sure payoffs to lotteries with expectations equal to those payoffs. It comes down to postulating increasing and concave cardinal utilities – integral kernels "responsible" for obtaining such an effect. If we restrict the set of kernels to (sufficiently) smooth functions, the above postulates may be formalized as $f'(x) > 0$, $f''(x) < 0$; $x \in R$ (characterizing the subsequent example of sup-type efficiency).

Especially important is the so called Laplace-transform order, having many applications (we will come back to them later) and quite nice formal properties (bringing, in turn, very interesting interpretations – which also will be mentioned in a further part of the paper). Remember that

$$P \prec_L Q \overset{DEF}{\Leftrightarrow} \int_o^\infty e^{-st} P(dt) \geq \int_o^\infty e^{-st} Q(dt) \quad \forall s > 0\,. \tag{21}$$

The properties of this order were studied by many authors, to mention only (Rolski, 1976; Stoyan, 1983; Reuter, Riedrich, 1981; Alzaid, Kim, Proschan, 1991). It turned out that the maximal set $\boldsymbol{F_M}$ for this order makes the set of all completely monotone functions defined on the positive half-line of reals (this topic was obtained by Reuter and Riedrich (1981)). We will remember here the definition of such a function and quote the part of the main theorem proved in the cited paper.

Definiton. *Let f be a real valued function, defined on the interval $(0, \infty)$. The function f is called completely monotone, if it is everywhere differentiable arbitrarily often ($f \in C^\infty(0, \infty)$) and the sequence of inequalities*

$$(-1)^n f^{(n)}(t) \geq 0; \quad n = 0, 1, ..., \quad t \in (0, \infty) \tag{22}$$

holds.

Remarks. In other words: the signs of subsequent derivatives alternate. Immediately from the definition follow the properties (much weaker than the condition defining the notion in mind, but important for interpreting "the contents" of Laplace order): f is non-negative, non-increasing and convex on $(0, \infty)$.

The cited authors took advantage of classical lemma of S. Bernstein (from 1923, see i.e. (Achiezer, 1965; Feller, 1969)) which gives the fundamental characterization of completely monotone functions as Laplace–Stieltjes transforms of positive measures (on the interval $\langle 0;\, \infty)$), and proved that the maximal set of functions, which generates the L-ordering (for distribution functions) coincides with the set of all real functions $f(t)$ defined on the interval $(0, \infty)$, for which the first derivative is completely monotonic. Therefore we can say that the above set of functions makes sup-type efficient element for the set of families of functions compatible with L-ordering. (Remember, to supplementing of arguments that the famous Bernstein lemma establishes that two following conditions are equivalent: for a given function $f : (0, \infty) \to R$

(i) f is completely monotone,

(ii) there is exactly one non-negative measure P on $\langle 0;\, \infty)$, such that

$$f(t) = \int_0^\infty e^{-xt} P(dx), \quad t \in (0, \infty)\ ). \tag{23}$$

4. Efficiency of orders and preorders

The subsequent formal models concern the efficiency of partial orders. There is a possibility of viewing the topic from two different perspectives. The first one qualifies as "better" (pre)orders, similar – in a sense – to linear orders. Our loose proposition is to quantify they "closeness to linearity" by means of properly defined measures. Formally, these measures might visualise "masses" of relations in mind (as an area or volume of appropriate subset of Cartesian product $X \times X$ – with its whole mass normed to unity). The maximal attainable masses would correspond to the least linear (pre)order containing a given order. The "antecesor" of such a comparison methodology comes from finite sets of reals (of cardinality, say, n). Then the "usual" inequality has (not normed) mass of $\frac{n(n+1)}{2}$, whereas partial order M has less cardinality, say m. So efficiency M may be expressed by the ratio

$$e_M = \frac{2m}{n(n+1)}. \tag{24}$$

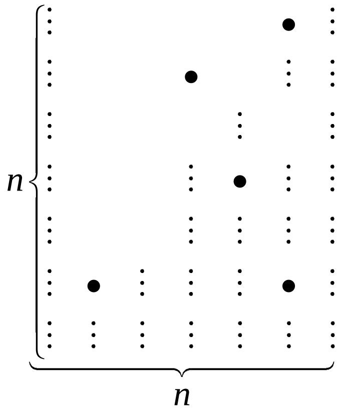

Fig. 2

In the "continuous version" of the above construction, we start from unit square $\langle 0;1\rangle \times \langle 0;1\rangle$ with (for instance) the Lebesgue measure (we may imagine simple area l of figures at the plane, subset of $X \times X$). In such a case the "natural" coefficient of efficiency of relaction M can be defined immediately as this area

$$e_M = l(M_1 \cup M_2 \cup M_3) = L(M_1) + L(M_2) + L(M_3). \tag{25}$$

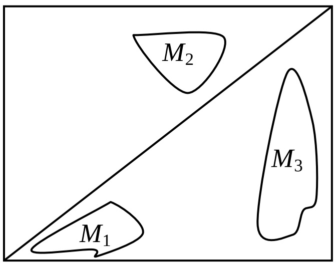

Fig. 3

It should be pointed out (once again) that both the above examples (special cases illustrating the general idea) play merely the role of proposals and suggest the direction of further investigation. These ideas demand precise definitions.

The second criterion of (pre)order efficiency we propose is the property which we called "informational capacity" (of a given order). As a preliminary announcement, which may help in understanding the idea, we give a hint: the "worst" – in a considered sense – order is an "ordinary" order on a real line, defining by "usual" (weak) inequality: $x \preceq y \overset{\text{det}}{\Leftrightarrow} x \le y \quad (x, y \in R)$. This relation is really the poorest one, because it tells us nothing but the statement that number x is no greater than number y. On the extreme polar there are such relations as preorders in functional spaces (general domains, (pre)ordered sets of values). We shall elaborate in some detail selected facts of stochastic dominances, Laplace order and in shortly, Lorenz order. They may be regarded as stochastic (pre)orders, however, other interpretations (of a formal as well as of a "practical" character) are also possible (it is, otherwise, a commonly known fact).

Let us remain for a moment in Euclidean spaces. It is obvious that the higher the dimension (of compared objects), the more subtle (richer) information can be brought and extracted (from data), which in turn, enables us to consider more comprehensive and deeper (pre)orders. The Pareto order informs us about mutual relations between elements of n pairs (corresponding) coordinates. Going a step ahead consider the other examples:

(α) $x \preceq_1 y \Leftrightarrow \sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} y_i, \quad x, y \in R_+^n$,

(β) $x \preceq_2 y \Leftrightarrow \sum_{i=1}^{n} x_i^2 \le \sum_{i=1}^{n} y_i^2, \quad x, y \in R_+^n$

(R_+^n denotes, as usual, positive orthant in R^n: $x \in R_+^n,\ x_i \ge 0;\ i = 1, ..., n$),

(γ) $x \preceq_l y$ (lexicographic order) $x, y \in R_x^n$.

All the above orders are "less informative" than the Paretian order (examples (α), (β) provide linear preorders, (γ) – linear order). So we observe the "reverse" (with respect to the property of "being similar to linearity") tendency. One may, carefully, conjecture that the linear-similarity and informative-capacity types of efficiency play mutually opposite roles. But let us leave the question open at the moment and notice subsequent, obvious (anyway, important) observation. It also could be useful to realize the "eternal" conflict between efficiency meant as coherency (compactness, economy) and as a capacity (the possibility of providing a significant amount of the information): any form of parametrization or aggregation of original, complete description of objects, results in allowing (improving, simplifying) comparison procedures, but at a cost of losing a part of characterization. So seeking for the "golden mean" has been a permanent, "historical" task of the subject.

Continuing the main discussed thread, let us pass to stochastic orders. First of all we can make the observation that they play a three-fold role (depending on point of view, purposes and interpretation). One of them we call: "static statements". We can deal with them when comparing the "shape of populations" or other "static" distribution of some quantities, mass, income or wealth. The second one appears when comparing "behaviour" of some random variable (especially: mean level of attaining values or their variability which is often identified with "riskiness"). The other is strictly connected with the previous, but is "dual for it", in a sense: it concerns the attitude of the decision maker towards risky situations, described (say) by real random variables or their distributions.

Let us begin the elaborating (of some feature) of stochastic orders, from (mentioned earlier) usual stochastic order (or – strong stochastic order, or – stochastic dominance of the first (stochastic) order). We may (in principle, equivalently) define it as a preorder in a space of random variables ("genuinely dynamic" setting) as well as – in a space of probability distributions (we restrict our consideration to the one-dimensional case). So, if P and Q will from now denote, respectively, distributions of random variables X or Y, then the notations

$$X \preceq_1 Y, \quad P \preceq_1 Q, \quad P \preceq_{st} Q \tag{26}$$

mean the same thing, namely

$$P(-\infty, x) \geq Q(-\infty, x) \quad \forall x \in R \tag{27}$$

$$\int u(x)P(dx) \leq \int u(x)Q(dx) \quad \forall x : R \to R \tag{28}$$

with u increasing, and such that the both integrals in (28) exist. In other words Q is preferred with respect to $\boldsymbol{P}$ by all agents preferring more (of wealth) than less.

It also known that by applying the so called technics of coupling (non-trivial mathematically (Szekli (1995)) these "measure-integral "comparisons can be changed by more convincing condition of point-wise comparisons

$$X \preceq_1 Y \Leftrightarrow \hat{X} \leq \hat{Y} \quad \text{almost sure,} \tag{29}$$

for some "artificially" constructed, defined on a "new" probability space, random variables, with distributions identical with the original phenomena.

The above several remarks clearly reflect the considerable amount of information contained in the discussed relation. By the way we may point out that if in condition (29) the family of functions u (utilities) would be substituted by the exactly one such a function, the "efficiency" of (pre)order drastically diminishes. Although we "gain" a complete order (instead of a partial preorder) but the relation turns out very "thin" in a sense that it describes the preferences of only one subject.

The next two orders, we comment, have even more interesting properties and interpretations – they are more informative-capacity efficient. Let us pass to variability orders and related notions. Remember that random variable X is called less than variable Y in a sense of convex order (we write $X \preceq_{cx} Y$ or $P \preceq_{cx} Q$) if

$$\int g(x)P(dx) \le \int g(x)Q(dx) \quad \forall g: R \to R, \quad g \text{ – convex,} \tag{30}$$

such that integrals in (30) exist.

The condition (30) means that the greater variability is preferred (because of the convexity of g, the "spread is highlighted and exponed" by the integral functional). The above condition is also equivalent (Goovaerts et al., 1990; Shaked, Shantihikumar, 1993; Rybicki, 2005a) to the so called stop-loss order, popular in actuarial context (after adding the condition that g has to increase), defined (for variables X and Y), by relation

$$E(X-a)^+ \le E(Y-a)^+ \quad \forall a \in R \tag{31}$$

(less – in average – excesses over each arranged level of the retention are better for insurance companies). So we can also write $X \preceq_{\text{var}} Y$ or $Y \preceq_{s-l} X$. Let us modify the point of view. The qualifying a random project Y as a "better" than less variable X reflects the subject's attitude for risk: it characterizes the so called risk seeking (or risk lovers) subjects who prefer riskier projects.

The contrasting philosophy presents the so called second (stochastic) dominance. The integral order which has been used in this case, requires (from integral kernel $g(x)$, which will be denoted traditionally, by letter $u(x)$ – "utility") to fulfil two conditions

$$\int u(x)P(dx) \le \int u(x)Q(dx) \tag{30'}$$

for each function $u: R \to R$, increasing and concave in its domain ($u(x)$ is, as a rule, continuous; in application $u(x)$ is twice differentiable and then one may write $u'(x) > 0$, $u''(x) < 0$). So the second dominance reflects a behaviour of "classical rational economic agents", preferring more (of wealth) than less and are risk averse – because the preorder (30') is in accordance with choices and rankings of member belonging to this group: concave utility "discriminates" the large deviations against central, "safe" values.

The next (commonly known) property of the convex order itself appears when applying to comparing statistical "static" populations. Assume that distributions of (say) incomes in populations ***X*** and ***Y*** coincide with distributions of X and Y. There are surprisingly close connections linking (on the level of "definitional mechanics") this order with the famous Lorenz order. Remember that the Lorenz order makes the key tool for the most universal comparing the grade of inequality in a distribution of given "mass" among the elements of a set in mind

$$X \preceq_{Lorenz} Y \Leftrightarrow \frac{X}{EX} \preceq_{cx} \frac{Y}{EX} \tag{32}$$

(Shaked, Shanthikumar, 1993; Arnold, 1986; Rybicki, 2005a, b).

The condition (32) turns out to be equivalent to the point-wise comparing of the respective Lorenz functions

$$X \preceq_{Lorenz} Y \Leftrightarrow L_X(u) \geq L_Y(u) \quad \forall u \in \langle 0, 1 \rangle .$$

So it leads to pass to (partial) preorder in a family of functions and (possibly) uncountable number of comparisons!

Concluding the last fragment of consideration we notice – at first glance – the large informative contents of the discussed order. The "informational richness" (and variety of interpretations) increases together with passing to higher orders of stochastic dominance and specifications of cardinal utilities "governing" these preferences. The popular DARA, CARA, HARA, INARA properties of attitudes of decision makers' towards the risk appear and, corresponding to them, analytical postulates concerning utility functions; see also standard risk aversion, *s*-convexity or sensivity to Dalton transfers (Kimbal, 1991, 1993; Le Breton, 1991; Denuit et al., 1999, 2001; Shorrock, Foster, 1987). We have to omit these very interesting subjects and will pass, immediately, to the "top step of the ladder". Such a role plays the stochastic dominance of infinite order – in other words: stochastic dominance for the class f completely monotonic (c.m.) utility functions (considered, among others, by Whitmore (1991) and Brocket, Golden (1987)). This order may be regarded as "most efficient in a sense of informative capacity" for many reasons. First of all, the cone of utility functions making its integral kernels consists of the class of c.m. functions (discussed by the author in an earlier part of the article). Alternating signs of subsequent derivatives indicate appearing all known "rational" properties of the decision-makers (acting according to this order). Finally, the celebrated Bernstein lemma (also cited in the article), guarantees that the first derivative of every $u \in U_\infty$ (from the above mentioned cone of infinitely many differentiable utility functions) is in fact the Laplace transform of some (Borel) measure B on $\langle 0, \infty)$

$$u'(x) = \int_0^\infty \exp(-ax)B(da) \qquad y \in (0, \infty) \tag{33}$$

It follows that the extreme points of U_∞ constitute themselves the family of exponential utilities too. So we observe the very close similarity between the Laplace transform order and considered "universal" stochastic orders. But the most significant observation (following from the fact that representation (33) implies

$$u(x) = \int_0^\infty \left[1 - \exp(-ax)\right] / a \; B(da) + u(0) \tag{33'}$$

is the following corollary: the order defined by kernels given by the formula (33') represents the *B*-mixture of preferences characterized by constant absolute risk aversions (CARA) for all positive constants a (see also, so called, mixed risk aversion (Caballé, Pomansky, 1996). In consequence: if the project X is preferred to Y in accordance to the Laplace order, then, at the same time, the reverse rankings hold for all risk averters with all positive constant absolute risk aversions. So the Laplace order as well as infinite rank stochastic order can be truly regarded as "strong efficient" in an informative-capacity sense.

5. The final remarks and conclusions

Let us begin this point from noticing "something else" on the matter of the essence of notion "efficiency" (it soon will turn out that the majority of observations and comments made in the article concern in equal measure the related terms: "effectiveness" or "economy"). We mean the kind of efficiency announced in the Introduction to be a "logic-type efficiency". It is, in a sense, a quite general approach to the discussed subject. So the next (the last one in the article) example of efficiency can be found in rudiments of classical logic. At the same time this case is of great practical importance. Very often the theorems have a "standard" structure of implication: "if A then T". This formalisation can be accompanied by a restriction that it is not possible to weaken conditions brought by A, since then the thesis T of the theorem will fail. On the other hand, it is impossible to "achieve" anything but the "original" thesis T from "original assumptions" A. Thus we deal with some form of "logical efficiency": both the so called minimal systems of assumptions and maximally strong conclusions genuinely reflect the idea of efficiency, again. The theorems of equivalence-type structures may be regarded as perfectly logically efficient.

It seems to be a proper time to present (in a very short, synthetical form) the piece of official terminology accepted by managers and accountants community (Nita, 2010; Pszczołowski, 1978; *CIMA Official Terminology*, 2005). The specialists of this field distinguish three related (but slightly different) notions: economy, efficiency and effectiveness. It is, however, worth noting that there is no unanimity in the meaning of these three (popular and even basic for the disciplines) terms. The generally recognized Polish specialist on praxeology and organization T. Pszczołowski (1978) proposed the following definitions: "Economy" (as a property of managed processes) means a relation between benefits (gains) and losses (costs) resulting in prevailing the former over the latter. In the opposite case, he says of "non-economy" (of the unit's or organisation's activity). Subsequently "efficiency" itself is characterized in the following way: act A is more efficient than act B, if it – at certain costs (losses) – gives the more valuable ("better") than B, final product. The third term of the discussed "triad", "effectiveness", is defined by him as characteristics of activities, leading to some positively assessed result regardless if it was aimed for or not.

The *CIMA Official Terminology* (2005) establishes somewhat different definitions of this system of notions. "Economy" denotes acquiring resources in the proper volume and quality at the lowest possible cost. "Efficiency" in turn is understood in the "classical" manner. There are two ("dual" in a sense) formalisations of it as principles: (α) gaining maximal useful effect for a given (in advance – for this aim) input (endowment); (β) attaining the desired effect at a minimal endowment (of resources). The last term, "effectiveness" is reserved for a modest task: making use of resources leading to achieving the aimed result. At this point the author feels obliged to make the reader sensitive to the fact that in mathematics, as well as in the wider-sense understood mathematical economics (and even in the area of economic theory), such classification is, to a considerable degree, pointless. The term "efficiency" is commonly accepted as a universal description of a whole family of various features of phenomena (sometimes substituted, without any misunderstanding, by "effectiveness"). So, "many a name" (and kinds) efficiency has, but only one "core" – there is an unquestionable agreement among researchers working at the "sufficiently serious and efficient" level of science! Some supplementary, instructive (at least, in the author's opinion) types of appearance of situations, in which one can ask the question concerning efficiency (effectiveness, economy) will be discussed in the second ("twin") paper submitted to the same issue of Mathematical Economics.

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