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PRODUCTION FUNCTION AS A MEASURE OF SCHOOL EDUCATION QUALITY

Abstract. This paper presents a model for measuring school (university) education quality on the basis of individual teacher (lecturer) quality, similarly to the model postulated by Bishop and Wößmann. Quality tuition shapes the recognition of the institution (school or university). The postulated model describes the behaviour of a given education institution as a whole. The quality of education in a given institution is represented as a process affected by all teachers involved. The paper also presents mathematical substantiation for the local maximum of the utility function resulting from the educational quality of the institution.

Key words: human capital, production function, education quality.

1. Introduction

Investment in human capital is a subject of increased interest in modern economic literature. One of the more notable approaches in this respect is to present the education process in the form of production function. A good example can be found in the works of Becker, in his analysis of time and goods allocation in human capital, among other notions. In Polish professional literature this approach is well represented by A. Niedzielski, postulating a system for cross-generation financing of education based on the CES production function. Bishop and Wößmann (J.H. Bishop, L. Wößmann (2004)), in „Institutional Effects in a Simple Model of Educational Production“, present a simple model of education quality in the form of a function representing education quality related to an individual student. Based on this model, the authors explain the influence of various institutions, such as the Central Examination Board.

This paper presents a model for measuring school (university) education quality on the basis of individual teacher (lecturer) quality, similarly to the model postulated by Bishop and Wößmann. The level of individual tuition quality is an important factor attracting the best students. Quality tuition shapes the recognition of the institution (school or university). The postulated model describes the behaviour of a given education institution as a whole. The quality of education in a given institution is represented as a process affected by all teachers involved. If educational quality of a single teacher widely contrasts with that of the remaining tutors, it affects the overall education quality of the institution under study. The paper also presents mathematical substantiation for the local maximum of the utility function resulting from the educational quality of the institution.

2. Form and theoretical assumptions of the function

Bishop and Wößmann reduce the school production function to a simple exponential Cobb-Douglas form. School quality in their approach is understood as educational effectiveness of the teaching process. Student education quality is expressed as:

$$Q_B = A_B E_B^\alpha (I_B R_B)^\beta, \quad \alpha + \beta < 1, \quad \alpha, \beta > 0, \quad (1)$$

where:

A_B – represents learning ability. This variable also accounts for any skills gathered at previous stages of the education process, as well as skills developed in relation to the student social background.

E_B – is defined as motivation and represents the time spent on learning. This is probably the most important variable of the education process, since even the least skillful students may succeed through dedication and hard work.

$I_B R_B$ – incorporates two factors that influence educational effectiveness of the institution. R_B represents the level of financial support from government directed to education, while I_B describes the efficiency of utilizing this support. The variable provides information whether the distribution of resources used for education, teaching methods and decisions is the most efficient per individual student and in a given time-frame. R_E represents only the indispensable education expenditures, as part of the governmental support for education may be utilized indirectly and independent of the education process as such. For this reason, the coefficient p is introduced, to represent this part of financial support that is not directly related to the

education process. If the overall level of government support for education is represented as X , then:

$$R_B = (1 - d)X. \quad (2)$$

Based roughly on the formula postulated by Bishop and Wößmann, the author suggests a slightly different approach to quality measurement. In this approach, production function is extended to cover the school as a whole. The postulated formula seeks to maximize the profits of a school under study in relation to individual effort of each teacher in the education process. It is assumed that all teachers act in the best interest of the school they are employed in. Consequently, the education quality of the school under study may be expressed as:

$$Q = AE_1^{\alpha_1} \cdot \dots \cdot E_n^{\alpha_n} (I_1 R_1)^{\beta_1} \cdot \dots \cdot (I_n R_n)^{\beta_n} \quad (3)$$

where:

n – is the number of teachers employed,

A_i – represents the ability to pass knowledge on the part of i 'th teacher. This factor also accounts for the ability of the teacher to adjust his/her performance to the criteria established by the education office, as well as additional skills and knowledge gathered in the course of the teacher's career. Moreover, it includes information on previous effects of education efforts of that teacher, such as the number of students taking part in contests on the subject or the percentage of top grades on school-leaving exams on the subject tutored by i .

$A = A(A_1, \dots, A_n)$ – represents the ability to pass knowledge to students on the level of school as a whole. This variable is related to abilities of all individual teachers.

E_i – is defined as motivation of the i 'th teacher to carry on his/her duties. This factor includes, among others, teacher preparation to lectures/classes.

$I_i R_i$ – are the coefficients $I_B R_B$ as defined in equation (1), but solely in relation to i 'th teacher. In other words, R_i represents this part of government expenditures that is used by the teacher with intensity I_i with the purpose of providing knowledge to students, such as e.g. the teacher's participation in state sponsored trainings or conferences.

It is also assumed that

$$\alpha_1 + \dots + \alpha_n + \beta_1 + \dots + \beta_n < 1,$$

$$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n > 0$$

and

$$A, I_1, \dots, I_n R_1, \dots, R_n, E_1, \dots, E_n > 0.$$

Hence, the effect of scale is decreasing.

The school aims to maximize its net utility (profit) in relation to individual teachers' input in the education process. Let W stand for school utility resulting from its level of education quality. This variable is meant to represent discounted value of future financial profit (such as additional financing for scientific projects or bonus paid to teachers) as well as discounted immaterial profits resulting from education process (such as parents' respect for student results or recognition from local authorities). Therefore, the expected utility of the school under study may be expressed as:

$$B_S = W A E_1^{\alpha_1} \dots E_n^{\alpha_n} (I_1 R_1)^{\beta_1} \dots (I_n R_n)^{\beta_n}. \quad (4)$$

assuming that $W > 0$. The cost of education process per i 'th teacher is expressed as:

$$C_i = c_i E_i^{\mu_i - 1}. \quad (5)$$

where c_i is a positive constant for $i \in \{1, \dots, n\}$. These include any and all costs incurred by the school and related to provision of extracurricular teacher activities, such as power consumption or the cost of making copies of education material. This also includes information on the cost of other educational activities of the school. These may involve the cost of individual training for teachers aimed at improving their educational skills. Other examples are periodic parental meetings or various integration activities and supplementary training for students. It is assumed that $\mu_i > 1$ since the final cost should be increasing. In mathematic terms, for the function

$$\frac{dC_i}{dE_i} = c_i \mu_i E_i^{\mu_i - 2}$$

to be increasing (parameter E_i), it suffices to provide $\mu_i > 1$ for $i \in \{1, \dots, n\}$. Thus, the school aims to maximize its net profits minus cost incurred, i.e. maximize the function:

$$F(E_1, \dots, E_n) = W A E_1^{\alpha_1} \dots E_n^{\alpha_n} (I_1 R_1)^{\beta_1} \dots (I_n R_n)^{\beta_n} - \sum_{i=1}^n C_i \quad (6)$$

in relation to E_i for $i \in \{1, \dots, n\}$. For maximization purposes, all other va-

¹ For clarity of notation, certain variables related to the expressions at hand will be omitted. Hence, C_i is an abridged form of $C_i(c_i, E_i, \mu_i)$.

riables are treated as given, so, for the sake of clarity, they are not included directly in function F . To maximize F , it is necessary to:

$$\frac{dF(E_1, \dots, E_n)}{dE_i} = 0 \quad \text{for } i \in \{1, \dots, n\}. \quad (7)$$

To further clarify the notation, the equation of $(I_1 R_1)^{\beta_1} \cdot \dots \cdot (I_n R_n)^{\beta_n} = V$ is introduced. Consequently, equations (7) take the forms of:

$$VWA\alpha_i E_i^{\alpha_i - 1} \prod_{k=1, k \neq i}^n (E_k^{\alpha_k}) - c_i \mu_i E_i^{\mu_i - 1} = 0 \quad \text{for } i \in \{1, \dots, n\}$$

which is equivalent to

$$VWA\alpha_i E_i^{\alpha_i - 1} \prod_{k=1, k \neq i}^n (E_k^{\alpha_k}) = c_i \mu_i E_i^{\mu_i - 1} \quad \text{for } i \in \{1, \dots, n\}.$$

Next, the logarithms for both sides are found, to obtain, respectively:

$$\ln(VWA\alpha_i) + (\alpha_i - 1)\ln(E_i) + \sum_{k \neq i} \alpha_k \ln(E_k) = \ln(c_i \mu_i) + (\mu_i - 1)\ln(E_i),$$

$i \in \{1, \dots, n\}$

$$\ln(VW\alpha_i) - \ln(c_i \mu_i) = (\mu_i - \alpha_i)\ln(E_i) - \sum_{k \neq i} \alpha_k \ln(E_k) \quad (8)$$

for $i \in \{1, \dots, n\}$.

Notations of $E_n = [E_1, \dots, E_n]'$ and $\ln(E_n) = [\ln(E_1), \dots, \ln(E_n)]'$ are introduced (logarithm operation for vector as an argument).

$$D_{V,W,A,\alpha,\mu,n} = [\ln(VWA)]\alpha_1 - \ln(c_1 \mu_1), \dots, \ln(VWA)\alpha_n - \ln(c_n \mu_n)']$$

$$P_{\alpha,\mu,n} = \begin{bmatrix} \mu_1 - \alpha_1 & -\alpha_2 & \dots & -\alpha_n \\ -\alpha_1 & \mu_2 - \alpha_2 & \dots & -\alpha_n \\ \vdots & & & \vdots \\ -\alpha_1 & -\alpha_2 & \dots & \mu_n - \alpha_n \end{bmatrix} \quad (9)$$

Consequently, the equation (8) may be transcribed to a matrix form:

$$D_{V,W,A,\alpha,\mu,n} = P_{\alpha,\mu,n} \ln(E_n).$$

Assuming that $\det P_{\alpha,\mu,n} \neq 0$, we arrive at:

$$\ln(E_n) = P_{\alpha,\mu,n}^{-1} D_{V,W,A,\alpha,\mu,n},$$

that is

$$E_n = \exp[P_{\alpha, \mu, n}^{-1} D_{V, W, A, \alpha, \mu, n}],$$

while \exp operation on matrix is meant as using \exp function in relation to each individual element of the matrix. Now it remains to demonstrate the reversibility of matrix $P_{\alpha, \mu, n}$ i.e. $\det P_{\alpha, \mu, n} \neq 0$. This gives:

$$\begin{aligned} \det P_{\alpha, \mu, n} &= \begin{vmatrix} \mu_1 - \alpha_1 & -\alpha_2 & \dots & -\alpha_n \\ -\alpha_1 & \mu_2 - \alpha_2 & \dots & -\alpha_n \\ \vdots & & & \vdots \\ -\alpha_1 & -\alpha_2 & \dots & \mu_n - \alpha_n \end{vmatrix} \begin{array}{l} \text{subtracting first row} \\ \text{from the remaining ones} \end{array} = \\ &= \begin{vmatrix} \mu_1 - \alpha_1 & -\alpha_2 & \dots & -\alpha_n \\ -\mu_1 & \mu_2 & \dots & 0 \\ \vdots & & & \vdots \\ -\mu_1 & 0 & \dots & \mu_n \end{vmatrix} = \\ &= -\mu_1 (-1)^{n+1} \underbrace{\begin{vmatrix} -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} & -\alpha_n \\ \mu_2 & 0 & \dots & & 0 \\ 0 & \mu_3 & \dots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \mu_{n-1} & 0 \end{vmatrix}}_I + (-1)^{n+n} \mu_n \underbrace{\begin{vmatrix} \mu_1 - \alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \\ -\mu_1 & \mu_2 & \dots & 0 \\ \vdots & & & \vdots \\ -\mu_1 & 0 & \dots & \mu_{n-1} \end{vmatrix}}_{II} \end{aligned}$$

Laplace expansion (last row).

Note that from assumption on positivity of α_n , all values of the first row of matrix I may be zeroed, save for the last one, that is by subtracting rescaled last column from all other columns (save for last). Such an operation will impact only the first row of matrix I , since the remaining values of the last column of matrix I are zeroed. Thus, the determinant of matrix I takes the form of:

$$I = \begin{vmatrix} 0 & 0 & \dots & 0 & -\alpha_n \\ \mu_2 & 0 & \dots & & 0 \\ 0 & \mu_3 & \dots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \mu_{n-1} & 0 \end{vmatrix} = (-1)^{n-1} \underbrace{\begin{vmatrix} \mu_2 & 0 & \dots & 0 & 0 \\ 0 & \mu_3 & \dots & & 0 \\ 0 & 0 & \dots & & 0 \\ \vdots & & & \mu_{n-1} & \vdots \\ 0 & 0 & & 0 & -\alpha_n \end{vmatrix}}_I = (-1)^n \alpha_n \mu_2 \cdot \dots \cdot \mu_{n-1}$$

The last determinant pass results from $n - 1$,th reshape of rows aimed at reshaping the matrix to diagonal form, hence the $(-1)^{n-1}$ (simple transform). In a diagonal matrix, the determinant is a product of elements of a diagonal.

As we can see, matrix **II** is matrix **I**, but „reduced by one dimension“. Consequently:

$$\det P_{\alpha,\mu,n} = -\alpha_n \mu_1 \cdot \dots \cdot \mu_{n-1} + \mu_{n-1} \det P_{\alpha,\mu,(n-1)}.$$

Using recurrent steps, we arrive at:

$$\begin{aligned} \det P_{\alpha,\mu,n} &= -\alpha_n \mu_1 \cdot \dots \cdot \mu_{n-1} + \mu_n \det P_{\alpha,\mu,(n-1)} = \\ &= -\alpha_n \mu_1 \cdot \dots \cdot \mu_{n-1} - \alpha_{n-1} \mu_1 \cdot \dots \cdot \mu_{n-2} \cdot \mu_n + \mu_n \mu_{n-1} \det P_{\alpha,\mu,(n-2)} = \\ &= \mu_1 \cdot \dots \cdot \mu_n - \sum_{i=1}^n (\alpha_i \prod_{k=1, k \neq i}^n \mu_k). \end{aligned}$$

Note that, from the underlying assumptions, the following holds true:

$$\mu_1 \cdot \dots \cdot \mu_n - \sum_{i=1}^n (\alpha_i \prod_{k=1, k \neq i}^n \mu_k) > 0.$$

Since $\alpha_1 + \dots + \alpha_n < 1$ and $\mu_i > 1$ for $i \in \{1, \dots, n\}$, then it is obvious that

$$\frac{\alpha_1}{\mu_1} + \dots + \frac{\alpha_n}{\mu_n} < 1. \tag{10}$$

After the multiplication of both sides of inequality (10) by $\mu_1 \cdot \dots \cdot \mu_n$ we arrive at:

$$\sum_{i=1}^n (\alpha_i \prod_{k=1, k \neq i}^n \mu_k) < \mu_1 \cdot \dots \cdot \mu_n.$$

It remains to provide substantiation for the existence of local maximum of function F . As seen from the equation $E_n = \exp[P_{\alpha\mu}^{-1} D_{V,W,A,\alpha,\mu,n}]$, the solution for the system of equations (7) consists of positive-valued E_i . Let $l_n = (l, l_2, \dots, l_n)$, with $l \in (0, \infty)$, while $l_2, \dots, l_n > 0$ are randomly chosen (constant). Let $f(l) = F(l_n)$. In consequence, function f takes the form of $f(l) = J_1 l^{\alpha_1} - J_2 l^{\mu_1} - J_3$ where J_1, J_2, J_3 are determined and positive (based on the equation (6)). The derivative of this function equals $f'(l) = J_1 \alpha_1 l^{\alpha_1-1} - \mu_1 J_2 l^{\mu_1-1}$. Thus $f'(l) = 0$ for $l_z = (J_1 \alpha_1 / (\mu_1 J_2))^{1/(\mu_1 - \alpha_1)}$.

At the same time, it can be seen that, as a result of underlying assumptions, $f'(l)$ is a sum of two decreasing functions, i.e.: $J_1 \alpha_1 l^{\alpha_1 - 1}$ ($\alpha_1 - 1 < 0$) and $-\mu_1 J_2 l^{\mu_1 - 1}$ ($\mu_1 - 1 > 0$). Hence, the function $f'(l)$ is also decreasing. It changes the sign from positive to negative passing the l_z point. Thus, point l_z represents the maximum of function f . The above reasoning seems to negate the existence of local minimum of function F . If function F were to reach local minimum at point $[E_1, \dots, E_n]$, then the function $F(l, E_2, \dots, E_n)$ (of one variable l) would reach local minimum. This was proven impossible. The limit of the function F at infinity equals

$$\lim_{\substack{x_1 \rightarrow \infty \\ \dots \\ x_n \rightarrow \infty}} F(x_1, \dots, x_n) = -\infty. \quad (11)$$

This results from the fact that for $x_1, \dots, x_n > 1$, simple inequalities occur:

$$\begin{aligned} F(x_1, \dots, x_n) &= WAV x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n} - \sum_{i=1}^n c_i x_i \leq \\ &\leq WAV \left[\max_{i \in \{1, \dots, n\}} \{x_i\} \right]^{\alpha_1 + \dots + \alpha_n} - \min_{i \in \{1, \dots, n\}} \{c_i\} \sum_{i=1}^n x_i \\ &\leq WAV \left[\max_{i \in \{1, \dots, n\}} \{x_i\} \right]^{\alpha_1 + \dots + \alpha_n} - \min_{i \in \{1, \dots, n\}} \{c_i\} \max_{i \in \{1, \dots, n\}} \{x_i\} \\ &= \max_{i \in \{1, \dots, n\}} \{x_i\} \left[WVA \left[\max_{i \in \{1, \dots, n\}} \{x_i\} \right]^{\alpha_1 + \dots + \alpha_n - 1} - \min_{i \in \{1, \dots, n\}} \{c_i\} \right] \end{aligned} \quad (12)$$

If

$$x_1 \rightarrow \infty, \dots, x_n \rightarrow \infty \quad \text{then} \quad \max_{i \in \{1, \dots, n\}} \{x_i\} \rightarrow \infty.$$

Thus, the first expression in square brackets of the equation (12) approaches zero, since $\alpha_1 + \dots + \alpha_n - 1 < 0$, while the second expression is constant and negative. Consequently, the whole expression of (12) approaches $-\infty$, which proves the validity of (11).

As seen from equation (11), we can select such a set of $T_{l,n} = [0, t]^n$, $t > 0$ that satisfies the condition of:

$$\forall_{t_1, \dots, t_n \in [0, t]^n} \forall_{i \in \{1, \dots, n\}} F(t_1, \dots, t_{i-1}, \underset{i\text{-th pos.}}{t}, t_{i+1}, \dots, t_n) < 0 \quad (13)$$

Let

$$x_n = (x_1, \dots, x_n), \quad x_i \in R, x_i \geq 0 \text{ for } i \in \{1, \dots, n\}.$$

$$S_{n,i}(x_n) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$$

(with 0 at the i -th position). Then, from equation (6) for any i we arrive at:

$$F(S_{n,i}(x_n)) = - \sum_{k \in \{1, \dots, n\}, k \neq i} c_k x_k^{\mu_k} \leq 0.$$

The above reasoning proves that function F at the border of set $T_{t,n}$ reaches only non-positive values.

Let

$$U_n = (u, \dots, u) \text{ (vector of length } n\text{), where } u \in (0, 1).$$

Then:

$$F(U_n) = WAV u^{\alpha_1 + \dots + \alpha_n} - \sum_{i=1}^n c_i u^{\mu_i} \geq WAV u^{\alpha_1 + \dots + \alpha_n} - \max_{i \in \{1, \dots, n\}} \{c_i\} n u^{\min\{\mu_i\}}$$

$$= u^{\alpha_1 + \dots + \alpha_n} \left[WAV - \max_{i \in \{1, \dots, n\}} \{c_i\} n u^{\min\{\mu_i\} - (\alpha_1 + \dots + \alpha_n)} \right].$$

The inequality results from the fact that

$$u^{\mu_i} < u^{\min\{\mu_i\}} \text{ for } u \in (0, 1), i \in \{1, \dots, n\}.$$

Since, in accordance with underlying assumptions,

$$\min\{\mu_i\} - (\alpha_1 + \dots + \alpha_n) > 0,$$

then

$$\lim_{u \rightarrow 0^+} \max_{i \in \{1, \dots, n\}} \{c_i\} n u^{\min\{\mu_i\} - (\alpha_1 + \dots + \alpha_n)} = 0$$

and

$$\max_{i \in \{1, \dots, n\}} \{c_i\} n u^{\min\{\mu_i\} - (\alpha_1 + \dots + \alpha_n)} > 0.$$

Consequently, we can select such $u_0 \in (0, 1)$ which satisfies the following, for $u \in (0, u_0)$:

$$\left[WAV - \max_{i \in \{1, \dots, n\}} \{c_i\} n u^{\min\{\mu_i\} - (\alpha_1 + \dots + \alpha_n)} \right] > 0.$$

Thus, the function F for a given $x \in T_{t,n}$ satisfies $F(x) > 0$. Each function on a closed and limited set will reach maximum. The set $T_{t,n}$ satisfies the conditions. Since there exist such points within the set $T_{t,n}$, for which F is positive, then the function cannot reach maximum values on the border of set $T_{t,n}$. Consequently, function F will reach maximum values within the confines of set $T_{t,n}$. As a result, point E_n ($E_n = \exp[P_{\alpha\mu}^{-1} D_{V,W,A,\alpha,\mu}]$) is a point, for which the function F reaches its local maximum².

The above calculations help establish the teacher's effort E_n that offers the most profit to the school under study. The mathematical steps presented above prove that the postulated form of school profit function has its maximum. Moreover, such a maximum may be calculated openly. Based on the ideas presented in „Institutional effects in a simple model of education production“, it is also possible to calculate the maximization of expected profits of the government in relation to government expenditure levels. Mathematically, such maximization would follow the reasoning described in this paper. Since the system of equations maximizing net profits of both school and government based on $E_1, \dots, E_n, R_1, \dots, R_n$ is solvable, then the school quality functions can be calculated at a level most profitable for both parties.

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² It is known that if a differentiable function F has a local extreme at any point, then partial derivatives of the first order zero at this point.