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PROBABILITIES OF A LOWER-LIMIT BARRIER IN THE PROBLEM OF THE IDENTIFICATION OF A BARRIER IN THE FUNCTIONING OF A CERTAIN INVENTORY STORAGE AND ISSUE SYSTEM

The paper investigates a barrier in the functioning of certain inventory system. Assuming that the storage input is a non-aggregated dynamic-parameter process, the paper derives a system of differential equations satisfied by the probabilities of a lower-limit barrier in subsysyem *L*. The system of equations expresses relations between the distributions of the probability of the lower-limit barrier on the one hand and the parameters of the product supply process and the parameters of the functioning of the transport subsystem on the other hand.

Keywords: *inventory system, barrier, transport, system of differential equations*

1. Introduction

This paper is a continuation of the investigations presented in papers [1]–[3]. The general operating principles of such a system, the definition of the notion of a barrier in its functioning, and an analysis of the behaviour of the subsystem *L* in intermediate states are given in [1] and [2]. Using the analytical forms of conditional probabilities in the case of a lower-limit barrier in this subsystem, obtained in [3], we will derive a system of differential equations satisfied by the probabilities of a lower-limit barrier in a process controlled by a non-aggregated input. The terminology and notations are the same as used in [1]–[3].

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2. Conditional probabilities in the case of a lower-limit barrier in the subsystem L

In order to obtain relations satisfied by the probabilities $Q_k^{ul}(\{0\},t)$ (cf. equation (1) in [1]), we will use the formulas for the conditional probabilities $q_{ik}^{rwl}(z,\{0\};\tau,t)$, $t \in T_l, t+\tau \in T_l$ derived in paper [3]:

for $x_i > 0, x_k < 0$,

$$q_{ik}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)} \dfrac{z+\tau x_k}{x_k - x_i} + o^{(l)}(\tau;z), & 0 \le z < -\tau x_k, \\ 0, & \text{other } z, \end{cases} \tag{1}$$

for $x_k > 0, x_i$ – any state,

$$q_{ik}^{11l}(z,\{0\};\tau,t) = o_1^{(l)}(\tau), 0 \le z \le V_1, \tag{2}$$

for any x_k, x_i,

$$q_{ik}^{00l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_0^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau), & 0 \le z < d\tau, \\ 0, & \text{other } z, \end{cases} \tag{3}$$

$$q_{kk}^{00l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_0^{*l}\tau)(1-\pi_k^{(l)}\tau) + o^{(l)}(\tau), & 0 \le z < d\tau, \\ 0, & \text{other } z, \end{cases} \tag{4}$$

for $x_k < 0$,

$$q_{kk}^{10l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) =$$

$$= \begin{cases} (1-\pi_k^{(l)}\tau)\pi_1^{*l}\tau + o^{(l)}(\tau), & d + x_k \le 0, 0 \le z < d\tau, \\ 0, & d + x_k \le 0, \text{ other } z, \\ (1-\pi_k^{(l)}\tau)\pi_1^{*l}\tau + o^{(l)}(\tau), & d + x_k > 0, 0 \le z < -x_k\tau, \\ (1-\pi_k^{(l)}\tau)\pi_1^{*l} \dfrac{d\tau - z}{d + x_k} + o^{(l)}(\tau;z), & d + x_k > 0, -x_k\tau < z < d\tau, \\ 0, & d + x_k > 0, \text{ other } z, \end{cases} \tag{5}$$

for $x_k > 0$,

$$q_{kk}^{10l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=\begin{cases}(1-\pi_k^{(l)}\tau)\pi_1^{*l}\dfrac{d\tau-z}{d+x_k}+o^{(l)}(\tau;z), & 0\le z<d\tau,\\ 0, & \text{other } z,\end{cases} \tag{6}$$

for $x_k=0$,

$$q_{kk}^{10l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=\begin{cases}(1-\pi_k^{(l)}\tau)\pi_1^{*l}\dfrac{d\tau-z}{d}+o^{(l)}(\tau;z), & 0\le z<d\tau,\\ 0, & \text{other } z,\end{cases} \tag{7}$$

for $x_k>0$,

$$q_{kk}^{11l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=0, 0\le z\le V_1. \tag{8}$$

for $x_k\le 0$,

$$q_{kk}^{11l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=\begin{cases}(1-\pi_k^{(l)}\tau)(1-\pi_1^{*l}\tau)+o^{(l)}(\tau), & 0\le z<\tau x_k,\\ 0, & \text{other } z,\end{cases} \tag{9}$$

for $x_k>0$,

$$q_{kk}^{01l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=0, 0\le z\le V_1, \tag{10}$$

for $x_k=0$,

$$q_{kk}^{01l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=\begin{cases}(1-\pi_k^{(l)}\tau)\pi_0^{*l}\left(\tau-\dfrac{z}{d}\right)+o^{(l)}(\tau;z), & 0\le z<\tau d,\\ 0, & \text{other } z,\end{cases} \tag{11}$$

for $x_k<0$,

$$q_{kk}^{01l}(z,\{0\};\tau,t)-o_1^{(l)}(\tau)=$$

$$=\begin{cases}(1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau+o^{(l)}(\tau), & d+x_k\le 0, 0\le z<d\tau,\\ 0, & d+x_k\le 0, \text{ other } z,\\ (1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau+o^{(l)}(\tau), & d+x_k>0, 0\le z<-x_k\tau,\\ (1-\pi_k^{(l)}\tau)\pi_0^{*l}\left(\tau-\dfrac{z+\tau x_k}{d+x_k}\right)+o^{(l)}(\tau;z), & d+x_k>0, -x_k\tau<z<d\tau,\\ 0, & d+x_k>0, \text{ other } z,\end{cases} \tag{12}$$

for $x_i < 0, x_k = 0$,

$$q_{ik}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\left(z + \dfrac{z}{x_i}\right) + o^{(l)}(\tau;z), & 0 \le z < -\tau x_i, \\ 0, & \text{other } z, \end{cases} \tag{13}$$

for $x_i < 0, x_k < 0$,

$$q_{ik}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) =$$

$$= \begin{cases} (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau), & x_i > x_k, 0 \le z < -\tau x_i, \\ (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\dfrac{z+\tau x_k}{x_k - x_i} + o^{(l)}(\tau;z), & x_i > x_k, -\tau x_i < z < -\tau x_k, \\ 0, & x_i > x_k, \text{ other } z, \\ (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau), & x_i < x_k, 0 \le z < -\tau x_k, \\ (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\left(\tau - \dfrac{z+\tau x_k}{x_k - x_i}\right) + o^{(l)}(\tau;z), & x_i < x_k, -\tau x_k < z < -\tau x_i, \\ 0, & x_i < x_k, \text{ other } z, \end{cases} \tag{14}$$

Using equation (6) from [3] and the relations (1)–(14) for $x_k < 0, t \in T_l, t + \tau \in T_l$, we get

$$\begin{aligned} Q_k^{1l}(\{0\},t+\tau) = \sum_i \Big\{ \int_0^{V_1} q_{ik}^{11l}(z,\{0\};\tau,t) f_i^{1l}(z,t)dz \\ + Q_i^{1l}(\{V_1\},t) q_{ik}^{11l}(V_1,\{0\};\tau,t) + Q_i^{1l}(\{0\},t) q_{ik}^{11l}(0,\{0\};\tau,t) \Big\} \\ + \int_0^{V_1} q_{kk}^{01l}(z,\{0\};\tau,t) f_k^{0l}(z,t)dz \\ + Q_k^{0l}(\{V_1\},t) q_{kk}^{01l}(V_1,\{0\};\tau,t) + Q_k^{0l}(\{0\},t) q_{kk}^{01l}(0,\{0\};\tau,t) \\ = B_{k,l}^{+1} + B_{k,l}^{01} + B_{k,l}^{-1}, \end{aligned} \tag{15}$$

where

$$B_{k,l}^{-1} = \sum_{\substack{i \ne k \\ x_i < 0}} \left\{ \int_0^{V_1} q_{ik}^{11l}(z,\{0\};\tau,t) f_i^{1l}(z,t)dz + Q_i^{1l}(\{0\},t) q_{ik}^{11l}(0,\{0\};\tau,t) \right.$$

$$\left. + Q_i^{1l}(\{V_1\},t) q_{ik}^{11l}(V_1,\{0\};\tau,t) \right\} = \sum_{\substack{i \ne k \\ x_i \ge x_k}} \left\{ \int_0^{-\tau x_i} [(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau)] f_i^{1l}(z,t)dz \right.$$

$$+\int_{-\tau x_i}^{-\tau x_k}\left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\frac{z+\tau x_k}{x_k-x_i}+o^{(l)}(\tau;z)\right]f_i^{1l}(z,t)dz+\int_{\tau x_k}^{V_1}o^{(l)}(\tau)f_i^{1l}(z,t)dz$$

$$\left.+Q_i^{1l}(\{0\},t)[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau+o^{(l)}(\tau)]+o^{(l)}(\tau)]Q_i^{1l}(\{V_1\},t)\right\}$$

$$\sum_{\substack{i\neq k\\ x_i<x_k}}\left\{\int_0^{-\tau x_k}[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau+o^{(l)}(\tau)]f_i^{1l}(z,t)dz\right.$$

$$+\int_{-\tau x_k}^{-\tau x_i}\left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\left(\tau-\frac{z+\tau x_k}{x_k-x_i}\right)+o^{(l)}(\tau;z)\right]f_i^{1l}(z,t)dz$$

$$\left.+\int_{\tau x_i}^{V_1}o^{(l)}(\tau)f_i^{1l}(z,t)dz+Q_i^{1l}(\{0\},t)[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau+o^{(l)}(\tau)]+o^{(l)}(\tau)Q_i^{1l}(\{V_1\},t)\right\},$$

$$B_{k,l}^{01}=$$

$$=\int_0^{V_1}q_{kk}^{11l}(z,\{0\};\tau,t)f_k^{1l}(z,t)dz+Q_k^{1l}(\{0\},t)q_{kk}^{11l}(0,\{0\};\tau,t)+Q_k^{1l}(\{V_1\},t)q_{kk}^{11l}(V_1,\{0\};\tau,t)$$

$$+\int_0^{V_1}q_{kk}^{01l}(z,\{0\};\tau,t)f_k^{0l}(z,t)dz+Q_k^{0l}(\{0\},t)q_{kk}^{01l}(0,\{0\};\tau,t)+Q_k^{0l}(\{V_1\},t)q_{kk}^{01l}(V_1,\{0\};\tau,t)$$

$$=\int_0^{-\tau x_k}[(1-\pi_k^{(l)}\tau)(1-\pi_1^{*l}\tau)+o^{(l)}(\tau)]f_k^{1l}(z,t)dz+\int_{-\tau x_k}^{V_1}o^{(l)}(\tau)f_k^{1l}(z,t)dz$$

$$+Q_k^{1l}(\{0\},t)[(1-\pi_k^{(l)}\tau)(1-\pi_1^{*l}\tau)+o^{(l)}(\tau)]+Q_k^{1l}(\{V_1\},t)o^{(l)}(\tau)$$

$$+\int_0^{-\tau x_k}[(1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau+o^{(l)}(\tau)]f_k^{0l}(z,t)dz$$

$$+\int_{-\tau x_k}^{\tau d}\left[(1-\pi_k^{(l)}\tau)\pi_0^{*l}\left(\tau-\frac{z+\tau x_k}{x_k+d}\right)+o^{(l)}(\tau;z)\right]f_k^{0l}(z,t)dz$$

$$+\int_{\tau d}^{V_1}o^{(l)}(\tau)f_k^{0l}(z,t)dz+Q_k^{0l}(\{0\},t)[(1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau+o^{(l)}(\tau)]+Q_k^{0l}(\{V_1\},t)o^{(l)}(\tau),$$

$$B_{k,l}^{+1} =$$

$$= \sum_{\substack{i\neq k \\ x_i \geq 0}} \left\{ \int_0^{V_1} q_{ik}^{11l}(z,\{0\};\tau,t) f_i^{1l}(z,t)dz + Q_i^{1l}(\{0\},t) q_{ik}^{11l}(0,\{0\};\tau,t) + Q_i^{1l}(\{V_1\},t) q_{ik}^{11l}(V_1,\{0\};\tau,t) \right\}$$

$$= \sum_{\substack{i\neq k \\ x_i \geq 0}} \left\{ \int_0^{-\tau x_k} \left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)} \frac{z+\tau x_k}{x_k - x_i} \right] f_i^{1l}(z,t)dz + \int_0^{V_1} o^{(l)}(\tau) f_i^{1l}(z,t)dz \right.$$

$$\left. + Q_i^{1l}(\{0\},t) \left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)} \frac{x_k \tau}{x_k - x_i} + o^{(l)}(\tau) \right] + Q_i^{1l}(\{V_1\},t) o^{(l)}(\tau) \right\}.$$

Let us now move $Q_k^{1l}(\{0\},t)$ to the left-hand side of equation (15). By dividing both sides of the resulting equation by τ and going to the limit for $\tau \to 0$, we obtain

$$\frac{\partial Q_k^{1l}(\{0\},t)}{\partial t} = \pi_0^{*l} Q_k^{0l}(\{0\}),t) - (\pi_k^{(l)} + \pi_1^{*l}) Q_k^{1l}(\{0\},t) - x_k f_k^{1l}(0,t)$$
$$+ \sum_{\substack{i\neq k \\ x_i < 0}} Q_i^{1l}(\{0\},t)\pi_{ik}^{(l)} + \sum_{\substack{i\neq k \\ x_i \geq 0}} \frac{x_k}{x_k - x_i} Q_i^{1l}(\{0\},t)\pi_{ik}^{(l)}. \tag{16}$$

As

$$Q_i^{1l}(\{0\},\, t) = 0 \,\text{for}\, x_i > 0,$$

equation (16) can be written in a simpler form. For $x_k = 0$ equation (16) is derived in a similar way. The counterpart of equation (16) for $Q_k^{0l}(\{0\},t)$ is derived analogously. Thus, the probabilities $Q_k^{ul}(\{0\},t)$ satisfy the following system of differential equations:

$$\frac{\partial Q_k^{1l}(\{0\},t)}{\partial t} = \pi_0^{*l} Q_k^{0l}(\{0\}),t) - (\pi_k^{(l)} + \pi_1^{*l}) Q_k^{1l}(\{0\},t) - x_k f_k^{1l}(0,t) + \sum_{\substack{i\neq k \\ x_i \leq 0}} Q_i^{1l}(\{0\},t)\pi_{ik}^{(l)},$$
$$x_k \leq 0, t \in T_l,$$
$$Q_k^{1l}(\{0\}),t) = 0, x_k > 0, t \in T_l, \tag{17}$$
$$\frac{\partial Q_k^{0l}(\{0\},t)}{\partial t} = \pi_1^{*l} Q_k^{1l}(\{0\}),t) - (\pi_k^{(l)} + \pi_0^{*l}) Q_k^{0l}(\{0\},t) + a f_k^{0l}(0,t) + \sum_{i\neq k} Q_i^{0l}(\{0\},t)\pi_{ik}^{(l)},$$
$$k = 1,2,\ldots,n, t \in T_l.$$

The obtained relations (17) will be used in the author's subsequent work for the quantitative identification of a barrier in the functioning of the system under consideration.

References

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Prawdopodobieństwa bariery dolnej procesu w zagadnieniu identyfikacji bariery funkcjonowania pewnego systemu gromadzenia i wydawania zapasów

Obiektem badania jest bariera działania pewnego systemu gospodarki zapasami. Przyjmując, że wejście magazynu–zbiornika jest procesem niezagregowanym o dynamicznych parametrach, wyprowadzono układ równań różniczkowych, który spełniają prawdopodobieństwa bariery dolnej podsystemu L. Układ ten wyraża powiązania między rozkładami prawdopodobieństwa bariery dolnej a parametrami procesu podaży produktu oraz parametrami funkcjonowania podsystemu transportowego.

Słowa kluczowe: *zapasy, bariera, transport, układ równań różniczkowych*