ON BAUTIN BIFURCATION IN A MONETARY DYNAMIC MODEL

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Abstract

In the paper a two-dimensional dynamic model of Asada (2011) describing the development of nominal rate of interest and expected rate of inflation is investigated. There are found conditions under which Bautin bifurcation (generalized Hopf bifurcation) arises. This kind of bifurcation enables the existence of so called corridor stability, when around the stable equilibrium two cycles emerge, the inner one which is unstable and the outer one which is stable. Numerical simulations of the model illustrating the achieved results are presented.

Key words: *dynamic model, equilibrium, stability, limit cycles*

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1. Introduction

In the paper we are going to deal with the T. Asada's model of central banking

$$Y = Y(r-\pi^e, G, \tau), Y_{r-\pi^e} = \frac{\partial Y}{\partial(r-\pi^e)} < 0, Y_G = \frac{\partial Y}{\partial G} > 0, Y_\tau = \frac{\partial Y}{\partial \tau} < 0 \quad (1)$$

$$\frac{M}{p} = L(Y, r, \pi^e), L_Y = \frac{\partial L}{\partial Y} > 0, L_r = \frac{\partial L}{\partial r} < 0, L_\pi = \frac{\partial L}{\partial \pi^e} < 0 \quad (2)$$

$$\pi = \varepsilon(Y - \bar{Y}) + \pi^e, \quad \bar{Y} > 0, \varepsilon > 0 \quad (3)$$

$$\dot{r} = \begin{cases} \alpha(\pi - \bar{\pi}) + \beta(Y - \bar{Y}) & \text{if} \quad r > 0 \\ \max(0, \alpha(\pi - \bar{\pi}) + \beta(Y - \bar{Y})) & \text{if} \quad r = 0 \end{cases} \quad (4)$$

$$\dot{\pi}^e = \gamma(\theta(\bar{\pi} - \pi^e) + (1-\theta)(\pi - \pi^e)), \gamma > 0, 0 \le \theta \le 1 \quad (5)$$

$$\sigma = \pi + \eta_\gamma \frac{\dot{Y}}{Y} - \eta_r \frac{\dot{r}}{r} - \eta_{\pi^e} \frac{\dot{\pi}^e}{\pi^e}, \quad \sigma = \frac{\dot{M}}{M} \quad (6)$$

where $\dot{r} = \frac{dr(t)}{dt}$, $\dot{\pi}^e = \frac{d\pi^e(t)}{dt}$, $\eta_\gamma = \frac{\frac{\partial L}{\partial Y}}{\frac{L}{Y}} > 0$, $\eta_r = -\frac{\frac{\partial L}{\partial r}}{\frac{L}{r}} > 0$, $\eta_{\pi^e} = -\frac{\frac{\partial L}{\partial \pi^e}}{\frac{L}{\pi^e}} > 0$ are elasticities of the real money demand with respect to changes of the real national income, nominal rate of interest and the expected rate of inflation, respectively. Variables in (1)-(6) have the following meanings:

Y - real national income (real output),
$\bar{Y}$ - natural output level corresponding to the natural rate of unemployment (fixed),
G - real government expenditures (fixed),
τ - marginal tax rate (fixed), $0 < \tau < 1$,
M - nominal money supply,
L - real money demand,
p - price level,
$\pi = \frac{\dot{p}}{p}$ - rate of inflation,
π^e - expected rate of inflation,
$\bar{\pi}$ - target rate of inflation,
r - nominal rate of interest,
$r - \pi^e$ - expected rate of interest,
σ - growth rate of nominal money supply M.

The model possesses five parameters $\alpha, \beta, \gamma, \varepsilon$ and θ.
Substituting equations (1) and (3) into equations (4), (5) and (6) we receive

$$\dot{r} = \begin{cases} f_1(r, \pi^e, \alpha, \beta, \varepsilon, G, \tau) & \text{if} \quad r > 0 \\ \max(0, f_1(r, \pi^e, \alpha, \beta, \varepsilon, G, \tau)) & \text{if} \quad r = 0 \end{cases} \tag{7}$$

$$\dot{\pi}^e = f_2(r, \pi^e, \gamma, \theta, \varepsilon, G, \tau), \tag{8}$$

$$\sigma = \varepsilon(Y(r - \pi^e, G, \tau) - \bar{Y}) + \pi^e + \eta_\gamma \frac{Y_{r-\pi^e}(\dot{r} - \dot{\pi}^e)}{Y(r - \pi^e, G, \tau)} - \eta_r \frac{\dot{r}}{r} - \eta_{\pi^e} \frac{\dot{\pi}^e}{\pi^e}, \tag{9}$$

where

$$\begin{aligned} f_1(r, \pi^e, \alpha, \beta, \varepsilon, G, \tau) &= \alpha(\varepsilon(Y(r - \pi^e, G, \tau) - \bar{Y}) + \pi^e - \bar{\pi}) \\ &\quad + \beta(Y(r - \pi^e, G, \tau) - \bar{Y}), \\ f_2(r, \pi^e, \gamma, \theta, \varepsilon, G, \tau) &= \gamma(\theta(\bar{\pi} - \pi^e) + (1 - \theta)\varepsilon(Y(r - \pi^e, G, \tau) - \bar{Y})). \end{aligned}$$

The system of equations (7)-(9) determines the dynamics of three variables (r, π^e, σ). The expression $\max(0, f_1(r, \pi^e, \alpha, \beta, \varepsilon, G, \tau))$ if $r = 0$ in (7) is so called nonnegative constraint. It prevents the nominal rate of interest to obtain negative values. In this paper we do not take this nonnegative constraint into consideration.

The system (7)-(9) is a decomposable one in the sense that the dynamics of r and π^e, which is determined by equations (7) and (8), does not depend on eq. (9). Equation (9) only plays the role to determine the growth rate of money supply σ. Therefore in the following considerations we deal only with the two-dimensional model represented by equations (7) and (8). Our aim is to find out if Bautin bifurcation can arise in the model. Bautin bifurcation which is also known as generalized Hopf bifurcation gives a complete phase portrait of solutions in a small neighborhood of the model equilibrium with respect to the values of the critical pair of parameters. This kind of bifurcation exhibits a special phenomenon that at some values of the critical pair of parameters two cycles emerge. The inner one which is unstable and the outer one which is stable. Solutions starting inside these two cycles cannot leave this domain, what means that this domain has the character of so called “corridor stability”. From economic point of view it is useful to study this phenomenon. The domain of “corridor stability” guarantees that solutions starting inside it, though they are going out from the equilibrium, do not leave this domain. The existence of Bautin bifurcation was also studied in the paper of Wu (2011), where a nonlinear Kaldor model of business cycle is analyzed, in the paper of Bella (2013) who analyzed the Goodwin's model of a class struggle, and also in the paper of Fan and Tang (2015) where a two-stage population model is studied.

2. The existence and stability of the equilibrium point

The question of the existence of an equilibrium point and its stability was rigorously solved in Asada (2016). As this part is necessary for our further considerations on the existence of Bautin bifurcation we present it also here in the whole scale.

The normal equilibrium point $E = (r^*, \pi^{e*})$ of the system (7)-(8) is determined by the relations

$$\dot{r} = 0, \;\; \dot{\pi}^e = 0, \;\; Y = \bar{Y}.$$

If we neglect the non-negative constraint of r, then we have

$$Y(r^* - \pi^{e*}, G, \tau) = \bar{Y},$$

$$\pi^{e*} = \bar{\pi}.$$

Denote

$$\rho^* = r^* - \bar{\pi}. \tag{10}$$

Then ρ^* is determined by the equation

$$Y(\rho^*, G, \tau) = \bar{Y}.$$

Solving this equation with respect to ρ^*, we have

$$\rho^* = \rho^*(G, \tau),$$

and from (10) we get the equilibrium value of r

$$r^* = \rho^* + \bar{\pi}.$$

Further we assume that $r^* > 0$ and $Y(r - \pi^e, G, \tau) \in C^7$ in a small neighbourhood of $E = (r^*, \pi^{e*})$.

Remark 1. As $G, \tau, \bar{Y}, \bar{\pi}$ are supposed to be fixed, then the value of the equilibrium $E = (r^*, \pi^{e*})$ does not depend on the values of parameters in the model.

First, transform the equilibrium $E = (r^*, \pi^{e*})$ into the origin by the shifting

$$x_1 = r - r^*, \quad x_2 = \pi^e - \pi^{e*}.$$

After the shifting the model is

$$\begin{aligned} \dot{x}_1 &= \alpha(\varepsilon(Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau) - \bar{Y}) + x_2 + \pi^{e*} - \bar{\pi}) \\ &\quad + \beta(Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau) - \bar{Y}) \equiv F_1(x_1, x_2; \alpha, \beta, \varepsilon) \\ \dot{x}_2 &= \gamma(\theta(\bar{\pi} - x_2 - \pi^{e*}) + (1 - \theta)\varepsilon(Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau) - \bar{Y}) \\ &\quad \equiv F_2(x_1, x_2; \gamma, \varepsilon, \theta) \end{aligned} \tag{11}$$

The Jacobian matrix of model (11) at the equilibrium $E_0 = (x_1^* = 0, x_2^* = 0)$ has the form

$$J(E_0;\alpha,\beta,\gamma,\varepsilon,\theta) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} F_{11} &= \tfrac{\partial F_1}{\partial x_1} = \alpha\varepsilon Y_{r-\pi^e} + \beta Y_{r-\pi^e} = (\alpha\varepsilon+\beta)Y_{r-\pi^e} < 0, \\ F_{12} &= \tfrac{\partial F_1}{\partial x_2} = -\alpha\varepsilon Y_{r-\pi^e} + \alpha - \beta Y_{r-\pi^e} = -(\alpha\varepsilon+\beta)Y_{r-\pi^e} + \alpha > 0, \\ F_{21} &= \tfrac{\partial F_2}{\partial x_1} = \gamma(1-\theta)\varepsilon Y_{r-\pi^e} < 0, \\ F_{22} &= \tfrac{\partial F_2}{\partial x_2} = \gamma(-\theta - (1-\theta)\varepsilon Y_{r-\pi^e}). \end{aligned}$$

The eigenvalues of Jacobian (12) are the roots of the characteristic equation of (12) of the form

$$\lambda^2 - \mathrm{Tr}J\lambda + \det J = 0,$$

which are given by the formula

$$\lambda_{1,2} = \frac{\mathrm{Tr}J \pm \sqrt{(\mathrm{Tr}J)^2 - 4\det J}}{2}, \tag{13}$$

where

$$\mathrm{Tr}J = (\alpha\varepsilon+\beta)Y_{r-\pi^e} + \gamma(-\theta - (1-\theta)\varepsilon Y_{r-\pi^e}) = (\alpha\varepsilon + \beta - \gamma(1-\theta)\varepsilon)Y_{r-\pi^e} - \gamma\theta,$$

$$\begin{aligned} \det J &= (\alpha\varepsilon+\beta)Y_{r-\pi^e}\gamma(-\theta-(1-\theta)\varepsilon Y_{r-\pi^e}) - (-(\alpha\varepsilon+\beta)Y_{r-\pi^e} + \alpha)\gamma(1-\theta)\varepsilon Y_{r-\pi^e} \\ &= -\gamma(\theta\beta + \alpha\varepsilon)Y_{r-\pi^e} > 0 \end{aligned}$$

Conditions for pure imaginary eigenvalues of (12) are:

$$\begin{aligned} 1.\quad \mathrm{Tr}J &= (\alpha\varepsilon+\beta-\gamma(1-\theta)\varepsilon)Y_{r-\pi^e} - \gamma\theta = 0 && (14) \\ 2.\quad \det J &= -\gamma(\theta\beta+\alpha\varepsilon)Y_{r-\pi^e} > 0 && (15) \end{aligned}$$

From (14) we have

$$Y_{r-\pi^e}\beta = (1-\varepsilon Y_{r-\pi^e})\gamma\theta + (\gamma-\alpha)\varepsilon Y_{r-\pi^e}.$$

From this we gain

$$\beta = \left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\theta + (\gamma-\alpha)\varepsilon. \tag{16}$$

Expression (16) is the equation of a line with respect to parameters θ and β. Its intersection $\tilde{\theta}$ with $\theta-$axis is given by the equation

$$\left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\tilde{\theta} + (\gamma-\alpha)\varepsilon = 0,$$

and

$$\tilde{\theta} = \frac{-(\gamma-\alpha)\varepsilon}{(\frac{1}{Y_{r-\pi^e}} - \varepsilon)\gamma} = \frac{\gamma-\alpha}{\gamma}\frac{-\varepsilon Y_{r-\pi^e}}{1-\varepsilon Y_{r-\pi^e}} < 1. \tag{17}$$

Performed considerations enable us to state the following theorem.

Theorem 1 *The following statements hold:*

1. *If* $0 < \alpha < \gamma$ *and* $0 < \theta < \tilde{\theta}, \tilde{\theta} = \frac{\gamma-\alpha}{\gamma}\frac{-\varepsilon Y_{r-\pi^e}}{1-\varepsilon Y_{r-\pi^e}}$, *then the equilibrium* $E = (r^*, \pi^{e*})$ *is*

   a) *asymptotically stable for* $\beta > \left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\theta + (\gamma-\alpha)\varepsilon$,

   b) *unstable for* $\beta < \left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\theta + (\gamma-\alpha)\varepsilon$.

2. *If* $0 < \alpha < \gamma$ *and* $\tilde{\theta} \le \theta < 1$, *then the equilibrium* $E = (r^*, \pi^{e*})$ *is asymptotically stable for all* β *and* ε.

3. *If* $\gamma \le \alpha$, *then the equilibrium* $E = (r^*, \pi^{e*})$ *is asymptotically stable for all* β, ε *and* $0 < \theta < 1$.

3. Existence of the generalized Hopf bifurcation (Bautin bifurcation)

The question of the existence of the simple Hopf bifurcation was rigorously analyzed in Asada (2016).

At the points (θ, β) lying on the segment

$$\beta = \left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\theta + (\gamma-\alpha)\varepsilon, \quad 0 < \theta < \tilde{\theta},$$

$$\tilde{\theta} = \frac{\gamma-\alpha}{\gamma}\frac{-\varepsilon Y_{r-\pi^e}}{1-\varepsilon Y_{r-\pi^e}} < 1, \quad \gamma-\alpha > 0$$

there is

$$\mathrm{Tr}J = (\alpha\varepsilon + \beta - \gamma(1-\theta)\varepsilon)Y_{r-\pi^e} - \gamma\theta = 0,$$
$$\mathrm{det}J = -\gamma(\theta\beta + \alpha\varepsilon)Y_{r-\pi^e} > 0,$$

and

$$\lambda_{1,2}(\theta;\alpha,\gamma,\varepsilon) = \pm i\sqrt{\mathrm{det}J} = \pm i\omega_0(\theta;\alpha,\gamma,\varepsilon).$$

Consider an arbitrary credibility parameter $\theta = \theta_0$, $0 < \theta < \tilde{\theta}$. Then at the pair (θ_0, β_0), $\beta_0 = \left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\theta_0 + (\gamma-\alpha)\varepsilon$ there is $\lambda_{1,2}(\theta_0) = \pm i\omega_0(\theta_0)$. We shall call this pair the critical pair of Jacobian (12). Let us fix parameters $\alpha, \gamma, \varepsilon$ in model (11). Further on we shall investigate the properties of the solutions of model (11) which respect to parameters θ and β from a small neighbourhood of the critical pair (θ_0, β_0). For this purpose it is suitable to transform θ_0 and β_0 in model (11) to the origin by shifting

$$\mu_1 = \theta - \theta_0, \quad \mu_2 = \beta - \beta_0.$$

We receive

$$\begin{aligned}
\dot{x}_1 &= \alpha(\varepsilon(Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau) - \bar{Y}) + x_2 + \pi^{e*} - \bar{\pi}) \\
&\quad + (\mu_2 + \beta_0)(Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau) - \bar{Y}) \\
&\equiv X_1(x_1, x_2; \mu_2, \alpha, \varepsilon) \\
\dot{x}_2 &= \gamma((\mu_1 + \theta_0)(\bar{\pi} - x_2 - \pi^{e*}) \\
&\quad + (1 - \mu_1 - \theta_0)\varepsilon(Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau) - \bar{Y})) \\
&\equiv X_2(x_1, x_2; \mu_1, \gamma, \varepsilon)
\end{aligned} \tag{18}$$

In shorten form model (18) can be written as

$$\dot{x} = \tilde{X}(x,\mu), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}. \tag{19}$$

Model (19) has the following properties:

1. $\tilde{X}(0,\mu) = 0$ for μ from a small neighbourhood of the critical value $\mu = 0$.
2. The eigenvalues of the Jacobian of (19) at $x = 0$ and μ from a small neighbourhood of $\mu = 0$ are
$$\lambda_{1,2}(\mu) = \delta(\mu) \pm i\omega(\mu), \quad \delta(0) = 0, \quad \omega(0) = \omega_0.$$

Perform Taylor expansion of model (19) at the equilibrium $E_0 = (x_1^*, x_2^*) = (0,0)$. We shall take into account that $Y = Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau)$ and

$$\frac{\partial Y}{\partial x_1} = \frac{\partial Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau)}{\partial (r - \pi^{e*})}, \frac{\partial Y}{\partial x_2} = -\frac{\partial Y(x_1 - x_2 + r^* - \pi^{e*}, G, \tau)}{\partial (r - \pi^{e*})}.$$

After Taylor expansion model (19) takes the form

$$\dot{x} = J(E_0;\mu)x + X(x,\mu), \tag{20}$$

where

$$J(E_0;\mu) = \begin{pmatrix} (\alpha\varepsilon + \beta_0 + \mu_2)Y_{r-\pi^e} & -(\alpha\varepsilon + \beta_0 + \mu_2)Y_{r-\pi^e} + \alpha \\ \gamma(1 - \mu_1 - \theta_0)\varepsilon Y_{r-\pi^e} & \gamma(-\mu_1 - \theta_0 - (1 - \mu_1 - \theta_0)\varepsilon Y_{r-\pi^e}) \end{pmatrix},$$

$$X(x,\mu) = \begin{pmatrix} X_1(x_1,x_2;\mu_2,\alpha,\varepsilon) \\ X_2(x_1,x_2;\mu_1,\gamma,\varepsilon) \end{pmatrix},$$

$$\begin{aligned} X_1(x_1,x_2;\mu_2,\alpha,\varepsilon) &= \frac{1}{2!}(a^{(2,0)}x_1^2 + 2a^{(1,1)}x_1x_2 + a^{(0,2)}x_2^2) \\ &+ \frac{1}{3!}(a^{(3,0)}x_1^3 + 3a^{(2,1)}x_1^2x_2 + 3a^{(1,2)}x_1x_2^2 + a^{(0,3)}x_2^3) \\ &+ \frac{1}{4!}(a^{(4,0)}x_1^4 + 4a^{(3,1)}x_1^3x_2 + 6a^{(2,2)}x_1^2x_2^2 + 4a^{(1,3)}x_1x_2^3 + a^{(0,4)}x_2^4) \\ &+ \frac{1}{5!}(a^{(5,0)}x_1^5 + 5a^{(4,1)}x_1^4x_2 + 10a^{(3,2)}x_1^3x_2^2 + 10a^{(2,3)}x_1^2x_2^3 + 5a^{(1,4)}x_1x_2^4 + a^{(0,5)}x_2^5) + \mathcal{O}(|x|^6), \end{aligned} \tag{21}$$

$$a^{(p,q)} = (-1)^q(\alpha\varepsilon + \beta_0 + \mu_2)\frac{\partial^k Y(r^* - \pi^{e*}, G, \tau)}{\partial (r - \pi^{e*})^k}, \quad p + q = k,$$

and

$$\begin{aligned} X_2(x_1,x_2;\mu_1,\gamma,\varepsilon) &= \frac{1}{2!}(b^{(2,0)}x_1^2 + 2b^{(1,1)}x_1x_2 + b^{(0,2)}x_2^2) \\ &+ \frac{1}{3!}(b^{(3,0)}x_1^3 + 3b^{(2,1)}x_1^2x_2 + 3b^{(1,2)}x_1x_2^2 + b^{(0,3)}x_2^3) \\ &+ \frac{1}{4!}(b^{(4,0)}x_1^4 + 4b^{(3,1)}x_1^3x_2 + 6b^{(2,2)}x_1^2x_2^2 + 4b^{(1,3)}x_1x_2^3 + b^{(0,4)}x_2^4) \end{aligned} \tag{22}$$

$$+\frac{1}{5!}(b^{(5,0)}x_1^5+5b^{(4,1)}x_1^4x_2+10b^{(3,2)}x_1^3x_2^2+10b^{(2,3)}x_1^2x_2^3+5b^{(1,4)}x_1x_2^4+b^{(0,5)}x_2^5)+\mathscr{O}(|x|^6),$$

$$b^{(p,q)}=(-1)^q\gamma(1-\mu_1-\theta_0)\varepsilon\frac{\partial^k Y(r^*-\pi^{e*},G,\tau)}{\partial(r-\pi^{e*})^k},\quad p+q=k.$$

Denoting $A=-(\alpha\varepsilon+\beta_0+\mu_2)Y_{r-\pi^e}, B=\gamma(1-\mu_1-\theta_0)\varepsilon Y_{r-\pi^e}$ then Jacobian $J(E_0;\mu)$ in (20) can be written in the form

$$J(E_0;\mu)=\begin{pmatrix}-A & A+\alpha\\ B & -B-\gamma(\mu_1+\theta_0)\end{pmatrix}. \tag{23}$$

In the next we shall express eigenvectors of Jacobian (23). An eigenvector $u_1=\begin{pmatrix}u_{11}\\ u_{21}\end{pmatrix}$ is determined by the equation

$$(-A-\lambda_1)u_{11}+(A+\alpha)u_{21}=0,\quad \lambda_1=\delta(\mu)+i\omega(\mu),\quad \mu=\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}. \tag{24}$$

Taking $u_{11}=A+\alpha$, we get from (24) that $u_{21}=A+\lambda_1$. An eigenvector $u_2=\begin{pmatrix}u_{12}\\ u_{22}\end{pmatrix}$ is determined by the equation

$$(-A-\lambda_2)u_{12}+(A+\alpha)u_{22}=0,\quad \lambda_2=\delta(\mu)-i\omega(\mu). \tag{25}$$

Taking $u_{12}=A+\alpha$, we get from (25) that $u_{22}=A+\lambda_2$.

Now, consider the matrix

$$\mathscr{M}=(u_1,u_2)=\begin{pmatrix}A+\alpha & A+\alpha\\ A+\lambda_1 & A+\lambda_2\end{pmatrix}.$$

Its inverse matrix $\mathscr{M}^{-1}$ is

$$\mathscr{M}^{-1}=\frac{1}{\det\mathscr{M}}\begin{pmatrix}A+\lambda_2 & -A-\alpha\\ -A-\lambda_1 & A+\alpha\end{pmatrix},$$

$$\det\mathscr{M}=(A+\alpha)(A+\lambda_2)-(A+\alpha)(a+\lambda_1)$$
$$=A(\lambda_2-\lambda_1)+\alpha(\lambda_2-\lambda_1)=-2i(A+\alpha)\omega(\mu).$$

Therefore

$$\mathscr{M}^{-1}=\frac{1}{-2i(A+\alpha)\omega(\mu)}\begin{pmatrix}A+\lambda_2 & -A-\alpha\\ -A-\lambda_1 & A+\alpha\end{pmatrix}.$$

Now, perform in (20) the transformation

$$x=\mathscr{M}y,\quad y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}.$$

We receive

$$\mathscr{M}\dot{y}=J(0;\mu)\mathscr{M}y+X(\mathscr{M}y,\mu),$$

and

$$\dot{y}=\mathscr{M}^{-1}J(0;\mu)\mathscr{M}y+\mathscr{M}^{-1}X(\mathscr{M}y,\mu), \tag{26}$$

where

$$\mathscr{M} = \begin{pmatrix} A+\alpha & A+\alpha \\ A+\lambda_1 & A+\lambda_2 \end{pmatrix},$$

$$\mathscr{M}^{-1} = \frac{1}{-2i(A+\alpha)\omega(\mu)} \begin{pmatrix} A+\lambda_2 & -A-\alpha \\ -A-\lambda_1 & A+\alpha \end{pmatrix},$$

$$J(0;\mu) = \begin{pmatrix} -A & A+\alpha \\ B & -B-\gamma(\mu_1+\theta_0) \end{pmatrix},$$

$$A = -(\alpha\varepsilon+\beta_0+\mu_2)Y_{r-\pi^e}, \quad B = \gamma(1-\mu_1-\theta_0)\varepsilon Y_{r-\pi^e}.$$

The relation between variables x_1, x_2 and the new variables y_1, y_2 is given by the formula $x = \mathscr{M}y$, what gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A+\alpha & A+\alpha \\ A+\lambda_1 & A+\lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (A+\alpha)(y_1+y_2) \\ (A+\lambda_1)y_1+(A+\lambda_2)y_2 \end{pmatrix}.$$

From this we have

$$x_1 = (A+\alpha)(y_1+y_2), \quad x_2 = (A+\lambda_1)y_1+(A+\lambda_2)y_2.$$

Putting these expressions into (21) and (22) instead of x_1 and x_2 we receive

$$X(\mathscr{M}y,\mu) = H(y,\mu) = \begin{pmatrix} H^1(y_1,y_2;\mu_2,\alpha,\varepsilon) \\ H^2(y_1,y_2;\mu_1,\gamma,\varepsilon) \end{pmatrix}.$$

System (26) gives

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \delta(\mu)+i\omega(\mu) & 0 \\ 0 & \delta(\mu)-i\omega(\mu) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2A_\alpha\omega(\mu)} \begin{pmatrix} iA_2H^1(y,\mu)-iA_\alpha H^2(y,\mu) \\ -iA_1H^1(y,\mu)+iA_\alpha H^2(y,\mu) \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \delta(\mu)+i\omega(\mu) & 0 \\ 0 & \delta(\mu)-i\omega(\mu) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2A_\alpha\omega(\mu)} \begin{pmatrix} H_1(y,\mu) \\ H_2(y,\mu) \end{pmatrix}, \tag{27}$$

where

$$y_2 = \bar{y}_1,\ A_\alpha = A+\alpha,\ A_1 = A+\lambda_1,\ A_2 = A+\lambda_2,\ A = -(\alpha\varepsilon+\beta_0+\mu_2)Y_{r-\pi^e},$$

$$\lambda_{1,2}(\mu) = \delta(\mu) \pm i\omega(\mu), \quad \delta(0) = 0, \quad \omega(0) = \omega_0,$$

$$H_1(y,\mu) = iA_2H^1(y,\mu) - iA_\alpha H^2(y,\mu),\ H_2(y,\mu) = -iA_1H^1(y,\mu) + iA_\alpha H^2(y,\mu).$$

The symbol $\overline{(\cdot)}$ means here and also in the following part of the paper the complex conjugate expression to an expression $(\cdot)$.

As the second equation in system (27) is complex conjugate to the first one, further we will work only with the first equation of (27) in the form

$$\dot{y} = \lambda(\mu)y + \frac{1}{2A_\alpha \omega(\mu)} H_1(y, \bar{y}, \mu), \tag{28}$$

$$y = y_1,\ \bar{y} = y_2,\ \lambda(\mu) = \delta(\mu) + i\omega(\mu),\ \delta(0) = 0, \quad \omega(0) = \omega_0 > 0.$$

In the following considerations we utilize the procedure which is described in Kuznetsov (2004).

Let us express equation (28) in the form

$$\dot{y} = \lambda(\mu)y + \sum_{2\le k+l\le 5} \frac{1}{k!l!} g_{kl}(\mu) y^k \bar{y}^l + \mathcal{O}(|y|^6). \tag{29}$$

Equation (29) can be transformed by an invertible parameter-dependent change of complex coordinate, smoothly depending on the parameters:

$$y = v + \sum_{2\le k+l\le 5} \frac{1}{k!l!} h_{kl}(\mu) v^k \bar{v}^l$$

for all sufficiently small $||\mu||$ into the equation

$$\dot{v} = \lambda(\mu)v + c_1(\mu)v^2\bar{v} + c_2(\mu)v^3\bar{v}^2 + \mathcal{O}(|v|^6), \tag{30}$$

$$c_1(\mu) = \frac{g_{21}}{2} + \frac{g_{20}g_{11}(2\lambda+\bar{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda-\bar{\lambda})},$$

where

$$\lambda = \lambda(\mu), \quad g_{kl} = g_{kl}(\mu),$$

$$\begin{aligned}
g_{21} &= \tfrac{i}{2\omega(\mu)}(A_1^2A_2(a_{12}-b_{03}) + A_2A_\alpha^2(a_{30}-b_{21}) - A_1A_\alpha(A_1b_{12}+2A_\alpha b_{21}) \\
&\quad +2A_1A_2A_\alpha(a_{21}-b_{12}) + 2A_1A_2^2a_{12} + A_2^2A_\alpha a_{21} - A_\alpha^3 b_{30}) + \tfrac{iA_1^2A_2^2a_{03}}{2\omega(\mu)A_\alpha}, \\
g_{20} &= \tfrac{i}{2\omega(\mu)}(2A_1A_2a_{11} + A_2A_\alpha a_{20} - A_1^2b_{02} - 2A_1A_\alpha b_{11} - A_\alpha^2 b_{20}) + \tfrac{iA_1^2A_2a_{02}}{2\omega(\mu)A_\alpha}, \\
g_{11} &= \tfrac{i}{2\omega(\mu)}(A_1A_2(a_{11}-b_{02}) + A_2A_\alpha(a_{20}-b_{11}) - A_1A_\alpha b_{11} + A_2^2a_{11} \\
&\quad -A_\alpha^2 b_{20}) + \tfrac{iA_1A_2^2a_{02}}{2\omega(\mu)A_\alpha}, \\
g_{02} &= \tfrac{i}{2\omega(\mu)}(A_2A_\alpha(a_{20}-2b_{11}) + A_2^2(2a_{11}-b_{02}) - A_\alpha^2 b_{20}) + \tfrac{iA_2^3a_{02}}{2\omega(\mu)A_\alpha}.
\end{aligned}$$

Remark 2. We do not present here the formula for the coefficient $c_2(\mu)$ as it is rather long expression.

The coefficients $c_1(\mu), c_2(\mu)$ are complex. They can be transformed into real expressions $l_1(\mu), l_2(\mu)$ by the following considerations.

Perform a time reparametrization

$$\tilde{\tau} = \omega(\mu)t$$

which gives

$$\frac{du}{d\tilde{\tau}} = (\nu(\mu)+i)\,v + d_1(\mu)\,v|v|^2 + d_2(\mu)\,v|v|^4 + \mathcal{O}(|v|^6) \tag{31}$$

with

$$\nu(\mu) = \frac{\delta(\mu)}{\omega(\mu)},\ d_1(\mu) = \frac{c_1(\mu)}{\omega(\mu)},\ d_2(\mu) = \frac{c_2(\mu)}{\omega(\mu)}.$$

Next, we introduce a new time κ such that

$$d\tilde{\tau} = (1 + e_1(\mu)\,|v|^2 + e_2(\mu)\,|v|^4)\,d\kappa,$$

where the real functions $e_{1,2}$ have yet to be defined. System (31) can be written as

$$\frac{dv}{d\kappa} = (\nu+i)\,v + ((\nu+i)\,e_1 + d_1)\,v|v|^2 + ((\nu+i)\,e_2 + e_1 d_1 + d_2)\,v|v|^4 + \mathcal{O}(|v|^6).$$

Therefore, setting

$$e_1(\mu) = -\mathrm{Im}[d_1(\mu)]$$
$$e_2(\mu) = -\mathrm{Im}[d_2(\mu)] + (\mathrm{Im}[d_1(\mu)])^2$$

yields

$$\frac{dv}{d\kappa} = (\nu(\mu)+i)\,v + l_1(\mu)\,v|v|^2 + l_2(\mu)\,v|v|^4 + \mathcal{O}(|v|^6), \tag{32}$$

where

$$l_1(\mu) = \mathrm{Re}[d_1(\mu)] - \nu(\mu)\,\mathrm{Im}[d_1(\mu)] = \frac{\mathrm{Re}[c_1(\mu)]}{\omega(\mu)} - \delta(\mu)\frac{\mathrm{Im}[c_1(\mu)]}{\omega^2(\mu)},$$

$$l_2(\mu) = \mathrm{Re}[d_2(\mu)] - \mathrm{Re}[d_1(\mu)]\,\mathrm{Im}[d_1(\mu)] + \nu(\mu)\,((\mathrm{Im}[d_1(\mu)])^2 - \mathrm{Im}[d_2(\mu)]).$$

The functions $\nu(\mu), l_1(\mu), l_2(\mu)$ are smooth and real-valued. The real function $l_1(\mu)$ is called the first Lyapunov coefficient. The real function $l_2(\mu)$ is called the second Lyapunov coefficient.

At point $\mu = 0$ there is

$$\begin{aligned}
l_1(0) &= \tfrac{\mathrm{Re}[c_1(0)]}{\omega(0)} = \tfrac{1}{2\omega_0}\,(\mathrm{Re}[g_{21}] - \tfrac{1}{\omega_0}\,\mathrm{Im}[g_{20}g_{11}]),\\
l_2(0) &= \tfrac{\mathrm{Re}[c_2(0)]}{\omega(0)} = \tfrac{1}{12}\,(\tfrac{1}{\omega_0}\mathrm{Re}[g_{32}] + \tfrac{1}{\omega_0^2}\mathrm{Im}[g_{20}\bar{g}_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \tfrac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12}]\\
&\quad + \tfrac{1}{\omega_0^3}(\mathrm{Re}[g_{20}(\bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02}(\bar{g}_{12} - \tfrac{1}{3}g_{30}) + \tfrac{1}{3}\bar{g}_{02}g_{03})\\
&\quad + g_{11}(\bar{g}_{02}(\tfrac{5}{3}\bar{g}_{30} + 3g_{12}) + \tfrac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30})] + 3\mathrm{Im}[g_{20}g_{11}]\mathrm{Im}[g_{21}])\\
&\quad + \tfrac{1}{\omega_0^4}(\mathrm{Im}[g_{11}\bar{g}_{02}(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2)] + \mathrm{Im}[g_{20}g_{11}](3\mathrm{Re}[g_{20}g_{11}] - 2|g_{02}|^2))),
\end{aligned}$$

where

$$g_{p,q} = \frac{i}{2A_\alpha\,\omega_0}\,d_1^{(p,q)}(0),\ p,q = 0,1,2,3,4,$$

$$\bar{g}_{p,q} = \frac{-i}{2A_\alpha\,\omega_0}\,d_2^{(p,q)}(0),\ p,q = 0,1,2,3,$$

$$d_k^{(p,q)}(0) = \frac{\partial^{p+q}H_k(y=0,\mu=0)}{\partial y_k^{p+q}},\ k = 1,2.$$

The point $\mu = 0$ is called the Bautin bifurcation point if $l_1(0) = 0$.

4. Numerical simulations

Consider model (7)-(8) consisting of the following functions

$$Y = C + I + G,\ C = c(Y - T) + C_0,\ T = \tau Y - T_0,\ I = \frac{\tilde{\kappa}}{e^{r-\pi^e}},$$

which gives

$$Y(r - \pi^e, G, \tau) = \frac{G + C_0 + cT_0}{1 - c(1-\tau)} + \frac{1}{1 - c(1-\tau)} \frac{\tilde{\kappa}}{e^{r-\pi^e}}.$$

Take the following values in this model:

$$\alpha = 0.1,\ \varepsilon = 1,\ \theta_0 = 0.4,\ \gamma = 2,\ c = 0.8,\ \tau = 0.4,\ G = 40,$$

$$C_0 = \frac{247}{25},\ T_0 = 2,\ \bar{\pi} = 0.02,\ \bar{Y} = 100,\ \tilde{\kappa} = \frac{13}{25} e^{0.03}.$$

After these specifications model (7)-(8) obtains the form

$$\begin{aligned} \dot{r} &= f_1(r, \pi^e, \beta) \\ \dot{\pi}^e &= f_2(r, \pi^e, \theta), \end{aligned}$$

where

$$\begin{aligned} f_1 &= \frac{1}{10}\left(\frac{e^{0.03}}{e^{r-\pi^e}} + \pi^e - \frac{51}{50}\right) + \beta\left(\frac{e^{0.03}}{e^{r-\pi^e}} - 1\right), \\ f_2 &= 2\left(\theta\left(\frac{1}{50} - \pi^e\right) + (1-\theta)\left(\frac{e^{0.03}}{e^{r-\pi^e}} - 1\right)\right). \end{aligned}$$

The equilibrium point of the model is $E = (r^* = 0.05, \pi^{e*} = 0.02)$. The intersection $\tilde{\theta}$ of the line $\beta = \left(\frac{1}{Y_{r-\pi^e}} - \varepsilon\right)\gamma\theta + (\gamma - \alpha)\varepsilon$ with θ-axis is $\tilde{\theta} = 0.475$. Critical pair of parameters (θ, β) has values $(\theta_0 = 0.4, \beta_0 = 0.3)$.

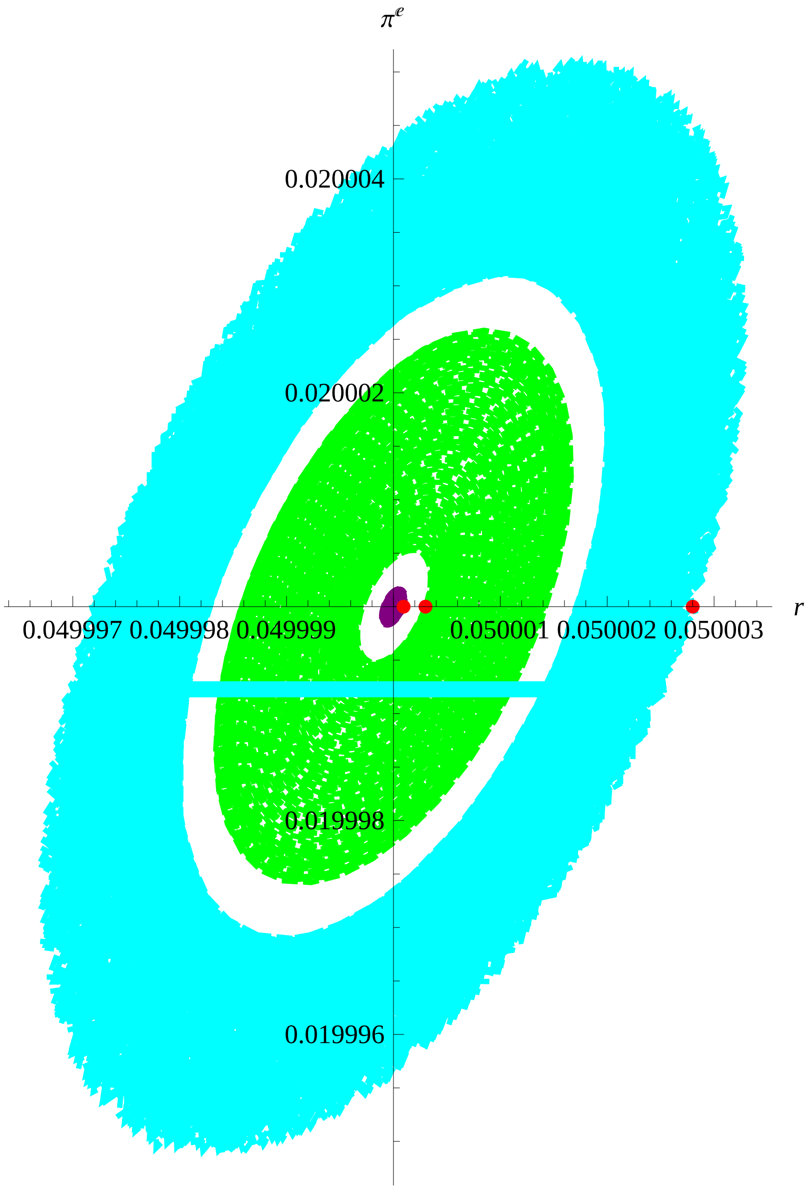
At the value $\mu = 0$ there is

$$l_1(0) = 0, \quad l_2(0) \doteq -0.0499549.$$

This means that $\mu = 0$ is the Bautin bifurcation point.

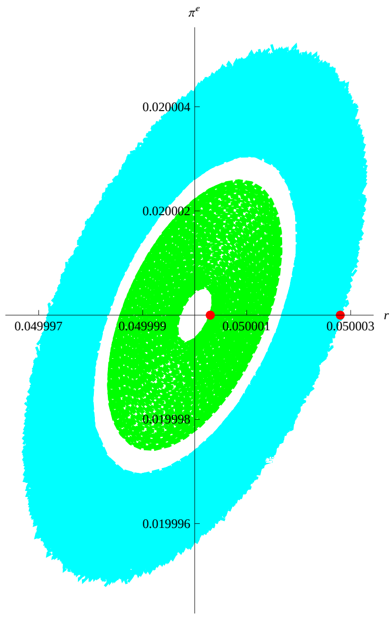
In the next part there are depicted three solutions of the model with different initial values and the same values of the critical parameters. In Fig.1 we can see all three solutions together which indicate two limit cycles - the stable one and an unstable cycle, which lies inside the first one. In Fig.2 and 3 there are depicted separated solutions. In Fig.2 there is a stable limit cycle between two solutions, in Fig.3 there is an unstable limit cycle.

Figure 1: Values of the critical parameters are $\theta = 0.401$ and $\beta = 0.299$. Initial values (red points): Inner solution starts at the point $(0.050000095, 0.02)$, middle solution at $(0.0500003, 0.02)$ and outer solution at $(0.0500028, 0.02)$.



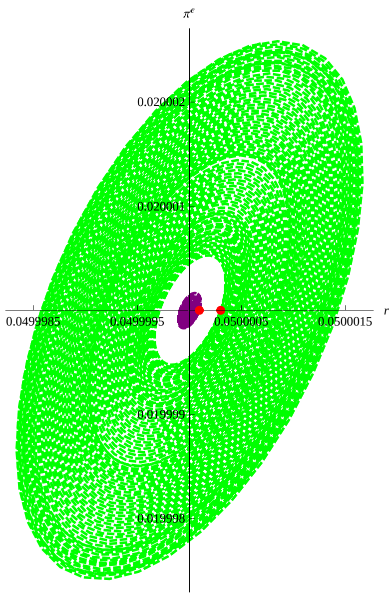
Source: The authors' work.

Figure 2: Values of the critical parameters are $\theta = 0.401$ and $\beta = 0.299$. Initial values (red points): Inner solution starts at the point $(0.0500003, 0.02)$ and outer solution at $(0.0500028, 0.02)$.



Source: The authors' work.

Figure 3: Values of the critical parameters are $\theta = 0.401$ and $\beta = 0.299$. Initial values (red points): Inner solution starts at the point $(0.050000095, 0.02)$ and outer solution at $(0.0500003, 0.02)$.



Source: The authors' work.

5. Conclusion

In the first part of the paper a two-dimensional dynamic model describing the development of nominal rate of interest and expected rate of inflation is introduced. The second part presents the results concerning the existence of an equilibrium of the model and its stability that were reached in Asada (2014). In the third part the question of the existence of Bautin bifurcation is studied. There are found conditions under which Bautin bifurcation can arise. Numerical example presented in the fourth section illustrates the reached results.

The model that is analyzed in this paper expresses monetary policy of a central bank. Its managers should control the development of nominal rate of interest and expected rate of inflation in the way to keep them in a stabilized domain. But Minsky (1982, 1986) showed that in an economy with developed financial system crises are inevitable. Managers of monetary and fiscal policy can only reduced them. But for doing this it is necessary to model the development of basic macroeconomic variables and to study their mutual relations. Such a knowledge can be very useful at controlling macroeconomic processes. The results that are reached in this paper contributes to this domain.

Asada in his paper Asada (2014) expanded the analyzed two-dimensional model to the four-dimensional one by adding to nominal rate of interest an expected rate of inflation also the developments of private firms debt and output, and to the six-dimensional model, adding to the four-dimensional model the developments of government bond and government expenditures. So far these models have been analyzed only with respect to the existence of equilibria and their stability, and the possible existence of cycles was only indicated. The question of the existence of cycles has not been rigorously analyzed yet.

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