DIDACTICS OF MATHEMATICS

No. 13 (17) 2016

FREE MEIXNER DISTRIBUTIONS

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Abstract. In this paper the most important distributions in free probability (and maybe in the universe) will be presented. It is worth emphasizing that these distributions are not known among economists.

Keywords: Meixner distributions.

Jel Classification: C40, C46.

DOI: 10.15611/dm.2016.13.02.

1. Introduction

In this paper we will focus on the presentation of free distributions of a Meixner type. By 'type' we understand measures defined up to offset and dilation. Below we describe the probabilistic measures for normalized measures, i.e. with zero mean and variance equal to one, which depend on two parameters. In general, free Meixner distributions depend on four parameters.

Classical Meixner distributions were introduced in terms of orthogonal polynomials in [Meixner 1934]. Meixner's system of orthogonal polynomials in free probability was established by Anshelevich [Anshelevich 2003], Saitoh and Yoshida [Saitoh, Yoshida 2001], and Bożejko and Wysoczański [Bożejko, Wysoczanski 2001]. Meixner free distributions can be classified as the following six types: Wigner's distribution, free Poisson distribution, free Pascal distribution (free negative binomial distribution), free gamma distribution, free binomial distribution, and pure free Meixner distribution. This classification was presented by Bożejko and Bryc [Bożejko, Bryc 2006], and was inspired by the fact that classical Meixner distributions, with similar parameters, satisfy the so-called Laha–Lukacs properties. The current paper is a review and does not present free probability theory to readers. The readers of this paper are assumed to be familiar with the basic ideas of free harmonic analysis.

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2. Meixner distributions

In this section we present the most important facts related to free Meixner distributions. A distribution which can be given by the Cauchy–Stieltjes transform of the form

$$G_{\mu_{a,b}}(z)=\int_{\boldsymbol{R}}\frac{1}{z-y}d\mu_{a,b}(dy)=$$
$$\frac{(1+2b)z+a-\sqrt{(z-a)^2-4(1+b)}}{2(bz^2+az+1)}=\cfrac{1}{z-\cfrac{1}{z-a-\cfrac{b+1}{z-a-\cfrac{b+1}{\ddots}}}},\tag{1}$$

where the branch of the square root should satisfy $\Im(G_\mu(z))\leq 0$ for $\Im(z)>0$ (see [Saitoh, Yoshida 2001]), is called a normalized free Meixner distribution $\{\mu_{a,b}: a\in\mathbb{R}, b\geq -1\}$.

Equation (1) describes the distribution with zero mean and variance equal to one. Absolutely continuous part of $\mu_{a,b}$ equals

$$\frac{\sqrt{4(1+b)-(x-a)^2}}{2\pi(bx^2+ax+1)}dx,\tag{2}$$

where $a-2\sqrt{(1+b)}\leq x\leq a+2\sqrt{(1+b)}$. This measure has one atom if $0\leq 4b<a^2$, and two atoms if $-1\leq b<0$. For a given parameterization, monic polynomials that are orthogonal with respect to the measure $\mu_{a,b}$, satisfy the relations

$$(x-a)p_n(x)=p_{n+1}(x)+(b+1)p_{n-1}(x), n=2,3,...,\tag{3}$$

where

$$p_0(x)=1, p_1(x)=x,\tag{4}$$

or equivalently, Jacobi parameters are of the form

$$J(\mu_{a,b})=\begin{pmatrix}0, & a, & a, & a, & \dots\\ 1, & b+1, & b+1, & b+1, & \dots\end{pmatrix}.\tag{5}$$

Assuming that $m_i(\mu)$ is the i^{th} moment of the measure μ, then the generating function of the moments corresponding to equation (1) is of the form

$$M_{\mu_{a,b}}(z)=\sum_{i=0}^{\infty}m_i(\mu_{a,b})=\frac{1}{z}G_{\mu_{a,b}}\left(\frac{1}{z}\right)=\frac{1+2b+az-\sqrt{(1-za)^2-4z^2(1+b)}}{2(z^2+az+b)},\tag{6}$$

for sufficiently small $|z|$. The R-transform corresponding to $M(z)$ is given by

$$\mathcal{R}_{\mu_{a,b}}(z) = \sum_{i=0}^{\infty} R_{i+1}\,(\mathrm{X}) z^i = G_{\mu_{a,b}}^{-1}(z) - 1/z = \frac{2z}{1-za+\sqrt{(1-za)^2-4z^2b}}, \quad (7)$$

where the square root should be chosen as $\lim_{z\to 0} \mathcal{R}_\mu\,(z) = 0$ (see [Saitoh, Yoshida 2001]). The numbers R_i are called free cumulants of the probability measure $\mu_{a,b}$.

Depending on the values of a and b, the distribution $\mu_{a,b}$ can become one of the following six types:

- Wigner's distribution, if $a = b = 0$;
- free Poisson distribution, if $b = 0$ and $a \neq 0$;
- free Pascal distribution (free negative binomial distribution), if $b > 0$ and $a^2 > 4b$;
- free gamma distribution, if $b > 0$ and $a^2 = 4b$;
- pure free Meixner distribution, if $b > 0$ and $a^2 < 4b$;
- free binomial distribution, if $-1 \leq b < 0$.

There are so-called Kesten's measures in this classification (see [Kesten 1959]) obtained with $b \neq 0$ and $a = 0$, i.e. their density is given by

$$\frac{\sqrt{4(1+b)-x^2}}{2\pi(bx^2+1)}. \quad (8)$$

Saitoh and Yoshida in [Saitoh, Yoshida 2001] proved that Meixner distributions are free infinitely divisible if and only if $b \geq 0$ (for a given parameterization). The Lévy–Khinchin representation in this case takes the beautiful form

$$\mathcal{R}_{\mu_{a,b}}(z) = \int_{\mathbb{R}} \frac{z}{1-xz} w_{a,b}(dx), \quad (9)$$

where $w_{a,b}$ is Wigner's measure with mean a and variance b. In particular, we obtain from (9) and (7)

$$R_{n+2}(\mu_{a,b}) = \int_{\mathbb{R}} x^n\, w_{a,b}(dx). \quad (10)$$

Another interesting formula for the cumulants of the free Meixner distributions is the following equation (from [Bożejko, Bryc 2006]):

$$R_{n+2}(\mu_{a,b}) = \sum_{\nu\in NC_{1,2}(n)} a^{s(\nu)}\, b^{|\nu|-s(\nu)}, \quad (11)$$

where $s(\nu)$ are blocks of size 1 with partitions, whereas $NC_{1,2}(n)$ is a set of all non-crossing partitions of the set $\{1, \dots, n\}$, such that each block of partitions is of size either 1 or 2, i.e. $|B_i| = 1$ or $|B_i| = 2$.

We proceed now to the main result of Bożejko and Bryc in [Bożejko, Bryc 2006] that supports the premise from the abstract. They proved that random variables with linear conditional first moment and quadratic conditional variance have free Meixner distributions.

Theorem 1. *Let us assume that* X, Y *are freely independent, self-adjoint, non-degenerate elements of a non-commutative probabilistic space, and that there are constants* $\alpha, \alpha_0, a, b, C \in \mathbb{R}$ *such that*

$$E(\mathrm{X}|\mathrm{X}+\mathrm{Y}) = \alpha(\mathrm{X}+\mathrm{Y}) + \alpha_0 \tag{12}$$

and

$$Var(\mathrm{X}|\mathrm{X}+\mathrm{Y}) = C[1 + a(\mathrm{X}+\mathrm{Y}) + b(\mathrm{X}+\mathrm{Y})^2]. \tag{13}$$

Then random variables X and Y have free Meixner distributions. In particular, when $E(\mathrm{X}) = E(\mathrm{Y}) = 0$ and $E(\mathrm{X}^2 + \mathrm{Y}^2) = 1$, then the distribution of X is one of the six types presented above.

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D I D A C T I C S O F M A T H E M A T I C S

No. 13 (17) 2016

SOME REMARKS ABOUT SERIES

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Abstract. This paper presents a proof of the classical theorem of the theory of series. This proof would be used in lectures on the series theory.

Keywords: series, convergent, divergent series, harmonic series.

Jel Classification: C00.

DOI: 10.15611/dm.2016.13.03.

1. Introduction

In the lectures of calculus the direct proof of the theorem that the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ is convergent for $1<\alpha$ is rarely introduced. The knowledge about the convergence of this series is used in exercises but the proof of the convergence of this series is presented on the whole by the integral criterion. In standard textbooks of calculus it is difficult to find a direct proof of convergence of the series. This paper presents a direct proof of convergence of the series. This text is a supplement for numerous books of calculus.

Theorem. *The series*

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$$

is divergence where $0<\alpha\le 1$ *and convergence where* $1<\alpha$.

For the proof of the theorem it is necessary to show a lot of lemmata.

Lemma 1. *The harmonic series* $\sum_{n=1}^{\infty}\frac{1}{n}$ *is divergent.*

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Proof. The harmonic series is equal such that:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{8}+\ldots+\frac{1}{2^n+1}+\ldots+\frac{1}{2^{n+1}}+\ldots.$$

It is possible to group the terms of harmonic series:

$$a_0=1+\frac{1}{2^1}=\frac{3}{2},\quad a_1=\frac{1}{2^1+1}+\frac{1}{2^2}=\frac{1}{3}+\frac{1}{4},$$

$$a_2=\frac{1}{2^2+1}+\frac{1}{2^2+2}+\frac{1}{2^2+3}+\frac{1}{2^3}=\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8},\ \ldots,$$

$$a_n=\frac{1}{2^n+1}+\ldots+\frac{1}{2^{n+1}}.$$

The harmonic series is equal

$$\sum_{n=1}^{\infty}\frac{1}{n}=\sum_{n=0}^{\infty}a_n.$$

It is obvious that $1+\frac{1}{2}>\frac{1}{2}$, and $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, and $\frac{1}{5}+\ldots+\frac{1}{8}>\frac{1}{8}+\ldots+\frac{1}{8}=\frac{1}{2}$, and $\frac{1}{2^n+1}+\ldots+\frac{1}{2^n+2^n}>2^n\cdot\frac{1}{2^{n+1}}=\frac{1}{2}$. From here the result below is true:

$$\sum_{n=1}^{\infty}\frac{1}{n}>\frac{1}{2}+\frac{1}{2}+\ldots,$$

i.e. the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Proof of the theorem: the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ is divergence for $0<\alpha\le 1$.

If α is a number such that $0<\alpha<1$ then for natural numbers the inequality $n^{\alpha}<n$ holds, hence the unequal $\frac{1}{n}<\frac{1}{n^{\alpha}}$ holds too, i.e. the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is a minorant of series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ for $0<\alpha<1$. Hence the minorant is a divergence series the series (1) is divergence for $0<\alpha<1$.

Lemma 2. *For each natural number n the expression is true:*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}.$$

Proof by induction.

For $n=1$ the left side is equal $\frac{1}{2}$ and the right side is equal $\frac{1}{2}$ too. Suppose that for some *n* the expression above holds. It is necessary to prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}.$$

The left side of the equality above by the induction assumption is equal

$$\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{n\cdot(n+2)+1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)},$$

it is obvious that

$$\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2},$$

so the proof of the lemma is complete.

Corollary. The series $\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)}$ is convergent, the sum is equal to 1.

Proof. Because $\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)}=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n\cdot(n+1)}\right)=\lim_{n\to\infty}\frac{n}{n+1}=1$, the thesis of the corollary is true.

Lemma 3. *The series* $\sum_{n=1}^{\infty}\frac{1}{n^2}$ *is convergent and the sum of this is less than or equal to* 2.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\ldots\leq 1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots$$

Proof. By this expression the conclusion below holds:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\le 1+\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)}=2\,.$$

The lemma is true.

Lemma 4. *The series* $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ *is convergent for* $\alpha=1+\frac{1}{2}$.

Proof. It is necessary to see the expressions:

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}=\frac{1}{1\cdot\sqrt{1}}+\frac{1}{2\cdot\sqrt{2}}+\frac{1}{3\cdot\sqrt{3}}+\frac{1}{4\cdot 2}+\frac{1}{5\cdot\sqrt{5}}+\ldots+\frac{1}{8\sqrt{8}}+$$

$$+\frac{1}{9\cdot 3}+\frac{1}{10\cdot\sqrt{10}}+\ldots+\frac{1}{15\cdot\sqrt{15}}+\frac{1}{16\cdot 4}+\frac{1}{17\cdot\sqrt{17}}+\ldots+\frac{1}{24\cdot\sqrt{24}}+\ldots+$$

$$+\frac{1}{k^2\cdot k}+\frac{1}{(k^2+1)\cdot\sqrt{k^2+1}}+\ldots+\frac{1}{(k^2+2k)\cdot\sqrt{k^2+2k}}+\ldots$$

i.e.

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}=\sum_{k=1}^{\infty}\left(\sum_{j=0}^{2k}\frac{1}{(k^2+j)\cdot\sqrt{k^2+j}}\right).$$

It is possible to write some obvious inequalities

$$\frac{1}{1\sqrt{1}}+\frac{1}{2\cdot\sqrt{2}}+\frac{1}{3\cdot\sqrt{3}}\le\frac{1}{1}+\frac{1}{1}+\frac{1}{1}=\frac{2\cdot 1+1}{1^3}=3,$$

and

$$\frac{1}{4\cdot 2}+\frac{1}{5\cdot\sqrt{5}}+\ldots+\frac{1}{8\cdot\sqrt{8}}\le\frac{1}{4\cdot 2}+\frac{1}{4\cdot 2}+\ldots+\frac{1}{4\cdot 2}=\frac{2\cdot 2+1}{2^3}=\frac{5}{8},$$

and

$$\frac{1}{9\cdot 3}+\frac{1}{10\cdot\sqrt{10}}+\ldots+\frac{1}{15\cdot\sqrt{15}}\le\frac{1}{9\cdot 3}+\frac{1}{9\cdot 3}+\ldots+\frac{1}{9\cdot 3}=\frac{2\cdot 3+1}{3^3}=\frac{7}{27},$$

and

$$\frac{1}{16\cdot 4}+\frac{1}{17\cdot\sqrt{17}}+\ldots+\frac{1}{24\cdot\sqrt{24}}\le\frac{1}{16\cdot 4}+\frac{1}{16\cdot 4}+\ldots+\frac{1}{16\cdot 4}=\frac{2\cdot 4+1}{4^3}=\frac{9}{64},$$

and generally

$$\frac{1}{k^2\cdot k}+\frac{1}{\left(k^2+1\right)\cdot\sqrt{k^2+1}}+\ldots+\frac{1}{\left(k^2+2k\right)\cdot\sqrt{k^2+2k}}$$
$$\le\frac{1}{k^2\cdot k}+\frac{1}{k^2\cdot k}+\ldots+\frac{1}{k^2\cdot k}=\frac{2k+1}{k^3}.$$

By this the inequality

$$\sum_{k=1}^{\infty}\left(\sum_{j=0}^{2k}\frac{1}{\left(k^2+j\right)\cdot\sqrt{k^2+j}}\right)\le\sum_{k=1}^{\infty}\frac{2k+1}{k^3}$$

holds i.e. the evaluation

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}\le\sum_{k=1}^{\infty}\frac{2k+1}{k^3}$$

is true. For every natural number *k* the expression

$$\frac{2k+1}{k^3}\le\frac{3k}{k^3}=\frac{3}{k^2}$$

holds, so the series $\sum_{n=1}^{\infty}\frac{4}{n^2}=3\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a majorant of the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}$ i.e.

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}\le 3\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}\,.$$

Because the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, by lemma 3, so the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}$ is convergent too. The sum of the series is less than or equal to 6.

Lemma 5. *If the series* $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ *is convergent for* $\alpha=1+\frac{1}{2^s}$ *where s is some natural number, then it is convergent for* $\alpha=1+\frac{1}{2^{s+1}}$ *too.*

Proof. Let for $s \in N$ to be $\beta(s) = \frac{1}{2^s}$. It is necessary to show that the series $\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\beta(s+1)}}$ is convergent if the series $\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\beta(s)}}$ is convergent. The series $\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\beta(s+1)}}$ is equal to $\sum_{n=1}^{\infty} \frac{1}{n \cdot \sqrt{n^{\beta(s)}}}$ i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\beta(s+1)}} = \frac{1}{1 \cdot \sqrt{1^{\beta(s)}}} + \frac{1}{2 \cdot \sqrt{2^{\beta(s)}}} + \frac{1}{3 \cdot \sqrt{3^{\beta(s)}}} + \ldots .$$

The inequalities below obviously hold:

$$\frac{1}{1 \cdot \sqrt{1^{\beta(s)}}} + \frac{1}{2 \cdot \sqrt{2^{\beta(s)}}} + \frac{1}{3 \cdot \sqrt{3^{\beta(s)}}} \leq \frac{1}{1} + \frac{1}{1} + \frac{1}{1} = \frac{2 \cdot 1 + 1}{1} = 3$$

and

$$\frac{1}{4 \cdot 2^{\beta(s)}} + \frac{1}{5 \cdot \sqrt{5^{\beta(s)}}} + \ldots + \frac{1}{8 \cdot \sqrt{8^{\beta(s)}}} \leq \frac{5}{4 \cdot 2^{\beta(s)}}$$

and

$$\frac{1}{9 \cdot 3^{\beta(s)}} + \frac{1}{10 \cdot \sqrt{10^{\beta(s)}}} + \ldots + \frac{1}{15 \cdot \sqrt{15^{\beta(s)}}} \leq \frac{7}{8 \cdot 3^{\beta[s]}}$$

and

$$\frac{1}{16 \cdot 4^{\beta(s)}} + \frac{1}{17 \cdot \sqrt{17^{\beta(s)}}} + \ldots + \frac{1}{24 \cdot \sqrt{24^{\beta(s)}}} \leq \frac{9}{16 \cdot 4^{\beta(s)}}$$

and generally

$$\frac{1}{k^2 \cdot k^{\beta(s)}} + \frac{1}{\left(k^2+1\right) \cdot \sqrt{\left(k^2+1\right)^{\beta(s)}}} + \ldots + \frac{1}{\left(k^2+2k\right) \cdot \sqrt{\left(k^2+2k\right)^{\beta(s)}}} \leq \frac{2k+1}{k^2 \cdot k^{\beta(s)}} .$$

The equality below is certain:

$$\frac{2k+1}{k^2 \cdot k^{\beta(s)}} \leq \frac{3k}{k^2 \cdot k^{\beta(s)}} = \frac{3}{k \cdot k^{\beta(s)}} ,$$

by this it is obvious that

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\beta(s+1)}} \leq 3 \cdot \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\beta(s)}} ,$$

by the comparison test for convergence of infinite series the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot n^{\beta(s+1)}}$ is convergent if the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot n^{\beta(s)}}$ is convergent. The sum of the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot n^{\beta(s+1)}}$ is less than or equal to $2\cdot 3^{s+1}$. By mathematical induction there is the finish of lemma 5.

Proof of the theorem: the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ is convergence where $1<\alpha$. If $1<\alpha$ that there is a natural number s such that $1+\frac{1}{2^s}<\alpha$ i.e. $1+\beta[s]<\alpha$ so for each natural n it is $n^{1+\beta(s)}<n^{\alpha}$ and consequently

$$\frac{1}{n^{\alpha}}<\frac{1}{n^{1+\beta(s)}},$$

by this the series $\sum_{n=1}^{\infty}\frac{1}{n\cdot n^{\beta(s)}}$ is a majorant of the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ and consequently the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ is convergent. The proof of the theorem is finished.

Usually the proof of the theorem is shown by **the integral test for convergence:**

On the interval $[m,\ \infty)$ where $m\in N$ the function $f(x)$ is positive and decreasing then the series $\sum_{n=m}^{\infty}f(n)$ and the integral $\int_{m}^{\infty}f(x)dx$ are both at the same time convergent or divergent.

Proof of the theorem with use the integral test for convergence.

The integral $\int_{1}^{\infty}\frac{1}{x}dx$ is divergence because $\lim_{t\to\infty}f(t)=\infty$. For $\alpha\neq 1$ the indefinite integral is equal: $\int\frac{1}{x^{\alpha}}=\frac{1}{1-\alpha}\cdot x^{1-\alpha}$. The value of the antiderivative at the point $x=1$ is equal $\frac{1}{\alpha-1}$, the limit $\lim_{x\to\infty}x^{1-\alpha}$ is equal to zero for

$1 < \alpha$ and it is infinity for $0 < \alpha < 1$. So $\int_1^\infty \frac{1}{x^\alpha} dx$ equals $\frac{1}{\alpha - 1}$ for $1 < \alpha$ and infinity for $0 < \alpha < 1$. This conclusion finishes the proof of the theorem.

The direct proof of the convergence of the series $\sum_{n=1}^\infty \frac{1}{n^\alpha}$ for $1 < \alpha$ in another way is presented in [Fihtenholz 1978, vol. 2, p. 227], the proof of the divergence is presented in the same way as in this paper.

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