O P E R A T I O N S   R E S E A R C H   A N D   D E C I S I O N S
No. 1 2013
DOI: 10.5277/ord130105

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SOLVING SOME DETERMINISTIC FINITE HORIZON INVENTORY MODELS

Two single-item, deterministic, continuous, finite horizon inventory models having no shortage have been considered. A demand rate function, which is relatively general, for the item has been assumed. In the first model, units of the item do not deteriorate; while in the second units deteriorate in a constant fraction rate. Some optimality conditions are shown for the models. Based on these properties, single variable search methods have been described to obtain globally optimal solutions. Numerical experiments indicate that the methods yield acceptable solutions within small time and are suitable for practical applications.

Key words: *inventory, finite horizon, globally optimal solution, search method*

1. Introduction

Inventory decisions are of utmost importance in the contexts such as, a manufacturing unit making an item, a retailer selling an item, or in a project (e.g. construction of a bridge or a building). In such situations, inventory decisions should take into account ordering costs for placing the orders for the materials with the suppliers and to procure the materials (transportation costs, taxes, etc.), inventory holding costs that represent the opportunity cost for financial capital being tied up in the materials, shortage costs incurred for the lack of a material, and other such relevant costs. There has been a very considerable discussion in the literature on various inventory models. We shall discuss a particular type of deterministic models that have a substantial scope of application.

Wagner and Whitin considered [9] a deterministic, single item, discrete model with demand concentrated on some time points. Orders may be placed only on such

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time points and supply lead times are zero. Shortages are not allowed. Ordering and inventory holding costs may vary from period to period but material price is constant over the periods. The authors presented an algorithm with which an optimal solution can be determined efficiently. Silver and Meal considered [5] a continuous version of the problem, imposing a cost on shortage. An approximate solution method was given, minimizing cost per unit separately for every cycle. Silver suggested [4] a modification of the method to give a heuristic for a linear demand rate. Donaldson proposed [2] an exact method, for the same demand rate function. Hariga considered [3] a log concave demand function $D(t)$, that is demand functions for which $D'(t)/D(t)$ is decreasing, and an exact solution method was outlined by him. A number of heuristic methods has been proposed subsequently, for different demand rate patterns and including a shortage cost or otherwise. Yang et al. suggested [8] a heuristic method for non-linear quadratic increasing demand rate, not having shortages. The authors also discussed a non-linear quadratic decreasing demand rate model and presented a solution method later [7]. Wang described [10] a heuristic method, entitled as consecutive improvement method, for the same model as in [8] and demonstrated its relative efficiency in comparison to some earlier methods, including that of the article mentioned.

In this paper, we consider models as in [10] but with a more general demand rate function. In our models, the demand rate need not be increasing, but only has to satisfy some conditions as continuity. As in that model [10], orders are not necessarily periodic or of the same size. Two variant models have been considered by us. In the first, there is no deterioration of the item; in the second, deterioration of the item occurs with time. Deterioration of the item has been considered in different inventory models, as exemplified by Teng et al. [6], Chern et al. [1]. Utilizing the properties as derived for the models presented in this paper, we suggest solution methods to obtain globally optimal solutions. Although, these are search methods, the methods are seen to be fast enough to be considered for practical applications. The analysis given by us is tractable and intuitive.

In the next section, the model without deterioration has been described. A solution method, a search method, based on a property derived by us has been given. The method was illustrated with some examples. In the next section, the model was modified to consider deterioration of the item. A solution method, akin to that of the first model, has been given for the model. We concluded with some relevant observations about the suitability of the models and the solution methods.

2. Model without deterioration of the item

2.1. Symbols and assumptions

H – end time point of planning horizon ($[0, H]$) ($H < \infty$),
$f(t)$ – instantaneous demand rate at time t, $0 \le t \le H$,

$D(t)$ – total demand in time $[0, t]$,
$I(t)$ – inventory (on-hand) position at time t,
c_1 – fixed cost of an order,
c_2 – inventory holding cost per unit of the material, per unit time,
n – number of orders placed in the planning horizon,
T_i – time of receipt of the i-th order,
Q_i – order quantity of the i-th order.

The following assumptions were made:

1. The system starts at time $t = 0$, with zero initial inventory. The first order (Q_1) is received at this time point ($T_1 = 0$). Final inventory at the end ($t = H = T_{n+1}$) is zero.
2. Demand rate $f(t)$ is continuous and non-negative, for $0 \le t \le H$.
3. Ordering quantities are continuous.
4. No shortage is allowed.
5. Lead time of supply is zero.
6. Ordering cost (c_1) is constant and positive ($c_1 > 0$).
7. Inventory holding cost is constant and positive ($c_2 > 0$).
8. Material price is constant and does not change over time, order quantities, etc.
9. The objective function is total cost incurred in the planning horizon that has to be minimized.

The above model, according to our perceptions, will have a substantial scope of application in the context of material requirements planning (MRP), in which the decision-makers are required to decide on the inventory decisions about ordering time points and ordering quantities, once production plans of the end-items are made. It is often felt in business contexts that shortage cost, considering both tangible and intangible costs, is high and shortage cannot be allowed. The model would also be applicable for planning inventory of materials in a construction projects etc.

2.2. Analysis

Without loss of generality, $f(t)$ is non-zero over an interval initially (except, possibly, $t = 0$) and, similarly, over an interval at the end (except, possibly, $t = H$).We may write, $D(t) = \int_0^t f(u)du$. In an optimal solution, any order must be received only when on-hand inventory becomes zero and we may consider only such solutions. For such a solution, the inventory cost in the ith cycle (from the receipt till the consumption of the i-th order),

$$K(i) = c_2 \int_{T_i}^{T_{i+1}} \left(Q_i - \left(D(t) - D(T_i)\right)\right) dt$$

Consider the problem of minimizing the total cost incurred in the planning horizon, given that there would be n orders. Total cost is obtained as

$$TC(n) = nc_1 + \sum_{i=1}^{n} K(i)$$

The following proposition can be made for the model.

Proposition 1. For an optimal solution, the time interval between two consecutive receipts and order quantities satisfy

$$Q_{i+1} = \left(T_{i+1} - T_i\right) f\left(T_{i+1}\right), \qquad i = 1, 2, ..., n-1 \tag{1}$$

Proof. First we take that, $f(t) > 0$, $0 \le t \le H$. Consider the rate of change of inventory holding cost, with ordering quantity Q_i, keeping T_i fixed ($i \ge 1$). This, since

$$\frac{dT_{i+1}}{dQ_i} = \frac{1}{f\left(T_{i+1}\right)}$$

exists, is given as

$$\frac{d}{dQ_i}\left(c_2 \int_{T_i}^{T_{i+1}} \left(Q_i - D(t) + D(T_i)\right) dt\right) = c_2\left(\left(T_{i+1} - T_i\right) + Q_i \frac{dT_{i+1}}{dQ_i} - \left(D\left(T_{i+1}\right) - D\left(T_i\right)\right)\frac{dT_{i+1}}{dQ_i}\right)$$

Noting that, $D(T_{i+1}) - D(T_i) = Q_i$, it may be written as $c_2(T_{i+1} - T_i)$. Also, consider the same rate of change but with T_{i+1} fixed ($i \ge 2$). This is as

$$\frac{d}{dQ_i} c_2\left(Q_i\left(T_{i+1} - T_i\right) - \int_{T_i}^{T_{i+1}} D(t)\, dt + D(T_i)\left(T_{i+1} - T_i\right)\right)$$

Since

$$Q_i = D\left(T_{i+1}\right) - D\left(T_i\right), \qquad \frac{dD\left(T_i\right)}{dQ_i} = -1 \qquad \text{and} \qquad \frac{dT_i}{dQ_i} = \frac{1}{f\left(T_i\right)}$$

the preceding derivative is $c_2Q_i/f(T_i)$.

The two derivatives must be same for two consecutive orders. Otherwise, because $f(t)$ is continuous, one of the order quantities can be increased, the other decreased, by a sufficiently small quantity, to get a better solution. This proves the proposition for $f(t) > 0$.

If the demand rate is zero at some instants or intervals, we may consider perturbing it by adding a sufficiently small quantity so that it is continuous and positive in $[0, H]$. Since the order quantities would change continuously with such a perturbation, the proposition holds for $f(t) \geq 0$. □

Proposition 1 states a necessary condition for an optimal solution. It is enlightening to note that, for an optimal solution, an ordering quantity is given with present demand rate and immediate past duration. The proposition suggests the following method, which requires search on one variable, given the number of orders. The working of the method is straightforward.

2.3. Solution method

The method has the following steps.

Step 0.

A. Set a suitable ΔQ (> 0), sufficiently small.

B. Set MinCost at a large value.

C. Define MinSoln to store optimum solution in terms of ordering quantities and time points of receipts.

D. Set TotalDemand = $\int_0^H f(u)du$.

E. Initialize $Q_1 = 0$, $T_1 = 0$.

Step 1. $Q_1 = Q_1 + \Delta Q$. Get $K(1)$, inventory holding cost for the first cycle. If $Q_1 \geq$ TotalDemand or $K(1) >$ MinCost $- nc_1$, give outputs and stop. Else, go to Step 2.

Step 2. i. For $i = 1, 2, \ldots, n-2$ do the following computations.

A. If $\sum_{k=1}^{i} Q_k <$ TotalDemand then, from Q_i calculate T_{i+1} using

$$Q_i = \int_{T_i}^{T_{i+1}} f(u)du, \qquad T_i < T_{i+1} < H$$

if such a solution exists. If $f(T_{i+1}) = 0$, shift T_{i+1} to the latest $T'_{i+1} \geq T_{i+1}$, such that $f(t) = 0$, $T_{i+1} \leq t \leq T'_{i+1}$. Go to Step 2(i)B.

If no feasible T_{i+1} is found, go to Step 1.
B. From T_{i+1}, calculate Q_{i+1}, using $Q_{i+1} = (T_{i+1} - T_i) f(T_{i+1})$.
ii. If feasible solutions are obtained earlier (or, if $n = 2$), set

$$Q_n = \text{TotalDemand} - \sum_{k=1}^{n-1} Q_k$$

Step 3. Calculate total cost (TC). If $TC <$ MinCost, MinCost = TC, update MinSoln. Go to Step 1.

We may note that, for any solution satisfying (1), Q_2, Q_3 etc., would change continuously with a change in Q_1. Since in the method all solutions satisfying the necessary condition of optimality are, in effect, checked, it yields a globally optimal solution, for a fixed n (>1). It may be used repeatedly, varying n, and storing the minimum solution. This may be continued to a sufficiently large value of n, when ordering cost (nc_1) becomes higher than the minimum total cost found. This ensures an exactly optimal solution (apart from the effect of granularity in the search) for the model, which has a general demand rate function.

2.4. Numerical examples

First, we solve an instance with increasing-constant-decreasing demand rate that may sometimes be appropriate in practical situations. The demand rate considered is:

$$f(t) = \begin{cases} at, & 0 \le t \le t_1 \\ at_1, & t_1 < t \le t_2 \\ at_1 \dfrac{H}{H - t_2} - at_1 \dfrac{t}{H - t_2}, & t_2 < t < H \end{cases}$$

where a, t_1, t_2, H are given parameters. The demand rate increase linearly till t_1, remains constant between t_1, t_2 and then decreases linearly to become zero at H. For the instance we take $a = 100.0$, $t_1 = 1.0$, $t_2 = 4.5$, $H = 5.0$. For the costs, we take ordering cost $c_1 = 25.0$, inventory holding cost rate $c_2 = 1.0$. The result is shown in Table 1. In it, ordering time points and ordering quantities for an optimal solution are given. There are 7 orders. We see that, during constant demand rate phase, ordering quantities remain constant. Optimal cost is 323.22. For an optimal solution, ordering cost and inventory holding cost are nearly equal (but not exactly equal). For such a demand rate, given Q_i, T_{i+1} is found solving a quadratic equation. The search variable (Q_1) has been varied with an increment of 0.05. The instance has been solved in 0.6 seconds.

Table 1. Instance with increasing–constant–decreasing demand rate for the model without deterioration

Order Number	Ordering time point	Ordering quantity	Order number	Ordering time point	Ordering quantity
1	0.0	31.9	5	2.7735	65.8251
2	0.7987	63.8	6	3.4318	65.8251
3	1.457	65.8251	7	4.0900	66.0
4	2.1153	65.8251			

Next, we, solve the instances as introduced in [8]. These 12 instances have quadratic increasing demand rate $f(t) = a + bt + ct^2\,(a \geq 0,\, b \geq 0,\, c \geq 0)$. Since the present method can also deal with other demand rate functions, we consider a quadratic decreasing-increasing demand rate, $f(t) = 190 - 60t + 10t^2 = 100 + 10(t - 3)^2$. So, it decreases till $t = 3$, and then increases. For these instances, we take $c_1 = 100$, $c_2 = 1$ and vary $H = 2$, 4 and 5.

The search variable (Q_1) has been varied, as before, with an increment of 0.05. With this, good quality solutions are obtained in reasonable time for the instances. For a comparison, solution values as given in [10] are mentioned for observations 10–12 in parentheses. The results are reported in Table 2.

Table 2. More problem instances and solutions for model without deterioration

Obs. No.	Problem parameters	Cost of an optimal solution	Number of orders	Time [s]
1	$a = 0, b = 900, c = 100, H = 1; c_1 = 9, c_2 = 2$	129.5338	7	1.9
2	$a = 0, b = 900, c = 100, H = 2; c_1 = 9, c_2 = 20$	367.7833	21	28.3
3	$a = 0, b = 100, c = 5, H = 3; c_1 = 100, c_2 = 2$	776.2956	4	0.9
4	$a = 0, b = 1600, c = 100, H = 4; c_1 = 9, c_2 = 2$	1598.9928	19	191.4
5	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 30, c_2 = 1$	293.6497	5	0.4
6	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 50, c_2 = 1$	381.1800	4	0.3
7	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 60, c_2 = 1$	421.1800	4	0.3
8	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 70, c_2 = 1$	455.1964	3	0.3
9	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 90, c_2 = 1$	515.1964	3	0.2
10	$a = 100, b = 150, c = 10, H = 1.0; c_1 = 30, c_2 = 2$	151.6122 (151.6123)	3	0.2
11	$a = 100, b = 150, c = 10, H = 1.5; c_1 = 30, c_2 = 2$	246.7411 (246.7411)	4	0.9
12	$a = 100, b = 150, c = 10, H = 2; c_1 = 30, c_2 = 2$	356.1620 (356.1619)	5	2.1
13	$a = 190, b = -60, c = 10, H = 2; c_1 = 100, c_2 = 1$	336.0935	2	0.2
14	$a = 190, b = -60, c = 10, H = 4; c_1 = 100, c_2 = 1$	615.6990	3	1.1
15	$a = 190, b = -60, c = 10, H = 5; c_1 = 100, c_2 = 1$	777.1678	4	2.0

We have used Newton's derivative method to solve the resultant equations to find T_{i+1} for given Q_i. It may be noted that, for $f(t) > 0$, there would be a unique solution for such an equation. The method mentioned is empirically seen to perform well for the problem instances. An alternative is bisection search method. It is not difficult to see that, this method would find the solution in a guaranteed manner, within some tolerance, for $f(t) > 0$. The method can be modified slightly for the case $f(t) \geq 0$.

The experiment has been conducted in Microsoft Excel, implementing the methods in Visual Basic. A Pentium 4, 1.86 GHz processor speed, 1 GB RAM personal computer, with Windows XP Prof essional operating system, has been used.

3. Model with deterioration

In this section, we consider a model which is same as in the earlier section, except that:

A. After a replenishment is received, there is a deterioration of the item. If inventory position at time t is $I(t)$, instantaneous deterioration rate is $\alpha I(t)$ per unit time ($0 < \alpha < 1$). That is, if I_0 units are there initially, after T time, $I_0 e^{-\alpha T}$ units are fit for use.

B. The price of the item is c_3 (> 0) per unit. There is no salvage value of a deteriorated unit.

3.1. Analysis

We have an analogous result for this model, which is given in the next proposition.

Proposition 2. For an optimal solution of the model with deterioration, the time interval between two consecutive receipts and order quantities satisfy

$$\frac{e^{\alpha(T_{i+1}-T_i)} f(T_{i+1})}{\alpha Q_{i+1} + f(T_{i+1})} = \frac{c_3 + \frac{c_2}{\alpha} \times \left(1 - \frac{f(T_{i+1})}{\alpha Q_{i+1} + f(T_{i+1})}\right)}{c_3 + \frac{c_2}{\alpha}\left(1 - e^{-\alpha(T_{i+1}-T_i)}\right)}, \qquad i = 1, 2, ..., n-1 \tag{2}$$

Proof. First assume that, $f(t) > 0,\ 0 \leq t \leq H$. Suppose that for a small change of ΔQ in Q_i, with T_i fixed, T_{i+1} changes by ΔT. We may write

$$\Delta Q \mathrm{e}^{-\alpha(T_{i+1}-T_i)} \approx \Delta T f\left(T_{i+1}\right)$$

This gives, with T_i fixed

$$\frac{dQ_i}{dT_{i+1}} = \mathrm{e}^{-\alpha(T_{i+1}-T_i)} f\left(T_{i+1}\right)$$

We also get, for T_{i+1} fixed

$$\frac{dQ_i}{dT_i} = -\left(\alpha Q_i + f(T_i)\right)$$

Let K_i denote the material cost and inventory holding cost in $[T_i, T_{i+1})$. To calculate inventory holding cost, we note that

$$\frac{dI(t)}{dt} = -(\alpha I(t) + f(t)), \qquad T_i < t < T_{i+1}, \qquad i = 0, 1, 2, \ldots, n \tag{3}$$

So, inventory holding cost in $[T_i, T_{i+1})$ is

$$c_2 \int_{T_i}^{T_{i+1}} I(t)dt = -\frac{c_2}{\alpha}\left(-Q_i + \int_{T_i}^{T_{i+1}} f\left(t\right)dt\right)$$

Consider the rate of change of K_i, with ordering quantity Q_i, keeping T_i fixed $(i \geq 1)$. This is given as

$$\left.\frac{dK_i}{dQ_i}\right|_{T_i} = c_3 + \frac{c_2}{\alpha}\left(1 - \mathrm{e}^{-\alpha(T_{i+1}-T_i)}\right)$$

The derivative with T_{i+1} fixed is

$$\left.\frac{dK_i}{dQ_i}\right|_{T_{i+1}} = c_3 + \frac{c_2}{\alpha}\left(1 - \frac{f\left(T_i\right)}{\alpha Q_i + f\left(T_i\right)}\right)$$

Thus, for an optimal solution, we must have the condition as stated in the proposition. This holds also for $f(t) \geq 0$. □

3.2. Solution method

The solution to the differential equation (3) is given as

$$I(t) = \int_{t}^{T_{i+1}} \mathrm{e}^{\alpha(u-t)} f(u)\,du, \qquad T_i < t < T_{i+1}, \qquad i = 0, 1, 2, \ldots, n \tag{4}$$

Having obtained $I(t)$, we can have a search method to obtain an optimal solution, which is similar to that in 2.3. To determine T_{i+1} for a Q_i, we may use Newton's derivative method, as is done in the earlier model, second type of instances. Given T_{i+1}, Q_{i+1} can be calculated from (2), in a single step.

3.3. Numerical examples

We solve the same problems as earlier, taking $\alpha = 0.1$ per unit time and $c_3 = 10.0$. The solutions are shown in Table 3. As may be seen, solutions have increased (sometimes markedly) number of orders, relative to the case when there is no deterioration. This, one may explain, is to reduce cost for deterioration. Inventory holding cost decreases, for more frequent ordering.

Table 3. Solutions for the model with deterioration

Obs. No.	Problem parameters	Cost of an optimal solution	Ordering and inventory holding cost	Number of orders	Time [s]
1	$a = 0, b = 900, c = 100, H = 1; c_1 = 9, c_2 = 2$	4990.96	132.08	9	34.2
2	$a = 0, b = 900, c = 100, H = 2; c_1 = 9, c_2 = 2$	21116.43	374.84	25	503.8
3	$a = 0, b = 100, c = 5, H = 3; c_1 = 100, c_2 = 2$	5900.16	800.13	5	14.2
4	$a = 0, b = 1600, c = 100, H = 4; c_1 = 9, c_2 = 2$	151023.09	1048.50	94	13189.1
5	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 30, c_2 = 1$	1802.79	433.74	9	9.0
6	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 50, c_2 = 1$	1958.66	564.31	7	6.6
7	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 60, c_2 = 1$	2027.32	613.43	6	6.1
8	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 70, c_2 = 1$	2087.32	673.43	6	5.5
9	$a = 6, b = 1, c = 0.005, H = 11; c_1 = 90, c_2 = 1$	2202.10	759.94	5	5.0
10	$a = 100, b = 150, c = 10, H = 1; c_1 = 30, c_2 = 2$	1966.81	152.32	3	3.8
11	$a = 100, b = 150, c = 10, H = 1.5; c_1 = 30, c_2 = 2$	3602.07	251.38	5	13.1
12	$a = 100, b = 150, c = 10, H = 2; c_1 = 30, c_2 = 2$	5704.03	361.58	7	30.9
13	$a = 190, b = -60, c = 10, H = 2; c_1 = 100, c_2 = 1$	3347.94	340.64	2	3.3
14	$a = 190, b = -60, c = 10, H = 4; c_1 = 100, c_2 = 1$	5826.70	646.68	4	16.8
15	$a = 190, b = -60, c = 10, H = 5; c_1 = 100, c_2 = 1$	7286.10	859.71	6	32.8

4. Conclusion

We have analyzed single item, deterministic, finite, continuous horizon inventory models that have substantial scope of application. We have been more intent on getting applicable models and solution methods. We have considered a relatively general demand rate function, which enhances the scope of application for such models. The analysis given by us is simple and intuitive. We have proposed single variable search methods, based on results as shown by us, to obtain globally optimal solutions for the models. As has been observed numerically, time requirement with the methods is within practicable limits. Time requirement may be further decreased with more efficient implementations of the methods. It would be better to use exactly optimal solutions for the models considered, rather than locally optimal/ approximate solutions. With the computing power that is easily available now, getting a globally optimal solution for the models would not pose much difficulty with the given methods.

Many generalizations are possible for the models that are considered in this article. If the planning horizon is large, we may need to consider variation of the material price, ordering cost etc. with time. Such costs may also vary with order quantities. There can be constraints on the amount of order quantities. The models can accommodate lead times which are not zero, but other deterministic constants. The models may also be extended for the cases, when material requirement and/ or lead time is not deterministic, but has some uncertain variations. There may be more than one item, and joint replenishments may need to be considered. It would be worthwhile if such generalizations are considered in future research, since such situations often arise in the practical contexts.

Acknowledgement

The author thanks very much the reviewers for their advice to improve the paper.

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