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CHOOSING WHAT TO PROTECT WHEN ATTACKER RESOURCES AND ASSET VALUATIONS ARE UNCERTAIN

The situation has been modelled where the attacker's resources are unknown to the defender. Protecting assets presupposes that the defender has some information on the attacker's resource capabilities. An attacker targets one of two assets. The attacker's resources and valuations of these assets are drawn probabilistically. We specify when the isoutility curves are upward sloping (the defender prefers to invest less in defense, thus leading to higher probabilities of success for attacks on both assets) or downward sloping (e.g. when one asset has a low value or high unit defense cost). This stands in contrast to earlier research and results from the uncertainty regarding the level of the attacker's resources. We determine which asset the attacker targets depending on his type, unit attack costs, the contest intensity, and investment in defense. A two stage game is considered, where the defender moves first and the attacker moves second. When both assets are equivalent and are treated equivalently by both players, an interior equilibrium exists when the contest intensity is low, and a corner equilibrium with no defense exists when the contest intensity is large and the attacker holds large resources. Defense efforts are inverse U shaped in the attacker's resources.

Keywords: *assets, defense, attack, game theory, uncertainty, resources, valuations, contest success function, optimization*

1. Introduction

This paper's contribution is to model the situation where the attacker's resources are unknown to the defender. The attacker's resources consist of money, property, competence, etc., which depend on skills, sex, age, cultural background, education, etc., all of which influence the attacker's capabilities. Defense intelligence communities work diligently to assess an attacker's capabilities, expressed in terms of resources and the attacker's intents based on e.g. valuations of the assets. Attackers

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usually camouflage both the amount and type of resources they possess (e.g. whether they have biological weapons), and how they value different assets they may attack. The paper addresses this challenge for defenders. We assume that the attacker's resources and valuations of the two assets are known to the attacker but unknown to the defender. These three characteristics are drawn from probability distributions. The attacker has resources which are converted into an attack effort against one of the two assets, where the unit cost of such an attack depends on the nature of the asset, as well as on the attacker's capabilities and mode of operation. Analogously, the defender has resources which are allocated to defend both assets, one asset, or neither asset, with appropriate unit costs. The efforts by the attacker and defender define the value of a contest success function for each asset, which determines the probability that an attacked asset is destroyed.

An attack can be of any kind made on an asset valuable to the defender. More generally, we consider any situation involving two players having incompatible goals. One example is a terrorist attack. Terrorists attack assets of economic, human and symbolic value. An attacker may target iconic buildings, airline security, food and water supply, launch an anthrax attack on a targeted population, or a cyber attack by breaking into computing devices, computer networks, or the internet to steal, gain access to, or destroy something of value to a defender. Alternatively, the launch of a new consumer good may be considered as a type of attack. The defender may counteract such a threat by enhancing the quality or lowering the price of its own good, try to isolate the market targeted by the attacker, or lobby to impose constraints on the attacker. An attempt to flood a market by smuggling e.g. in containers, may also be considered as a form of attack. If there is more than one port for container freight, the defender (customs) needs to allocate defensive inspection resources to multiple ports. When conducting this resource allocation, the defender does not usually know either the attacker's valuations of these multiple assets, or the resources available to the attacker.

To position this paper within the literature and illustrate its contributions, first consider Bier et al.'s [4] paper which differs from the current paper in two ways. First, Bier et al. [4] assume that the probability of a successful attack against an asset depends only on the defense resources allocated to that asset. In contrast, this paper assumes that the probability of a successful attack against an asset depends on both the defense and attack resources allocated to that asset, as well as the contest intensity for that asset which expresses the effectiveness of the technology used. Second, Bier et al. [4] do not model the attack effort, i.e. the amount of resources allocated to an attack but assume that the attacker can be of n unknown types, one for each asset, which expresses the attacker's valuation of each asset. In contrast, this paper models the attacker's valuation of each asset, and additionally assumes that the attacker has specified resources to be directed at an asset, with varying unit attack costs dependent on the nature of each asset.

Second, Nikoofal and Zhuang [18] consider resource allocation for a defender encountering an attacker who has private information about the valuation of the targets. The defender knows that the attacker's valuations belong to bounded distribution free intervals. The defender leaves no target undefended, in contrast to Bier et al. [4], and in contrast to Levitin and Hausken [16, 17] who consider false targets. Nikoofal and Zhuang [18] apply robust optimization and determine the impact of the defender's assumptions regarding bounds on the unknown parameters and the attack effectiveness ratio on the robustness of the solution.

Third, Powell [19] analyzes a sequential game where a defender allocates resources between two assets. The vulnerability of asset 1 is the defender's private information. The defender moves first and thus may signal information about the vulnerability of a target to the uninformed attacker. In contrast, we assume that the defender is uninformed about the attacker's resources and valuations of the assets. The defender moves first when facing an unknown threat, which is often realistic in practice. The attacker moves second, he has complete information but knows that the defender protects its assets while not knowing the attacker's resources and valuations of the assets, which causes an interesting realistic dilemma.

Fourth, Fey [7] considers a contest between two players who each have private information about the costs of their own efforts and choose their strategies simultaneously. In contrast, we assume that the defender moves first and the attacker moves second. The defender knows neither the attacker's resources nor the attacker's valuations of the assets. We also consider two contests. Realistically, an attacker does not attack all the assets that the defender protects, so we assume that the attacker attacks one asset.

Fifth, Wang and Zhuang [32] consider how to balance congestion and security when strategic players have private information.

Further research has considered deception by the defender. Zhuang et al. [28] determine the balance between capital and expense for defensive investments. They show that defenders can achieve more cost effective security in a multiple stage game through secrecy and deception. In each stage, the defender may choose truthful disclosure, secrecy, or deception. The attacker updates his information after observing the defender's signals and the result of a contest. Zhuang and Bier [30,31] determine why a defender might prefer secrecy or deception regarding her allocation of defensive resources, rather than disclosure, in a homeland security context. Bernhardt and Polborn [2] find that when a country values targets similarly, it should conceal defenses and distribute defense resources randomly.

More generally, see Fey [7] and the references therein for research on incomplete information Tullock games with a contest success function (so called Tullock games), Sandler and Siqueira's [23] review, and the references classified under "incomplete information" in Hausken and Levitin's [13] review.

For related research not involving incomplete information, Kunreuther and Heal [15], Sandler and Lapan [22], and Frey and Luechinger [8] consider the defense of multiple assets which involve, for example, substitution effects. Arce and Sandler [1], Bier et al. [3], Hausken and Zhuang [14], and Powell [20] consider a strategic attacker. Shan and Zhuang [24] consider how a defender strikes a balance (tradeoff) between equity and efficiency. Zhuang and Bier [29] determine how a defender balances defense against terrorism and natural disasters. For research on Blotto games of allocative strategic mismatch, see Golman and Page [9]. Powell [21] shows that in the first move of a sequential Blotto game, the defender defends all sites, the attacker then attacks all sites that are not well defended and refrains from attacking sites that are well protected. The attacker can be deterred. In contrast, using Tullock's [27] contest success function, Hausken [11] shows that the attacker can never be deterred in the Blotto game when both players have fixed resources but can be deterred when there are variable resources (i.e., no upper bounds exist on the amount of resources that can be used).

This paper considers a goal oriented attacker, i.e. it assumes that the attacker is strategic. An alternative approach is to consider an opportunistic attacker who has no clearly predefined goal but adapts his actions depending on the arising opportunities. Comparing these two different kinds of attacker, Shan and Zhuang [25] analyze a defender facing an attacker who may be strategic (maximizes the defender's expected loss) or non-strategic (attacks with an exogenously determined probability).

We assume that the probability of a successful attack against an asset depends on both the defense and attack resources allocated to that asset. This is, first, a theoretical assumption common in the contest success literature [5] and the defense and attack literature [13] which is supported empirically (in parts of the same literature). Second, it is also supported by experience and common sense. For example, if an asset is defended and not attacked, it is preserved. Conversely, if an asset is not defended but it is attacked, the defender loses the asset.

The rent seeking literature (see e.g. [5]) usually assumes competition for one so called rent, and some research generalizes to two rents or arbitrarily many rents. Many of the insights generated by considering two rents are confirmed by analyzing n rents, and some new insights from analyzing n rents pertain to the actual number of rents available. In this paper, we confine attention to two assets which reveals interesting insights depicted e.g. graphically along two dimensions. Analyzing more than two assets complicates the analysis and is suitable for future research.

As is common in the systems defense and attack literature [13], we assume that the defender moves first and the attacker moves second. The reasoning is usually that the defender seeks to preserve the status quo, it may hold more resources than the attacker, and it designs a defense system in preparation for a possible future attack. The attacker, on the other hand, may seek to circumvent the status quo, may probe for weaknesses and in doing so may take the current defense system as given when

designing its optimal attack. The 9/11 attack may thus be perceived as the defender moving first and designing a defense with an exploitable weakness, and the attacker moving second with an overwhelming attack. An example of the opposite situation, not considered in this paper, is the attacker moving first with a surprise attack, and the defender moving second with an emergency response. Hausken et al. [12] compares three games where the defender moves first, the attacker moves first, or both players move simultaneously, respectively.

The model assumes various parameters such as effort, cost, valuation of assets, etc. The empirical values of some of these parameters are available from various records such as governmental budget allocations, or can be established by interviewing experts in the appropriate areas or defectors from terrorist organizations. In the field of cyber security, Gordon and Loeb [10] have written a book oriented at practical applications seeking to establish the costs and benefits of managing cybersecurity resources. Model validation is left for future research.

Section 2 presents a model with a description of the players, technology, strategies, payoffs, sequential equilibrium, and an example of a cumulative distribution function used to define the defender's prior assessment regarding the type of the attacker. Section 3 analyzes the model focusing on the attacker, the defender, and equilibrium, providing examples and graphical illustrations. Section 4 concludes with a brief summary and results.

2. The model

2.1. Notation

s_i	–	defender's effort for asset i, $i = 1, 2$
a_i	–	defender's unit effort cost for asset i, $i = 1, 2$
v_i	–	defender's valuation of asset i, $i = 1, 2$
$u(s_1, s_2, K)$	–	defender's utility
$R \in \mathbf{R} \subset \mathfrak{R}_+$	–	attacker's resources
S_i	–	attacker's effort for asset i, $i = 1, 2$
A_i	–	attacker's unit effort cost for asset i, $i = 1, 2$
$V_i \in \mathbf{V}_i \subset \mathfrak{R}_+$	–	attacker's valuation of asset i, $i = 1, 2$
$U(s_1, s_2, K)$	–	attacker's utility
$F = F_{R,V1, V2}$	–	cumulative distribution function describing the defender's prior assessment of the attacker's type
$f = f_{R,V1,V2}$	–	density function describing the defender's prior assessment of the attacker's type
$\mathbf{K} \rightarrow \{0, 1/2, 1\}$	–	attacker's type
$G(s_1, s_2, K)$	–	probability that the attacker attacks asset 1

$q_i(S_i, s_i, m_i)$	–	contest success function for asset i, $i = 1, 2$
m_i	–	parameter for the contest technology for asset i, $i = 1, 2$
γ	–	parameter for the isoutility condition
ψ	–	parameter for the isocost condition

2.2. The players

The defender exerts effort $s_i \geq 0$ at unit cost $a_i > 0$ to defend asset i valued at $v_i > 0$, $i = 1, 2$, where a_1, a_2, v_1, v_2 are common knowledge.

Assumption 1. The attacker has resources $R \in \mathbf{R} \subset \mathfrak{R}_+$ which are used to attack one of the assets valued at $V_i \in \mathbf{V}_i \subset \mathfrak{R}_+$ with effort S_i at unit cost A_i, where A_i is common knowledge.

We assume that only one asset is attacked, since this is often common and realistic in practice, for example the 1995 bombing of the Alfred P. Murrah Federal Building in Oklahoma City. Even when multiple assets are attacked, such as in the 9/11 attack, these can be understood as an attack on one collection of assets, often of the same or similar nature, or belonging to one branch of government, as opposed to other collections of assets, e.g. belonging to different branches of government. The attacker might want to attack both targets[1] but in practice this may lead to the attacker being detected and disabled if simultaneity is impossible. For example, if only the World Trade Center had been attacked using an airplane on September 11, 2001, it would have been much harder to attack, e.g. the Pentagon, in the same manner on September 12, 2001, since substantial defense efforts (scrambling jets, etc.) would have been mounted to screen for exactly such attacks. Furthermore, the logistics of multiple simultaneous attacks in, for example, geographically dispersed locations requires additional coordination resources, which may not be available to the attacker. That is, we assume that one attack exhausts the attacker's resources.

The attacker's resources R and valuations V_1 and V_2 are known to the attacker but unknown to the defender. Thus we have a game with incomplete information where the triple $(R, V_1, V_2) \in \mathbf{R} \times \mathbf{V}_1 \times \mathbf{V}_2 = \mathbf{K}$ describes the type of the attacker. This type is a random variable with cumulative distribution function $F = F_{R,V1,V2} : \mathbf{K} \rightarrow \{0, 1/2, 1\}$ which is common knowledge, where $f = f_{R,V1,V2}$ is the density function. $\mathbf{K} = 1$ means an attack on asset 1, $\mathbf{K} = 0$ means an attack on asset 2, and $\mathbf{K} = 1/2$ means an attack on both assets. Thus $F_{R,V1,V2}$ describes the defender's prior assessment of the attacker's type.

Assumption 2. The cumulative distribution function F is twice continuously differentiable, with density f.

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[1]For example, for a contest success function in ratio form attacking all assets is optimal.

The function F may attach high probability to values close to the defender's valuations v_1 and v_2 but this variable's support R can be a wide interval that include values far from v_1 and v_2.

2.3. Technology

Nature first draws the type of the attacker, i.e. draws values R, V_1, V_2 from F. This draw is observed by the attacker but not by the defender. The defender then chooses s_1 and s_2 which are observed by the attacker. Finally, the attacker chooses one asset to attack.

If asset i is attacked, then the attack effort is $S_i = R/A_i$. The probability that asset i is destroyed, given that it is attacked, is determined by the contest success function

$$q_i\left(S_i, s_i, m_i\right) = \int\limits_{R\in\Re_+} CSF\left(S_i, s_i, m_i\right) f dR = \int\limits_{R\in\Re_+} CSF\left(\frac{R}{A_i}, s_i, m_i\right) f dR \qquad (1)$$

where $\dfrac{\partial q_i}{\partial s_i} < 0$ and $\dfrac{\partial q_i}{\partial S_i} > 0$, and m_i is a parameter describing the contest technology.

Since the attacker's resources are not fixed, as in the contest literature but is drawn from a probability density f, using Eq. (1), we determine the expected value of the contest success function. To generate analytical results, the most commonly used example of a contest success function is the ratio form [26, 27]

$$q_i\left(S_i, s_i, m_i\right) = \int\limits_{R\in\Re_+} \frac{S_i^{m_i}}{S_i^{m_i} + s_i^{m_i}} f dR = \int\limits_{R\in\Re_+} \frac{\left(\dfrac{R}{A_i}\right)^{m_i}}{\left(\dfrac{R}{A_i}\right)^{m_i} + s_i^{m_i}} f dR \qquad (2)$$

where $0 \le m_i \le 1$ is the contest intensity for asset i [2].

The environment is symmetric if $a_1 = a_2$, $A_1 = A_2$, $v_1 = v_2$, $m_1 = m_2$, and $F_{R,V1,V2} = F_{R,V2,V1}$ for all $(V1, V2) \in \Re_+^2$.

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[2]The contest intensity $m_i = 0$ gives an egalitarian distribution, where the players' efforts have no impact on q_i. When $0 < m_i < 1$, there is a disproportional advantage to investing less than one's opponent, $m_i = 1$ gives a proportional distribution, $m_i > 1$ gives a disproportional advantage to investing more effort than one's opponent (economies of scale), and $m_i = \infty$ gives a step function where *the winner-takes-all*.

2.4. Strategies and payoffs

A pure strategy for the defender is given by a pair $(s_1, s_2) \in \mathfrak{R}_+^2$. A pure strategy for the attacker is given by a choice $K: \mathfrak{R}_+^3 \times [0, 1]^3 \to \{1, 0\}$, which specifies which asset to attack given the type $(R, V_1, V_2) \in \mathfrak{R}_+^3$ of the attacker and the observed efforts (s_1, s_2) of the defender. We let $K(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2) = 1$ denote an attack on asset 1, $K(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2) = 0$ denote an attack on asset 2, and $K(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2) = 1/2$ denote attack on both assets. This means that K depends on the attacker's parameters, the conflict technology parameters m_1 and m_2, the defender's strategies s_1 and s_2 but not on the defender's parameters a_1, a_2, v_1, v_2. We consider a two stage game of incomplete information where the defender chooses (s_1, s_2) in stage 1, and the attacker chooses $K(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2)$ in stage 2.

The probability that the attacker attacks asset 1 is

$$G\left(s_1, s_2, K\right) = \int_{R \in \mathfrak{R}_+} \int_{V_1 \in \mathfrak{R}_+} \int_{V_2 \in \mathfrak{R}_+} K\left(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2\right) f dR dV_1 dV_2 \qquad (3)$$

The attacker's ex ante expected utility, from the defender's perspective of not knowing the attacker's type, is $V_1 q_1$ if asset 1 is attacked, and $V_2 q_2$ if asset 2 is attacked, i.e.

$$U(s_1, s_2, K) = \int_{R \in \mathfrak{R}_+} \int_{V_1 \in \mathfrak{R}_+} \int_{V_2 \in \mathfrak{R}_+} \left(K\left(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2\right) \frac{V_1 \left(\dfrac{R}{A_1}\right)^{m_1}}{\left(\dfrac{R}{A_1}\right)^{m_1} + s_1^{m_1}} \right.$$

$$\left. + \left(1 - K\left(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2\right)\right) \frac{V_2 \left(\dfrac{R}{A_2}\right)^{m_2}}{\left(\dfrac{R}{A_2}\right)^{m_2} + s_2^{m_2}} \right) f dR dV_1 dV_2 \qquad (4)$$

The attacker's expected utility from his own perspective can be calculated by substituting his type (R, V_1, V_2) into Eq. (4). The defender's expected utility is $v_1 + v_2 - v_1 q_1 - a_1 s_1 - a_2 s_2$ if asset 1 is attacked, and $v_1 + v_2 - v_2 q_2 - a_1 s_1 - a_2 s_2$ if asset 2 is attacked, i.e.

$$u\left(s_1, s_2, K\right) = v_1 + v_2 - G\left(s_1, s_2, K\right) v_1 q_1 - \left(1 - G\left(s_1, s_2, K\right)\right) v_2 q_2 - a_1 s_1 - a_2 s_2 \qquad (5)$$

2.5. Perfect Bayesian equilibrium

Both players maximize their expected utilities. We determine a pure-strategy perfect Bayesian sequential equilibrium.

Definition 1. An equilibrium is a pair of strategies (s_1^*, s_2^*) and $K^*(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2)$ such that

$$K = K(s_1, s_2) = \arg\max_{K \in [0,1]^3} U(s_1, s_2, K), \quad (s_1, s_2) = \arg\max_{(s_1, s_2) \in \Re_+^2} u(s_1, s_2, K(s_1, s_2)) \quad (6)$$

The consistency condition on beliefs is satisfied for this sequential equilibrium, since the attacker is the only player with private information, and the attacker chooses his strategy after the defender.

2.6. Example of the cumulative distribution function *F*

As an example, we assume that V_1 and V_2 are uniformly and independently distributed with support $[0, V_{iM}]$, i.e.

$$F_{V_i}(V_i) = \begin{cases} \dfrac{V_i}{V_{iM}} \text{ if } 0 \le V_i \le V_{iM} \\ 1 \text{ if } V_i \ge V_{iM} \end{cases} \Leftrightarrow f_{V_i}(V_i) = \begin{cases} \dfrac{1}{V_{iM}} \text{ if } 0 \le V_i \le V_{iM} \\ 0 \text{ if } V_i \ge V_{iM} \end{cases} \Rightarrow E(V_i) = \dfrac{V_{iM}}{2} \quad (7)$$

and that *R* is uniformly distributed, independently of V_1 and V_2, with support $[0, R_M]$, i.e.

$$F_R(R) = \begin{cases} \dfrac{R}{R_M} \text{ if } 0 \le R \le R_M \\ 1 \text{ if } R \ge C_M \end{cases} \Leftrightarrow f_R(R) = \begin{cases} \dfrac{1}{R_M} \text{ if } 0 \le R \le R_M \\ 0 \text{ if } R \ge R_M \end{cases} \Rightarrow E(R) = \dfrac{R_M}{2} \quad (8)$$

3. Analyzing the model

3.1. The attacker

The attacker moves second with perfect information. We confine our attention to pure strategies[3].

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[3] We ignore cases with the measure zero where the inequality signs in Eq. (7) are replaced by equality signs, $R = 0$, $V_1 = 0$, or $V_2 = 0$.

Lemma 1. The strategy K^* is optimal for the attacker if and only if, for all $(s_1, s_2) \in \Re_+^2$,

$$
\frac{V_1 \left(\dfrac{R}{A_1} \right)^{m_1}}{\left(\dfrac{R}{A_1} \right)^{m_1} + s_1^{m_1}} > \frac{V_2 \left(\dfrac{R}{A_2} \right)^{m_2}}{\left(\dfrac{R}{A_2} \right)^{m_2} + s_2^{m_2}} \Rightarrow K^* \left(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2 \right) = 1
$$

$$
\frac{V_1 \left(\dfrac{R}{A_1} \right)^{m_1}}{\left(\dfrac{R}{A_1} \right)^{m_1} + s_1^{m_1}} < \frac{V_2 \left(\dfrac{R}{A_2} \right)^{m_2}}{\left(\dfrac{R}{A_2} \right)^{m_2} + s_2^{m_2}} \Rightarrow K^* \left(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2 \right) = 0 \qquad (9)
$$

$$
\frac{V_1 \left(\dfrac{R}{A_1} \right)^{m_1}}{\left(\dfrac{R}{A_1} \right)^{m_1} + s_1^{m_1}} = \frac{V_2 \left(\dfrac{R}{A_2} \right)^{m_2}}{\left(\dfrac{R}{A_2} \right)^{m_2} + s_2^{m_2}} \Rightarrow K^* \left(R, V_1, V_2, A_1, A_2, m_1, m_2, s_1, s_2 \right) = \frac{1}{2}
$$

Proof. This follows from comparing the two terms in the integrand in Eq. (4). The third line in Eq. (9) expresses the fact that the attacker is equally likely to attack both assets when indifferent regarding which asset to attack.

To illustrate, first, if $m_1 = m_2 = 1$, then the attacker attacks asset 1 if $\dfrac{V_1}{V_2} > \dfrac{R + A_1 s_1}{R + A_2 s_2}$.

If $s_1 = s_2$ and $A_1 = A_2$, then the attacker attacks the most valuable asset. If s_1/s_2 or A_1/A_2 decreases, then the attacker becomes more likely to attack asset 1, which is either the worst defended or has the lowest unit attack cost. However, even if asset 1 is undefended ($s_1 = 0$), asset 2 will still be attacked if it is sufficiently more valuable $V_2 > V_1 \dfrac{R + A_2 s_2}{R}$. The attacker attacks only one target and does not waste its resources on undefended assets of low value. Second, if $m_2 = A_1 = A_2 = 1$ and $V_1 = V_2$, then the attacker attacks asset 1 if $R > \left(\dfrac{s_1^{m_1}}{s_2} \right)^{1/(m_1 - 1)} \Leftrightarrow m_1 > \dfrac{\ln(R/s_2)}{\ln(R/s_1)}$, as illustrated in Fig. 1 when $s_2 = 1$. Since the assets are equally valuable to the attacker, the unit costs of attacking both assets are the same, and $m_2 = 1$, the attacker makes its decision based

exclusively on its resources R, the contest intensity m_1 of asset 1, and the defender's defenses s_1 and s_2.

Assume that $s_2 = 1$. If $s_1 = 1$, so that both assets are equally well defended, four cases are possible. First, the attacker attacks asset 1 if $R > s_1$ and $m_1 > 1$. The condition $R > s_1$ means that the attack resources are greater than the defense effort, and the logic of the contest success function in Eq. (2) when the exponent satisfies $m_1 > 1 = m_2$ is that the attacker exploits its superiority by attacking asset 1. Second, when $R > s_1$ and $m_1 < 1$, having superior resources, the attacker prefers to exploit the higher contest intensity $m_2 = 1$ and attacks asset 2. Third, when $R < s_1$ and $m_1 > 1$, having inferior resources, the attacker prefers to exploit the lower contest intensity $m_2 = 1$ and attacks asset 2. Fourth, when $R < s_1$ and $m_1 < 1$, having inferior resources, the attacker prefers to exploit the lower contest intensity of asset 1 by attacking it. Fig. 1 illustrates these results, assuming that $m_2 = A_1 = A_2 = 1, V_1 = V_2, s_2 = 1$, for the more general case that the defense effort invested in asset 1 differs from $s_1 = 1$. In the left panel the defender invests less effort in asset 1 than asset 2, $s_1 = 0.9$. The attacker thus attacks asset 1 when $R = m_1 = 1$. However, when the attacker holds more resources and m_1 is low, or the attacker holds less resources and m_1 is high, then asset 2 is attacked. The dotted vertical line at $R = s_1 = 0.9$ is the asymptote of the corresponding hyperbolic function. As s_1 decreases towards zero, the upper left region and the lower right region shrink, so at the limit when $s_1 = 0$, asset 1 is guaranteed to be attacked. In the right panel the defender invests higher defense effort in asset 1 than asset 2, $s_1 = 1.1$. Hence, conversely to the case above, the attacker attacks asset 2 when $R = m_1 = 1$, and attacks asset 1 in the bottom left and upper right regions of the parameter space, with a vertical asymptote at $R = s_1 = 1.1$. As s_1 increases to 2, the two regions in which asset 1 is attacked disappear and asset 2 is guaranteed to be attacked.
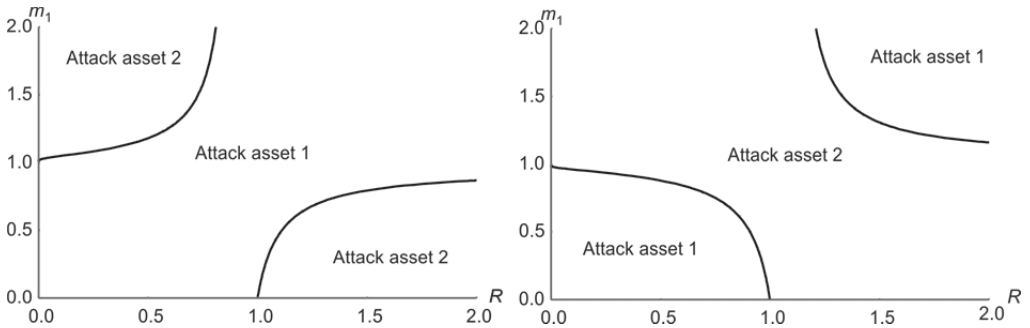

Fig. 1. Which asset to attack when $m_2 = A_1 = A_2 = 1, V_1 = V_2, s_2 = 1$.
Left panel: $s_1 = 0.9$, right panel: $s_1 = 1.1$

3.2. The defender

The defender chooses s_1 and s_2 in stage 1 to maximize its expected utility in (5), given the attacker's optimal choice of $K = K^*$ in stage 2, i.e.

$$(s_1, s_2) = \arg \max_{(s_1, s_2) \in \Re_+^2} u\left(s_1, s_2, K(s_1, s_2)\right) \tag{10}$$

where $K = K(s_1, s_2) = K^*$ is determined from (6). Using (5), the defender's isoutility set is

$$\left((s_1, s_2) \in \Re_+^2 : v_1 + v_2 - G(s_1, s_2, K)v_1 q_1 \right.$$
$$\left. -\left(1 - G(s_1, s_2, K)\right)v_2 \, q_2 - a_1 \, s_1 - a_2 \, s_2 = \gamma\right) \tag{11}$$

The isocost curves $\{(s_1, s_2) \in \Re_+^2 : a_1 s_1 + a_2 s_2 = \psi\}$ are linear. The isobenefit curves (isoutility minus isocost) are convex.

Lemma 2. The isoutility condition in (11) implicitly defines a function $s_2 = s_2(s_1, \gamma)$ at $s_1 = 0$, where

$$\frac{ds_2(0, \gamma)}{ds_1} = \frac{-\dfrac{dG}{ds_1}(v_1 - v_2 q_2) - a_1}{\dfrac{dG}{ds_2}(v_1 - v_2 q_2) + v_2 \dfrac{dq_2}{ds_2} + a_2} \tag{12}$$

An analogous result holds when $s_2 = 0$.

Proof. When $s_1 = 0$ (the case $s_2 = 0$ is analogous), the attacker attacks asset 1 giving the defender utility $v_2 - a_2 s_2 = \gamma$, and thus a unique $s_2 = (v_2 - \gamma)/a_2$. Differentiating (11) implicitly gives

$$\frac{dG}{ds_1}(v_1 q_1 - v_2 q_2) + G v_1 \frac{dq_1}{ds_1} + a_1 + \left(\frac{dG}{ds_2}(v_1 q_1 - v_2 q_2) + (1 - G)v_2 \frac{dq_2}{ds_2} + a_2\right)\frac{ds_2}{ds_1} = 0 \tag{13}$$

Substituting $s_1 = 0$, $G(0, s_2, K) = 0$, and $q_1(S_1, s_1, m_1) = q_1(S_1, 0, m_1) = 1$ into (13) gives (12). QED.

The right hand side of (12) can be positive or negative, giving upward or downward sloping isoutility curves when $s_1 = 0$. From (12), $dG/ds_1 < 0$ and $dG/ds_2 > 0$. The

contest success function ratio, q_2 in (12), is lower than one when $s_2 > 0$, according to (2). Hence, the denominator in (12) is positive, e.g. when $v_1 = v_2$, and the numerator is positive, e.g. when $v_1 = v_2$ and a_1 is small. This confirms that isoutility curves can be upward sloping. This means that increasing s_1 from $s_1 = 0$ causes the defender's utility to fall unless accompanied by a higher s_2. This results from the attacker's substitution effect, whereby a higher s_1 leads to an increased probability of an attack on asset 2, which thus requires more defense. Bier et al. ([4], p. 569) demonstrate that universally upward sloping isodamage curves near the axes (when the probability of an attack on one of the targets is small) occur when costs are ignored. This means that, in some cases, the defender prefers to waste resources rather than decrease the probability of the success of an attack on an asset. In contrast, Lemma 2 and (12) show that isoutility curves can be downward sloping near the axes, which is more common.

This result about the slope of the isoutility curves differs from the conclusions from Bier et al.'s [4] model and is mainly due to two factors. First, this paper assumes probabilistic uncertainty about the attacker's resources, in contrast to Bier et al. [4] who do not model the attacker's resources. Second, this paper assumes that both the defense efforts and attack efforts effect the probability of a successful attack through a contest success function, whereas Bier et al. [4] do not consider such a function but consider defender's resources indirectly through modeling how the defender determines the probability of the success of an attack on each asset.
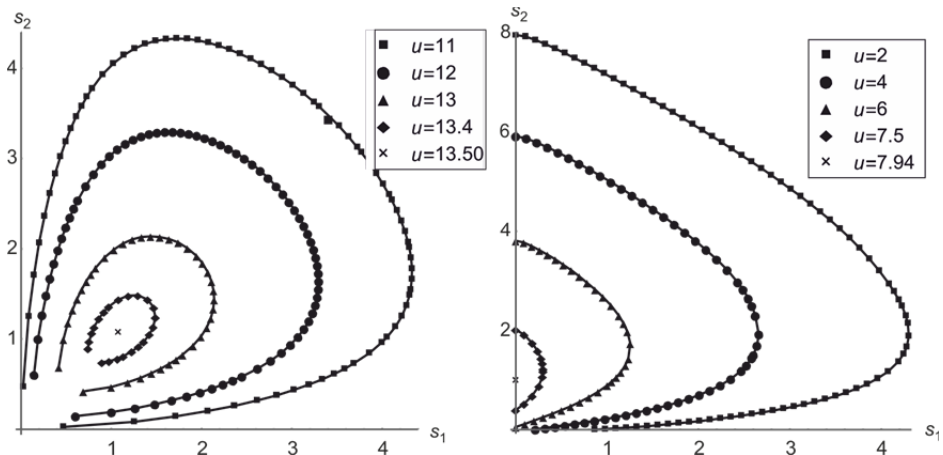

Fig. 2. Isoutility curves for the defender when $a_1 = a_2 = A_1 = A_2 = m_1 = m_2 = 1$.
$R_M = 2$, $v_2 = 10$. Left panel: $v_1 = 10$, right panel: $v_1 = 0.5$

Many parameter combinations in this paper's model can cause downward sloping isoutility curves near the axes. One example occurs when v_1 is small compared with v_2, which makes the denominator in (12) negative. Another example occurs when a_1 is

large, which can make the numerator in (12) negative. The values of R, A_i, and m_i also have an impact.

Assuming that the cumulative distribution function F is given by (7) and (8) in the example in section 2.5, Fig. 2 shows the defender's isoutility curves for two examples when $a_1 = a_2 = A_1 = A_2 = m_1 = m_2 = 1$, $R_M = 2$, and $v_2 = 10$. The graph in the left panel was obtained by considering a symmetric example where the value of asset 1 to the defender is $v_1 = 10$. The graph in the right panel was obtained by considering an asymmetric example where $v_1 = 0.5$. Curves with the increasing distance from the origin ($s_1 = 0$, $s_2 = 0$) have lower utilities. We consider the following four cases: ($s_1 > 0$, $s_2 > 0$), ($s_1 = 0$, $s_2 > 0$), ($s_1 > 0$, $s_2 = 0$) and ($s_1 = s_2 = 0$). First, for the symmetric example when $s_1 = s_2$, and for the asymmetric example when s_2 is large, an increase in s_1 is accompanied by a decrease in s_2 to ensure the same utility. For the symmetric example, when just one of s_1 and s_2 is small, the isoutility curves are upward sloping. For the asymmetric example, the isoutility curves are upward sloping only when s_2 is small. Second, when $s_1 = 0$, for the symmetric example the isoutility curves are upward sloping regardless of s_2 but for the asymmetric example the isoutility curves are upward sloping when s_2 is small, and downward sloping when s_2 is large. The low value of asset 1 causes the substitution effect to be inoperative when $s_1 = 0$ and s_2 is large. Increasing investment in the defense of asset 1 from zero is costly for the defender when asset 1 has low value, and the defender decreases investment in its defense of asset 2 to earn the same utility. Third, when $s_2 = 0$, in both examples the isoutility curves are upward sloping regardless of s_1. For the asymmetric example this follows since, when $s_2 = 0$, increased investment in the defense of asset 1 must be accompanied by increased investment in the defense of the more valuable asset 2, otherwise the attacker would become more likely to attack asset 2. Fourth, ($s_1 = s_2 = 0$) is a special case. For both examples, if both s_1 and s_2 increase at the same rate, the isoutility curves are downward sloping (as in the case 1 above), if $s_1 = 0$ and s_2 increases we get the second case (where the isoutility curves are upward sloping), and if $s_2 = 0$ and s_1 increases, we get the third case (where the isoutility curves are upward sloping).

3.3. Equilibrium

Proposition 1. A pure equilibrium (s_1^*, s_2^*, K^*) exists. The attacker's equilibrium strategy is pure. If there exists a mixed equilibrium strategy for the defender, then for any (s_1^*, s_2^*) in the mixture's support, a pure equilibrium exists where the defender plays (s_1^*, s_2^*) and receives the same utility as at the mixed equilibrium.

Proof. The attacker plays a pure strategy $K^* = K^*(s_1, s_2)$ described by (9) in Lemma 1, as a best response to s_1 and s_2. Upper limits $\bar{s}_1$ and $\bar{s}_2$, for s_1 and s_2 respectively, are chosen so that $u(\bar{s}_1, s_2^*, K^*) < 0$ and $u(s_1^*, \bar{s}_2, K^*) < 0$. The defender confines s_1 and s_2 to $(s_1, s_2) \in [0, \bar{s}_1] \times [0, \bar{s}_2]$, since $s_1 > \bar{s}_1$ or $s_2 > \bar{s}_2$ would be suboptimal. Since $u(s_1, s_2, K^*)$ is a continuous function of (s_1, s_2) on a compact set, an equilibrium defense strategy exists. The defender is indifferent between any pairs (s_1, s_2) in the support of an equilibrium mixture. QED.

Differentiating (5), the first order conditions for an interior solution, where $s_1 > 0$ and $s_2 > 0$, of the defender's optimization problem are

$$\frac{du}{ds_1} = -\frac{dG}{ds_1}(v_1 q_1 - v_2 q_2) - G v_1 \frac{dq_1}{ds_1} - a_1 = 0$$

$$\frac{du}{ds_2} = -\frac{dG}{ds_2}(v_1 q_1 - v_2 q_2) - (1-G) v_2 \frac{dq_2}{ds_2} - a_2 = 0$$

(14)

The second order conditions are considered in the Appendix.

Proposition 2a. In a symmetric environment, a pure unique equilibrium exists where $s = s_1 = s_2$. **2b.** In a symmetric environment when $m \leq m_T$, where m_T, $m_T \geq 1$, is a threshold value of the contest intensity, an interior equilibrium exists where $\lim_{R \to \infty} s = 0$. In a symmetric environment, when $m > m_T$, an interior equilibrium exists when $R < R_T$, where R_T is a threshold resource value, and a corner equilibrium $s = 0$ exists when $R \geq R_T$. **2c.** For any $v_1 > 0$, a lower limit $g_2(v_1) > 0$ exists such that asset 2 is undefended at any equilibrium if $v_2 < g_2(v_1)$. Analogously, $g_1(v_2) > 0$ exists for $v_2 > 0$ such that asset 1 is undefended at any equilibrium if $v_1 < g_1(v_2)$.

Proof. 2a. If the defender were to attach positive probability to $s_1 \neq s_2$, the attacker would attack the asset which gives the highest probability of success, i.e. the asset with the lowest investment in defense effort. The defender can increase its expected utility by making the attacker indifferent regarding which asset to attack, i.e. setting $s_1 = s_2$. Applying the assumptions of symmetry into (14), i.e. setting $a_i = a$, $v_i = v$, $A_i = A$, $V_i = V$, and $m_i = m$, gives

$$-\frac{dG}{ds_1}(v q_1 - v q_2) - G v \int_{R \in \Re_+} \frac{-\left(\frac{R}{A}\right)^m m s_1^{m-1}}{\left(\left(\frac{R}{A}\right)^m + s_1^m\right)^2} f dR - a = 0$$

$$-\frac{dG}{ds_2}(vq_1 - vq_2) - (1-G)v \int_{R\in\Re_+} \frac{-\left(\dfrac{R}{A}\right)^m ms_2^{m-1}}{\left(\left(\dfrac{R}{A}\right)^m + s_2^m\right)^2} fdR - a = 0 \tag{15}$$

which has a unique solution $s = s_1 = s_2$ and $q_1 = q_2$ when $G = 1/2$.

2b. Using symmetry and taking the limit as $R\to\infty$, (14) becomes

$$\lim_{R\to\infty}\frac{du}{ds} = -\frac{vms^{m-1}}{2}\lim_{R\to\infty}\int_{R\in\Re_+} \frac{-\left(\dfrac{R}{A}\right)^m}{\left(\left(\dfrac{R}{A}\right)^m + s^m\right)^2} fdR - a \tag{16}$$

Using L'Hopital's rule gives

$$\lim_{R\to\infty}\frac{du}{ds} = \frac{vms^{m-1}}{4}\lim_{R\to\infty}\int_{R\in\Re_+} \frac{1}{\left(\dfrac{R}{A}\right)^m + s^m} fdR - a \ . \tag{17}$$

when $m = 1$, $vms^{m-1}/4 = v/4$, and equating (17) to zero gives an interior solution $s > 0$, where s decreases as R increases. The same follows as m increases marginally above 1. As m increases above m_T, where $m_T \geq 1$, s^{m-1} becomes arbitrarily small. That is, $\lim_{m\to\infty, R\to\infty} s^{m-1} = 0$ when $m > m_T$ and $m_T \geq 1$. Substituting this into (17) gives $\lim_{R\to\infty}\frac{du}{ds} = 0 - a < 0$, i.e. a corner solution with $s = 0$.

2c. From the first order condition for asset 2 in (14), $dG/ds_2 > 0$. For fixed $v_1 > 0$, the term in brackets multiplied by dG/ds_2 can be made positive by decreasing v_2. This gives an overall negative impact on du/ds_2. The term with the coefficient $(1 - G)$ is negative but its absolute value can be made arbitrarily close to zero by decreasing v_2. The term with the coefficient a_2 is negative. Thus a positive value v_2 satisfying $v_2 < g_2(v_1)$ exists such that $du/ds_2 < 0$, which means that no interior solution exists for s_2 and thus $s_2 = 0$. The proof for the threshold $g_1(v_2)$ is analogous.

Example. Solving (15) for the example in section 2.5 under symmetry gives

$$
\frac{v}{2as}\left(\frac{1}{1+\left(\dfrac{As}{R_M}\right)^m} - \frac{\left(\dfrac{As}{R_M}\right)^{-m}}{1+m}\right.
$$

(18)

$$
\left.\times\text{Hypergeometric2}F1\left(1,1+\frac{1}{m},2+\frac{1}{m},-\left(\frac{As}{R_M}\right)^{-m}\right)\right)=1
$$

which simplifies to

$$
\left(\frac{1}{R_M}\ln\left(1+\frac{R_M}{As}\right)-\frac{1}{R_M+As}\right)\frac{Av}{2a}=1 \qquad \text{when } m=1
$$

where Hypergeometric2$F1$ is the hypergeometric function. Figure 3 shows s, u and U as functions of R_M with the reference (base) case being $a = A = m = 1$, $v = 10$, $V_{iM} = 20$, denoted by shaded squares. When R_M is small, due to the attacker's inferior resources, the defender earns high utility close to $2v = 20$, this utility is decreasing in R_M, and the defender exerts modest effort. The attacker earns low utility, which is increasing in R_M. As R_M increases, s reaches a maximum at $R_M = 2.34$, and thereafter it decreases as the defender grows weaker and finds it too costly to compete against an attacker with superior resources. The curve with shaded circles assumes a weaker attacker with $v/a = 5$ for the effort curve and $v = 5$ for the utility curves[4], while the other parameter values are the same as in the reference case. The defender exerts lower effort and earns lower utility, while the attacker earns higher utility due to its higher valuation $V_{iM} = 20$. The curve with shaded triangles assumes a weaker attacker with a three times greater unit attack cost $A = 3$. The defender's effort s increases more slowly, reaching a maximum at $R_M = 7.01$ (outside the plot), and thereafter decreases. The curve with shaded diamonds assumes a large contest intensity $m = m_1 = m_2 = 5$. When R_M is low, the defender's strength and attacker's weakness are both amplified. The opposite occurs when R_M is large, to the extent that the defender is deterred and does not defend when $R_M > 5.35$. This gives the corner solution $s = s_1 = s_2 = 0$ as in Proposition 2b. The left panel in Fig. 2 illustrates an interior equilibrium $s = s_1 = s_2$

---

[4]Equation (18) shows that v and a only appear once (in the ratio v/a), and thus only this ratio matters in determining s, while v and a both independently effect u and U.

= 1.08. The right panel in Fig. 2 shows a corner equilibrium $(s_1, s_2) = (0, 1.00)$, where asset 1 is undefended.
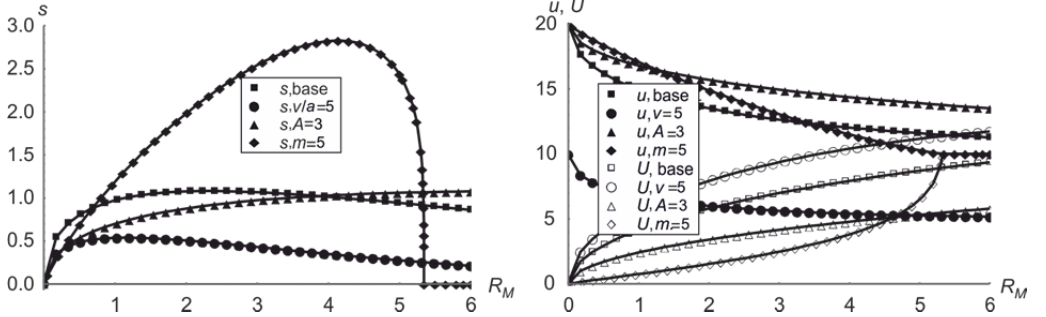

Fig. 3. s, u and U as functions of R_M with the baseline $a = A = m = 1$, $v = 10$, $V_{iM} = 20$, and for cases where one of the values v, A, m is shifted

We define

$$F \succ_1 \tilde{F} \Leftrightarrow \int_{R \in \mathfrak{R}_+} \int_{V_1 \in \mathfrak{R}_+} \int_{V_2 \in \mathfrak{R}_+} f(R, V_1, V_2) dR dV_1 dV_2 > \int_{R \in \tilde{\mathfrak{R}}_{1+}} \int_{V_1 \in \tilde{\mathfrak{R}}_{2+}} \int_{V_2 \in \tilde{\mathfrak{R}}_{3+}} \tilde{f}(R, V_1, V_2) dR dV_1 dV_2 \quad (19)$$

where $\succ_1$ induces a partial order over distributions of the attacker's resources based on first order stochastic dominance, and $\tilde{\mathfrak{R}}_{i+} \subseteq \mathfrak{R}_+$. Verbally, replacing F by $\tilde{F}$ means that the attacker gains resources.

Lemma 3. Replacing F by $\tilde{F}$ as defined in (19), which means the attacker gains resources, may induce the defender to increase or decrease its efforts s_1 and s_2.

Proof. Assuming a symmetric environment, we rewrite (15) as

$$\frac{vms^{m-1}}{2} \int_{R \in \mathfrak{R}_+} \frac{\left(\dfrac{R}{A}\right)^m}{\left(\left(\dfrac{R}{A}\right)^m + s^m\right)^2} f dR = a \quad (20)$$

Replacing F by $\tilde{F}$ can be accomplished in the case of a uniform distribution as in the example in section 2.5 by increasing R_M, which amounts to integrating (20) for larger values of R. But we know from the inverse U shaped curves in Fig. 3 (left

panel) that a larger R_M can cause a larger or smaller effort s dependent on whether the defender is weak or strong. QED.

Lemma 3 means that a clear-cut reaction to an attacker with increased resources does not exist. First, a defender with low unit costs of defense and high valuations of the assets defends less due to its strength. Second, a defender with high unit costs of defense and low valuations of the assets also defends less but then due to weakness. Third, a defender with intermediate unit costs of defense and high valuations of the assets may invest strongly in defense, in contrast to points 1 and 2. A specific form of F has to be specified to determine when the defender increases or decreases its effort, as illustrated in Fig. 3.

4. Conclusion

Determining an attacker's resource capabilities is essential for protecting assets. This paper analyzes an attacker attacking one of two assets. The attacker's resources and valuations of the two assets are known to the attacker but unknown to the defender. These parameters constitute the attacker's type, which is drawn from a three dimensional probability distribution. We specify how the attacker determines which asset to attack depending on his type, his unit attack costs, the contest intensity for each asset, and how well each asset is defended in a two stage game where the defender moves first and the attacker moves second. Bier et al. [4] show that isodamage curves are upward sloping near the axes, which means that the defender prefers to invest less in defense, thus resulting in higher probabilities of success for attacks on both assets. In contrast, we provide an analytical expression for a case where the isoutility curves are downward sloping. The latter occurs when one asset has a low value or a high unit defense cost. The difference between these results follows since we account for uncertainty regarding the attacker's resources and model investment in both defense and attack. Both defense efforts and attack efforts influence the probability of a successful attack. We show that a pure equilibrium exists. In a symmetric environment, which means that both assets are equivalent and are treated equivalently by both players, an interior equilibrium exists when the contest intensity is not too large, and a corner equilibrium with no defense exists when the contest intensity is large and the attacker holds plentiful resources. We specify the conditions under which an asset remains undefended. Increasing the attacker's resources can cause greater or smaller investment in defense effort depending on whether the defender defends due to weakness or strength.

Appendix.
Second order conditions

$$\frac{d^2u}{ds_1^2} = -\frac{d^2G}{ds_1^2}(v_1q_1 - v_2q_2) - 2\frac{dG}{ds_1}v_1\frac{dq_1}{ds_1} - Gv_1\frac{d^2q_1}{ds_1^2}$$

$$\frac{d^2u}{ds_2^2} = -\frac{d^2G}{ds_2^2}(v_1q_1 - v_2q_2) + 2\frac{dG}{ds_2}v_2\frac{dq_2}{ds_2} - (1-G)v_2\frac{d^2q_2}{ds_2^2} \qquad (A1)$$

$$\frac{d^2u}{ds_1s_2} = -\frac{d^2G}{ds_1s_2}(v_1q_1 - v_2q_2) + \frac{dG}{ds_1}v_2\frac{dq_2}{ds_2} - \frac{dG}{ds_2}v_1\frac{dq_1}{ds_1} - Gv_1\frac{d^2q_1}{ds_1s_2}$$

The relative magnitudes of $d^2G/ds_is_j < 0$, $i, j = 1, 2$, seem difficult to determine in general, so we consider a symmetric environment where ($A1$) becomes

$$\frac{d^2u}{ds^2} = vms^{m-1}$$

$$\times \int_{R\in\Re_+}\int_{V\in\Re_+}\int_{V\in\Re_+}\left(\frac{dK^*}{ds}\frac{2\left(\frac{R}{A}\right)^m}{\left(\left(\frac{R}{A}\right)^m + s^m\right)^2} + K^*\left(\frac{R}{A}\right)^m\frac{(m-1)\left(\frac{R}{A}\right)^m - (m+1)s^m}{s\left(\left(\frac{R}{A}\right)^m + s^m\right)^3}\right)fdRdVdV$$

$$\frac{d^2u}{ds^2} = 2ms^{m-1}\frac{dG}{ds}v\int_{R\in\Re_+}\frac{\left(\frac{R}{A}\right)^m}{\left(\left(\frac{R}{A}\right)^m + s^m\right)^2}fdR$$

$$(A2)$$

$$+ Gv\left(\frac{R}{A}\right)^m ms^{m-2}\int_{R\in\Re_+}\frac{(m-1)\left(\frac{R}{A}\right)^m - (m+1)s^m}{\left(\left(\frac{R}{A}\right)^m + s^m\right)^3}fdR$$

Since $dG/ds < 0$, the first term is negative. The second term is negative when $(m-1)(R/A)^m \le (m+1)s^m$, which is satisfied when $m \le 1$.

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