OPERATIONS RESEARCH AND DECISIONS

No. 4 2014
DOI: 10.5277/ord140405

Hannu NURMI[1]

ARE WE DONE WITH PREFERENCE RANKINGS? IF WE ARE, THEN WHAT?

Intransitive, incomplete and discontinuous preferences are not always irrational but may be based on quite reasonable considerations. Hence, we pursue the possibility of building a theory of social choice on an alternative foundation, viz. on individual preference tournaments. Tournaments have been studied for a long time independently of rankings and a number of results are therefore just waiting to be applied in social choice. Our focus is on Slater's rule. A new interpretation of the rule is provided.

Keywords: *preferences, Slater's rule, social choice*

1. Introduction

A standard version of what is known as thin rationality defines rationality as acting in accordance with one's preferences: Faced with the choice of either A or B, a rational actor chooses A if and only if A is strictly preferred to B by the actor. This much seems indisputable. The standard view also deals with a situation involving more than two alternatives. For such situations two requirements are imposed on the preferences of a rational chooser: (1) transitivity, and (2) completeness. It is well-known that if these assumptions are fulfilled, rational behavior can be represented as utility maximizing. Transitivity seems a plausible property when a common standard is being resorted to in each pairwise preference evaluation: if A is faster than B and B faster than C, it is obvious that A is faster than C. However, in general this may not be the case; the "reason" for which A is preferred to B may well be different from the reason for B being preferred over C. Hence, it is purely contingent whether A is pre-

[1]Department of Political Science and Contemporary History, University of Turku, 20500 Turku, Finland, e-mail address: hnurmi@utu.fi

ferred to C [14]. More specifically, there is no reason to argue that it is more rational that A is preferred to C than that C is preferred to A.

One goal of this paper is to argue that it is in general misleading to associate complete and transitive preference relations with rationality, especially if the latter refers to having good reasons for choices. There may be good reasons for having intransitive individual preference relations. Moreover, there may be good reasons for having incomplete preferences. I have elaborated these claims somewhat more extensively elsewhere [17], but will summarize the elaborations in the next section. The main goal of this paper is to provide some insight into how to proceed in looking for alternative foundations to the theory of voting. It should be stated at the outset that this is not to play down the importance of standard decision theory. The latter is based on assumptions and proceeds to give results that hold under those assumptions. Our aim is to discuss possible avenues for building theories of choice using assumptions that differ from the standard ones and are at the same time plausible ways of dealing with intransitive and incomplete preference relations.

2. Reasonable intransitivity and incompleteness

It is not difficult to envision situations where it makes perfect sense to have an intransitive preference relation. Consider a multiple criterion decision making (MCDM) situation with three criteria and three alternatives *x*, *y* and *z*. Suppose that the first criterion ranks the alternatives so that *x* is first, *y* second and *z* third, or $x \succ y \succ z$. Let the second criterion produce the ranking $y \succ z \succ x$ and the third $z \succ x \succ y$. If each criterion is roughly of equal importance to the others, it makes sense to form an overall preference relation over the alternatives by placing alternative *i* ahead of alternative *j* only in the case that *i* is ranked higher than *j* by at least two criteria out of three.

This setting is obviously identical with the well-known Condorcet's paradox, where criteria are substituted for voters. Examples from real world decision making are not difficult to come by. In the academic world, universities are often ranked in terms of multiple criteria, e.g., quality of research, quality of instruction and external impact. As long as the ranking according to criteria coincides with that described above, there is nothing unreasonable in ending with an intransitive ranking over three universities $x, y, z : x \succ y \succ z \succ x ...$ The same is potentially true in any situation involving a choice between three options (e.g. job applicants) and three criteria (linguistic skills, work experience, flexibility (working overtime, availability to travel)).

The 3-criteria, 3-alternatives setting may give the erroneous idea that the margin of the alternatives' being considered preferable to one another is of the order 2 to 1. Using Saari's ranking wheel [21] it is, however, easy to construct examples where the preferability margin becomes ($n - 1$) to 1 where n is the number of criteria and alternatives.

Incompleteness of preferences is usually considered of minor practical importance. Ever since the discovery of the preference reversal phenomenon by Lichtenstein and Slovic [12] it has, however, been taken more seriously. Still, the observation that a person who states that her preference is $x \succ y$, would be willing to pay more for y than for x, is usually considered a result of some sort of confusion on the part of the individual. It is, however, easy to show that this preference reversal is a straightforward extension of Ostrogorski's paradox [20] to MCDM contexts. Table 1 illustrates this phenomenon, which is related to the doctrinal paradox [13].

Table 1. Ostrogorski's paradox in an MCDM setting

Issue	Issue 1	Issue 2	Issue 3	Row choice
Criterion A	X	X	Y	X
Criterion B	X	Y	X	X
Criterion C	Y	X	X	X
Criterion D	Y	Y	Y	Y
Criterion E	Y	Y	Y	Y
Column				overall choice
Choice	Y	Y	Y	?

There are two candidates X and Y for an office that involves activities in three kinds of issues, i.e., issues 1−3. These might be social welfare policy, foreign policy and cultural policy. The nominating authority uses five criteria – criteria A−E – in determining the best choice for the office. In Table 1, the entry in each cell indicates which candidate is judged to be better on the corresponding issue and criterion. So, for example, Y in row B and column 2 signifies that Y is regarded better than X on criterion B and issue 2.

If all issues and all criteria are deemed roughly equally important, it makes sense to determine the choice by first picking on each issue the candidate that is better on more criteria than its competitor, and then choose the candidate picked on more issues than the other. This is column first aggregation [3]. Since Y is picked on each of the three issues, the choice would reasonably be Y.

If, on the other hand, we look at the decision problem from another angle and aggregate first over rows – i.e., pick on each criterion, the candidate that is better than the other on more issues – and then over the columns, we end up with X, since it is picked on more criteria than Y. This is the result of the rows first aggregation.

Taken together, these two ways of aggregation lead to the conclusion that the choice cannot be X – because it would contradict the columns first aggregation – but it cannot be Y either – since this would contradict the rows first aggregation. Hence, it is quite reasonable to have an incomplete preference relation between X and Y. The point of the preceding discussion is to argue that intransitive and incomplete preference relations can be quite reason based, hence rational. The remainder of the paper is devoted to discussing an alternative starting point for social choice and voting theory.

3. Rankings from tournaments

If complete and transitive preference relations are neither necessary nor sufficient conditions for rationality, then it follows that we cannot represent rational behavior as utility maximizing. But what would then be a plausible alternative foundation for choice theory? Obviously we should start from something less demanding than individual preference rankings. A weaker notion that comes to mind is that of a tournament, i.e., a complete and asymmetric relation over alternatives. It obviously does away with only the transitivity assumption, but let us see whether something useful can be built on individual preference tournaments.

The idea of using tournaments in social choice is not new (cf. [11]). Typically, however, earlier works have converted individual preference rankings into majority voting tournaments. The resulting $k \times k$ matrices consist of 0's and 1's with 1 indicating that the alternative represented by the corresponding row is majority preferred to the alternative represented by the column. The eventual Condorcet winners, thus, can immediately be spotted by looking for a row with all non-diagonal entries equal to 1. Similarly, the Condorcet loser – if one exists – can be found by singling out the row with all non-diagonal entries equal to 0. Another often used solution concept, the Copeland winner, is also easily recognizable as the row with the largest sum of entries. Obviously, this may not be unique.

Taking majority voting tournaments as the point of departure is, however, putting the carriage before the horse. Our main interest is in dealing with, and aggregation of, individual preference tournaments. Majority voting tournaments are just one way of aggregating individual tournaments. In fact, using individual tournaments as the starting point offers a rather rich variety of aggregation methods. We shall illustrate this point with reference to a method known as Slater's rule [22].

The original aim of Slater's rule was to address the problem of inconsistency in paired comparisons in certain types of psychological experiments. In these experiments, the subjects were shown a set of m objects in pairs so that each subject was confronted with $m(m-1)/2$ binary choices. The goal was to determine whether the subjects made their choices as if these were determined by one of the $m!$ possible dimensions or rankings of the objects. Slater's focus is on the number of inconsistent choices made by subjects. Slater's main target was the way inconsistencies were dealt with by the dominant method in the early 1960's, the Kendall–Babbington–Smith technique [8]. This technique focuses on intransitive triples of alternatives, i.e., choices that express a cyclic preference over a triple of alternatives. Instead, Slater looks at the minimal number of inconsistent choices. This minimum refers to the number of different paired choices from those stemming from the nearest adjoining order. To illustrate, consider three options *x*, *y* and *z* and the following observed responses: $x \succ y$, $x \succ z$, and $z \succ y$. Now, there are six possible strict rankings of the options:

$$x \succ y \succ z \quad (1)$$
$$x \succ z \succ y \quad (0)$$
$$y \succ x \succ z \quad (2)$$
$$y \succ z \succ x \quad (3)$$
$$z \succ x \succ y \quad (1)$$
$$z \succ y \succ x \quad (2)$$

The numbers in parentheses indicate the number of differences between pairwise orders of the observed responses and each of the six orders. It is clear that the observed responses are consistent with the $x \succ z \succ y$ ranking.

Suppose that instead of the above observations we observe the following schedule: $x \succ y$, $y \succ x$, and $z \succ x$.

$$x \succ y \succ z \quad (1)$$
$$x \succ z \succ y \quad (2)$$
$$y \succ x \succ z \quad (2)$$
$$y \succ z \succ x \quad (1)$$
$$z \succ x \succ y \quad (1)$$
$$z \succ y \succ x \quad (2)$$

Now we have three nearest adjoining orders, each at a unit distance from the observed schedule. So, Slater's rule seeks the nearest adjoining order and determines the number of choices that are inconsistent with it.

The rule is, indeed, a natural way to find the nearest consistent (i.e., complete and transitive) relation once a schedule of pairwise comparisons has been observed. As we have just seen, it may result in a tie between two or more rankings.

The way Slater's rule has been interpreted in the social choice literature is that, given the set of preference rankings of a set *N* of individuals, we form the tournament matrix, i.e., a $k \times k$ matrix as described above. This is then compared with all $k \times k$ tournament matrices corresponding to complete and transitive rankings over the *k* alternatives. Again the distance between the observed and generated matrices is determined.

The nearest rankings, i.e., those with the same minimum distance from the observed matrix, are considered to be the Slater rankings and their top alternatives the Slater winners. The criterion used in forming the tournament matrices is the majority rule, hence the term majority based Slater's rule will be used from here onwards.

4. Some important results on Slater's rule

As was pointed above, Slater's rule may result in a tie between several rankings. This, of course, may be the case with other rules as well. Some ties are, however, more problematic than others. To illustrate this, let us focus on an example devised by Stob [23], see also [4].

	A	*B*	*C*	*D*	*E*	*F*	*G*
A	0	0	1	1	1	1	1
B	1	0	1	1	0	0	0
C	0	0	0	1	1	1	0
D	0	0	0	0	1	1	1
E	0	1	0	0	0	1	1
F	0	1	0	0	0	0	1
G	0	1	1	0	0	0	0

There are two Slater rankings here:

$$B \succ A \succ C \succ D \succ E \succ F \succ G$$

and

$$A \succ C \succ D \succ E \succ F \succ G \succ B$$

One noteworthy observation to be made is that the alternative ranked first in one Slater ranking is placed last in the other ranking. Surely, a bizarre outcome from the point of view of applications.

This is not the only problem pertaining to Slater's rule as an analogue to social choice. Others relate to its compatibility or rather incompatibility with respect to other approaches to such problems. The following list is not exhaustive, but illustrative:

- The Slater winner may be in any position in the Dodgson ranking [9]. The Dodgson method, it will be recalled, counts the number of preference inversions required to make any alternative the Condorcet winner. The starting point of this method is usually a preference profile, but it can be implemented with tournaments as well.
- The Slater and Copeland rankings can be far from each other [5]. The Copeland procedure determines the ranking of alternatives by counting the number of pairwise

victories. The more contestants a given alternative defeats, the higher its rank. This result says that Slater's rule may end up with a blatant discrepancy with the (Copeland) procedure that is based on counting pairwise victories.

• One solution known as a prudent order [1] may be the exact opposite of a unique Slater ranking. This result was shown by Lamboray [10]. He also demonstrates that this extreme discrepancy with a prudent order can also hold for Kemeny's ranking. Given a preference profile, a prudent order can be found by conducting pairwise comparisons of alternatives, starting from unanimity and then proceeding by gradually diminishing the required qualified majority threshold. At some threshold value q, the comparisons end up with a cycle, while at value $q + 1$ they do not. A prudent order is now any linear extension of the non-cyclic relation obtained by using the threshold $q + 1$. Clearly, prudent ordering often results in a set of rankings rather than a single one.

• A related result pertaining to prudent orders and Slater's rule is also due to Lamboray [10]. It states that a unique Slater winner may be in any position of a prudent order.

• Östergård and Vaskelainen [18] demonstrate the possibility of a discrepancy between the Banks and Slater sets when the number of alternatives is at least 14. In profiles with 14 or more alternatives, these sets may have an empty intersection. The Banks [2] set, it will be recalled, consists of the end points of Banks chains, each of which is constructed by starting from an alternative and looking for another one that defeats it by a majority of votes. Then another alternative defeating both previous alternatives is sought. Continuing in this manner we eventually end up with an alternative that beats all the previous ones, but is unbeaten by any other that would defeat all the preceding ones.

5. A reinterpretation of Slater's rule

There is some justification for calling the method outlined above Slater's rule but it could also be argued that associating the rule with the majority principle (that is used in the construction of the pairwise comparison matrix) is a new element, not present in Slater's rule itself. The distance measure – the Hamming distance – on the other hand, is the same both in Slater's article and in its standard social choice interpretation. This measure simply counts the number of elements that have to be changed from 0 to 1 or vice versa in order to render two matrices of equal dimension identical.

Another interpretation would be the following. Given a set of k alternatives, construct all $k!$ rankings R_j ($j = 1, ..., k!$) and convert them into tournaments. Call these generated tournament matrices, G_j, $j = 1, ..., k!$. The matrix G_m thus corresponds to the ranking R_m. The set of generated tournament matrices is denoted by G and the set of the corresponding rankings by R. Now consider the individual preference tournaments

I_i, i = 1, ..., n as reported by the individuals and a generic generated matrix G_m. For each individual i compute the Hamming distance between his matrix I_i and G_m. We denote this by

$$d_H\left(I_i, G_m\right)$$

Let us define the distance score $DS(m)$ of Gm with respect to I_i as follows:

$$DS\left(m\right) = \sum_i d_H\left(I_i, G_m\right)$$

The new interpretation of Slater's rule now calls for the choice of the generated ranking with the smallest distance from the reported tournaments. Since this may not be unique we define the set SR of Slater rankings as follows:

$$SR = \left\{ R_l \in R \mid \sum_i dH\left(I_i, G_l\right) \leq \sum_i dH\left(I_i, G_t\right), \forall G_t \in G \right\}$$

In other words, the set of Slater rankings, as understood here, consists of the rankings that correspond to those generated matrices that are nearest to the observed tournaments in the sense of minimizing the sum of Hamming distances from the latter. We thus pick those rankings that correspond to tournaments that – if unanimously adopted – would require the minimal number of changes in the binary preferences of individuals (assuming that their reported preference tournament is truthful – sometimes a questionable assumption, no doubt (cf., e.g. [19]).

To illustrate the difference between the standard version of Slater's rule and our version, consider a setting of 11 individuals and three alternatives x, y and z. Let the observed pairwise comparison matrix be as follows:

	x	y	z
x	–	9	5
y	2	–	9
z	6	2	–

It translates into the following majority tournament matrix:

	x	y	z
x	–	1	0
y	0	–	1
z	1	0	–

The majority based (standard) interpretation of Slater's rule would end up with the set of three rankings: $x \succ y \succ z$, $y \succ z \succ x$, and $z \succ x \succ y$.

To determine the Slater ranking in our unanimity based interpretation, we generate all six rankings, convert them into tournament matrices and change the entries equal to 1 into 11. For example, the unanimity ranking $x \succ y \succ z$ becomes:

	x	y	z
x	–	11	0
y	0	–	11
z	11	0	–

Comparing this with the observed pairwise comparison matrix, we see that 2 individuals would have to change their minds in favor of *x* for it to be unanimously preferred to *y*. Similarly, 6 individuals would need to change their minds to make *x* unanimously preferred to *z* and so on.

The number of changes in binary preferences required to make each of the six possible rankings unanimous is listed below:

$$x \succ y \succ z: \quad (10)$$
$$x \succ z \succ y: \quad (17)$$
$$y \succ x \succ z: \quad (17)$$
$$y \succ z \succ x: \quad (17)$$
$$z \succ x \succ y: \quad (16)$$
$$z \succ y \succ x: \quad (23)$$

Hence, in contrast to the majority based Slater's rule, the unanimity based one produces in this example a unique winning ranking, viz. $x \succ y \succ z$. By changing the majority margins – but not the majority relations – one can bring about different unanimity based outcomes, e.g., the ranking $y \succ z \succ x$ is the sole winning ranking if the pairwise majority matrix is:

	x	y	z
x	–	6	0
y	5	–	11
z	11	0	–

So, obviously the unanimity based version of Slater's rule is different from the standard, majority based, one. But is the former interpretation consistent with the one outlined in [22]? It certainly seems to be. It deals with Hamming distances between consistent tournaments and observation schedules (reported opinions expressed as tournaments). The main difference between the standard social choice interpretation of Slater's rule and the one advocated here is in the way that tournaments are formed and in the principle used in their aggregation. The standard version starts from majority rule based tournaments, thereby already performing one stage of aggregation before calculating the Hamming distances. In the version advocated here, there is only one stage involving distance measurement and that involves individual preference tournaments and their distance to "consistent" ones. In this sense we are perhaps closer to Slater's idea than the standard version.

There is another aspect which would seem to support our version of Slater's rule. To wit, most social choice rules can be characterized in terms of an ideal or goal state, where the choice or ranking is unambiguous, and a metric used to measure the distance between the observed profile and the goal state (see [15] and [6]). For example, the Borda count can be characterized by a goal state where all individuals have the same alternative ranked first and by the inversion metric (see also [16]). In assessing a voting rule's applicability in the context of certain decisions, it may be helpful to look at the plausibility of the underlying goal state. Our version of Slater's rule has the underlying goal state where all individuals are unanimous about the collective ranking. In this respect, it is identical to Kemeny's rule [7]. It differs from Kemeny's rule in using a different voter input, as well as a different method for measuring the distance from the observations to the goal state.

6. Conclusion

We have shown that complete and transitive preference relations are neither sufficient nor necessary for rationality if the latter is interpreted as reason based decision making. We have addressed the question of constructing social choice rules based on less demanding assumptions than that of individual preference rankings. Individual preference tournaments provide one plausible way to proceed. Fortunately, much work has already been done on the study of tournaments. However, the aggregation of tournaments has received practically no attention thus far. The most common way to apply tournament theory to social choice is via paired comparisons using the simple majority rule. Using Slater's rule as an example, we have argued that this is not necessarily the best approach. Our proposed interpretation of Slater's rule is based on the principle of unanimity rather than the majority principle. Admittedly, showing that a new interpretation of Slater's rule exists does not say anything at all about its plausibility. This is something to be determined by systematic comparisons of the proposed version with other solutions based on preference tournaments. We already know that the standard version of Slater's rule exhibits significant incompatibilities with other such solutions.

In this regard, the performance of the suggested version of Slater's rule is unlikely to be worse.

Acknowledgement

The author thanks two anonymous reviewers for constructive comments on an earlier version.

References

[1] ARROW K., RAYNAUD H., *Social choice and multicriterion decision making*, MIT Press, Cambridge 1986.
[2] BANKS J.T., *Sophisticated voting outcomes and agenda control*, Social Choice and Welfare, 1985, 4, 295–306.
[3] BEZEMBINDER T., VAN ACKER P., *The Ostrogorski paradox and its relation to nontransitive choice*, Journal of Mathematical Sociology, 1985, 11, 131–158.
[4] BRUNK H.D., *Mathematical methods for ranking from paired comparisons*, Journal of the American Statistical Association, 1960, 55, 503–520.
[5] CHARON L., HUDRY O., *Maximum distance between Slater orders and Copeland orders of tournaments*, Order, 2011, 28, 99–119.
[6] ELKIND E., FALISZEWSKI P., SLINKO A., *Distance rationalization of voting rules*. *Manuscript*, 2014.
[7] KEMENY J., *Mathematics without numbers*, Daedalus, 1959, 88, 571–591.
[8] KENDALL M.G., BABBINGTON SMITH B., *On the method of paired comparisons*, Biometrika, 1939, 31, 324–345.
[9] KLAMLER C., *The Dodgson ranking and its relation to Kemeny's method and Slater's rule*, Social Choice and Welfare, 2004, 23, 91–102.
[10] LAMBORAY C., *A comparison between the prudent order and the ranking obtained with Borda's, Copeland's, Slater's and Kemeny's rules*, Mathematical Social Sciences, 2007, 54, 1–16.
[11] LASLIER J.- F., *Tournament solutions and majority voting*, Springer, Berlin 1997.
[12] LICHTENSTEIN S., SLOVIC P., *Reversal of preferences between bids and choices*, Journal of Experimental Psychology, 1971, 89, 4655.
[13] LIST C., PETTIT P., *Aggregating sets of judgments: Two impossibility results compared*, Synthese, 2004, 140, 207–235.
[14] MAY K.O., *Intransitivity, utility and aggregation of preference patterns*, Econometrica, 22, 1954, 1–13.
[15] MESKANEN T., NURMI H., *Distance from consensus: a theme and variations*, [in:] B. Simeone, F. Pukelsheim (Eds.), *Mathematics and Democracy. Recent Advances in Voting Systems and Collective Choice*, Springer, Berlin 2006, 117–132.
[16] NITZAN S., *Some measures of closeness to unanimity and their implications*, Theory and Decision, 1981, 13, 129–138.
[17] NURMI H., *Making sense of intransitivity, incompleteness and discontinuity of preferences*, [in:] P. Zaraté, G. Kersten, J. Hernández (Eds.), *Group decision and negotiation. A process-oriented view*, Springer, Heidelberg 2014, 184–192.
[18] ÖSTERGÅRD P.R., VASKELAINEN V.P., *A tournament of order 14 with disjoint Banks and Slater sets*, Discrete Applied Mathematics, 2010, 5158, 88–591.
[19] PAULY M., *Can strategizing in round-robin subtournaments be avoided?*, Social Choice and Welfare, 2014, 43, 29–46.

[20] RAE D., DAUDT H., *The Ostrogorski paradox. A peculiarity of compound majority decision*, European Journal of Political Research, 1976, 4, 391–398.
[21] SAARI D.G., *Decisions and elections. Explaining the unexpected*, Cambridge University Press, Cambridge 2001.
[22] SLATER P., *Inconsistencies in a schedule of paired comparisons*, Biometrika, 1961, 48, 303–312.
[23] STOB M., *Rankings from round-robin tournaments*, Management Science, 1985, 31, 1191–1195.

Received 16 September 2014
Accepted 23 October 2014