OPERATIONS RESEARCH AND DECISIONS
No. 1 2015
DOI: 10.5277/ord150102

Yusuke KAMISHIRO[1]

ON THE CORE OF A COST ALLOCATION PROBLEM UNDER ASYMMETRIC INFORMATION[2]

We study a cost allocation problem under asymmetric information, and show that the ex ante incentive compatible core is non-empty. We also obtain a non-emptiness result for the incentive compatible coarse core, which is one concept of an interim core.

Keywords: *cost allocation, asymmetric information, core, incentive compatibility*

1. Introduction

This study examines the non-emptiness of the core in a cost allocation problem where agents are asymmetrically informed about the cost of options (e.g., networks, equipment to produce goods) they can choose. Cost allocation problems have been important research topics in cooperative game theory and operations research (see [11, 23] for surveys of the area).

A group of agents chooses one of several options, and its aggregate cost is allocated among the agents. The agents have private information on the state of nature that determines the cost of the options they can choose. They form a coalition, which uses a mechanism to choose one option (as a function of the reported private information) and share its cost. We assume that the actual cost of the option chosen is publicly verifiable (before the cost is shared among agents), but the actual costs of the other unchosen options are assumed to be unverifiable. The latter assumption implies that some

[1]College of Economics, Kanto Gakuin University, 1-50-1 Mutsuura-Higashi, Kanazawa-ku, Yokohama, Kanagawa, 236-8501, Japan, email address: ykamishi@kanto-gakuin.ac.jp
[2]Some of the results of this paper were presented at the 2014 Spring National Conference of the Operations Research Society of Japan under the same title (in Japanese). I thank participants of this conference and the SING 10 conference for valuable comments. The comments and suggestions of two anonymous referees have also been helpful.

agent might report his type falsely to avoid paying a higher cost (see the example in Section 3), and we impose incentive compatibility conditions on the mechanism.

The theory regarding the non-emptiness of the core when information is asymmetric can be divided into two cases (see [7, 8] for surveys of this area). The first case occurs when a coalition is formed before each player receives private information (ex ante), and the second occurs when a coalition is formed after each player receives private information (interim). Suppose that some agents can acquire information about the cost of some option by inspection (without any fee). The ex ante scenario corresponds to the case where a coalition is formed before such inspections; whereas the interim scenario corresponds to the case where a coalition is formed after such inspections.

The non-emptiness of the core under asymmetric information has been analyzed, mainly in exchange economies. Forges, Mertens, and Vohra [6] show that the ex ante incentive compatible core and the incentive compatible coarse core, one concept of an interim core, might be empty in quasilinear exchange economies. Vohra [20] gives several sufficient conditions for which such cores are non-empty. Forges [5] considers assignment games with asymmetric information and obtains several non-emptiness results regarding the cores. In this study, we prove that in cost allocation situations, both the ex ante incentive compatible core and the incentive compatible coarse core are non-empty if the core in each state is non-empty.

The rest of the paper is organized as follows. In Section 2, we describe the model and present several examples. We define the mechanisms that each coalition uses and incentive compatibility conditions in Section 3. Sections 4 and 5 present non-emptiness results for the ex ante and interim scenarios, respectively. Finally, Section 6 concludes our study.

2. The model

2.1. Notation

Let $N = \{1, 2, ..., n\}$ be a set of agents and X_S be the set of options coalition $S(\subseteq N)$ can choose. The set of states that determine the cost of each option is denoted by Ω and the set of player i's private information (an information partition on Ω) by T_i. For notational convenience, we denote $T_S := \prod_{i \in S} T_i$ and $T_{-i} := T_{N \setminus \{i\}}$. Let $C(x_S, \omega)$ be the cost of $x_S \in X_S$ when the true state is $\omega \in \Omega$, and π be the probability distribution of

$\Omega \times T_N$. We assume that agents are risk-neutral with respect to cost, i.e., agents care only about the expected cost[3].

2.2. Examples

We can consider the following applications of the model.

Example 1. Agent *i* needs q_i units of a (homogeneous) good. When a coalition *S* is formed, the coalition produces $\sum_{i \in S} q_i$ units of the good, and shares the cost of production. To produce the good, they need to use one of several items of equipment, each with different cost functions (increasing, concave). Agents have different information about their cost functions. The questions are what equipment to choose and how to allocate the total cost among agents.

Example 2. A group of agents plans to set up a public facility (e.g., a park) in some location. Let X_S be the set of locations where coalition *S* can set up a facility (we assume that if $S \subseteq S'$, then $X_S \subseteq X_{S'}$). Agents have different information about cost when the facility is set up at x_S (e.g., only some player in coalition *S* knows the cost of $k \in X_S$). The questions are where do they set up the facility and how to allocate the total cost.

Example 3. Minimum cost spanning tree games[4]. A group of agents, geographically separated, want some particular service provided by a common supplier, which we call the source. Then, we need to find a network that connects the agents to the source. Minimum cost spanning tree (MCST) games, introduced by Claus and Kleitman [2], analyze how the aggregate cost should be allocated to the agents.

We consider a situation where the costs of links are not fully known to agents. Then, for each $S \subseteq N$, X_S is the set of networks such that all the agents in *S* are connected to the source, and Ω is the set of cost profiles of links.

---

[3]In exchange economies, Vohra [20] shows that if agents' utilities are linear, then both the ex ante incentive compatible core and the incentive compatible coarse core are non-empty. However, our results cannot be directly inferred from his argument because it relies on the convexity of feasibility sets, which our setup does not satisfy. However, we use a similar approach to show the non-emptiness of the incentive compatible coarse core.

[4]Kamishiro [12] shows that the ex ante incentive compatible core of the minimum cost spanning tree game is non-empty, and the current study can be considered an extension of this result. We also add more results about the interim stage.

3. Incentive compatible mechanisms

Let us consider a mechanism μ_S for coalition $S(\subseteq N)$. The mechanism $\mu_S := (z_S, c_S)$ chooses one of the feasible options for *S* according to the reported types of the members of *S*, and allocates the aggregate cost among agents in *S*, depending on the option it has chosen, reported types, and the observed event. Here, z_S and c_S are mappings from T_S to X_S and $X_S \times T_S \times \Omega$ to R_+^S, respectively.

The mechanism μ_S is implemented according to the following scenario.

1. After a vector of type $t_N \in T_N$ and a state $\omega \in \Omega$ are selected according to *π*, every agent $i \in N$ is informed of his own type $t_i \in T_i$.

2. If *S* has formed, each member of *S* reports his type to μ_S (although not necessarily truthfully)[5]. If the type profile t_S is reported, then μ_S's interim probability of the state being $\omega \in \Omega$ is given by $\pi(\omega \mid t_S) = \pi(\omega, t_S) / \pi(t_S)$. If the reported type profile t_S is such that $\pi(t_S) = 0$, then we define $\pi(\omega \mid t_S) = 0$ for all $\omega \in \Omega$.

3. Afterward, μ_S chooses an option $z_S(t_S) \in X_S$ according to t_S.

4. Then, μ_s knows the actual cost $C(z_S(t_S), \omega)$ and allocates the total cost to members of *S* (we denote $c_{S,i}(z_S(t_S), t_S, \omega)$ as the cost each agent *i* must pay).

The mechanism μ_S must satisfy the following feasibility conditions.

• Measurability

$$c_S(z_S(t_S), t_S, \omega) = c_S(z_S(t_S), t_S, \omega') \qquad \text{if} \qquad C(z_S(t_S), \omega) = C(z_S(t_S), \omega')$$

• Cost-coverage

$$\sum_{i \in S} c_{S,i}(z_S(t_S), t_S, \omega) \geq C(z_S(t_S), \omega) \qquad \text{for all } t_S \in T_S \text{ and all } \omega \in \Omega$$

The measurability condition implies that each coalition can only use information based on the observed event. The equality in the measurability condition means that although both ω and ω' lead to a different cost profile other than $z_S(t_S)$, the mechanism must allocate the aggregate cost to the members of *S* in the same way. The cost--coverage condition implies that the sum of the payments made by the agents in *S* is at

[5]By the revelation principle, all implementable decisions of *S* in Bayesian Nash equilibrium can be represented as the outcome of the incentive compatible mechanism, which we defined above (see [14] or [15]).

least the aggregate cost. Let M_S denote the set of mechanisms for S satisfying these two conditions.

Next, we define the incentive compatibility of μ_S. Loosely speaking, the incentive compatibility condition means that even if agents falsely report their private information, they do not gain (i.e., they cannot decrease (expected) costs). Consider $i \in S$ and let t_i be the true type of agent i and r_i the reported type. Then, agent i's expected cost is

$$Ec_i(\mu_S \mid t_i, r_i) := \sum_{t_{S\setminus\{i\}}, \omega} c_{S,i}(z_S(r_i, t_{S\setminus\{i\}}), (r_i, t_{S\setminus\{i\}}), \omega)\pi(t_{S\setminus\{i\}}, \omega \mid t_i)$$

We denote $Ec_i(\mu_S \mid t_i)$ as the (interim) expected cost of agent i when he truthfully reports his type:

$$Ec_i(\mu_S \mid t_i) := Ec_i(\mu_S \mid t_i, t_i).$$

The mechanism μ_S satisfies incentive compatibility if

$$Ec_i(\mu_S \mid t_i) \leq Ec_i(\mu_S \mid t_i, r_i)$$

for all $i \in S$ and all $t_i, r_i \in T_i$. Let M_S^* denote the set of feasible and incentive compatible mechanisms for S[6]. By definition, $M_S^* \subseteq M_S$.

Note that in this game, we consider cost instead of utility; thus, the inequality sign is reversed compared to standard game or implementation theory such as in Holmström and Myerson [10] and Mas-Colell, Whinston and Green [14]. There is one more difference between our definition and theirs: In our definition, the cost profile depends upon the option and an observed event, while in their definition agents' utilities depend upon the option (or allocation bundles) and their (unobserved) private information. Although our definition allows each coalition to use information from an observed event, incentive constraints still matter, as the following example shows.

Example. MCST game. Let us consider an MCST game to observe the conditions (particularly, incentive compatibility) imposed on the mechanisms. Let $N = \{1, 2\}$ and 0 be the source. The costs of possible links are shown below, where the cost of the link {01} may be 4 or 16. Each occurs with probability 1/2. We assume that only agent 1 knows the true cost of the link {01} as private information. Then, $\Omega = \{\omega, \omega'\}$ and

[6]The asterisk indicates incentive compatibility such as that in Holmström and Myerson [10].

$T_1 = \{t_1, t_1'\}$ where ω and t_1 correspond to the case where the cost of the link $\{01\}$ is 4, and ω' and t_1' correspond to the case where it is 16.

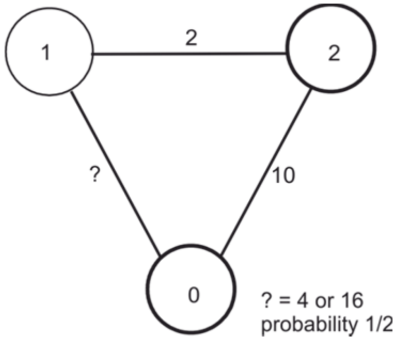

Fig. 1. MCST game under asymmetric information

When the information structure is complete, the allocation rule suggested by Bird [1] (which we call the Bird mechanism) gives a cost profile in the core [9]. In the case where the information structure is incomplete, we can check that the Bird mechanism violates incentive compatibility[7].

The Bird [1] mechanism connects the links so that the total cost is minimized, and each agent pays the cost of the link incident to him on the unique path from the source. In this example, each agent's cost according to the reported type of T_1 and the true state implemented by the Bird mechanism is represented by Table 1.

Table 1. Cost allocation rule
by the Bird mechanism

t	ω	ω'
t_1	4, 2	16, 2
t_1'	2, 10	

Agent 1's reported type is used to label the rows in the table, and the state is used to label the columns. The two numbers in each cell are the agents' costs: the first component is the cost paid by agent 1, while the second is that paid by agent 2. If t_1' is reported, then the mechanism does not connect the link $\{01\}$. In this case, we are not able to assign different cost profiles according to the state (by measurability).

[7]The same point was also shown by Kamishiro [12]. This result is due to the fact that the Bird mechanism violates cost monotonicity [3].

Consider the situation in which the state is ω, i.e., the true type of agent 1 is t_1. If agent 1 announces his private information truthfully (i.e., reports t_1), agent 1's cost is 4 (the cost of link {01} would be 4, because the state of Ω is ω). Whereas, if agent 1 reports t_1', his cost is only 2. Therefore, this mechanism violates incentive compatibility.

4. Ex ante stage

In this section, we consider the situation where a coalition is formed at the ex ante stage. We denote $Ec_i(\mu_S)$ as the (ex ante) expected cost of agent $i \in S$ from the mechanism μ_S:

$$Ec_i(\mu_S) = \sum_{t_S,\omega} \pi(t_S, \omega) c_{S,i}(z_S(t_S), t_S, \omega)$$

Let $\mu_N \in M_N^*$ and $\nu_S \in M_S^*$ for $S \subseteq N$. The mechanism ν_S ex ante dominates μ_N for coalition S if $Ec_i(\nu_S) < Ec_i(\mu_N)$ for all $i \in S$. The ex ante incentive compatible core (ex ante IC core) mechanism is the set of all mechanisms $\mu_N \in M_N^*$ that are not ex ante dominated by any mechanism $\nu_S \in M_S^*$ for any coalition S. The ex ante IC core is the set of all expected cost vectors $(x_i)_{i \in N}$ such that $x_i \geq Ec_i(\mu_N)$ for all $i \in N$, where μ_N is an ex ante incentive compatible core mechanism.

We define a cooperative game corresponding to the ex ante cost allocation situation, i.e., the cost function C_A^* in this setup. A coalition S achieves a cost vector $x = (x_i)_{i \in S} \in R^S$ (i.e., $x \in C_A^*(S)$), if there exists an incentive compatible mechanism $\mu_S \in M_S^*$ such that $x_i \geq Ec_i(\mu_S)$ for every $i \in S$.

We can readily check that if for $x = (x_i)_{i \in S} \in C_A^*(S)$ and $x' = (x_i')_{i \in S} \in R^S$, $\sum_{i \in S} x_i = \sum_{i \in S} x_i'$ holds, then $x' \in C_A^*(S)$. In fact, x' can be achieved by modifying the cost allocations used for x independently of types, i.e., in an incentive compatible way. Thus, the cooperative game is a game with side payments (hereafter called a TU game), and we can describe the cost function as

$$c_A^*(S) := \min_{\mu_S \in M_S^*} \sum_{i \in S} \sum_{t_S \in T_S} \pi(t_S) Ec_i(\mu_S \mid t_i).$$

Here, the minimum is taken over all incentive compatible mechanisms μ_S. This defines the ex ante cost allocation game.

The core of a TU game over N is the set of all cost vectors, which can be achieved by the grand coalition, $x = (x_i)_{i \in N}$, $\sum_{i \in N} x_i \geq c^*(N)$ and cannot be improved on by any coalition, i.e., $\sum_{i \in S} x_i \leq c^*(S)$ for every S. The ex ante IC core is given by the set of expected cost vectors $x = (x_i)_{i \in N}$ that belong to the core of the ex ante cost allocation game $((c_A^*(S))_{S \subseteq N}$.

As a benchmark, we consider the cost function without incentive compatibility constraints

$$c_A(S) := \min_{\mu_S \in M_S} \sum_{i \in S} \sum_{t_S \in T_S} \pi(t_S) E c_i(\mu_S \mid t_i).$$

By definition, $c_A^*(S) \geq c_A(S)$.

We also define the core for each state of nature. If the state of nature is specified, the core could be defined in the standard way (complete information case): for each $\omega \in \Omega$, let $c(S, \omega) := \min_{x_S \in X_S} c(x_S, \omega)$. The core of the cost allocation game in state $\omega \in \Omega$ is given by the set of allocations $((y_i(\omega))_{i \in N} \in R^N$, satisfying $\sum_{i \in N} y_i(\omega) = c(N, \omega)$ and $\sum_{i \in S} y_i(\omega) \leq c(S, \omega)$ for every $S \subset N$.

Without incentive constraints, this model can be interpreted as a cost allocation game with symmetric uncertainty, and we can easily obtain a non-emptiness result for the ex ante core (without IC) when the core in each state is non-empty. However, when incentive compatibility conditions are imposed, then showing that the core is non-empty is not trivial. In exchange economies, Vohra [20] and Forges, Mertens and Vohra [6] give examples where the ex ante IC core is empty. In our setup, we can show that the ex ante IC core is non-empty if the core in each state is non-empty.

Theorem 1. If the core of the cost allocation game in each state $\omega \in \Omega$ is non-empty, then the ex ante IC core is non-empty.

Proof of Theorem 1. To prove the theorem, it suffices to show $c_A(N) = c_A^*(N)$, i.e., there exists an incentive compatible mechanism that achieves first-best efficiency [6].

We consider a proportional mechanism for coalition N. This mechanism allocates the total cost among agents at a constant rate, whichever the reported type and observed event. This mechanism is denoted by $\mu_N^P[k_N]$, where $k_N = (k_i)_{i \in N}$ is an

N-dimensional vector satisfying $k_i \geq 0$ for all $i \in N$ and $\sum_{i \in N} k_i = 1$. Using this mechanism, each agent i pays a fraction k_i of the aggregate cost. The mechanism $\mu_N^P[k_N] := (z_N^P, c_N^P)$ is represented by the following:

$$z_N^P(t_N) := \arg\min_{x_N \in X_N} \sum_{\omega} \pi(\omega \mid t_N) C(x_N, \omega)$$ [8]

$$c_{N,i}^P(z_N(t_N), t_N, \omega) := k_i C(z_N(t_N), \omega)$$

This mechanism clearly satisfies the feasibility conditions. It also satisfies incentive compatibility, because

$$\begin{aligned}
& Ec_i(\mu_N^P \mid t_i, r_i) \\
& = \sum_{t_{-i}, \omega} \pi(t_{-i}, \omega \mid t_i) c_{N,i}^P(r_i, t_{-i}, \omega) \\
& = \sum_{t_{-i}, \omega} \{\pi(t_{-i}, \omega \mid t_i) k_i C(z_N(r_i, t_{-i}), \omega)\} \\
& = k_i \sum_{t_{-i}, \omega} \{\pi(t_{-i} \mid t_i) \pi(\omega \mid t_N) C(z_N(r_i, t_{-i}), \omega)\} \\
& \geq k_i \sum_{t_{-i}, \omega} \{\pi(t_{-i} \mid t_i) \pi(\omega \mid t_N) C(z_N(t_i, t_{-i}), \omega)\} \\
& = \sum_{t_{-i}, \omega} \pi(t_{-i}, \omega \mid t_i) c_{N,i}^P(t_i, t_{-i}, \omega) \\
& = Ec_i(\mu_N^P \mid t_i)
\end{aligned}$$

■

Remarks

1. In Examples 1 through 3, the core of the cost allocation game in each state $\omega \in \Omega$ is non-empty. Hence the ex ante IC cores of these games are non-empty.

---

[8]To simplify the argument, we assume that the minimizer is a singleton. If not, then by taking probability distribution on minimizers, the same argument can be applied.

2. Forges [5] shows that to obtain ex ante characteristic functions in assignment games with asymmetric information, one cannot restrict the set of mechanisms to deterministic ones without loss of generality. In our setup, however, we can restrict our argument to deterministic mechanisms to obtain ex ante cost functions, which can be derived in a similar way to the one used in the above proof by constructing a proportional mechanism for each coalition.

3. In this cost allocation game, we find that one of the proportional mechanisms belongs to the ex ante IC core. The appropriate proportion can be determined as follows. Let $(c_i)_{i\in N}$ be an element of the core of the ex ante cost allocation game $((c_A(S))_{S\subseteq N}$. Then, if we define the rate $(k_i)_{i\in N}$ as $k_i := \frac{c_i}{\sum_{j\in N} c_j}$, we can show that the mechanism leads to the ex ante IC core allocation.

For example, let $N = \{1, 2\}$. There are two states of the world, $\Omega = \{\omega, \omega'\}$, each of which occurs with probability 1/2. Only agent 1 knows the state from his private information ($T_1 = \{t_1, t_1'\}$). Let $X_1 = \{x_1\}$, $X_2 = \{x_2\}$ and $X_{1,2} = \{x_1, x_2\}$. The cost profile is $C(x_1, \omega) = 6$, $C(x_1, \omega') = 14$, and $C(x_2, \omega) = C(x_2, \omega') = 10$.

Then, the ex ante cost function is $c_A(\{1\}) = 10$, $c_A(\{2\}) = 10$, and $c_A(\{1, 2\}) = 8$. We can check that the allocation $(c_1, c_2) = (4, 4)$ belongs to the core of the cost function, and then $k_1 = k_2 = 1/2$. The allocation rule corresponding to this proportion is shown in Table 2.

Table 2. An example of the ex ante IC core allocations

t	ω	ω'
t_1	3, 3	7, 7
t_1'	5, 5	

We can check that this allocation rule belongs to the ex ante IC core.

5. Interim stage

At the interim stage, agents already possess private information when they engage in coalitional negotiations. Hence, to define notions of the core at the interim stage, we need to specify to what extent agents can exchange information in a coalition, and depending on this, several concepts of the core can be considered. If we assume that agents are not allowed to exchange information until they are in a coalition, then the

IC coarse core defined by Vohra [20] is an appropriate notion of the core. It is larger than other interim cores in which the blocking of agreements can be facilitated by information transmission among agents (e.g., the credible core in [3]).

5.1. The incentive compatible coarse core

First, we consider the non-emptiness of the IC coarse core in cost allocation situations. According to this concept of the core, agents base their objections on events that are common knowledge inside the coalition at the interim stage. For an event $E \subseteq T_N$, let E_i be the corresponding set of types for agent *i*, i.e., $E_i := \{t_i \mid t_N \in E\}$. An event $E \subseteq T_N$ is common knowledge for S if $\pi(t'_{-i} \mid t_i) = 0$ for all $i \in S$, $t_i \in E_i$ and $(t'_{-i}, t_i) \notin E$. If S is a singleton, then common knowledge is synonymous with "knowledge" (because each agent already knows his own type at the interim stage).

Let $\mu_N \in M_N^*$ be a feasible and incentive compatible mechanism. Coalition *S* has an incentive compatible coarse objection to μ_N if there exists an event *E* that is common knowledge for *S* and an incentive compatible mechanism $\nu_S \in M_S^*$ such that $Ec_i(\nu_S \mid t_i) < Ec_i(\mu_N \mid t_i)$. The incentive compatible coarse core (IC coarse core) mechanism is the set of all mechanisms $\mu_N \in M_N^*$ that do not have any incentive compatible coarse objection. The incentive compatible coarse core is the set of all interim expected cost vectors $(x_i(t_i))_{i\in N, t_i \in T_i}$ such that $x_i(t_i) \geq Ec_i(\mu_N \mid t_i)$ for all $i \in N$ and all $t_i \in T_i$, where μ_N is an incentive compatible coarse core mechanism.

Theorem 2. If the core of the cost allocation game in each state $\omega \in \Omega$ is non-empty, then the IC coarse core is non-empty.

To simplify mathematical expressions, we limit our discussion to the case where $\pi(\omega \mid t_N)$ is either 1 or 0 for all $t_N \in T_N$ and all $\omega \in \Omega$, i.e., collecting all the agents' private information together resolves all uncertainty. (Since we assume that agents are risk-neutral, this assumption does not affect the argument below.)

Wilson [21] showed that the coarse core (without incentive constraints) of a standard exchange economy is non-empty, since it is the standard core of an appropriately defined balanced NTU cooperative game with players (i, t_i). Vohra [20] extended his argument to exchange economies with linear utility functions, and Forges [5] extended this to assignment games. We proceed in a similar way and establish that the IC coarse core is the core of an NTU function.

The key factor in showing the non-emptiness of the core is the lemma given below (similar lemmas are used in Vohra [20] and Forges [5]).

Lemma. Suppose that the core of the cost allocation game in each state $\omega \in \Omega$ is non-empty. Let **S** be a balanced family of coalitions with associated weights $\lambda_S (S \in \mathbf{S})$, and let μ_S be a feasible mechanism for *S*. Consider the following mechanism for *N* $\mu_N := (z_N, c_N)$:

$$z_N(t_N) := \arg\min_{x_N \in X_N} \sum_{\omega} \pi(\omega \mid t_N) C(x_N, \omega)$$

$$c_{N,i}(z_N(t_N), t_N, \omega) := \sum_{S \in \mathbf{S},\, S \ni i} \lambda_S c_{S,i}(z_S(t_S), t_S, \omega)$$

where t_S is the projection of t_N onto the *S*-coordinate. Then, μ_N is feasible and

$$Ec_i(\mu_N \mid t_i, r_i) = \sum_{S \in \mathbf{S},\, S \ni i} \lambda_S Ec_i(\mu_S \mid t_i, r_i)$$

In particular, if every μ_S is incentive compatible, so is μ_N.

Proof of Lemma. We use the Bondareva–Shapley [19] theorem, which states that a TU game has a nonempty core if and only if it is balanced, to prove the lemma. Recall that a family **S** of coalitions is balanced if there exist weights λ_S $(S \in \mathbf{S})$ such that for all $i \in N$, $\sum_{S \in \mathbf{S}, S \ni i} \lambda_S = 1$ and that a game *c* (represented by its cost function) is balanced if $\sum_{S \in \mathbf{S}} \lambda_S c(S) \geq c(N)$ for every such family.

Since the core of the cost allocation game in each state $\omega \in \Omega$ is non-empty, by balancedness, the following holds: for each $\omega \in \Omega$, for all $S \in \mathbf{S}$, and for all $x_S \in X_S$, there exists $x_N \in X_N$ such that $C(x_N, \omega) \leq \sum_{S \in \mathbf{S}} \lambda_S C(x_S, \omega)$.

Using the above, we can show that μ_N satisfies the cost-coverage condition, because

$$C(z_N(t_N), \omega) \leq \sum_{S \in \mathbf{S}} \lambda_S C(z_S(t_S), \omega)$$

$$\leq \sum_{S \in \mathbf{S}, S \ni i} \lambda_S c_{S,i}(z_S(t_S), t_S, \omega) = \sum_{i \in N} c_{N,i}(z_N(t_N), t_N, \omega)$$

If μ_S is incentive compatible for every *S*, then so is μ_N, since the reports of agents in *N**S* play no role. ■

Proof of Theorem 2. As in [21], let us construct an auxiliary NTU game, in which the players are of the types from the original model. A typical player is denoted (i, t_i) and there are $\sum_i |T_i|$ players. The cost function of player (i, t_i) is $Ec_i(\cdot\,|\,t_i)$. The grand coalition is (N, T_N), and other allowable coalitions are restricted to the form $(S, E) = \{(i, t_i)\,|\, i \in S, t_i \in T_i\}$, where $S \subseteq N$ and E is a common knowledge event for S. The feasible cost set for S is derived by applying the cost functions $Ec_i(\cdot\,|\,t_i)$ to the set of feasible mechanisms. We can show by a standard argument that for any balanced collection of coalitions and corresponding mechanisms, the mechanism constructed as in the Lemma above is feasible for the grand coalition. The NTU game is, therefore, balanced and by Scarf's theorem [17], it has a non-empty core. ■

5.2. Interim core notions with information transmission

As a different concept of the core, we consider the core based on endogenous information transmission in which each blocking move is identified with an equilibrium based on a communication mechanism used by coalitions. This approach was proposed by Serrano and Vohra [18]. They showed that this notion of core encompasses other notions proposed in the literature such as the credible core of Dutta and Vohra [4] and the virtual utility core of Myerson [16], depending on the constraints upon the mechanisms that each coalition can use. If random mechanisms are allowed and there is no restriction on randomization, the appropriate notion of a core is the randomized mediated core of Serrano and Vohra [18]. It is known that the randomized mediated core is a subset of the virtual utility core and the credible core.

We can show that the randomized mediated core might be empty in cost allocation situations. Since there is no restriction on the mechanisms in the definition of the randomized mediated core, such a core would be almost identical to the ex post core with (interim) incentive compatibility, where ex post refers to the case where the type profile is commonly known by all agents. In this section, we give an example and show that the ex post core with (interim) incentive compatibility in this example is empty[9]. Using almost the same steps as those in Example 4.1 from Kamishiro and Serrano [13], we can show that the randomized mediated core is empty in the given example. However, we omit the proof here.

[9]This numerical example is based on Young [22], which shows that there is no core allocation rule that is coalitionally monotonic (in complete information settings).

Example. There are five agents, $N = \{1, 2, 3, 4, 5\}$. There are two states of the world, ω and ω'. Only agent 2 is informed of the true state as his private information $(T_2 = \{t_2, t'_2\})$.

Let $\hat{\mathbf{S}} := \{\{1, 5\}, \{1, 2, 4\}, \{1, 4, 5\}, \{1, 2, 4, 5\}, N\}$. In each coalition, the set of options and cost of each option is determined as follows.

• If $S \notin \hat{\mathbf{S}}$, then the coalition has only one option (i.e., X_S is a singleton) and the cost of this option is independent of the true state. The cost profile of the family of sets other than $\hat{\mathbf{S}}$ is

$$C(x_{S_1}, \omega) = C(x_{S_1}, \omega') = 3 \quad (S_1 = \{3, 5\})$$

$$C(x_{S_2}, \omega) = C(x_{S_2}, \omega') = 3 \quad (S_2 = \{1, 2, 3\})$$

$$C(x_{S_3}, \omega) = C(x_{S_3}, \omega') = 9 \quad (S_3 = \{1, 3, 4\})$$

$$C(x_{S_4}, \omega) = C(x_{S_4}, \omega') = 9 \quad (S_4 = \{2, 4, 5\})$$

• For $S \subset S_j$ $(j = 1, 2, 3, 4)$, define $C(x_S, \omega) = \min_j \{C(x_S, \omega) : S \subset S_j\}$ for $\omega \in \Omega$.

• For $S \in \hat{\mathbf{S}} \setminus N$, there are two options, x_S and x'_S. The cost profile of the options in each state is given as $C(x_S, \omega) = 9$, $C(x_S, \omega') = 15$, and $C(x'_S, \omega) = C(x'_S, \omega') = 12$.

• For N, $C(x_N, \omega) = 11$, $C(x_N, \omega') = 15$, and $C(x'_N, \omega) = C(x'_N, \omega') = 12$.

Then, every ex post core allocation rule needs to satisfy the conditions $c_N(z_N(t_2), t_2, \omega) = (0, 1, 2, 7, 1)$ and $c_N(z_N(t'_2), t'_2, \omega') = (3, 0, 0, 6, 3)$. Hence, agent 2 has an incentive to falsely identify his type when his true type is t_2. This means that the ex post core with (interim) incentive compatibility is empty, although the core in each state is non-empty. ■

6. Conclusion

We have defined cost allocation problems under asymmetric information and showed that both the ex ante IC core and the IC coarse core are non-empty in these situations. Hence, our framework can be seen as deriving sufficient conditions for the non-emptiness of these cores. Obtaining necessary conditions for the non-emptiness of these cores would be an interesting open problem.

An example has also been presented in which the randomized mediated core is empty in cost allocation situations. It is not clear whether we can obtain any results on the non-emptiness of the core when there are some restrictions on the mechanisms available (the credible core and the virtual utility core would be appropriate notions of the core). However, this would be a topic for future research.

As a different (but somewhat similar) situation, we could also consider the case where a group of agents chooses one of several options, and its aggregate "benefit" is allocated among the agents. (The agents have private information on the state of nature, which determines their benefits from the options they can choose.) If the cores are defined in appropriate ways, we can similarly obtain results on the non-emptiness of the cores. In this situation, if the information structures are complete, then the framework can be embedded into classical TU games. The extension to environments without side payments also remains a significant topic.

References

[1] BIRD C.G., *On cost allocation for a spanning tree: A game theory approach*, Networks, 1976, 6 (4), 335–350.

[2] CLAUS A., KLEITMAN D.J., *Cost allocation for a spanning tree*, Networks, 1973, 3 (4), 289–304.

[3] DUTTA B., KAR A., *Cost monotonicity, consistency and minimum cost spanning tree games*, Games and Economic Behavior, 2004, 48 (2), 223–248.

[4] DUTTA B., VOHRA R., *Incomplete information, credibility, and the core*, Mathematical Social Sciences, 2005, 50 (2), 148–165.

[5] FORGES F., *The ex ante incentive compatible core of the assignment game*, Mathematical Social Sciences, 2004, 47 (2), 135–151.

[6] FORGES F., MERTENS J.-F., VOHRA R., *The ex ante incentive compatible core in the absence of wealth effects*, Econometrica, 2002, 70 (5), 1865– 1892.

[7] FORGES F., MINELLI E., VOHRA R., *Incentive and the core of an exchange economy: a survey*, Journal of Mathematical Economics, 2002, 38 (1–2), 1–41.

[8] FORGES F., SERRANO R., *Cooperative games with incomplete information: some open problems*, International Game Theory Review, 2013, 15 (2), 301–317.

[9] GRANOT D., HUBERMAN G., *Minimum cost spanning tree games*, Mathematical Programming, 1981, 21 (1), 1–18.

[10] HOLMSTRÖM B., MYERSON R., *Efficient and durable decision rules with incomplete information*, Econometrica, 1979, 51 (6), 1799–1819.

[11] HOUGAARD J.L., *An introduction to allocation rules*, Springer, Berlin 2009.

[12] KAMISHIRO Y., *The ex ante incentive compatible core of the minimum cost spanning tree games*, Keizaikei (Kanto Gakuin Economics Journal), 2013, 257, 11–21.

[13] KAMISHIRO Y., SERRANO R., *Equilibrium blocking in large quasilinear economies*, Mathematics of Operations Research, 2011, 36 (3), 552–567.

[14] MAS-COLELL A., WHINSTON M.D., GREEN J.R., *Microeconomic Theory*, Oxford University Press, New York 1995.

[15] MYERSON R.B., *Incentive compatibility and the bargaining problem*, Econometrica, 1979, 47 (1), 61–74.

[16] MYERSON R.B., *Virtual utility and the core for games with incomplete information*, Journal of Economic Theory, 2007, 136 (1), 260–285.
[17] SCARF H., *The core of an n-person game*, Econometrica, 1967, 35 (1), 50–69.
[18] SERRANO R., VOHRA R., *Information transmission in coalitional voting games*, Journal of Economic Theory, 2007, 134 (1), 117–137.
[19] SHAPLEY L.S., *On balanced sets and core*, Naval Research Logistics Quarterly, 1967, 14 (4), 453–460.
[20] VOHRA R., *Incomplete information, incentive compatibility, and the core*, Journal of Economic Theory, 1999, 86 (1), 123–147.
[21] WILSON R., *Information, efficiency, and the core of an economy*, Econometrica, 1978, 46 (4), 807–816.
[22] YOUNG H.P., *Monotonic solutions of cooperative games*, International Journal of Game Theory, 1985, 14 (2), 65–72.
[23] YOUNG H.P., *Cost allocation*, [in:] R. Aumann, S. Hart (Eds.), *Handbook of Game Theory with Economic Applications*, Vol. 2, Elsevier Science, 1994, 1193–1235.

Received 13 September 2014
Accepted 27 January 2015