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CHOQUET INTEGRAL CALCULUS ON A CONTINUOUS SUPPORT AND ITS APPLICATIONS

The results of the calculation of the Choquet integral of a monotone function on the nonnegative real line have been described. Next, the authors prepresented Choquet integral of nonmonotone functions, by constructing monotone functions from nonmonotone ones by using the increasing or decreasing rearrangement of a nonmonotone function. Finally, this paper considers some applications of these results to the continuous agregation operator OWA, and to the representation of risk measures by Choquet integral.

Keywords: *Choquet integral, distorted Lebesgue measure, risk measure, OWA operator*

1. Introduction

The notion of measure is a very important concept in mathematics, particularly for the theory of integrals. These measures are based on the property of additivity. This property is not required in many areas such as decision theory and the theory of cooperative games, where it has become essential to define nonadditive measures, which are usually called capacities [2], or fuzzy measures [17]. A fundamental concept that uses such nonadditive measures is the Choquet integral [2], defined as an integral with respect to a capacity.

The Choquet integral is a nonadditive integral of a function with respect to a capacity (or nonadditive measure, or fuzzy measure). It was characterized mathematically by Schmeidler [15], and then by Murofushi and Sugeno [8] using the concept of a capacity introduced by Choquet. Later it was used in utility theory [16], leading to the so-called Choquet expected utility.

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So far, many studies have focused on the theory and the applications of the Choquet integral defined on a discrete set [5, 7]. In the discrete case, the Choquet integral of a function with respect to a capacity is easy to calculate. However, this is not the case for the Choquet integral of functions on a continuous support. Recent developments in the theory regarding the Choquet integral of real functions [18, 19] appear to open up new horizons.

This paper is a continuation of the seminal work of Sugeno [18, 19] and results and applications already established by Narukawa and Torra [10, 12]. In particular, on the theoretical side, we provide methods for the calculation of the Choquet integral for nonmonotone functions, first by providing an analytic calculation for functions with a single maximum or minimum, and second by providing a general method based on the increasing or decreasing rearrangement of a function. In the second part of the paper, we give some possible applications of these new methods, e.g., for computing the continuous version of OWA operators, and for computing distortion risk measures.

This paper is organized as follows. Section 2 is devoted to the concepts of measures and capacities, and to the presentation of the Choquet integral, and essentially focuses on definitions. In Section 3, we present the results obtained by Sugeno [19] for calculating the Choquet integral of a monotone function on the nonnegative real line with respect to a capacity, in particular with respect to distorted Lebesgue measure. Then we consider the Choquet integral of nonmonotone functions, exploring methods of analytical calculation with examples, and by using the increasing or decreasing rearrangement of a nonmonotone function to turn it into a monotone function. In Section 4, we focus on the application of the results obtained from the previous sections, to the continuous aggregation operator OWA. Also, we make some links between the Choquet integral, in the context of distorted probabilities, and concepts used in finance, such as the notion of a risk measure. Finally, we end this paper with some concluding remarks.

2. Preliminaries

In this section, we present some basic definitions and properties of measures and the Choquet integral.

2.1. Additive and nonadditive measures

We recall some definitions and properties of additive measures and capacities.

Let Ω be a set, and let $\mathcal{A}$ be a collection of subsets of Ω. $\mathcal{A}$ is a σ-algebra if it satisfies the following conditions:

1) $\Omega \in \mathcal{A}$,
2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
3) $\forall n \in \mathbb{N}$, $A_n \in \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$.

The pair $(\Omega, \mathcal{A})$ is called a measurable space.

Let $(\Omega, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \to \mathbb{R}^+$ is a σ-additive measure if it satisfies the following conditions:

1) $\mu(\varnothing) = 0$,
2) $\mu(\bigcup_{n=1}^{+\infty} A_i) = \sum_{n=1}^{+\infty} \mu(A_i)$ for every countable sequence $(A_n)_{n \in \mathbb{N}}$ of $\mathcal{A}$ of pairwise disjoint sets ($A_i \cap A_j = \varnothing$ for all $i \neq j$).

The triplet $(\Omega, \mathcal{A}, \mu)$ is called a measure space.

A probability measure P on $(\Omega, \mathcal{A})$ is an additive measure such that $\mu(\Omega) = 1$. The triplet $(\Omega, \mathcal{A}, P)$ is called a probability space.

Let $\mathcal{B}$ be the smallest σ-algebra including all the closed intervals of $\mathbb{R}$. There is a measure λ on $(\mathbb{R}, \mathcal{B})$ such that: $\lambda([a, b]) = b - a$ for every interval $[a, b]$ with $-\infty < a \leq b < \infty$. This measure is called the Lebesgue measure.

Consider two measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$. A function $f : \Omega_1 \to \Omega_2$ is measurable if $\forall E \in \mathcal{A}_2$, $f^{-1}(E) \in \mathcal{A}_1$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A measurable function f from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ is called a random variable.

The distribution function of a random variable X is defined to be the function $F : \mathbb{R} \to [0, 1]$ given by:

$$\forall x \in \mathbb{R},\ F(x) = P(X \leq x)$$

The notion of capacity was introduced by Choquet [2] in his theory of capacities. A similar concept was proposed by Sugeno [17] under the name fuzzy measure, and by Denneberg [3] under the name nonadditive measure.

Definition 1. Let $(\Omega, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \to [0, 1]$ is called a capacity [2] or fuzzy measure [17] if it satisfies the following conditions:

1) $\mu(\varnothing) = 0$,
2) $\mu(A) \leq \mu(B)$, if $A \subseteq B$, $A, B \in \mathcal{A}$.

For any capacity μ, the dual capacity $\bar{\mu}$ is defined by $\bar{\mu}(A) = 1 - \mu(A^c)$ for any $A \in \mathcal{A}$. The capacity μ is normalized if, in addition, $\mu(\Omega) = 1$.

Definition 2. Let μ be a capacity on $(\Omega, \mathcal{A})$. μ is called concave or submodular, if $\mu(A)+\mu(B) \geq \mu(A \cup B)+\mu(A \cap B)$ for all $A, B \in \mathcal{A}$.

A capacity μ is called convex or supermodular, if it satisfies the previous property with the reverse inequality.

2.2. Distortion measures

We call any nondecreasing function $m:[0,1] \to [0,1]$ with $m(0)=0$ and $m(1)=1$ a distortion function.

Let P be a probability measure on $(\Omega, \mathcal{A})$ and let m be a distortion function. The set function $m \circ P$ defined by $m \circ P(A) = m(P(A)), \forall A \in \mathcal{A}$ is called a distorted probability.

Definition 3 [19]. Consider a Lebesgue measure λ on $(\mathbb{R}, \mathcal{B})$ and let $m:\mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable nondecreasing function such that $m(0)=0$. Then $\mu_m = m \circ \lambda$ is a distorted Lebesgue measure. We have $\mu_m([a,b]) = m(b-a)$.

2.3. Choquet integral

Definition 4 [2, 8]. Let μ be a capacity on $(\Omega, \mathcal{A})$, and let $f:\Omega \to \mathbb{R}^+$ be a measurable function. The Choquet integral of f with respect to μ is defined by:

$$(C)\int f d\mu = \int_0^\infty \mu_f(r)dr$$

where $\mu_f(r) = \mu(\{\tau \mid f(\tau) \geq r\})$.

Let $A \subset \Omega$. The Choquet integral of f with respect to μ on A is defined by [18]:

$$C\int_A f d\mu = \int_0^\infty \mu(\{\tau \mid f(\tau) \geq r\} \cap A)dr$$

Below, we give some well-known properties of the Choquet integral (see, e.g., Denneberg [3])

Proposition 1. Let μ be a capacity on $(\Omega, \mathcal{A})$. Let f and g be two measurable functions on $(\Omega, \mathcal{A})$. We have the following properties:

1) if $f \le g$ then $(C)\int f d\mu \le (C)\int g d\mu,$

2) $(C)\int \lambda f d\mu = \lambda (C)\int f d\mu$ for every $\lambda \in \mathbb{R}_+,$

3) $(C)\int (f + c) d\mu = (C)\int f d\mu + c\mu(\Omega)$, for every constant $c \in \mathbb{R},$

4) $(C)\int - f d\mu = -(C)\int f d\overline{\mu}.$

Proposition 2. Let μ be a capacity on $(\Omega, \mathcal{A}).$ Let f and g be two measurable functions on $(\Omega, \mathcal{A}).$

- If μ is supermodular, then the Choquet integral with respect to μ is superadditive:

$$(C)\int (f+g) d\mu \ge (C)\int f d\mu + (C)\int g d\mu$$

- If μ is submodular, then the Choquet integral with respect to μ is subadditive:

$$(C)\int (f+g) d\mu \le (C)\int f d\mu + (C)\int g d\mu$$

Proposition 3 [10]. Let μ be a submodular capacity on $(\Omega, \mathcal{A}).$ Let f and g be two measurable functions on $(\Omega, \mathcal{A}).$ The Choquet integral satisfies the following inequality:

$$[(C)\int (f+g)^2 d\mu]^{1/2} \le [(C)\int f^2 d\mu]^{1/2} + [(C)\int g^2 d\mu]^{1/2}$$

3. The calculus of Choquet integrals

In this section, we introduce methods for calculating continuous Choquet integrals with respect to distorted Lebesgue measures on the nonnegative real line. The calculus of continuous Choquet integrals was recently studied by Sugeno [18, 19] for nonnegative monotonic functions, and mainly nondecreasing functions. Based on the results in [18, 19], we will start by calculating the Choquet integral of nonnegative monotonic functions with respect to distorted Lebesgue measures. Next, we construct monotonic functions from nonmonotone ones, in order to calculate the Choquet integral of nonmonotonic functions.

Let $\mathcal{F} = \{ f \mid f : \mathbb{R}^+ \to \mathbb{R}^+,\ f$: measurable, derivable, monotone$\}$ be the class of measurable, differentiable, nonnegative and monotone functions.

We denote the nondecreasing functions of $\mathcal{F}$ by $\mathcal{F}^+$, and the nonincreasing functions of $\mathcal{F}$ by $\mathcal{F}^-$.

Sugeno [18, 19] used the Laplace transform in order to establish the basis of Choquet integral calculus for nondecreasing functions.

Let *f* be a function defined for all real numbers $t \geq 0$. The Laplace transform of *f* is the function $F(s) = \mathcal{L}[f(t)]$ defined by

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

for those $s \in \mathbb{C}$ for which the integral is defined.

The inverse Laplace transform is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds$$

where γ is a real number such that the contour path of integration is in the region of convergence of $F(s)$.

Without any assumption on *f*, the integral $\int_0^{\infty} f(t)e^{-st}dt$ does not necessarily exist. Sufficient condition for the existence of the Laplace transform are:

1. f is a piecewise continuous function.
2. $|f(t)| \leq Me^{\alpha t}$, $M > 0$ and $\alpha \in \mathbb{R}$.

We list below some properties of the Laplace transform which will be useful in the sequel:

1) $\mathcal{L}\int_0^t f dt = \dfrac{F(s)}{s}$,

2) $\mathcal{L}\dfrac{df}{dt} = sF(s) - f(0)$,

3) $\mathcal{L}(f \cdot g) = F(s)G(s)$, where $\cdot$ is the convolution product.

3.1. Choquet integral of monotonic functions

Let μ be a capacity, and assume that $\mu([\tau, t])$ is differentiable with respect to τ on $[0, t]$ for every $t > 0$.

Theorem 1 [18]. Let $g \in \mathcal{F}^+$, then the Choquet integral of *g* with respect to μ on $[0, t]$ is given by:

$$C\int_{0,\ t} g d\mu = -\int_0^t \mu'([\tau, t])g(\tau)d\tau$$

In particular, if $\mu = \mu_m$, we have:

$$C\int_{0,\ t} g d\mu_m = \int_0^t m'(t-\tau)g(\tau)d\tau$$

Remark 1. $C\int_{0,\ t} g d\mu_m = m'\cdot g(t)$, where $\cdot$ is the convolution product.

The next proposition shows how to calculate the Choquet integral by using the Laplace transformation.

Theorem 2 [18]. Let $g \in \mathcal{F}^+$, and let μ_m be a distorted Lebesgue measure. The Choquet integral of g with respect to μ_m on $[0, t]$ is given by:

$$C\int_{0,\, t} g d\mu_m = \mathcal{L}^{-1}[sM(s)G(s)],$$

where $M(s)$ is the Laplace transform of m, $G(s)$ is the Laplace transform of g, and $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Example 1. Let $m(t) = t$, and $g(t) = e^t - 1$, then:

$$C\int_{0,\, t} g d\mu_m(\tau) = C\int_{0,\, t} (\mathrm{e}^\tau - 1)d\mu_m(\tau) = \mathcal{L}^{-1}[sM(s)G(s)]$$

where $M(s) = \mathcal{L}[m(t)] = \dfrac{1}{s^2}$, and $G(s) = \mathcal{L}[g(t)] = \dfrac{1}{s-1} - \dfrac{1}{s}$.

Hence, $sM(s)G(s) = \dfrac{1}{s(s-1)} - \dfrac{1}{s^2}$ and therefore:

$$C\int_{0,\, t} g d\mu_m(\tau) = \mathrm{e}^t - t - 1$$

Theorem 3 [18]. Let μ_m be a distorted Lebesgue measure, let f be a nondecreasing and continuous function with $f(0) = 0$, then the solution of the Choquet integral equation $f(t) = C\int_{0,t} g(\tau)d\mu_m(\tau)$ is given by:

$$g(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{sM(s)}\right]$$

where $g \in \mathcal{F}^+$.

Theorem 4. [18] Let $g \in \mathcal{F}^-$, then the Choquet integral of g with respect to μ on $[0, t]$ is given by:

$$C\int_{0,t} gd\mu = t\int_0^t \mu'([0, \tau])g(\tau)d\tau$$

In particular, if $\mu = \mu_m$, we have:

$$C\int_{0,t} gd\mu_m = t\int_0^t m'(\tau)g(\tau)d\tau$$

3.2. Choquet integral of nonmonotonic functions

In the previous section, we described methods for calculating the continuous Choquet integral of monotonic functions on the nonnegative real line. In this section, we will use nondecreasing and nonincreasing rearrangements of a nonmonotonic function to transform it into a monotonic function. This issue was addressed by Ralescu and Sugeno [14], as well as Ralescu [13], and recently studied by Sugeno [19] and Narukawa et al. [12].

Let $\mathcal{G}$ be the class of nonnegative and continuous functions on $[0, t]$ for some fixed $t \in \mathbb{R}^+$: $\mathcal{G} = \{g \mid g : [0, t] \to \mathbb{R}^+, g \text{ continuous}\}$.

3.2.1. Analytic calculation

In this section, we provide an explicit expression for the Choquet integral of functions with one maximum or minimum, that is, for functions which are increasing then decreasing on an interval (or the converse).

Let $g \in \mathcal{G}$. We calculate the Choquet integral of g on $[0, t]$, when there exists $\tau_m \in [0, t]$ such that $\overline{g} = g(\tau_m) = \max_{0\le\tau\le t} g(\tau)$ (or $\underline{g} = g(\tau_m) = \min_{0\le\tau\le t} g(\tau)$).

We assume first that the function g is nondecreasing on $[0, \tau_m]$, and nonincreasing on $[\tau_m, t]$, with $g(\tau_m) = \overline{g}$ (Fig. 1).

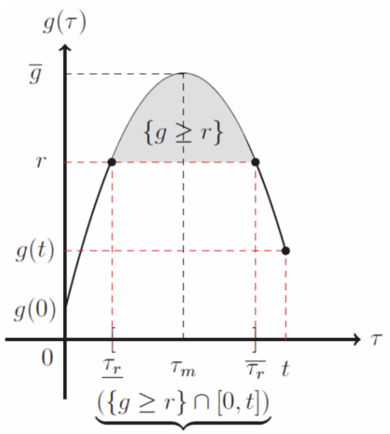

Fig. 1. The case of a nonmonotonic function

Let $g_1(\tau) = g(\tau)$ on $[0, \tau_m]$, and $g_2(\tau) = g(\tau)$ on $[\tau_m, t]$. If $g(t) \ge g(0)$ (respectively, $g(t) \le g(0)$), for each $r \in [g(t), \overline{g}]$ (respectively, $r \in [g(0), \overline{g})$, there exists a unique pair $(\underline{\tau_r}, \overline{\tau}_r)$, such that: $\underline{\tau_r} = g_1^{-1}(r)$, and $\overline{\tau}_r = g_2^{-1}(r)$ (Fig. 1).

• If $g(t) \ge g(0)$, the Choquet integral of g with respect to the measure μ is:

$$
\begin{aligned}
(C)\int_{[0,t]} g d\mu &= \int_0^{\infty} \mu(\{\tau \mid g(\tau) \ge r\} \cap [0, t]) dr \\
&= \int_0^{g(0)} \mu([0, t]) dr + \int_{g(0)}^{g(t)} \mu([g_1^{-1}(r), t]) dr + \int_{g(t)}^{g} \mu([g_1^{-1}(r), g_2^{-1}(r)]) dr \\
&= \mu([0, t]) g(0) + \int_{g(0)}^{g(t)} \mu([g_1^{-1}(r), t]) dr + \int_{g(t)}^{\overline{g}} \mu([g_1^{-1}(r), g_2^{-1}(r)]) dr
\end{aligned}
$$

In particular, if $\mu = \mu_m$, we have:

$$C\int_{0,t} g d\mu_m = m(t)g(0) + \int_{g(0)}^{g(t)} m(t - g_1^{-1}(r))dr + \int_{g(t)}^{\bar{g}} m(g_2^{-1}(r) - g_1^{-1}(r))dr \qquad (1)$$

• If $g(t) \le g(0)$, the Choquet integral of g with respect to the measure μ is given by:

$$C\int_{0,t} g d\mu = \mu([0, t])g(t) + \int_{g(t)}^{g(0)} \mu([0, g_2^{-1}(r)])dr + \int_{g(0)}^{\bar{g}} \mu([g_1^{-1}(r), g_2^{-1}(r)])dr$$

In particular, if $\mu = \mu_m$, we have:

$$C\int_{0,t} g d\mu_m = m(t)g(t) + \int_{g(t)}^{g(0)} m(g_2^{-1}(r))dr + \int_{g(0)}^{\bar{g}} m(g_2^{-1}(r) - g_1^{-1}(r))dr \qquad (2)$$

We assume now that g is nonincreasing on $[0, \tau_m]$, and nondecreasing on $[\tau_m, t]$, with $\underline{g} = g(\tau_m) = min_{0\le\tau\le t} g(\tau)$.

Proceeding similarly, we find:

• If $g(t) \ge g(0)$, the Choquet integral of g with respect to the measure μ is:

$$C\int_{[0,t]} g d\mu = \mu([0, t])\underline{g} + \int_{\underline{g}}^{g(0)} \mu([0, g_1^{-1}(r)] \cup [g_2^{-1}(r), t])dr + \int_{g(0)}^{g(t)} \mu([g_2^{-1}(r), t])dr$$

In particular, if $\mu = \mu_m$, we have:

$$C\int_{0,7} g d\mu_m = m(t)\underline{g} \int_{\underline{g}}^{g(0)} m(t + g_1^{-1}(r) - g_2^{-1}(r)]dr + \int_{g(0)}^{g(t)} m(t - g_2^{-1}(r)]dr \qquad (3)$$

• If $g(t) \le g(0)$, the Choquet integral of g with respect to the measure μ is given by:

$$C\int_{0,t} g d\mu = \mu([0, t])\underline{g} + \int_{\underline{g}}^{g(t)} \mu([0, g_1^{-1}(r)] \cup [g_2^{-1}(r), t])dr + \int_{g(t)}^{g(0)} \mu([0, g_1^{-1}(r)])dr$$

In particular, if $\mu = \mu_m$, we have:

$$C\int_{0,t} g d\mu = m(t)\underline{g} + \int_{\underline{g}}^{g(t)} m(t + g_1^{-1}(r) - g_2^{-1}(r))dr + \int_{g(t)}^{g(0)} m(g_1^{-1}(r))dr \qquad (4)$$

Example 2. Let $m(r) = r^2$, and $g(r) = r^2 - 2r + 2, \forall r \in [0, 2]$.

We have $g_1^{-1}(r) = 1 + \sqrt{r-1}, g_2^{-1}(r) = 1 - \sqrt{r-1}$, and for all $t \in [1, 2]$, $g(t) \le 0$. From (4), we obtain

$$C\int_{0,\,t} g d\mu_m = t^2 + \int_1^{t^2-2t+2} (t - 2\sqrt{r-1})^2 dr + \int_{t^2-2t+2}^{2} (1 - \sqrt{r-1})^2 dr$$

$$= \frac{4}{3}t^4 - \frac{28}{3}t^3 + 14t^2 - 8t + 2$$

3.2.2. Choquet integral of nonmonotonic functions and rearrangement

The principle behind the theory of monotone equimeasurable rearrangements of functions [6] is that, for any function *f* defined on the real line, there exists a nondecreasing (respectively, nonincreasing) function that has the same distribution function as the function *f* with respect to the Lebesgue measure. This function is called the nondecreasing (respectively, nonincreasing) rearrangement of the function *f*.

Based on the results obtained in [19] and [12], we shall further explore the calculation of the Choquet integral of nonmonotonic functions on the nonnegative real line.

Let $g \in \mathcal{G}$ be a continuous and nonnegative function on $[0, t]$, with $\tau_m \in [0, t]$ such that *g* is nondecreasing (respectively, nonincreasing) on $[0, \tau_m]$, and nonincreasing (respectively, nondecreasing) on $[\tau_m, t]$, and $g(t) \ge g(0)$, with $\overline{g} = g(\tau_m) = \max_{0\le\tau\le t} g(\tau)$ (respectively, $\underline{g} = g(\tau_m) = \min_{0\le\tau\le t} g(\tau)$).

Let $\lambda_g : [g(0), \overline{g}] \to [0, t]$ (respectively, $\lambda_g : [\underline{g}, g(t)] \to [0, t]$) be a function defined by:

$$\lambda_g(r) = \lambda(\{\tau \mid g(\tau) \ge r\})$$

The function λ_g is continuous, and nonincreasing on $[g(0), \overline{g}]$ (respectively, $[\underline{g}, g(t)]$).

We define the function $g^* : [0, t] \to [g(0), \overline{g}]$ (respectively, $g^* : [0, t] \to [\underline{g}, g(t)]$) by:

$$g^*(\tau) = \lambda_g^{-1}(t - \tau) \tag{5}$$

where g^* is called the nondecreasing rearrangement of g on $[0, t]$. The function g^* is nondecreasing and continuous on $[0, t]$.

Remark 2. For all $\tau \in [0, \tau_t]$ (respectively, $\tau \in [\tau_0, t]$), where $g(\tau_t) = g(t)$ (respectively, $g(\tau_0) = g(0)$), $g^*(\tau) = g(\tau)$.

If $g(t) \leq g(0)$, we define the function $g^* : [0, t] \rightarrow [g(t), \bar{g}]$ (respectively, $g^* : [0, t] \rightarrow [g(0), \underline{g}]$) by:

$$g^*(\tau) = \lambda_g^{-1}(\tau) \tag{6}$$

The function g^* is the nonincreasing rearrangement of g on $[0, t]$. It is continuous and nonincreasing on $[0, t]$.

Remark 3. For all $\tau \in [\tau_0, t]$ (respectively, $\tau \in [\tau_t, t]$), where $g(\tau_0) = g(0)$ (respectively, $g(\tau_t) = g(t)$), $g^*(\tau) = g(\tau)$.

Proposition 4. Let $g \in \mathcal{G}$, and let g^* be a rearrangement of g on $[0, t]$, then:

$$\lambda(\{\tau \mid g^*(\tau) \geq r\}) = \lambda(\{\tau \mid g(\tau) \geq r\})$$

Corollary 1. Let $g \in \mathcal{G}$, and let g^* be a rearrangement of g on $[0, t]$, then the Choquet integral of g with respect to measure μ_m on $[0, t]$ can be written as:

$$C\int_{0,\, t} g d\mu_m = C\int_{0,\, t} g^* d\mu_m$$

Hence, by Theorems 1 and 4 we find:

• If $g(t) \geq g(0)$, the Choquet integral of g with respect to the measure μ_m on $[0, t]$ becomes:

$$C\int_{0,\, t} g d\mu_m = \int_0^t m'(t-\tau) g^*(\tau) d\tau$$

where g^* is given by (5).

• If $g(t) \leq g(0)$, the Choquet integral of g with respect to the measure μ_m on $[0, t]$ becomes:

$$C\int_{0,\,t} g d\mu_m = \int_0^t m'(\tau) g^*(\tau) d\tau$$

where g^* is given by (6).

Example 3. Let $m(\tau) = \tau^2$, and $g(\tau) = 4 - (\tau - 2)^2$. The function g is nonnegative, continuous, and nonmonotonic on $[0, t]$, where $t \in [2, 4]$. The function g is nondecreasing on $[0, 2]$, and nonincreasing on $[2, t]$. We have: $g(t) \geq g(0)$, thus the nondecreasing rearrangement g^* of the function g on $[0, t]$ is defined by:

$$g^*(\tau) = \begin{cases} g(\tau), & \text{if } 0 \leq \tau \leq 4 - t \\ \dfrac{4 - (t - \tau)^2}{4}, & \text{if } 4 - t \leq \tau \leq t \end{cases}$$

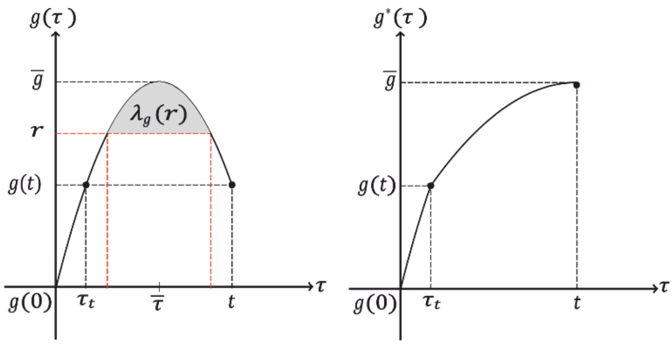

Fig. 2. The function g and its nondecreasing rearrangement g^* on $[0, t]$

The Choquet integral of g with respect to the measure μ_m on $[0, t]$ is given by:

$$(C)\int_{0,\,t} g d\mu_m = \int_0^t m'(t - \tau) g^*(\tau) d\tau = 2\int_0^t (\tau - t) g^*(\tau) d\tau$$

$$= 2\int_0^{4-t} (\tau - t) g(\tau) d\tau + 2\int_{4-t}^{t} (\tau - t)\,\frac{4 - (t - \tau)^2}{4}\, d\tau$$

$$= 2(t^4 - 8t^3 + 16t^2 - 16) - \frac{1}{6(t-4)^2(7t^2 - 16)}$$

4. Some applications

In this section, we review some applications of the Choquet integral on the nonnegative real line.

4.1. OWA operator on the real line

4.1.1. OWA operator

The ordered weighted averaging (OWA) operator was introduced by Yager [24]. In this section, we define $\Omega = \{1, ..., n\}$, and $\mathcal{A} = 2^{\Omega}$.

Definition 5. [24] Let $w = (w_1, \dots, w_n)$, such that $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$. The ordered weighted averaging operator (OWA) with respect to w is defined for the vector $a = (a_1, ..., a_n) \in \mathbb{R}_+^n$ by:

$$\mathrm{OWA}_w(a) = \sum_{i=1}^{n} w_i a_{\sigma(i)}$$

where σ is a permutation of $\{1, ..., n\}$, i.e. $(a_{\sigma(1)} \geqslant \dots \geqslant a_{\sigma(n)})$ is a permutation of the vector a.

The capacity μ is said to be symmetric if $\mu(A) = \mu(B)$ whenever $|A| = |B|$, $\forall A, B \in \mathcal{A}$.

Proposition 5 [9]. For each OWA_w operator, there exists a symmetric capacity μ given by $\mu(\{1\}) = w_1$, and $\mu(\{1, ..., i\}) = w_i$, $i = 1, ..., n$, such that:

$$\mathrm{OWA}_w(a) = C\int a d\mu$$

for any $a \in \mathbb{R}_+^n$.

4.1.2. Continuous OWA operator

In this section, we will use the Choquet integral to define the continuous OWA operator (COWA) [11]. Consider the Lebesgue measure λ, and let μ be a capacity on $([0, 1], \mathcal{B})$. μ is said to be symmetric if $\lambda(A) = \lambda(B)$ implies that $\mu(A) = \mu(B)$.

Definition 6 [11]. Let μ be a symmetric capacity, and $f:\mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. The continuous OWA operator is defined by:

$$\mathrm{COWA}_{\mu}(f) = (C)\int f d\mu$$

Let $m:[0,1]\to[0,1]$ be a distortion function. Thus a distorted Lebesgue measure μ_m is a symmetric capacity. It follows that we can consider the Choquet integral with respect to μ_m as a COWA operator.

Corollary 2. Let $f:[0,t]\to\mathbb{R}^+$ be a differentiable and nondecreasing function. It follows that:

$$\mathrm{COWA}_{\mu_m}(f) = \mathcal{L}^{-1}[sM(s)F(s)]$$

where $F(s)=\mathcal{L}[f(t)],$ and $M(s)=\mathcal{L}[m(t)].$

Corollary 3. Let $f:[0,t]\to\mathbb{R}^+$ be a differentiable and nonincreasing function. It follows that:

$$\mathrm{COWA}_{\mu_m}(f) = \int_0^t m'(t) f(\tau) d\tau$$

Example 4 [12]. Let $f:[0,1]\to\mathbb{R}^+$ be a differentiable and nondecreasing function. We define the sequence of functions f_k by: $f_1(t)=\int_0^t f(\tau)d\tau,\quad f_{k+1}(t)=\int_0^t f_k(\tau)d\tau,$ for $t\in[0,1]$, $k\in\{1, 2, ..., n\}.$ Let us show that

$$\mathrm{COWA}_{\mu_m}(f) = n! f_n(t)$$

where $m(t)=t^n.$

Since f is nondecreasing, the continuous OWA operator is given by:

$$C\int_{0,t} f d\mu_m = \mathcal{L}^{-1}[sM(s)F(s)]$$

where $F(s)=\mathcal{L}[f(t)]$, and $M(s)=\mathcal{L}[m(t)]=\dfrac{n!}{s^{n+1}}$. Thus

$$(C)\int_{0,\,t} fd\mu_m = n!\mathcal{L}^{-1}\left[\frac{F(s)}{s^n}\right]$$

Note that $\dfrac{F(s)}{s}=\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]=\mathcal{L}[f_1(t)]=F_1(s)$, hence $\dfrac{F(s)}{s^n}=\mathcal{L}[f_n(t)]$, therefore:

$$\mathrm{COWA}_{\mu_m}(f)=n!\mathcal{L}^{-1}[\mathcal{L}[f_n(t)]]=n!f_n(t)$$

To complete this example, let us take the function $f(x)=\mathrm{e}^x-1$ (from Example 1), with $m(t)=t^n$. We have $f_n(x)=e^x-\dfrac{t^n}{n!}-\dfrac{t^{n-1}}{(n-1)!}\ldots\dfrac{t^2}{2}-t-1,$ thus

$$\mathrm{COWA}_{\mu_m}(f)=n!(e^t-t-1)-t^n-nt^{n-1}\ldots\ (n(n-1)\ \ldots\ 3)t^2,\ x\in[0,1]$$

4.2. Risk measures and Choquet integrals

Risk management is a subject of concern in finance and insurance. One of the most significant problems in managing risk is determining a measure that can take into account various characteristics of the distribution of losses. For this goal, there are tools, called risk measures, to quantify and predict risk. They enable risk assessment and comparison of different risks. To manage such risks, several risk measurements have been proposed, each having its own advantages and disadvantages.

Artzner et al. [1] sought to characterize what would determine whether a risk measurement is "effective". For this goal, they introduced the notion of a coherent risk measurement.

4.2.1. Risk measures

Let $(\Omega,\mathcal{A})$ be a measurable space. Denote by $\mathcal{X}$ the space of measurable functions X such that $\|X\|_\infty=\sup_{w\in\Omega}|X(w)|$ is bounded:

$$\mathcal{X}:=\{X:\Omega\to\mathbb{R},\ \mathcal{A}\text{-measurable},\ \|X\|_\infty<\infty\}$$

Definition 7 [4]. A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a coherent risk measure if it satisfies the following conditions for all $X, Y \in \mathcal{X}$:

1) monotonicity: $X \le Y \Rightarrow \rho(X) \le \rho(Y)$,

2) translation invariance: $\rho(X + c) = \rho(X) + c, \ \forall c \in \mathbb{R}$,

3) positive homogeneity: $\rho(\lambda X) = \lambda \rho(X), \ \forall \lambda \ge 0$,

4) subadditivity: $\rho(X + Y) \le \rho(X) + \rho(Y)$.

Note that we adopt the definition which is used in the case where random variables are interpreted as losses.

The definition of a risk measure given by Artzner and al. [1] corresponds to the definition referred to above. However, in the definition of Artzner et al. [1], the sign + in property (2) is changed to the sign –, and property (1) becomes:

$$X \le Y \Rightarrow \rho(X) \ge \rho(Y)$$

Definition 8 [4]. A mapping $\rho : X \in \mathcal{X} \to \mathbb{R}$ is called a monetary risk measure if it satisfies the axioms of monotonicity and translation invariance.

Definition 9 [4]. A monetary risk measure ρ is called a convex risk measure if it satisfies the following property, $\forall X, Y \in \mathcal{X}, \ \forall \lambda \in [0, 1]$:

$$\rho(\lambda X + (1 - \lambda)Y) \le \lambda \rho(X) + (1 - \lambda)\rho(Y)$$

Let $(\Omega, \mathcal{A})$ be a measurable space, and let μ be a capacity. We define the function $\rho_\mu : \mathcal{X} \to \mathbb{R}$ by:

$$\rho_\mu(X) = C\int X d\mu, \qquad X \in \mathcal{X}$$

In the case where random variables are interpreted as gains, we define the function ρ_μ as follows:

$$\rho_\mu(X) = C\int - X d\mu, \qquad X \in \mathcal{X}$$

By the properties of the Choquet integral, we know that the function ρ_μ is a positive homogeneous measure of monetary risk, and if μ is a submodular capacity, then the function ρ_μ is a convex risk measure.

4.2.2. Distortion risk measures

Distortion risk measures were introduced in Yaari's paper [23], they aim to measure the risks based on a distortion of probabilities. Wang et al. [21] developed the concept of a distortion risk measure by calculating expected losses based on a nonlinear transformation of the cumulative distribution function of the risk factor. Distortion risk measures allow the production of better risk measures by distorting the original measure.

Definition 10 [21]. Let $g:[0,1]\to[0,1]$ be a distortion function, and $X\in\mathcal{X}$ be a random variable. The distortion risk measure for X associated with the distortion function g is defined by:

$$\rho_g(X)=\int_0^{+\infty} g(G_X(x))dx+\int_{-\infty}^{0}[g(G_X(x))-1]dx$$

where $G_X = P(X>x)=1-P(X\le x)=1-F_X(x)$, and $F_X(x)$ is the distribution function of X.

Several popular risk measures belong to the family of distortion risk measures. For example, the value-at-risk (VaR), the tail value-at-risk (TVaR) and the Wang distortion measure.

Remark 4. When the distortion function is concave, the distortion risk measure is also subadditive [20, 22].

If X is nonnegative, then the distortion risk measure associated with the distortion function g is defined by:

$$\rho_g(X)=\int_0^{+\infty} g(G_X(x))dx$$

Remark 5. $\rho_g(X)$ is the Choquet integral of X with respect to the distorted probability $\mu_g = g\circ P$.

Remark 6. If the distortion function g is differentiable, and the distribution function F is continuous (and strictly increasing), then the distortion risk measure can be written as follows:

$$\rho_g(X)=\int_0^{1} F_X^{-1}(1-x)g'(x)dx \tag{7}$$

Indeed, using the substitution $u = F_X(x)$, and integrating by parts:

$$\rho_g(X) = \int_0^{+\infty} g(G_X(x))dx + \int_{-\infty}^{0} [g(G_X(x)) - 1]dx$$
$$= \int_0^{+\infty} g(1 - F_X(x))dx + \int_{-\infty}^{0} [g(1 - F_X(x)) - 1]dx$$
$$= \int_{F_X(0)}^{1} g(1-u)(F_X^{-1})'(u)du + \int_0^{F_X(0)} [g(1-u) - 1](F_X^{-1})'(u)du$$
$$= \int_{F_X(0)}^{1} g'(1-u)F_X^{-1}(u)du + \int_0^{F_X(0)} g'(1-u)F_X^{-1}(u)du$$
$$= \int_0^1 F_X^{-1}(1-u)g'(u)du$$

Proposition 6. Let the distribution function of X, F_X, be continuous and strictly increasing. If X is nonnegative, then $\rho_g(X) = C\int_{0,t} F_X^{-1} d\mu_g$, where $\mu_g = g \circ P$.

Proof. Let X be a nonnegative random variable, then we have:

$$\rho_g(X) = \int_0^1 F_X^{-1}(1-u)g'(u)du = \int_0^1 g'(1-u)F_X^{-1}(u)du = C\int_{0,1} F_X^{-1} d\mu_g$$

by theorem 1.

Remark 7. The value at risk (VaR) is one of the most popular risk measures, due to its simplicity and intuitiveness. However, it is known that the distortion function associated to this risk measure is not differentiable. Therefore, our results do not apply in this case.

One way to obviate this problem may be to consider a differentiable approximation to the discontinuous function that yields the VaR. This approach, which may be very interesting to study, is left for future work.

Proposition 7. For every $X \in \mathcal{X}$, $\rho_g(X) = -\rho_{\bar{g}}(-X)$, where $\bar{g}(x) = 1 - g(1-x)$ is the dual function of g.

Proof. For every $X \in \mathcal{X}$, we have:

$$\begin{aligned}\rho_g(X) &= C\int -(-X)d\mu_g = -C\int -Xd\overline{\mu_g} \\ &= -\ C\int -Xd\mu_{\bar{g}}, \qquad (\overline{\mu_g} = \mu_{\bar{g}}) = -\rho_{\bar{g}}(-X)\end{aligned}$$

5. Conclusion

The work presented in this paper revolves around two axes. The first axis focused on methods of calculating the Choquet integral with respect to a Lebesgue distortion measure. The second purpose was to apply these calculations.

After introducing basic concepts of the Choquet integral, we presented methods of calculating the Choquet integral of monotone and nonmonotone functions on the positive real line.

In the case of nonmonotone functions, we used nondecreasing and nonincreasing rearrangements of a nonmonotone function to turn it into a monotone function, in order to apply results regarding the calculation of the Choquet integral of monotone functions.

We applied the Choquet integral to calculating the continuous aggregation operator OWA, and we represented risk measures by Choquet integrals to facilitate the verification of convexity risk measures.

Many aspects of this work remain obviously to be deepened. An interesting subject of study would be to try to obtain results about the calculus of the Choquet integral with respect to a general capacity, not limited to distorted Lebesgue measures.

It is also important to find a general representation for the calculation of the Choquet integral of a monotone or nonmonotone function on the real line. Applying these calculations would allow us to address many areas.

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