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## A NEWLY DEVELOPED METHOD FOR COMPUTING RELIABILITY MEASURES IN A WATER SUPPLY NETWORK

A reliability model of a water supply network has been examined. Its main features are: a topology that can be decomposed by the so-called state factorization into a (relatively) small number of derivative networks, each having a series-parallel structure (1), binary-state components (either operative or failed) with given flow capacities (2), a multi-state character of the whole network and its sub-networks – a network state is defined as the maximal flow between a source (sources) and a sink (sinks) (3), all capacities (component, network, and sub-network) have integer values (4). As the network operates, its state changes due to component failures, repairs, and replacements. A newly developed method of computing the inter-state transition intensities has been presented. It is based on the so-called state factorization and series-parallel aggregation. The analysis of these intensities shows that the failure-repair process of the considered system is an asymptotically homogenous Markov process. It is also demonstrated how certain reliability parameters useful for the network maintenance planning can be determined on the basis of the asymptotic intensities. For better understanding of the presented method, an illustrative example is given.

**Keywords:** *water supply network, reliability, series-parallel aggregation, factorization*

### 1. Introduction

Reliability of water supply networks (WSN) is an important topic studied by many researchers in various aspects. Two main approaches can be distinguished in this field: analytical (derivation of appropriate formulas based on analysis of the network structure) and statistical (computation of reliability indices by means of statistical methods applied to operational or simulation data). The approach pursued in this paper belongs to the former class, and can be referred to as topology and flow oriented. However, the literature in this area is rather scarce – a short survey is given below. A comprehensive

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synopsis of analytical methods used in reliability analysis of WSN can be found in [3] and [6], where the idea of topological reductions, also applied in the current paper, is explained in detail. The authors of [7] give a general view on the issue of WSN reliability, with a focus on reliability measures, also related to topology and flow distribution. A method similar to that presented in this paper is described in [8], where simple facts from graph theory are used to compute the mechanical reliability and hydraulic availability of a water distribution system. However, the method from [8] becomes highly complex in the computational sense as the network size increases. As far as statistical approach is concerned, the reader is referred to [5] which provides a comprehensive outlook on statistical methods in WSN reliability analysis, [2], where Poisson and Cox regression is applied for the survival analysis of pipes in a WSN, and [1], where an interesting case study is presented along with the application of non and semi-parametric methods to the failure dataset.

A water supply network can be modeled by a graph whose vertices represent the nodes, and edges represent the pipes. Three types of nodes are distinguished, i.e. intake points, supply points, and branching nodes. In Figure 1, an example network model is presented. Intake points are marked with arrows directed towards a node, supply points – with arrows directed away from a node, branching points are not marked.

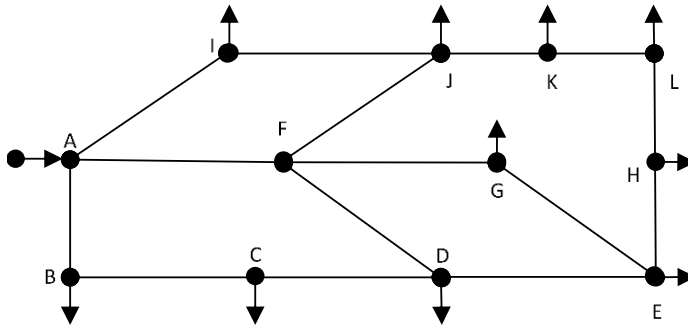


Fig. 1. The structure of a simple exemplary network

A component, i.e. a node or a pipe, is characterized by its capacity defined as the maximal amount of water that can flow through the component in a unit of time. Components can be either in a working or a failed state. Clearly, a failed component has zero capacity. A state of the network is a non-binary function of the states of its components, called the network state function. It can be defined in several ways, as further presented. Both the component capacities and the network state are positive integers, which in practice does not limit the model's applicability.

The network state changes as the components fail or undergo repairs and/or replacements. The failure-repair (F-R) process of each component is a Markov one, because

a component's failure and repair intensities have constant values. It is plausible to assume that these processes are independent, hence the network state as a function of time is also a Markov process. However, it is only asymptotically homogenous, while the F-R processes of individual components are fully homogenous.

The main result presented in this paper is a method of computing the intensities of transitions between the network states. These intensities are basic reliability parameters that can be used to find other frequently used reliability indices. The considered method is a three-step one. First, the network structure is decomposed, by the so-called state factorization, into a small number of derivative networks, each having a series-parallel structure. The probability distribution of the state function of the main network's is a linear combination of the probability distributions of the state functions of the derivative networks. Second, the respective probability distribution for each derivative network is computed using a method of series-parallel aggregation developed by the author of this work. Third, the inter-state transition intensities for the main network are computed from a formula based on the probability distribution of the main network's state function.

## 2. Notation and definitions

Throughout the paper, the following notation will be used:

- $E$  – the set of network components,  $E = \{e_1, \dots, e_n\}$ ,
  - $X_i(t)$  – state of  $e_i$  at time  $t$  (operative or failed),
  - $p_i(t), q_i(t)$  – probability that  $e_i$  is operative/failed at time  $t$ ,
  - $\lambda_i, \mu_i$  – failure/repair intensity of  $e_i$  (constant values),
- As  $X_i(t)$  is a two-state homogenous Markov process, we have:

$$p_i(t) = \frac{\mu_i}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} \exp[-(\lambda_i + \mu_i)t] \quad (1)$$

$$q_i(t) = \frac{\lambda_i}{\lambda_i + \mu_i} - \frac{\lambda_i}{\lambda_i + \mu_i} \exp[-(\lambda_i + \mu_i)t]$$

hence the asymptotic ( $t \rightarrow \infty$ ) values of  $p_i(t)$  and  $q_i(t)$  are given by:

$$p_i = \frac{\mu_i}{\lambda_i + \mu_i}, \quad q_i = \frac{\lambda_i}{\lambda_i + \mu_i} \quad (2)$$

The notation section continued:

$c_i$  – capacity (throughput) of operative  $e_i$ ;  $c_i$  has an integer value,

$e_{i1}, \dots, e_{im}$  – supply points,  
 $d_{i1}, \dots, d_{im}$  – demands at  $e_{i1}, \dots, e_{im}$ , respectively,  
 $S_{i1}, \dots, S_{im}$  – supply capabilities at  $e_{i1}, \dots, e_{im}$ , respectively (random variables),  
 $\Phi$  – a function expressing the network state w.r.t. the states of its components and the given supply/demand relations,

$A(\alpha) = \Pr(\Phi \geq \alpha)$ ;  $\alpha$  availability; a stochastic measure of the network reliability,  
 $\Phi$  can be defined in several ways, i.e.:

1. Percentage of adequately supplied points:

$$\Phi = \frac{1}{m} \sum_{i: S_i \geq d_i} 1 \quad (3)$$

2. Weighted percentage of adequately supplied points

$$\Phi = \frac{1}{d_1 + \dots + d_m} \sum_{i: S_i \geq d_i} d_i \quad (4)$$

3. Percentage of total demand fulfilled

$$\Phi = 1 - \frac{\sum_{i: d_i > S_i} (d_i - S_i)}{d_1 + \dots + d_m} \quad (5)$$

4. Supply capability at a selected supply point

$$\Phi = S_i \quad (6)$$

Finally, we give the definitions of fault, recovery, their intensities, and present the formulas from which these intensities can be computed.

Fault – an event that causes  $\Phi$  to fall from a level  $\geq \alpha$  to a level  $< \alpha$

Recovery – an event that causes  $\Phi$  to rise from a level  $< \alpha$  to a level  $\geq \alpha$

The adopted assumptions imply that the fault-recovery (F-R) process is a Markov one, thus the fault and recovery intensities are respectively defined as follows:

$$\Lambda^-(\alpha, t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \Pr [\Phi(t + \Delta t) < \alpha \mid \Phi(t) \geq \alpha] \quad (7)$$

$$\Lambda^+(\alpha, t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \Pr [\Phi(t + \Delta t) \geq \alpha \mid \Phi(t) < \alpha] \quad (8)$$

In the long run, the above intensities converge to constant values denoted as  $\Lambda^-(\alpha)$  and  $\Lambda^+(\alpha)$ , hence the F-R process is asymptotically homogenous. It can be shown [4] that

$$\Lambda^-(\alpha) = \frac{1}{\Pr(\Phi \geq \alpha)} \sum_{i \in E} \lambda_i p_i I_i(\alpha) \quad (9)$$

$$\Lambda^+(\alpha) = \frac{1}{\Pr(\Phi < \alpha)} \sum_{i \in E} \mu_i q_i I_i(\alpha) \quad (10)$$

where  $I_i(\alpha)$  is the (generalized) Birnbaum importance of  $e_i$ , here referred to as  $\alpha$ -importance, that can be found from the following formula:

$$I_i(\alpha) = \Pr(\Phi \geq \alpha) |_{p_i=1} - \Pr(\Phi \geq \alpha) |_{p_i=0} \quad (11)$$

It will be further demonstrated how other commonly used reliability parameters can be found using the intensities  $\Lambda^+(\alpha)$  and  $\Lambda^-(\alpha)$ . As follows from Eqs. (9)–(11), they can be computed using the failure/repair intensities of individual components, and  $A(\alpha)$ . Thus, the ability to compute  $A(\alpha)$ , given  $p_1, \dots, p_n$ , has a key significance for many other reliability calculations. An efficient method to find  $A(\alpha)$  is presented in the next two chapters.

### 3. Transforming the main network into derivative networks by means of component-state factorization

Component-state factorization consists in two-variant modification of the network structure with respect to a selected (pivotal) component which is assumed to be either operative or failed, depending on the variant. This operation is first performed on the main network, and then repeated on derivative networks until a series-parallel structure is obtained. The diagram in Fig. 2 shows the binary tree of networks obtained as a result of this procedure applied to the example network in Fig. 1. The state of the component  $e$  is denoted as  $x_e$ ;  $x_e = 1$  if  $e$  is operative, otherwise  $x_e = 0$ . For the sake of this example, it is assumed that only pipes are subject to failure, and  $\Phi = S_E$ , i.e.  $\Phi$  is the supply capability of the node  $E$ . A factorization step w.r.t. a pipe is performed as follows: If the pipe is failed, it is removed from the network diagram, if the pipe is operative, the two nodes which it connects are merged into one node. The main network  $N$  and the derivative networks  $N_1, \dots, N_6$  are shown in Figs. 3 and 4. As can be seen,  $N_3, N_4, N_5$ , and  $N_6$  have series-parallel structures, thus further factorization is not performed on them.

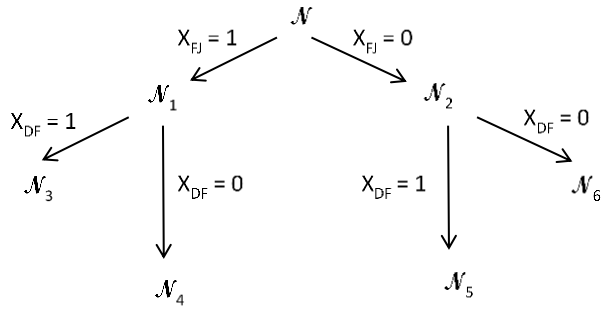


Fig. 2. The factorization tree for  $N$

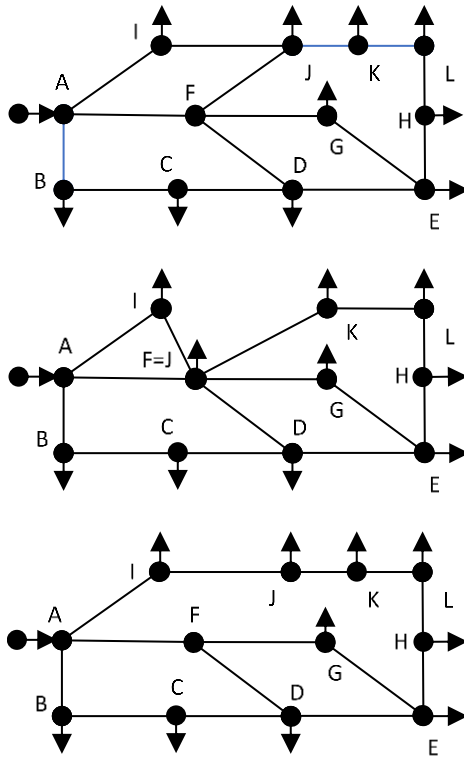


Fig. 3. The main network  $N$  and its derivative networks  $N_1, N_2$  ordered from top to bottom

From the law of total probability we obtain:

$$A(\alpha) = p_{FJ} p_{DF} A_3(\alpha) + p_{FJ} q_{DF} A_4(\alpha) + q_{FJ} p_{DF} A_5(\alpha) + q_{FJ} q_{DF} A_6(\alpha) \quad (12)$$

where  $A(\alpha)$  is the  $\alpha$ -availability of  $N$ ,  $A_3(\alpha)$ , ...,  $A_6(\alpha)$  are the  $\alpha$ -availabilities of the derivative networks  $N_3$ , ...,  $N_6$  respectively, i.e. the probabilities that the supply capabilities of the node  $E$  in these networks are greater or equal to  $\alpha$ .

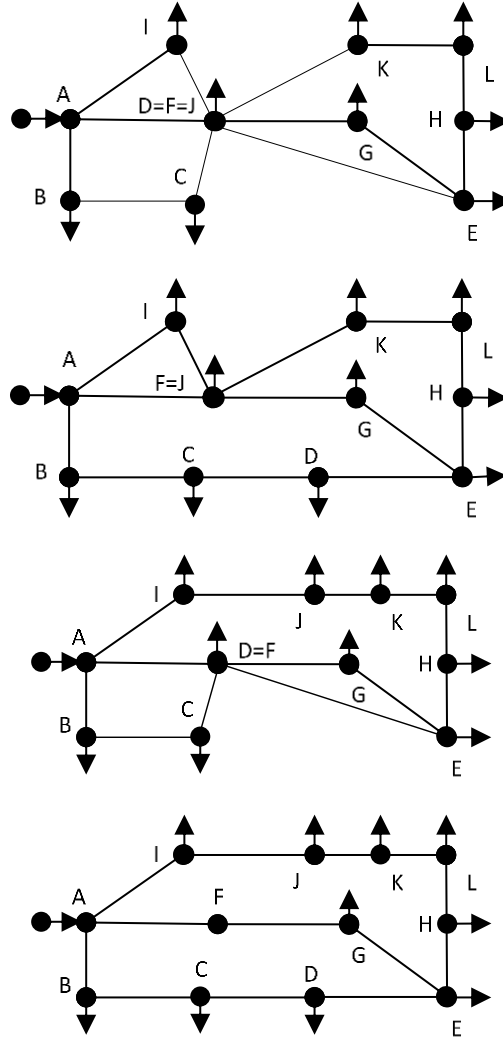


Fig. 4. The derivative networks  $N_3$ , ...,  $N_6$  ordered from top to bottom

Clearly, the assumption that only pipes can fail is unrealistic and cannot be made in practice. Thus, it can be necessary to perform factorization w.r.t. both nodes and pipes. The example diagram of such a procedure is presented in Fig. 5. The main network  $N$  is

composed of the nodes A, B, C, D, and the pipes connecting these nodes. It is assumed that  $\Phi = S_D$ , i.e.  $\Phi$  is the supply capability of the node D. A factorization step w.r.t. to a node is performed as follows: if the node is failed, it is removed from the network diagram along with all the connected pipes, if the node is operative, the diagram remains unchanged. To calculate  $A(\alpha)$  – the  $\alpha$ -availability of  $N$ , we have to use the following formula:

$$A(\alpha) = q_B A_1(\alpha) + p_B q_C A_3(\alpha) + p_B p_C q_{BC} A_5(\alpha) + p_B p_C p_{BC} A_6(\alpha) \tag{13}$$

where  $A_1(\alpha)$ ,  $A_3(\alpha)$ ,  $A_5(\alpha)$ , and  $A_6(\alpha)$  are the  $\alpha$ -availabilities of the networks located in the leaves of the factorization tree.

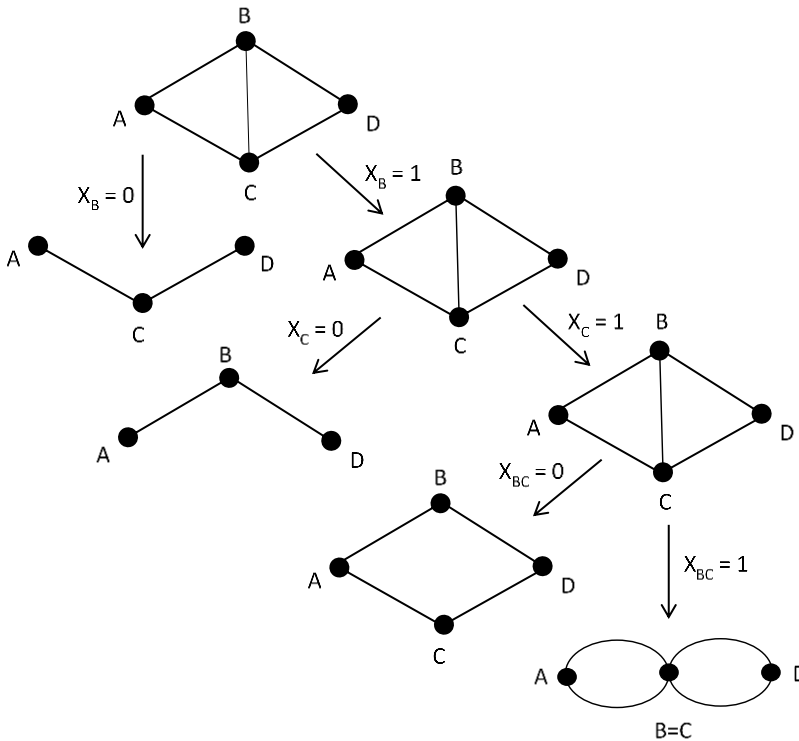


Fig. 5. The component-state factorization of a simple network w.r.t. nodes and pipes

Let us note that a factorization step w.r.t. a pipe is only possible if both nodes connected by this pipe are operative. Let us also note that a certain problem occurs in such a case, namely, how to calculate the capacity of the node merged of the nodes connected by this pipe? Below, a set of straightforward rules for this calculation is given. A merged node is illustrated in Fig. 6.



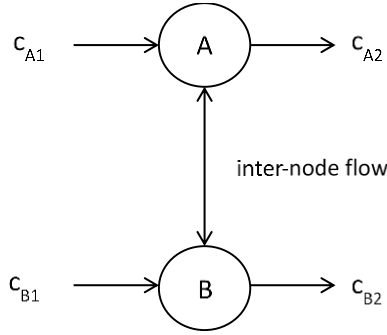


Fig. 6. The diagram of a merged node

We have:  $c_{A1}$ ,  $c_{A2}$ ,  $c_{B1}$ ,  $c_{B2}$  – flow capacities of respective inputs/outputs,  $c_{AB}$  – capacity of the internode flow,  $c_{A=B}$  – flow capacity of A merged with B

$$\left. \begin{array}{l} c_{A1} \leq c_{A2} \\ c_{B1} \leq c_{B2} \end{array} \right\} \rightarrow c_{A=B} = c_{A1} + c_{B1} \quad (14)$$

$$\left. \begin{array}{l} c_{A1} \leq c_{A2} \\ c_{B1} \geq c_{B2} \end{array} \right\} \rightarrow c_{A=B} = c_{A1} + c_{B2} + \min(c_{A2} - c_{A1}, c_{B1} - c_{B2}, c_{AB}) \quad (15)$$

$$\left. \begin{array}{l} c_{A1} \geq c_{A2} \\ c_{B1} \geq c_{B2} \end{array} \right\} \rightarrow c_{A=B} = c_{A2} + c_{B2} \quad (16)$$

$$\left. \begin{array}{l} c_{A1} \geq c_{A2} \\ c_{B1} \leq c_{B2} \end{array} \right\} \rightarrow c_{A=B} = c_{A2} + c_{B1} + \min(c_{A1} - c_{A2}, c_{B2} - c_{B1}, c_{AB}) \quad (17)$$

It should be remarked that in the above figure, the arrows marked with  $c_{A1}$ ,  $c_{A2}$ ,  $c_{B1}$ , and  $c_{B2}$  represent total inflows and outflows, i.e. possibly through more than one pipe.

#### 4. Series-parallel aggregation

This technique consists in stepwise aggregation of components and/or sub-modules arranged in series or in parallel into a module subsequently regarded as one component. In the last step the whole system is aggregated into a single component. In Figure 7, the reliability block diagram (RBD) of the derivative network  $N_4$  is presented. Below the figure, the aggregation steps are specified.

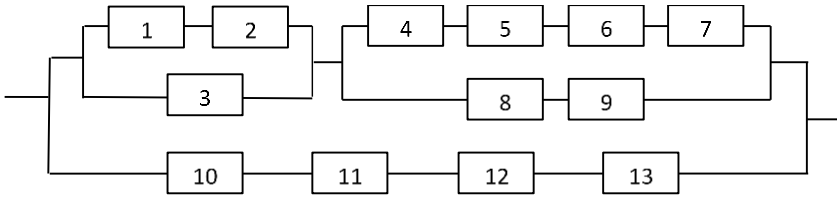


Fig. 7. The RBD of the derivative network  $N_4$

Aggregation steps for  $N_4$ :

- Step 1.  $e_1, e_2 \rightarrow M_1 = \text{ser}(e_1, e_2)$
- Step 2.  $M_1, e_3 \rightarrow M_2 = \text{par}(M_1, e_3)$
- Step 3.  $e_4, e_5, e_6, e_7 \rightarrow M_3 = \text{ser}(e_4, e_5, e_6, e_7)$
- Step 4.  $e_8, e_9 \rightarrow M_4 = \text{ser}(e_8, e_9)$
- Step 5.  $M_3, M_4 \rightarrow M_5 = \text{par}(M_3, M_4)$
- Step 6.  $M_2, M_5 \rightarrow M_6 = \text{ser}(M_2, M_5)$
- Step 7.  $e_{10}, e_{11}, e_{12}, e_{13} \rightarrow M_7 = \text{ser}(e_{10}, e_{11}, e_{12}, e_{13})$
- Step 8.  $M_6, M_7 \rightarrow M_8 = \text{par}(M_6, M_7)$

Along with structural aggregation, maximum effective capacities and  $\alpha$ -availabilities of modules obtained by aggregation of components or sub-modules have to be calculated. The respective formulas are given in Lemmas 1–5. The first two lemmas give the formulas for maximum effective capacities of modules composed of single components connected in series or in parallel. Let  $M$  denote such a module, and let  $e_1, \dots, e_k$  be components of  $M$ ,  $c_1, \dots, c_k$  – capacities of operative  $e_1, \dots, e_k$ ,  $d_1, \dots, d_k$ : demands at  $e_1, \dots, e_k$  (if  $e_j$  is not a supply point, then  $d_j = 0$ ),  $C_M$  – maximum effective capacity of  $M$ .

**Lemma 1.** For  $M = \text{ser}(e_1, \dots, e_k)$  it holds that

$$C_M = \min \left[ \max \left( c_1 - \sum_{i=1}^k d_i, 0 \right), \max \left( c_2 - \sum_{i=2}^k d_i, 0 \right), \dots, \max(c_k - d_k, 0) \right] \quad (18)$$

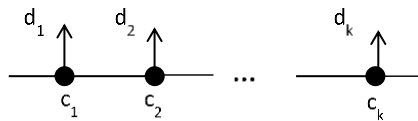


Fig. 8. A module composed of  $k$  components connected in series

Sketch of the proof: analyzing Fig. 8, we conclude that  $c_j > \sum_{i=j, \dots, k} d_i$ ,  $j = 1, \dots, k$  is the necessary condition for the effective capacity of  $\text{ser}(e_1, \dots, e_k)$  to be positive. If this condition is fulfilled, we have:

$$C_{\text{ser}(e_1, \dots, e_{j+1})} = \min[C_{\text{ser}(e_1, \dots, e_j)}, c_{j+1}] - d_{j+1} \quad (19)$$

wherefrom (18) is obtained by induction.

**Lemma 2.** For  $M = \text{par}(e_1, \dots, e_k)$  it holds that

$$C_M = \sum_{j=1}^k \max[(c_j - d_j), 0] \quad (20)$$

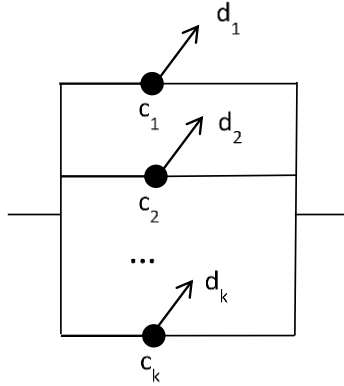


Fig. 9. A module composed of  $k$  components connected in parallel

Sketch of the proof: analyzing Fig. 9, we conclude that  $C_M$  given by (20) is the sum of the effective capacities of  $e_1, \dots, e_k$ , where such capacity for  $e_j$  is equal to  $\max[(c_j - d_j), 0]$ .

The two following lemmas show how to compute  $\alpha$ -availabilities of the modules presented in Figs. 8 and 9. Let  $\Phi_M$  and  $A_M(\alpha) = \Pr(\Phi_M \geq \alpha)$  denote respectively the (random) effective capacity and the effective  $\alpha$ -availability of the module  $M$ .

**Lemma 3.** For  $M = \text{ser}(e_1, \dots, e_k)$  it holds that

$$A_M(\alpha) = \begin{cases} 1, & \alpha \leq 0 \\ p_1 \dots p_k, & 1 \leq \alpha \leq C_M \\ 0, & \alpha \geq C_M + 1 \end{cases} \quad (21)$$

Let us note that the effective throughput of  $\text{ser}(e_1, \dots, e_k)$  can be equal to either 0 or  $C_M$ , where  $C_M$  is given by (18).

**Lemma 4.** For  $M = \text{par}(e_1, \dots, e_k)$ , the following recursive formula holds:

$$A_{\{1, \dots, j\}}(\alpha) = p_j A_{\{1, \dots, j-1\}} \left[ \alpha - \max(c_j - d_j, 0) \right] + q_j A_{\{1, \dots, j-1\}}(\alpha), \quad j = 2, \dots, k$$

$$A_M(\alpha) = A_{\{1, \dots, k\}}(\alpha) \quad (22)$$

where

$$A_{\{1\}}(\alpha) = \begin{cases} 1, & \alpha \leq 0 \\ p_1, & 0 < \alpha \leq \max(c_1 - d_1, 0) \\ 0, & \alpha > \max(c_1 - d_1, 0) \end{cases} \quad (23)$$

$A_M(\alpha)$  should be computed for all integer  $\alpha \in [1, C_M]$ , where  $C_M$  is given by (20). The proofs of lemmas 3 and 4 are based on formulas for the distribution functions of the minimum and sum of independent, integer-valued random variables.

To end this chapter, it will be demonstrated how  $\alpha$ -availabilities of modules composed of sub-modules rather than single components are calculated. Let  $M = \text{ser}(M_1, \dots, M_k)$  or  $M = \text{par}(M_1, \dots, M_k)$ , where  $M_1, \dots, M_k$  are sub-modules of  $M$ .

**Lemma 5.** If  $M = \text{ser}(M_1, \dots, M_k)$  then we have:

$$A_M(\alpha) = \prod_{j=1}^k \Pr \left[ \Phi_{M_j} \geq \alpha + \sum_{i=j+1}^k d_{M_i} \right] = \prod_{j=1}^k A_{M_j} \left( \alpha + \sum_{i=j+1}^k d_{M_i} \right) \quad (24)$$

where  $d_{M_i}$  is the total demand of all components of  $M_i$ .  $A_M(\alpha)$  should be computed for all integer  $\alpha$  such that

$$1 \leq \alpha \leq \min_{i=1, \dots, k} \left( C_{M_i} - \sum_{i=j+1}^k d_{M_i} \right)$$

which is equivalent to the following condition, better suited for computational purposes:

$$A_{M_i} \left( \alpha + \sum_{i=j+1}^k d_{M_i} \right) > 0, \quad i = 1, \dots, k$$

**Lemma 6.** If  $M = \text{par}(M_1, \dots, M_k)$ , then we have the following recursive formula:

$$\begin{aligned}
 A_{M_1 \cup \dots \cup M_j}(\alpha) &= \sum_{\beta=0}^{C_{M_j}} \Pr(\Phi_{M_j} = \beta) \Pr\left(\sum_{i=1}^{j-1} \Phi_{M_i} \geq \alpha - \beta\right) \\
 &= \sum_{\beta=0}^{C_{M_j}} [A_{M_j}(\beta) - A_{M_j}(\beta+1)] A_{M_1 \cup \dots \cup M_{j-1}}(\alpha - \beta), \quad j = 2, \dots, k \quad (25)
 \end{aligned}$$

$A_M(\alpha)$  should be computed for all integer  $\alpha \in [1, C_{M_1} + \dots + C_{M_k}]$ .

The above lemma is proved in a similar way as formulas (21) and (22). To illustrate the results of this chapter, let us find  $A(\alpha)$  for a small series-parallel network whose RBD is presented in Fig. 10.

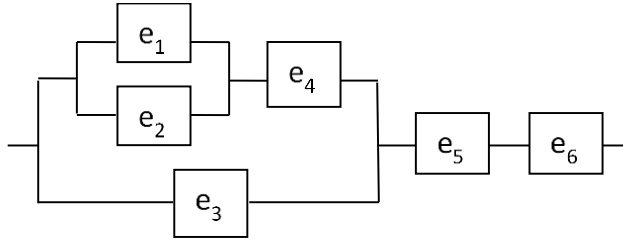


Fig. 10. The RBD of a small series-parallel network

We assume that  $c_1 = 3, c_2 = 4, c_3 = 5, c_4 = 6, c_5 = 8, c_6 = 7$ , and  $d_1 = 1, d_2 = 1, d_3 = 2, d_4 = 2, d_5 = 1, d_6 = 1$ . The arrow denotes are aggregated to.  $A(\alpha)$  is computed in the following aggregation steps:

**Step 1.**  $e_1, e_2 \rightarrow M_1 = \text{par}(e_1, e_2)$ . According to (22), we have:

$$A_{\{1,2\}}(\alpha) = p_2 A_{\{1\}}(\alpha - c_2 + d_2) + q_2 A_{\{1\}}(\alpha) = p_2 A_{\{1\}}(\alpha - 3) + q_2 A_{\{1\}}(\alpha)$$

Hence

$$A_{\{1,2\}}(\alpha) = 1, \quad \alpha \leq 0$$

$$A_{\{1,2\}}(\alpha) = A_{\{1,2\}}(2) = p_2 + q_2 p_1$$

$$A_{\{1,2\}}(3) = p_2$$

$$A_{\{1,2\}}(4) = A_{\{1,2\}}(5) = p_2 p_1$$

$$A_{\{1,2\}}(\alpha) = 0, \quad \alpha > 5$$

**Step 2.**  $M1, e4 \rightarrow M2 = \text{ser}(M1, e4)$ . According to (24) we have:

$$A_{\{1,2,4\}}(\alpha) = A_{\{1,2\}}(\alpha + d_4)A_{\{4\}}(\alpha) = A_{\{1,2\}}(\alpha + 2)A_{\{4\}}(\alpha), \quad \alpha \geq 1$$

Hence

$$\begin{aligned} A_{\{1,2,4\}}(\alpha) &= 1, \quad \alpha \leq 0 \\ A_{\{1,2,4\}}(1) &= p_2 p_4 \\ A_{\{1,2,4\}}(2) &= A_{\{1,2,4\}}(3) = p_2 p_1 p_4 \\ A_{\{1,2,4\}}(\alpha) &= 0, \quad \alpha > 3 \end{aligned}$$

**Step 3.**  $M2, e3 \rightarrow M3 = \text{par}(M2, e3)$ . According to (25) we have:

$$A_{\{1,2,3,4\}}(\alpha) = q_3 A_{\{1,2,4\}}(\alpha) + p_3 A_{\{1,2,4\}}(\alpha - 3)$$

Hence

$$\begin{aligned} A_{\{1,2,3,4\}}(\alpha) &= 1, \quad \alpha \leq 0 \\ A_{\{1,2,3,4\}}(1) &= q_3 p_2 p_4 + p_3 \\ A_{\{1,2,3,4\}}(2) &= A_{\{1,2,3,4\}}(3) = q_3 p_2 p_1 p_4 + p_3 \\ A_{\{1,2,3,4\}}(4) &= p_3 p_2 p_4 \\ A_{\{1,2,3,4\}}(5) &= A_{\{1,2,3,4\}}(6) = p_3 p_2 p_1 p_4 \\ A_{\{1,2,3,4\}}(\alpha) &= 0, \quad \alpha > 6 \end{aligned}$$

**Step 4.**  $M3, e5, e6 \rightarrow M4 = \text{ser}(M3, e5, e6)$ . According to (24), we have:

$$A_{\{1,2,3,4,5,6\}}(\alpha) = A_{\{1,2,3,4\}}(\alpha + d_5 + d_6)A_{\{5\}}(\alpha + d_6)A_{\{6\}}(\alpha), \quad \alpha \geq 1$$

Hence

$$\begin{aligned} A_{\{1,2,3,4,5,6\}}(\alpha) &= 1, \quad \alpha \leq 0 \\ A_{\{1,2,3,4,5,6\}}(1) &= (q_3 p_2 p_1 p_4 + p_3) p_5 p_6 \\ A_{\{1,2,3,4,5,6\}}(2) &= p_3 p_2 p_4 p_5 p_6 \\ A_{\{1,2,3,4,5,6\}}(3) &= A_{\{1,2,3,4,5,6\}}(4) = p_3 p_2 p_1 p_4 p_5 p_6 \\ A_{\{1,2,3,4,5,6\}}(\alpha) &= 0, \quad \alpha > 4 \end{aligned}$$

## 5. Numerical computation of $\alpha$ -availability and other reliability parameters

As indicated at the end of Chapter 2, computing the  $\alpha$ -availability of a considered network is necessary for finding other important reliability parameters.  $A(\alpha)$  should be computed for each integer  $\alpha$  such that  $0 < A(\alpha) < 1$ . For this purpose, the component-state factorization, presented in Chapter 3, and the series-parallel aggregation, based on the formulas (21)–(25), are applied. Let us note that for a larger network, only the numerical values of  $A(\alpha)$  should be found. Deriving a closed formula, as in the last example, is practically impossible in such a case.

Once  $A(\alpha)$  is calculated for all appropriate  $\alpha$ , then the fault and recovery intensities can be found from (9) and (10). Let us recall that fault and recovery are defined in Chapter 2. Prior to applying these formulas, the  $\alpha$ -importances  $I_i(\alpha)$  have to be computed from (11), for  $i = 1, \dots, n$ . All these computations require  $p_i$ ,  $i = 1, \dots, n$  to be the known data. If only the failure and repair intensities of the network's components are given, the respective probabilities can be obtained from (2).

For a more complete reliability characteristic of a water supply network, we also need some time-related parameters. Before defining them, let us introduce some more notation:

$\Phi(t)$  – the supply capability of a selected supply point at time  $t$ ,

$T_{k,\alpha}$  – length of the  $k$ th period during which  $\Phi(t) \geq \alpha$ ,

$U_{k,\alpha}$  – length of the  $k$ th period during which  $\Phi(t) < \alpha$ .

As the F-R process is an asymptotically homogenous Markov one,  $T_{k,\alpha}$  and  $U_{k,\alpha}$  are asymptotically ( $k \rightarrow \infty$ ) exponential random variables with the parameters  $\Lambda^-(\alpha)$  and  $\Lambda^+(\alpha)$ , respectively. The first two of the aforementioned parameters are  $\tau_\alpha$  and  $\rho_\alpha$  – the asymptotic expected lengths of  $T_{k,\alpha}$  and  $U_{k,\alpha}$  respectively. We have:

$$\tau_\alpha = \lim_{k \rightarrow \infty} E(T_{k,\alpha}) = \frac{1}{\Lambda^-(\alpha)} \quad (26)$$

$$\rho_\alpha = \lim_{k \rightarrow \infty} E(U_{k,\alpha}) = \frac{1}{\Lambda^+(\alpha)} \quad (27)$$

because the expected value of an exponential random variable is equal to the inverse of its parameter.

The last two parameters are defined as follows:

$n_\alpha^-(t)/n_\alpha^+(t)$  – the expected number of faults/recoveries in a time period of length  $t$ , provided that the system has long been in operation. It can be shown that

$$n_{\alpha}^{-}(t) = n_{\alpha}^{+}(t) = \frac{t}{\frac{1}{\Lambda^{+}(\alpha)} + \frac{1}{\Lambda^{-}(\alpha)}} = \frac{t}{\rho_{\alpha} + \tau_{\alpha}} \quad (28)$$

Clearly, we only have to know  $\Lambda^{-}(\alpha)$  and  $\Lambda^{+}(\alpha)$  to apply the formulas (26)–(28). Let us stress that time units should be used cautiously! For example, if the intensities are given in  $h^{-1}$ , then  $n_{\alpha}^{-}(8760)$  is the expected number of faults in a year.

## 6. Conclusion

A new method of computing selected reliability parameters has been presented, characterizing the fault-and-recovery process of a water supply network. The approach to reliability analysis of water supply systems is similar to that found in [2], [6], and [8]. In the examples given in section 3 it is assumed that the network has only one intake point, and  $\Phi$  – the network-state function – is defined as the supply capability of a selected supply point. It is possible to apply the presented methodology also for networks with multiple intake points. For this purpose a multiple-sources network has to be transformed into a single-source one by adding one virtual collective source node and connecting it to the actual source nodes with virtual pipes of appropriate capacities. Modifying the presented results so as to cover the cases of the other network-state functions defined in section 2 will be the subject of the further research.

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