The Fifth-Order Fieldaberration Coefficients for the Spherical Surfaces

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In a paper published in Applied Optics [3] the wave aberration for sagittal focus for arbitrary surface of rotational symmetry has been carried out and then rearranged into three terms which, in the Seidel region, go over into astigmatism (the first) and into the Petzval curvature (the second) while the third disappears. After simplification for the spherical surfaces they can be used as basic equations. In the present paper the formulas for the parameters of a given principal ray and the paraxial sagittal quantities H_s, h_s, with accuracy to the fifth order were derived. Substituting them to the basic equations and recalculating simple relations for the fifth order fieldaberration coefficients were obtained. The limits of these relations when $h \to 0$ have been discussed.

Let us take an astigmatic beam from an object point $\bar{Q}_1(0, \eta_1)$ and denote the intersection point between the first surface and the principal ray by $\bar{P}_1(\bar{y}_1, \bar{z}_1)$. It should be emphasized that the principal ray is here defined as a ray passing through the center of the aperture stop from which the meridional calculations are carried out in both (image and object space) directions.

By analogy to the paraxial aperture angle u usually defined as an angle between a marginal ray and the optical axis we introduce the paraxial sagittal aperture angle u_s (Fig. 1) defined as that between marginal ray in the astigmatic beam sagittal section and the principal ray. Using he same analogy, h_s denotes the paraxial sagittal height of incidence understood as the distance

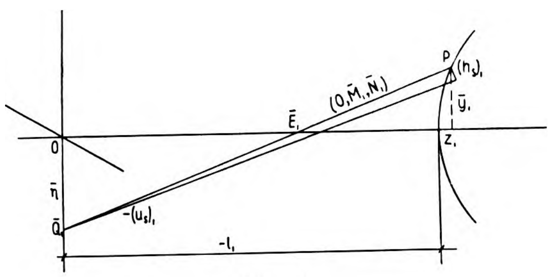

Fig. 1

of the astigmatic marginal ray intersection point (sagittal section), in the surface from the principal ray. When the principal ray approaches the optical axis, both u_s and h_s go over into the corresponding paraxial quantities [1]. We shall need the direction cosines, they are also shown on the figure (Fig.1).

In the paraxial region none of the quantities (h or u) are strictly determined. Only the ratios of heights and aperture angles, as well as height to angles, are determined. The same may be said about h_s and u_s. To avoid that either one of the angles or one of the heights is usually normalized. For our purpose it will be convenient to assume the paraxial incidence height and the paraxial sagittal incidence height equal to each other in the stop space [1]:

$$(h_s)_d = h_d. \tag{1}$$

The invariant D_s

$$D_s = n u_s d_s \tag{2}$$

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was used [2] to calculate the exact formula for the wave aberration in the sagittal focus of an arbitrary surface of rotational symmetry, d_s denotes the distance of the projection of the sagittal focus on the optical axis from the curvature center C_s corresponding to the given incidence height of the principal ray $\bar{y}$

$$\Delta[\bar{N}(\bar{H}/\bar{H}_s)W_s] = \frac{1}{2}A\left[\bar{z}\Delta(u_s)+h_s\Delta(\bar{N})+\frac{1}{2}h\Delta(nu_s)(1-\bar{z}c-r_s c\cos\bar{G})\right], \tag{3}$$

where

$\Delta f(x) = f(x')-f(x)$,

$\bar{H} = nu\bar{\eta}$, $A = ni$,

$\bar{\eta}$ = distance of the intersection point between the image plane and the principal ray from the optical axis,

i = incident angle of the paraxial aperture ray,

$(0, \bar{M}, \bar{N})$ = direction cosines of the principal ray,

$\bar{x}$ = distance from the projection on the optical axis of the point of intersection of the principal ray with the optical surface to the surface vertex,

$\bar{G}$ = angle between the optical axis and the normal to the surface at its intersection point with the principal ray,

W_s = wave aberration of the sagittal focus understood as the distance between two spheres passing through the intersection point of the principal ray and the optical axis and measured along the ray of the astigmatic beam, the spheres centers being located in the point of the intersection of the principal ray with the focusing plane and sagittal focus, respectively.

The expression $\bar{H}_s = nu_s\bar{\eta}_s$ (where $\bar{\eta}_s$ is the distance of the sagittal focus from the optical axis) is an invariant discovered by H. H. Hopkins and is a simple generalisation of that attributed to Lagrange-Helmholtz. Equation (3) is written for one surface; the general equation for the whole system may be obtained on the basic of the relation

$$\bar{N}'_k(n'_k u'_k \bar{\eta}'_k/\bar{H}_s)(W'_s)_k = \bar{N}_{k+1}(n_{k+1}u_{k+1}\bar{\eta}_{k+1}/\bar{H}_s)(W_s)_{k+1}. \tag{4}$$

The corresponding equation for a system of p surfaces will be

$$\bar{N}'_p\bar{H}'_p(W'_s)_p-\bar{N}_1\bar{H}_1(W_s)_1 = \frac{1}{2}\bar{H}_s\sum_{l=1}^{p}A_l[(\bar{z}_l\Delta_l(u_s)+(h_s)_l\Delta_l(\bar{N})]+$$

$$+\frac{1}{2}\sum_{l=1}^{p}h_l\Delta_l(nu_s)[1-\bar{z}_l c_l-(r_s)_1 c_l\cos\bar{G}_l]. \tag{5}$$

Usually the calculation is easier when computing first for one surface and then for the system of p surfaces. Equation 3 will be the basic equation for futher considerations. The quantity $\Delta(u_s)$ may be expressed by invariant $\bar{H}_s$ and the paraxial sagittal height h_s. Using the geometrical properties of the astigmatic beam and invariancy of $\bar{H}_s$ after a few recalculations we get [3] three terms: the first of which, in the Seidel region, goes over into astigmatism, the second into the Petzval curvature and the third disappears

$$S = (2\bar{z}/\bar{y}\bar{h})BB_s hh_s\cos\bar{G}\Delta(1/n)+2(Ah_s/\bar{y})(\bar{y}\cos\bar{G}+\bar{z}\sin\bar{G})\Delta(\cos\bar{I}),$$

$$K = (2\bar{z}/\bar{y}\bar{h})H\bar{H}_s\Delta[-(1/n)], \tag{6}$$

$$F = -(2\bar{z}/\bar{h})A_s Bh\Delta(1/n)+2(Ah_s/\bar{y})B_s\Delta(1/n)(\bar{y}\sin\bar{G}-\bar{z}\cos\bar{G}),$$

$$\Delta(\bar{W}_s) = \frac{1}{4}(K+S+F), \tag{7}$$

where

$$\bar{W}_s = \bar{N}\bar{H}W_s/\bar{H}_s;\ B_s = n\sin\bar{I}.$$

The above expressions (Eq. 6 and 7) were used to calculate the fifth order fieldaberration coefficients for the spherical surfaces and it was assumed that the diaphragm is not in the tangent plane of the surface. For the spherical surfaces it is easily seen from the figure (Fig. 2) that

$$\sin\bar{G} = c\bar{y};\ \cos\bar{G} = 1-\bar{z}c;\ c = \frac{1}{r}.$$

$$\bar{y}\sin\bar{G}-\bar{z}\cos\bar{G} = c\bar{y}^2-\bar{z}(1-c\bar{z}) = c(\bar{y}^2+\bar{z}^2)-\bar{z} = \bar{z}. \tag{8}$$

The last relation follows from the equation of the spherical surfaces with the vertex in the origin of the coordinates axes. Analogically we have

$$\bar{y}\cos\bar{G}+\bar{z}\sin\bar{G} = \bar{y}. \tag{9}$$

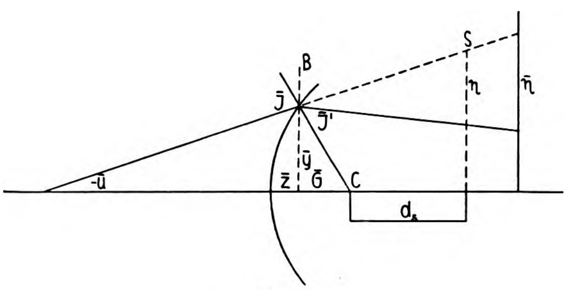

Fig. 2

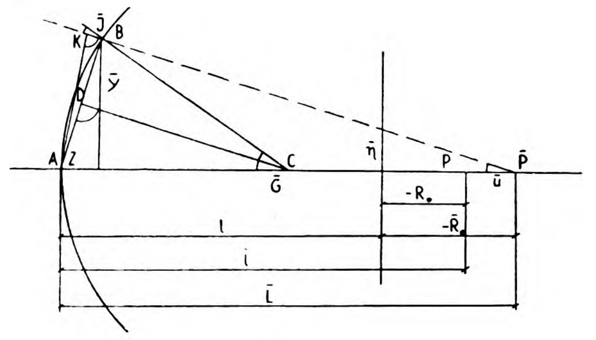

Fig. 3

In the Equation (7) we have still an unknow quantity A_s

$$A_s = ni_s = n(h_s c\cos\bar{I}-u_s), \tag{10}$$

which can be obtained from the invariant H_s,

$$\bar{H}_s = nu_s\eta_s = nu_s(\bar{y}+\bar{M}s) = nu_s\bar{y}+nh_s\bar{M} = n\bar{y}(h_s c\cos\bar{I}-i_s)+nh_s\bar{M},$$

where u_s we calculate from the (Eq. 10).

$$\bar{H}_s = nh_s\bar{y}c\cos\bar{I}-\bar{y}A_s+nh_s\bar{M} = nh_s\sin G\cos\bar{I}-\bar{y}A_s-nh_s\sin\bar{U}$$
$$= nh_s\sin\bar{G}\cos\bar{I}-\bar{y}A_s-nh_s(\sin\bar{G}\cos\bar{I}-\sin\bar{I}\cos\bar{G}) = nh_s\sin\bar{I}\cos\bar{G}-\bar{y}A_s = B_s h_s\cos\bar{G}-\bar{y}A_s,$$

where

$$B_s = n\sin\bar{I}.$$

Therefore

$$A_s = \frac{1}{y}(B_s h_s\cos\bar{G}-\bar{H}_s). \tag{11}$$

Substituting the relations (8), (9) and (11) to (6) we get

$$K = (2\bar{z}/\bar{y}\bar{h})H\bar{H}_s\Delta\left(-\frac{1}{n}\right),$$

$$S = (2\bar{z}/\bar{y}\bar{h})BB_s hh_s\cos\bar{G}\,\Delta(1/n)+2(Ah_s)\,\Delta(\cos\bar{I}), \tag{12}$$

$$F = (2\bar{z}/\bar{y}\bar{h})B_s h_s(Ah-hB\cos\bar{G})+hB\bar{H}_s\Delta(1/n),$$

where

$$A = ni;\ B = n\bar{i}.$$

The next question that comes to mind is that the exact formulas for the parameters of the given principal ray should be replaced by the quantities with the fifth order accuracy. From here on, l will represent a length, that is the distance from the object plane to the surface (Fig. 3),

$\bar{l}$ that from the paraxial diaphragm plane to the surface and $\bar{L}$ a distance from the diaphragm plane to the surface for the ray which makes the angle $\bar{U}$ with the optical axis. The distance from the point P ($\bar{P}$) to the object plane is denoted as $-R_0$ ($-\bar{R}_0$). It is easily seen from the Fig. 3 that

$$\frac{\bar{M}}{\bar{N}} = \tan(-\bar{U}) = \frac{\bar{\eta}}{\bar{R}_0} = \frac{\bar{\eta}}{R_0} + \bar{\eta}\left(\frac{1}{\bar{R}_0} - \frac{1}{R_0}\right),$$

$$\bar{\eta} = \bar{\eta}_{\max}\cdot\tau + \delta\eta,$$

$$0 \leqslant \tau \leqslant 1.$$

The second term we will calculate for the principal ray from the spherical wave aberration [4].

$$\bar{W} = -\frac{n}{2}\bar{\eta}^2\left(\frac{1}{\bar{R}_0} - \frac{1}{R_0}\right) = -\frac{n}{2}\bar{\eta}^2_{\max}\tau^2\left(\frac{1}{\bar{R}_0} - \frac{1}{R_0}\right) + O(\tau^6).$$

where $O(\tau^n)$ denotes the quantities of order τ^n and higher.

For the third order aberration

$$\bar{W} = \frac{1}{8}\bar{S}_\mathrm{I}\tau^4.$$

Therefore

$$\bar{\eta}\left(\frac{1}{\bar{R}_0} - \frac{1}{R_0}\right) = -\frac{1}{n\bar{\eta}_{\max}}\frac{\partial\bar{W}}{\partial\tau} = -\frac{1}{2n\bar{\eta}_{\max}}\bar{S}_\mathrm{I}\tau^3 = -\frac{\sin g_s}{2H_s}\bar{S}_\mathrm{I}\tau^3,$$

where

$$\bar{\eta}\sin g_s = u_s\bar{\eta}_s.$$

After substituting it into the relation for $\bar{M}/\bar{N}$ we have

$$\frac{\bar{M}}{\bar{N}} = \tau\frac{\bar{\eta}_{\max}}{R_0}\left(1 + \frac{\delta\eta}{\eta_p}\right) - \frac{\sin g_s}{2H_s}\bar{S}_\mathrm{I}\tau^3 + O(\tau^6),$$

where $\eta_p = \tau\cdot\eta_{\max}$.

The above quantities should be calculated with accuracy to the τ^4 because for the fifth order aberration we do not take into consideration the term equal or higher than τ^5. In this approximation

$$\frac{\sin g_s}{H_s} = \frac{u}{H} + O(\tau^2),$$

$$\frac{\delta\eta}{\eta_p} = \frac{1}{2H}S_\mathrm{V}\tau^2 + O(\tau^4).$$

Finally we have

$$\frac{\bar{M}}{\bar{N}} = -\bar{u}\tau\left(1 + \frac{1}{2H}S_\mathrm{V}\tau^2\right) - \frac{u}{2H}\bar{S}_\mathrm{I}\tau^3 + O(\tau^5),$$

$$\frac{\bar{M}}{\bar{N}} = -\bar{u}\tau - \frac{1}{2H}(\bar{u}S_\mathrm{V} + u\bar{S}_\mathrm{I}) + O(\tau^5). \tag{13}$$

When the number of the surfaces is t then the symbols S_V and $\bar{S}_\mathrm{I}$ denote

$$S_\mathrm{V} = \sum_{k=1}^{t-1}(S_\mathrm{V})_k, \quad \bar{S}_\mathrm{I} = \sum_{k=1}^{t-1}(\bar{S}_\mathrm{I})_k - \sum_{k=1}^{d-1}(\bar{S}_\mathrm{I})_k. \tag{14}$$

The second relation results from the definition of the principal ray. On the first surface $\bar{S}_\mathrm{I} = 0$ only if the center of the acting diaphragm lies in the vertex of the surface. The second term must be substracted for every surface and it is more convenient to calculate it separately and subtract afterwards.

From the expression (13) it is easy to calculate $\bar{M}$ and $\bar{N}$ with accuracy to the τ^4. At first we calculate $\bar{M}$ [1]

$$\bar{M} = (\bar{M}/\bar{N})(1-\bar{M}^2)^{1/2} = (\bar{M}/\bar{N})\left(1-\frac{1}{2}\bar{M}^2-\frac{1}{8}\bar{M}^4\right)$$

$$= \bar{M}/\bar{N}\left[1-\frac{1}{2}(\bar{M}/\bar{N})^2(1-\bar{M}^2)-\frac{1}{8}(\bar{M}/\bar{N})^4(1-\bar{M}^2)^2\right]$$

$$= (\bar{M}/\bar{N})\left[1-\frac{1}{2}(\bar{M}/\bar{N})^2+\frac{1}{2}(\bar{M}/\bar{N})^2\bar{M}^2-\frac{1}{8}(\bar{M}/\bar{N})^4+O(\bar{M}^6)\right]$$

$$= (\bar{M}/\bar{N})\left[1-\frac{1}{2}(\bar{M}/\bar{N})^2+\frac{3}{8}(\bar{M}/\bar{N})^4+O(\bar{M}^6)\right],$$

because

$$\frac{1}{2}(\bar{M}/\bar{N})^2\bar{M}^2 = \frac{1}{2}(\bar{M}/\bar{N})^4(1-\bar{M}^2) = \frac{1}{2}(\bar{M}/\bar{N})^4+O(\bar{M}^6).$$

Using the relation (13) and recalculating we have

$$\bar{M} = -\bar{u}\tau+\frac{1}{2}\left[\bar{u}^3-\frac{1}{H}(\bar{u}S_{\mathrm{V}}+u\bar{S}_{\mathrm{I}})\right]\tau^3+O(\tau^5). \tag{15}$$

Dividing the expression (15) by (13) and recalculating we obtain a relation for $\bar{N}$.

$$\bar{N} = 1-\frac{1}{2}\left[\bar{u}^2\tau^2+\frac{\bar{u}}{H}(\bar{u}S_{\mathrm{V}}+u\bar{S}_{\mathrm{I}}-\frac{3}{4}H\bar{u}^3)\tau^4\right]+O(\tau^6). \tag{16}$$

All the quantities in this and following relation are the paraxial ones. The terms with τ^2 are connected with the third order aberration, those with τ^4 with the fifth order respectively. Now we want to find formulas for $\sin\bar{I}$ and $\cos\bar{I}$. It is easily seen from the Fig. 3 that

$$\sin\bar{I} = \frac{\bar{L}-r}{r}\sin\bar{U} = \frac{\bar{l}-r}{r}\sin\bar{U}+\frac{R_0-\bar{R}_0}{r}\sin\bar{U}$$

and

$$\bar{L} = l-\bar{R}_0 = l-R_0+R_0-\bar{R}_0 = l+R_0-\bar{R}_0.$$

Substituting $(-\sin\bar{U})$ by $\bar{M}$ from Eq. (15) we get the formula for $\sin\bar{I}$

$$\sin\bar{I} = \frac{\bar{l}-r}{r}\left[\bar{u}\tau+\frac{1}{2H}(\bar{u}S_{\mathrm{V}}+u\bar{S}_{\mathrm{I}}-H\bar{u}^3)\tau^3\right]+\frac{1}{r}\left(\frac{1}{\bar{R}_0}-\frac{1}{R_0}\right)\bar{R}_0R_0\sin\bar{U}$$

$$= (\bar{h}c-\bar{u})\left[\tau+\frac{1}{2H}(S_{\mathrm{V}}-H\bar{u}^2)\tau^3\right]+\frac{\bar{l}-r}{2rH}u\bar{S}_{\mathrm{I}}\tau^3-\frac{1}{r}\bar{R}_0R_0\frac{\sin\bar{U}\,\bar{S}_{\mathrm{I}}}{2n\eta_{\max}^2}\tau^2,$$

where

$$\bar{h} = \bar{u}\bar{l};\ \sin\bar{U} = -\frac{\bar{\eta}}{\bar{R}_0} = -\frac{\eta_{\max}\cdot\tau}{R_0};$$

$$\bar{h}c = \bar{g};\ \bar{u}+\bar{i} = \bar{g}.$$

Therefore

$$\sin\bar{I} = \bar{i}\tau+\frac{\bar{i}}{2H}(S_{\mathrm{V}}-H\bar{u}^2)\tau^3+\frac{\bar{S}_{\mathrm{I}}}{2rH}[(\bar{l}-r)u+R_0u]\tau^3+O(\tau^5)$$

$$= \bar{i}\tau+\frac{\bar{i}}{2H}(S_{\mathrm{V}}-H\bar{u}^2)\tau^3+(\bar{S}_{\mathrm{I}}/2H)(hc-u)+O(\tau^5)$$

$$= \bar{i}\tau+\frac{\bar{i}}{2H}(S_{\mathrm{V}}-H\bar{u}^2)\tau^3+\frac{i}{2H}\bar{S}_{\mathrm{I}}\tau^3+O(\tau^5) = \bar{i}\tau-\frac{1}{2H}(H\bar{u}^2\bar{i}-\bar{i}S_{\mathrm{V}}-i\bar{S}_{\mathrm{I}})\tau^3+O(\tau^5). \tag{17}$$

For $\cos\bar{I}$ we will use the identity

$$\cos\bar{I} = (1-\sin^2\bar{I})^{1/2}$$

After some recalculation with the accuracy to the $O(\tau^5)$ we obtain

$$\cos\bar{I} = 1-\frac{1}{2}\bar{i}^2\tau^2+\frac{\bar{i}}{2}\left[\bar{u}^2\bar{i}-\frac{1}{4}\bar{i}^3-\frac{1}{H}(\bar{i}S_V+i\bar{S}_I)\right]\tau^4+O(\tau^5). \tag{18}$$

The relations for $\bar{y}$ and $\bar{z}$ will be obtained in two different ways, each of them being a kind of check for the other. To start with the first way let us take, from the point O, the bisector of an angle between the radius r of the spherical surface and the optical axis. It cuts the secant AB at the point D (Fig. 3)

$$\sphericalangle ACD = \frac{\bar{I}+\bar{U}}{2}; \quad \sphericalangle KAB = \frac{\bar{I}-\bar{U}}{2},$$

$$AB = \frac{\bar{y}}{\cos\frac{\bar{I}+\bar{U}}{2}} = \frac{\bar{L}\sin\bar{U}}{\cos\frac{\bar{I}-\bar{U}}{2}}.$$

Therefore

$$\bar{y} = \frac{\bar{L}\sin\bar{U}\cos\frac{\bar{U}+\bar{I}}{2}}{\cos\frac{\bar{U}-\bar{I}}{2}} = \bar{L}\sin\bar{U}\,\frac{\cos\left(\bar{U}-\frac{\bar{U}-\bar{I}}{2}\right)}{\cos\frac{\bar{U}+\bar{I}}{2}} = \bar{L}\sin\bar{U}\cos\bar{U}-\tan\frac{\bar{I}-\bar{U}}{2}\sin\bar{U},$$

$$\tan\frac{\bar{I}-\bar{U}}{2} = \frac{\sin\frac{\bar{I}-\bar{U}}{2}\cos\frac{\bar{I}-\bar{U}}{2}}{\cos\frac{\bar{I}-\bar{U}}{2}\cos\frac{\bar{I}-\bar{U}}{2}} = \frac{\sin(\bar{I}-\bar{U})}{1+\cos(\bar{I}-\bar{U})} = \frac{\sin\bar{I}\cos\bar{U}-\sin\bar{U}\cos\bar{I}}{1+\cos\bar{I}\cos\bar{U}+\sin\bar{I}\sin\bar{U}},$$

$$\bar{y} = \bar{L}\sin\bar{U}\cos\bar{U}-\frac{1}{2}\bar{L}\sin^2\bar{U}(\sin\bar{I}-\sin\bar{U})+O(\tau^5) = \bar{l}\sin\bar{U}\cos\bar{U}-\frac{1}{2}\bar{l}\sin^2\bar{U}(\sin\bar{I}-\sin\bar{U})+(R_0-\bar{R}_0)\cdot$$
$$\sin\bar{U}\cos\bar{U}+O(\tau^5),$$

since

$$\frac{1}{2}(R_0-\bar{R}_0)\sin^2\bar{U}(\sin\bar{I}-\sin\bar{U}) = O(\tau^5).$$

We want to find $R_0-\bar{R}_0$ with the accuracy $O(\tau^4)$. Because $\sin\bar{U} = O(\tau)$ we have

$$R_0-\bar{R}_0 = \left(\frac{1}{\bar{R}_0}-\frac{1}{R_0}\right)\bar{R}_0R_0 = -\frac{1}{2n\bar{\eta}_{\max}^2}\bar{S}_I\bar{R}_0R_r\tau^2,$$

$$\bar{y} = \bar{l}\sin\bar{U}\cos\bar{U}-\frac{1}{2}\bar{l}\sin^2\bar{U}(\sin\bar{I}-\sin\bar{U})-\frac{\bar{R}_0R_0\sin\bar{U}\cos\bar{U}}{2n\bar{\eta}_{\max}^2}\bar{S}_I\tau^2.$$

With the help of the relations (Eq. 15–17) we will obtain

$$\bar{y} = \bar{h}\tau+\frac{1}{2}\bar{l}\left[-2\bar{u}^3+\frac{1}{H}(\bar{u}S_V+u\bar{S}_I)\tau^3\right]-\frac{1}{2}\bar{l}\bar{u}^2\tau^2(\bar{i}-\bar{u})+u(l-\bar{l})(\bar{S}_I/2H)\tau^3+O(\tau^5)$$
$$= \bar{h}\tau-\frac{1}{2}\left[\bar{h}\bar{u}\bar{g}-\frac{1}{H}(\bar{h}S_V+h\bar{S}_I)\right]\tau^3+O(\tau^5). \tag{19}$$

The following expressions were used above:

$$\bar{\eta}_{\max} = -\bar{u}R_0; \quad H = nu\bar{\eta}_{\max};$$
$$h = lu; \quad \bar{h} = \bar{l}\bar{u}; \tag{20}$$
$$\bar{g} = \bar{u}+\bar{i}.$$

The second way will consist in derivation of the relation (19) from the equality $c\bar{y} = \sin\bar{G}$ (see Fig. 3).

$$c\bar{y} = \sin\bar{G} = \sin(\bar{U}+\bar{I}) = \sin\bar{U}\cos\bar{I}+\cos\bar{U}\sin\bar{I} = -\bar{M}\cos\bar{I}+\bar{N}\sin\bar{I}.$$

After substituting the expressions for the $\bar{M}$, $\bar{N}$, $\sin\bar{I}$, $\cos\bar{I}$ and a few recalculations it may be obtained

$$c\bar{y} = \bar{g}\tau - \frac{1}{2}\left[\bar{g}^2\bar{u} - \frac{1}{H}(\bar{g}S_{\mathrm{V}}+g\bar{S}_{\mathrm{I}})\right]\tau^3 + O(\tau^5). \tag{21}$$

Therefore

$$\bar{y} = \bar{h}\tau - \frac{1}{2}\left[\bar{g}\bar{h}\bar{u} - \frac{1}{H}(\bar{h}S_{\mathrm{V}}+h\bar{S}_{\mathrm{I}})\right]\tau^3 + O(\tau^5).$$

The expression (21) may be used to calculate $\sin\bar{G}$. The equation for $\bar{z}$ may be obtained (see Fig. 3) from the following relations

$$\bar{z} = \bar{y}\tan\frac{\bar{U}+\bar{I}}{2} \quad \text{or} \quad \bar{z}c = 1-\cos\bar{G}.$$

In the same way we receive:

$$\bar{z} = \frac{1}{2}\bar{g}\tau^2\left\{\bar{h} + \left[\frac{1}{4}\bar{h}\bar{g}^2 - \bar{h}\bar{g}\bar{u} + \frac{1}{H}(\bar{h}S_{\mathrm{V}}+h\bar{S}_{\mathrm{I}})\right]\tau^2\right\} + O(\tau^5). \tag{22}$$

Expressions for K, S and F in (Eq. 12) all contain the term $(2\bar{z}/\bar{y})$. This term may be obtained when dividing relation (22) by (21). After a simple recalculation

$$\frac{2\bar{z}}{\bar{y}} = \bar{g}\tau + \frac{1}{2}\left[\frac{1}{2}\bar{g}^2(\bar{i}-\bar{u}) + \frac{1}{H}(\bar{g}S_{\mathrm{V}}+g\bar{S}_{\mathrm{I}})\right]\tau^3 + O(\tau^5). \tag{23}$$

The quantity $\cos\bar{G}$ may be received from the relations

$$\cos\bar{G} = 1-\bar{z}c$$

or

$$\cos\bar{G} = (1-\sin^2\bar{G})^{1/2},$$

$$\cos\bar{G} = 1-\frac{1}{2}\bar{g}\tau^2\left\{\bar{g} + \left[\frac{1}{4}\bar{h}^2\bar{g} - \bar{h}^2\bar{u} + \frac{1}{H}(\bar{g}S_{\mathrm{V}}+g\bar{S}_{\mathrm{I}})\right]\tau^2\right\} + O(\tau^5). \tag{24}$$

In the Eq. (12) we have the difference between $\cos\bar{I}$ after and before refraction

$$\Delta(\cos\bar{I}') = \cos\bar{I}' - \cos\bar{I}.$$

Calculating $\cos\bar{I}'$ we have to use the expressions for $\sum_{k=1}^{t}(S_{\mathrm{V}})_k$ and $\sum_{k=1}^{t}(\bar{S}_{\mathrm{I}})_k$, but these would be not convenient for computing and programming. We could avoid this difficulty using the law of refraction $n'\sin\bar{I}' = n\sin\bar{I}$

$$\sin\bar{I}' = \frac{n}{n'}\left\{\bar{i}\tau - \frac{1}{2}\left[\bar{u}^2\bar{i} - \frac{1}{H}(\bar{i}S_{\mathrm{V}}+i\bar{S}_{\mathrm{I}})\right]\tau^3\right\} + O(\tau^5),$$

$$\cos\bar{I}' = (1-\sin^2\bar{I}')^{1/2} = 1-\frac{1}{2}\left(\frac{n}{n'}\right)^2\left\{\bar{i}^2\tau^2 - \bar{i}\left[\bar{u}^2\bar{i} - \frac{1}{H}(\bar{i}S_{\mathrm{V}}+i\bar{S}_{\mathrm{I}}) - \frac{1}{4}\left(\frac{n}{n'}\right)^2\bar{i}^3\right]\tau^4\right\} + O(\tau^6).$$

Therefore

$$\Delta(\cos\bar{I}) = -\left[\left(\frac{n}{n'}\right)^2-1\right]\left\{\bar{i}^2\tau^2 - \bar{i}\left[\bar{u}^2\bar{i} - \frac{1}{4}\left|\left(\frac{n}{n'}\right)^2+1\right|\bar{i}^3 - \frac{1}{H}(\bar{i}S_{\mathrm{V}}+i\bar{S}_{\mathrm{I}})\right]\tau^4\right\} + O(\tau^6). \tag{25}$$

We still need the expression for h_s and $\bar{H}_s$. The quantity $\bar{H}_s$ is an invariant and we can choose for calculations any space. The most convenient is the stop space $(h_s)_d = h_d$. In the plane of the diaphragm $\bar{y} = 0$ and $\bar{h} = 0$

$$\bar{H}_s = -n_d(h_s)_d\bar{M}_d,$$
$$H\tau = n_d h_d \bar{u}_d.$$

Dividing these two relations we obtain

$$\bar{H}_s = -H\tau\frac{\bar{M}_d}{\bar{u}_d} = H\tau\left\{1-\frac{1}{2}\left[\bar{u}_{d+1}^2-\frac{1}{H}\sum_{k=1}^{d}(S_{\rm V})_k\right]\tau^2\right\}+O(\tau^4). \tag{26}$$

Denoting

$$V_d = \bar{u}_{d+1}^2-\frac{1}{H}\sum_{k=1}^{d}(S_{\rm V})_k,$$

we have (27)

$$\bar{H}_s = H\tau\left(1-\frac{1}{2}V_d\tau^2\right).$$

In the formula we have not any term with $\bar{S}_I$ because in this space $\bar{S}_I$ is equal to zero from the definition.

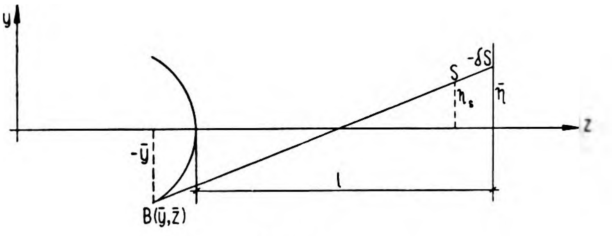

Fig. 4

A more complicated task is to derive an expression for h_s. We shall start with the calculating of the quantity $\bar{\eta}/\eta_s$ (Fig.4)

$$\frac{\bar{\eta}}{\eta_s} = \frac{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}}{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}+\delta S\bar{N}}.$$

Now we recalculate the left side in the following manner:

$$\frac{\bar{\eta}}{\eta_s} = \frac{\eta_p+\delta\eta}{\eta_s} = \frac{\eta_p}{\eta_s}+\frac{\delta\eta}{\eta_s}\approx\frac{\eta_p}{\eta_s}+\frac{\delta\eta}{\eta_p},$$

$$\frac{\eta_p}{\eta_s} = \frac{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}}{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}+\delta S\bar{N}}-\frac{\delta\eta}{\eta_p}. \tag{28}$$

From the definition of the $\bar{H}_s$ and H we have

$$\frac{H\tau}{\bar{H}_s} = \frac{nu\eta_p}{nu_s\eta_s}.$$

Therefore

$$u_s = \frac{\bar{H}_s u\eta_p}{H\tau\eta_s}.$$

After substituting the relation (28)

$$u_s = \frac{\bar{H}_s u}{H\tau}\left[\frac{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}}{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}+\delta S\bar{N}} - \frac{\delta\eta}{\eta_p}\right]. \tag{29}$$

It is easily seen from the Fig. 1 that

$$h_s = u_s s = u_s\left(\frac{l-\bar{z}}{\bar{N}}+\delta S\right) = \frac{H_s u}{H\tau}\left(\frac{l-\bar{z}}{\bar{N}}+\delta S\right).$$

$$\left[\frac{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}}{l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}+\delta S\bar{N}} - \frac{\delta\eta}{\eta}\right] = \frac{\bar{H}_s(h-\bar{z}u)}{H\tau\bar{N}}\left[1+\frac{\bar{N}^2\delta S\bar{y}}{\bar{M}\left((l-\bar{z})(l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}+\delta S\bar{N})\right)} - \frac{\delta\eta}{\eta_p}\right]. \tag{30}$$

We omit the small term $\delta S\,\delta\eta$. In the relation for K, S and F the term h_s is multiplied by the factor τ^2 and it is sufficient to calculate it with accuracy τ^3. The quantity δS is of the order $O(\tau^2)$. Now, we calculate the second term in the bracket

$$\frac{\bar{N}^2\bar{y}\delta S}{\bar{M}(l-\bar{z})\left(l-\bar{z}+\bar{y}\dfrac{\bar{N}}{\bar{M}}+\delta S\bar{N}\right)} = \frac{\bar{N}^2\bar{y}\delta S}{\bar{M}(l-\bar{z})^2+\bar{y}\bar{N}(l-\bar{z})+\bar{M}(l-\bar{z})\bar{N}\delta S}$$

$$= \frac{\bar{h}\delta S\tau+O(\tau^5)}{-\bar{u}\tau l^2+\bar{h}\tau l+O(\tau^3)} = \frac{\bar{h}\delta S+O(\tau^4)}{-\bar{u}l^2+\bar{h}l+O(\tau^2)}. \tag{31}$$

The aberration δS may be calculated from the following relation [1, 2]:

$$W_s = -\frac{1}{2}nuu_s\delta S\frac{\bar{H}_s}{H}.$$

Therefore

$$\delta S = -\frac{2\bar{H}W_s}{nuu_s\bar{H}_s}.$$

The quantities $\bar{H}$, $\bar{H}_s$, u_s and W_s can be estimated from the following relations [1]:

$$\bar{H} = nu\bar{\eta} = nu(\eta+\delta\eta) = H\tau+O(\tau^2),$$

$$\bar{H}_s = H\tau+O(\tau^2),$$

$$u_s = u+O(\tau^2),$$

$$W_s = \tfrac{1}{4}(S_{\mathrm{III}}+S_{\mathrm{IV}})\tau^2.$$

After a few manipulations it may be obtained

$$\delta S = -\frac{S_{\mathrm{III}}+S_{\mathrm{IV}}}{2nu^2}\tau^2. \tag{32}$$

The equation (30) may be rewritten as follows:

$$-\frac{\bar{h}(S_{\mathrm{III}}+S_{\mathrm{IV}})\tau^2}{2nh(\bar{h}u-h\bar{u})}+O(\tau^4) = -\frac{\bar{h}(S_{\mathrm{III}}+S_{\mathrm{IV}})}{2hH}\tau^2+O(\tau^4). \tag{33}$$

With the accuracy $O(\tau^4)$ we will have

$$\frac{\delta\eta}{\eta_p} = \frac{1}{2H}S_{\mathrm{V}}\tau^2+O(\tau^4). \tag{34}$$

After substituting the relations (33) and (34) to the expression for h_s (Eq. 30) we obtain

$$h_s = \frac{\bar{H}_s(h-\bar{z}u)}{H\tau\bar{N}}\left[1-\frac{\bar{h}\left(S_{\mathrm{III}}+S_{\mathrm{IV}}+\frac{h}{\bar{h}}S_{\mathrm{V}}\right)}{2Hh}\tau^2\right]+O(\tau^4)$$

$$= \frac{\bar{H}_s}{H\tau}\left\{h-\frac{1}{2}\left[\bar{h}\bar{g}u-h\bar{u}^2+\frac{1}{H}(\bar{h}S_{\mathrm{III}}+\bar{h}S_{\mathrm{IV}}+hS_{\mathrm{V}})\right]\tau^2\right\}+O(\tau^4) \quad (35)$$

or

$$h_s = h-\frac{1}{2}\left\{\bar{h}\bar{g}u-h\bar{u}^2+hV_d+\frac{1}{H}[\bar{h}(S_{\mathrm{III}}+S_{\mathrm{IV}})+hS_{\mathrm{V}}]\right\}\tau^2+O(\tau^4). \quad (36)$$

Thus, we have calculated all quantities which are present in the Eq. 12. In expression for K, which is the simplest and also in the expressions for S and F we have the term $\bar{H}_s$ (direct or trough h_s). Substituting (Eq. 27 and 23) for $\bar{H}_s$ and $(2\bar{z}/\bar{y})$ we receive

$$K = H^2c\Delta\left(-\frac{1}{n}\right)\tau^2\left(1-\frac{1}{2}V_d\tau^2\right)\left\{1+\frac{1}{2}\left[\frac{1}{2}\bar{g}(\bar{i}-\bar{u})+\frac{1}{H}\left(S_{\mathrm{V}}+\frac{h}{\bar{h}}\bar{S}_{\mathrm{I}}\right)\right]\tau^2\right\}+O(\tau^6)$$

$$= H^2c\Delta\left(-\frac{1}{n}\right)\tau^2\left\{1+\frac{1}{2}\left[\frac{1}{2}\bar{g}(\bar{i}-\bar{u})-V_d+\frac{1}{H}\left(S_{\mathrm{V}}+\frac{h}{\bar{h}}\bar{S}_{\mathrm{I}}\right)\right]\tau^2\right\}+O(\tau^6). \quad (37)$$

We shall now calculate explicitly the expression for F. It is given by

$$F = \frac{2\bar{z}}{\bar{y}\bar{h}}\Delta\left(\frac{1}{n}\right)\bar{H}_s\left[B_s\frac{h_s}{\bar{H}_s}(A\bar{h}-hB\cos\bar{G})+hB\right].$$

After using the previous relations for $2\bar{z}/\bar{y}$, $\bar{H}_s$, $B_s = n\sin\bar{I}$, h_s, $\cos\bar{G}$ it may be obtained

$$F = ncH\Delta\left(\frac{1}{n}\right)\tau^2\left(1-\frac{1}{2}V_d\tau^2\right)\left\{1+\frac{1}{2}\left[\frac{1}{2}\bar{g}(\bar{i}-\bar{u})+\frac{1}{H}\left(S_{\mathrm{V}}+\frac{h}{\bar{h}}\bar{S}_{\mathrm{I}}\right)\right]\right\}\times$$

$$\times\left\langle\frac{n}{H}\left\{\bar{i}-\frac{1}{2}\left[\bar{u}^2\bar{i}-\frac{1}{H}(\bar{i}S_{\mathrm{V}}+i\bar{S}_{\mathrm{I}})\right]\tau^2\right\}\left\{h-\frac{1}{2}\left[\bar{h}\bar{g}u-h\bar{u}^2+\frac{1}{H}(\bar{h}S_{\mathrm{III}}+\bar{h}S_{\mathrm{IV}}+hS_{\mathrm{V}})\right]\tau^2\right\}\times\right.$$

$$\left.\times\left\{\bar{h}i-h\bar{i}\left(1-\frac{1}{2}g^2\tau^2\right)\right\}+h\bar{i}\right\rangle+O(\tau^6)=\frac{1}{2}n\bar{g}\Delta\left(\frac{1}{n}\right)\left\{\bar{g}\bar{i}(n\bar{i}hg+uH)+\bar{i}(S_{\mathrm{III}}+S_{\mathrm{IV}})-\frac{h}{\bar{h}}i\bar{S}_{\mathrm{I}}\right\}\tau^4+$$

$$+O(\tau^6). \quad (38)$$

In this relation we have no term with τ^2, because F is a disturbance expression which vanishes in the Seidel region.

The last relation may be obtained from the following formula (Eq. 12)

$$S = \frac{2\bar{z}}{\bar{y}\bar{h}}BB_shh_s\cos\bar{G}\Delta\left(\frac{1}{n}\right)+2Ah_s\Delta(\cos\bar{I}),$$

$$S = n^2\Delta\left(\frac{1}{n}\right)\bar{i}\tau^2\left\{h-\frac{1}{2}\left[\bar{h}\bar{g}u-h\bar{u}^2+hV_d+\frac{1}{H}(\bar{h}S_{\mathrm{III}}+\bar{h}S_{\mathrm{IV}}+hS_{\mathrm{V}})\right]\tau^2\right\}\left\langle ch\left\{1+\frac{1}{2}\left[\frac{1}{2}\bar{g}(\bar{i}-\bar{u})+\right.\right.\right.$$

$$\left.\left.+\frac{1}{H}\left(S_{\mathrm{V}}+\frac{h}{\bar{h}}\bar{S}_{\mathrm{I}}\right)\right]\tau^2\right\}\left\{\bar{i}-\frac{1}{2}\left[\bar{u}^2\bar{i}-\frac{1}{H}(\bar{i}S_{\mathrm{V}}+i\bar{S}_{\mathrm{I}})\right]\tau^2\right\}\left\{1-\frac{1}{2}\bar{g}^2\tau^2\right\}-ni\left(\frac{1}{n'}+\frac{1}{n}\right)\times$$

$$\left.\times\left\{\bar{i}-\left[\bar{u}^2\bar{i}-\frac{1}{4}\left|\left(\frac{n}{n'}\right)^2+1\right|\bar{i}^3-\frac{1}{H}(\bar{i}S_{\mathrm{V}}+i\bar{S}_{\mathrm{I}})\right]\tau^2\right\}\right\rangle+O(\tau^6).$$

After multiplication and simplification using the refraction law it may be received with the accuracy to $O(\tau^6)$.

$$S = n^2\Delta\left(\frac{1}{n}\right)\bar{i}\tau^2\Big\langle -h\bar{i}(u-i')+\frac{1}{2}\Big\{-h\bar{i}\bar{u}^2(i+i')-\frac{1}{2}\bar{h}g^2\bar{i}(\bar{g}+2\bar{u})-(u-i')\bar{i}(hV_d+\bar{h}\bar{g}u)-$$

$$-\frac{1}{2}h(i+i')\left[\left(\frac{n}{n'}\right)^2+1\right]\bar{i}^3+\frac{1}{H}\Big[\bar{i}(u-i')(hS_{\mathrm{V}}-\bar{h}S_{\mathrm{IV}}-\bar{h}S_{\mathrm{III}})+$$

$$+h\left(\bar{i}(u-i-2i')+\frac{h}{\bar{h}}\bar{i}g\right)\bar{S}_{\mathrm{I}}\Big]\Big\}\tau^2\Big\rangle+O(\tau^6). \qquad (39)$$

This relation may be simplified by using (Eq. 38) for F

$$S = n^2\Delta\left(\frac{1}{n}\right)\bar{i}\tau^2\Big\langle h\bar{i}(u-i')+\frac{\tau^2}{2}\Big\{-\frac{1}{2}h(i+i')\bar{i}\left[2\bar{u}^2-\left[\left(\frac{n}{n}\right)^2+1\right]\bar{i}^2\right]-\frac{1}{2}\bar{h}g^2\bar{i}(\bar{g}+2\bar{u})-h\bar{i}(u-i')\times$$

$$\times\left[V_d-\frac{n\bar{h}\bar{g}g\bar{i}}{H}\right]+\frac{1}{H}\left[\bar{i}(u-i')hS_{\mathrm{V}}-\bar{h}i(i+i')-h\bar{i}g)\frac{h}{\bar{h}}\bar{S}_{\mathrm{I}}\right]\Big\}\Big\rangle-\frac{nr\bar{i}(u-i')}{H}F+O(\tau^6). \qquad (40)$$

In the previous expressions for K, F and S we have three terms

$$\begin{aligned}\hat{S}_{\mathrm{I}} &= \frac{h}{\bar{h}}\bar{S}_{\mathrm{I}},\\ \hat{h} &= h(i+i'),\\ \hat{i} &= \bar{i}(u-i'),\end{aligned} \qquad (41)$$

which may be used for further simplification

$$K = -H^2\Delta\left(\frac{1}{n}\right)\tau^2 c\left\{1+\frac{1}{2}\left[\frac{1}{2}\bar{g}(\bar{i}-\bar{u})-V_d+\frac{1}{H}(S_{\mathrm{V}}+\hat{S}_{\mathrm{I}})\right]\tau^2\right\}+O(\tau^6),$$

$$F = \frac{1}{2}n\bar{g}\Delta\left(\frac{1}{n}\right)\tau^4\{\bar{g}\bar{i}(n\bar{i}hg+Hu)+\bar{i}(S_{\mathrm{III}}+S_{\mathrm{IV}})-\bar{i}\hat{S}_{\mathrm{I}}\}+O(\tau^6), \qquad (42)$$

$$S = n^2\Delta\left(\frac{1}{n}\right)\bar{i}\tau^2\Big\langle h\hat{i}+\frac{\tau^2}{2}\Big\{-\bar{u}^2\bar{i}h-\frac{1}{2}\bar{h}g^2\bar{i}(\bar{g}+2\bar{u})-h\hat{i}V_d-\frac{1}{2}\hat{h}\left[\left(\frac{n}{n'}\right)^2+\right.$$

$$\left.+1\right]\bar{i}^3+\frac{\bar{h}h\bar{g}g\bar{i}\hat{i}n}{H}+\frac{1}{H}\left[\hat{i}hS_{\mathrm{V}}-(\bar{h}i(i+i')-h\bar{i}g)\hat{S}_{\mathrm{I}}\right]\Big\}\Big\rangle-\frac{\hat{i}rn}{H}F+O(\tau^6).$$

These formulas are not difficult in computing and programming. They are indeterminate when the plane of the diaphragm covers with the tangent plane to the surface (both $\bar{h}$ and $\bar{S}_{\mathrm{I}}$ tend to zero). This is not a serious difficulty because $\bar{S}_{\mathrm{I}}$ tends to zero more rapidly than $\bar{h}$ (with $\bar{h}^4$) and

$$\lim_{\bar{h}\to 0}\hat{S}_{\mathrm{I}} = 0. \qquad (43)$$

We only ought to verify if $\bar{h}$ is equal to zero and in this case to put $\hat{S}_{\mathrm{I}} = 0$.

After programming these relations have been used to investigate the optical systems. The results are now in preparation and will be published soon.

Полевая абберация V ряда для сферических поверхностей

В работе, опубликованной в журнале „Аплайд оптик” (3), выведена зависимость волновой абберации сагитального фокуса для любой поверхности с вращательной симметрией. Затем зависимость эта была преобразована так, что были получены три выражения, первое из которых в области Зейдела переходит в астигматизм, второе — в кривизну Петзвала, а третье — исчезает. Эти уравнения после упрощения становятся основными для сферических поверхностей. Выведены зависимости для параметров главного чула и сагитальных величин h_s и H_s с точностью до пятого ряда.

После подставления их в основные уравнения и соответствующих преобразований получены простые зависимости для коэффициентов полевых абберациий пятого ряда.

Выведен и обсуждён предел выражения, когда $h \to 0$.

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