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Conditions for Bruns Eikonal Transformation

In the paper some general conditions to be fulfilled when constructing the Bruns eikonals with the help of the Legendre method have been determined. The results obtained have been illustrated by an example of an optical system of rotational symmetry in the paraxial region.

I. Introduction

In geometrical optics the properties of the imaging systems can be described by means of the so called eikonal function (named also a characteristic function, see for instance [1] and [2]). Depending on the type of coordinates used, which define the light ray positions in both the object and image space, we have to do with various eikonals The Bruns eikonals used in practice (point characteristic, angle characteristic and that of mixed type) may usually be transformed into each other by means of the Legendre transformations (see, for instance, [4]). The singularities occurring are examined separately in the particular cases.

In this paper some general conditions to be satisfied have been established to make the construction of a new eikonal function possible. The starting function in our considerations is the point characteristic.

II. Donkin theorem — Legendre transformations

The Donkin theorem plays an important part in solving the problem (see, for instance, [7] and [8]). It concerns the function of several variables containing, in general, also some parameters. We cite the theorem here in the form convenient for further considerations. In particular the function properties with respect to the parameters are of importance here.

Given is a function X of several variables

$$X = X(x_1, \ldots, x_i, \ldots, x_n, a_1, \ldots, a_k, \ldots, a_m) = X(x_i, a_k) \tag{1}$$

where a_k denotes the parameters. Let the Hess' determinant D of the function X be different from zero

$$D = \det\left(\frac{\partial^2 X}{\partial x_i \partial x_l}\right)^n_{i,l=1} \neq 0. \tag{2}$$

For the transformations determined by the formula

$$y_i = \frac{\partial X}{\partial x_i} \qquad i = 1, 2, \ldots n \tag{3}$$

there exists a reversed transformation

$$x_i = \frac{\partial Y}{\partial y_i} \tag{4}$$

defined by the function

$$Y = Y(y_1, \ldots, y_i, \ldots, y_n, a_1, \ldots, a_k, \ldots, a_m)$$

$$= Y(y_i, a_k) = \sum_{i=1}^{n} x_i y_i - X. \tag{5}$$

With respect to the parameters a_k the functions $X(x_i, a_k)$ and $Y(y_i, a_k)$ meet the following conditions:

$$\frac{\partial Y}{\partial a_k} = -\frac{\partial X}{\partial a_k} \qquad k = 1, 2, \ldots m. \tag{6}$$

Any transformation of type (3) and (4) of the variables x_i into y_i along with the change in the transformed function due to (5) is called the Legendre transformation (see, for example, [7], [8] and [9]). In theoretical physics the Legendre transformations are in commen use.

III. Point eikonal

Let us consider an optical system of rotational symmetry. In both the object and the image spaces the reference systems are the rectangular Cartesian coordinate systems with the X and X' axes directed along the optical axis of the system, respectively. The coordinates of the ray origin are: $x = 0, y, z$;

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while those of its end: $x' = 0, y', z'$ (Fig. 1). The planes YZ and Y'Z' are not optically conjugate with each other.

The optical path L of the ray as a function of the coordinates y, z, y', z' is called a point characteristic (see, for instance, [2] and [3])

$$L(y, z, y', z') = E_1. \tag{7}$$

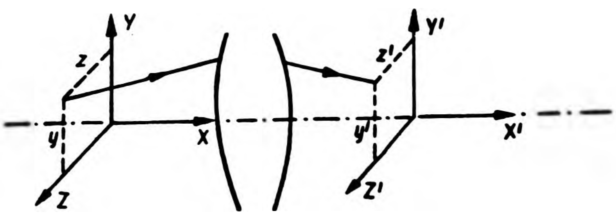

Fig. 1 Tracing a ray through an optical system

It may be shown that the function E_1 satisfies the following conditions:

$$\begin{aligned} n\mu &= -\frac{\partial E_1}{\partial y}, & n'\mu' &= \frac{\partial E_1}{\partial y'}, \\ n\nu &= -\frac{\partial E_1}{\partial z}, & n'\nu' &= \frac{\partial E_1}{\partial z'}, \end{aligned} \tag{8}$$

where n and n' denote the indices of refraction of the media in the object and image space, respectively, while (μ, ν) and (μ', ν') are the directional cosines of the ray in the same spaces.

In this way the partial derivatives of the point characteristic define the direction of the light ray in both spaces of the optical system.

Also it is worth noticing that

$$dE_1 = -n(\mu dy + \nu dz) + n'(\mu' dy' + \nu' dz'). \tag{9}$$

IV. Bruns eikonals

The point characteristic due to Bruns is for us a starting function for constructing other eikonals conjugate with the first one. According to the type of the independent variables used we have different examination possibilities of the imaging procedure.

1. Angle characteristic. As independent variables the directional cosines of the ray are accepted, i.e. μ, ν in the object space and μ', ν' in the image space. In agreement with the formula (5) we have

$$E_2(\mu, \nu, \mu', \nu') = -n(\mu y + \nu z) + n'(\mu' y' + \nu' z') - E_1 \tag{10}$$

Consequently

$$dE_2 = -n(y d\mu + z d\nu) + n'(y' d\mu' + z' d\nu') \tag{11}$$

and

$$\begin{aligned} \frac{\partial E_2}{\partial \mu} &= -ny, & \frac{\partial E_2}{\partial \mu'} &= n'y' \\ \frac{\partial E_2}{\partial \nu} &= -nz, & \frac{\partial E_2}{\partial \nu'} &= n'z' \end{aligned} \tag{12}$$

The construction of angle characteristic $E_2(\mu, \nu, \mu', \nu')$ is possible, however, only when in accordance with the Donkin theorem the Hess' determinant o, the point characteristic is different from zero. Thusf the following condition has to be fulfilled

$$\det\left(\frac{\partial^2 E_1}{\partial \xi_k \partial \xi_l}\right)_{k,l=1}^{4} \neq 0 \tag{13}$$

where

$$\xi_1 = y, \ \xi_2 = z, \ \xi_3 = y', \ \xi_4 = z'$$

2. Mixed eikonal E_3. The independent variables are the variables y, z of the ray origin and the directional cosines μ', ν' in the image space. In the Legendre transformation, when applied to the function E_1, the magnitudes y and z are to be considered as parameters. Then, in accordance with formula (5) we get

$$E_3(y, z, \mu', \nu') = n'(\mu' y' + \nu' z') - E_1. \tag{14}$$

For the differential dE_3 we have

$$dE_3 = n(\mu dy + \nu dz) + n'(y' d\mu' + z' d\nu'). \tag{15}$$

The partial derivatives of the eikonal E_3 determine the values μ, ν and y', z', respectively:

$$\begin{aligned} \frac{\partial E_3}{\partial y} &= n\mu, & \frac{\partial E_3}{\partial \mu'} &= n'y', \\ \frac{\partial E_3}{\partial z} &= n\nu, & \frac{\partial E_3}{\partial \nu'} &= n'z'. \end{aligned} \tag{16}$$

The condition for existance of the function E_3 is, in this case, the following:

$$\det\left(\frac{\partial^2 E_1}{\partial \xi_k \partial \xi_1}\right)^2 \tag{17}$$

$$k, l = 1 \neq 0$$

where $\quad \xi_1 = y', \quad \xi_2 = z'$.

3. Mixed eikonal E_4. Here, the directional cosines in the object space μ, ν and the coordinates of the ray end y', z' are accepted as independent variables. Thus, the parameters of the function E_1 in the Legendre transformation are the magnitudes y' and z'. On the base of formula (5) we get for the function E_4 the following relation:

$$E_4 = -n(\mu y + \nu z) - E_1. \tag{18}$$

The remaining properties of the eikonal E_4 may be expressed with the help of the formulae

$$dE_4 = -n(yd\mu+zdv)-n'(\mu' dy'+v' dz'), \quad (19)$$

$$\frac{\partial E_4}{\partial \mu} = -ny, \qquad \frac{\partial E_4}{\partial y'} = -n'\mu',$$
$$\frac{\partial E_4}{\partial v} = -nz, \qquad \frac{\partial E_4}{\partial z'} = -n'v'. \quad (20)$$

In accordance to (2) the existance condition for the eikonal E_4 is the following

$$\det\left(\frac{\partial^2 E_1}{\partial \xi_k \partial \xi_l}\right)^2_{k,l=1} \neq 0 \quad (21)$$

where $\xi_1 = y, \quad \xi_2 = z.$

V. Gaussian optics

The above general considerations will be illustrated by Gaussian optics as an example. In accordance with the assumption the coordinates y, z and y', z' are small and the rays are in the paraxial region. Then, the point characteristic may be expended into a series with respect to the rotation invariants

$$A = y^2+z^2, \; B = 2(yy'+zz'), \; C = y'^2+z'^2. \quad (22)$$

For the sake of simplicity we restrict ourselves to the first terms (see [2]) and write

$$E_1 = E_1^0+E_1^1 = E_1^0+\frac{1}{2}a_1A+\frac{1}{2}a_2B+\frac{1}{2}a_3C \quad (23)$$

where a_1, a_2 and a_3 are the constants characteristic of an optical system.

Imaging is stigmatic in this case (a perfect system).

The focal lengths may then be expressed in the form

$$f = \frac{na_2}{a_2^2-a_1a_3}, \qquad f' = \frac{n'a_2}{a_2^2-a_1a_3}. \quad (24)$$

The positions of the foci determine the formulae

$$x_F = x-\frac{a_3}{a_2}f, \qquad x'_{F'} = \bar{x}'-\frac{a_1}{a_2}f' \quad (25)$$

where $x' = \bar{x}'-\frac{n'}{a_3}$ is an image coordinate of a point object located at x.

The existance condition for an angle characteristic (13) takes the following form

$$\det\left(\frac{\partial^2 E_1}{\partial \xi_k \partial \xi_l}\right)^4_{k,l=1} = \begin{vmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_2 & 0 & a_3 & 0 \\ 0 & a_2 & 0 & a_3 \end{vmatrix} = a_1a_3(a_1a_3-a_2^2)^2 \neq 0 \quad (26)$$

If $a_1a_3-a_2^2 = 0$, the angle characteristic can not be used the optical system being afocal (telescopic). The given values of μ and v are then associated with the determined values of μ' and v', the same for all the rays of a parallel bundle. Also, for $a_3 = 0$, $a_1 \neq 0$, and $a_2 \neq 0$ (or for $a_3 \neq 0$, $a_1 = 0$ and $a_2 \neq 0$) the condition for angle characteristic existence is not fulfilled. Then $x = x_F$ (or $x' = x'_{F'}$). The original point (or end point) of the ray lies in the focal plane and all the rays starting from that point have the same direction in the image space (or the object space).

For a mixed eikonal F_3 the condition (17) has the form

$$\begin{vmatrix} a_3 & 0 \\ 0 & a_3 \end{vmatrix} = a_3^2 \neq 0. \quad (27)$$

If $a_3 = 0$, then $x = x_F$. All the rays starting from the object point are parallel in the image space.

For the mixed eikonal F_4 the condition (21) takes the form

$$\begin{vmatrix} a_1 & 0 \\ 0 & a_1 \end{vmatrix} = a_1^2 \neq 0. \quad (28)$$

If $a_1 = 0$, then $x' = x'_{F'}$. All the rays parallel to each other in the object space hit the image point.

Resumé

Dans ce travail on a établi les conditions générales qui sont à satisfaire pour qu'on puisse construire les fonctions iconales de Bruns à l'aide de la transformation de Legendre. Les résultats obtenus sont appliqués pour étudier le cas d'un systéme centré dans le domaine paraxial.

Условия преобразования эйконалов Брунса

В этой работе устанавливаются общие условия для образования эйконалов Брунса при помощи преобразования Лежандра. Полученные результаты используются для исследования центрированной оптической системы в области параксиальных лучей.

References

[1] Born M., Wolf E.; *Principles of Optics*, ed. 2, Ed. Pergamon Press, Oxford-London-Edinburgh-NewYork-Paris-Frankfurt, 1964.

[2] Tudorowski A. I.; *Tieoria opticzeskich priborow*, T. 1, Izd. Akad. Nauk ZSRR, Moskwa-Leningrad, 1948.

[3] Chrétien H.; *Cours du calcul des combinaisons optiques*, Ed. Revue d'Optique, Paris, 1934.

[4] Landau L., Lifchitz E.; *Théorie du champ*, éd. 2, Éd. Mir, Moscou, 1966.

[5] Maréchal A.; *Traité d'optique instrumentale* T. 1, *Imagérie geometrique-aberrations*, Éd. Revue d'Optique, Paris, 1952.

[6] Kurant R., Hilbert D.; *Metody matematiczeskoj fiziki*, T. 2, izd. 2, Tzd. Gos. Tiechn., Moskwa-Leningrad, 1951.

[7] Gantmacher F.; *Lekcji po analiticzeskoj mechanikie*, izd. 2, Izd. Nauka-Gławnaja Red. Fiziko-mat. Lit., Moskwa, 1966.

[8] Wojewoda H.; Prace Naukowe Instytutu Fizyki Technicznej Politechniki Wrocławskiej, Studia i Materiały Nr 3, Wrocław 1971.

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