*Toshimitsu Asakura and Hiromichi Mishina**

Partially Coherent Two-Point Resolution by Annular Aperture

Two-point resolution with partially coherent light is investigated in the annular aperture imaging system by using the Sparrow resolution criterion. The resolution is discussed as a function of two parameters, the spatial coherence condition of illumination and the central obstruction of an annular aperture. The two ratios of the measurable to the real quantities of the point separation and peak intensity are also studied as functions of the above two parameters.

1. Introduction

In recent years, many investigators have studied image formation by partially coherent light. When a performance property of optical imaging systems is evaluated under partially coherent illumination, either the two-point resolution criterion or the optical transfer function characteristic is generally employed. Though it is nowadays well known that an optical transfer function is superior as a performance criterion to two-point resolution, its use is usually limited to incoherent imaging systems because imaging systems become nonlinear both in amplitude and in intensity under the partially coherent illumination. In this respect, the criterion of two-point resolution in partially coherent imaging has been recently investigated [1–11], because it is very easily treated without any modifications under partially coherent illumination.

Most papers have treated partially coherent two-point resolution in imaging systems with slit and circular apertures. From the viewpoint of the two-point resolution, an imaging system with an annular aperture has received much attention, since it is known that the introduction of a central opaque obstruction into the circular aperture decreases the diameter of the diffraction pattern, thus increasing the two-point resolution in a Rayleigh sense. We can refer to the papers [12–13] for a comprehensive treatment of the annular aperture. The annular aperture imaging system has two defects: there always appears a considerable loss of intensity in the diffraction pattern due to the central obstruction within the circular aperture, and some deterioration in image quality occurs due to the increased intensity of the secondary and higher maxima in the diffraction pattern with the increase of the central obstruction. Despite these defects present in the annular aperture imaging system, it is worth to investigate it in respect of two-point resolution under the partially coherent illumination. Asakura [16] first studied two-point resolution of the annular aperture imaging system for two limiting cases of completely coherent and incoherent illumination. Very recently, Gupta *at al.* [2] in their short paper have initiated investigation of the subject under partially coherent illumination. The present paper will investigate in some detail partially coherent two-point resolution in the annular aperture imaging system by using the Sparrow resolution criterion [16–19].

2. Image of Two-Points by Partially Coherent Light

The general formula for a spatially stationary imaging system under partially coherent illumination is given by [20]

$$I(x) = \iint \Gamma(\xi_1,\xi_2) O(\xi_1) O^*(\xi_2) \times \\ \times K(x-\xi_1) K^*(x-\xi_2) d\xi_1 d\xi_2,$$

where ξ and x are the coordinates in object and image spaces, respectively, $\Gamma(\xi_1,\xi_2)$ is the mutual coherence function of light illuminating the object,

* Research Institute of Applied Electricity, Hokkaido University Sapporo, Hokkaido, Japan.

$O(\xi)$ is the object transmittance, and $K(x-\xi)$ is the amplitude impulse response of the imaging system. The considered object transmittance $O(\xi)$ in the two-point resolution consists of two equally bright points and can be simply written in a form

$$O(\xi) = I_0^{1/2}\{\delta(\xi-b)+\delta(\xi+b)\}, \tag{2}$$

where it is assumed without loss of generality that two points having a separation $2b$ are situated at equal distances $\pm b$ from the optical axis, they have equal intensities I_0, and are co-phasal. By substituting eq. (2) for eq. (1) and using the complex degree of coherence $\gamma(\xi_1,\xi_2) = \Gamma(\xi_1,\xi_2)/I_0$ for $\Gamma(\xi_1,\xi_2)$, and the amplitude impulse response of $K(x/q-\xi/p)$ giving an erect image (where p and q are the object and image distances from the system) we get

$$I(x) = I_0^2 \int\int \gamma(\xi_1,\xi_2)\{\delta(\xi_1-b)+\delta(\xi_1+b)\}$$
$$\{\delta(\xi_2-b)+\delta(\xi_2+b)\}\times$$
$$\times K\left(\frac{x}{q}-\frac{\xi_1}{p}\right)K^*\left(\frac{x}{q}-\frac{\xi_2}{p}\right)d\xi_1 d\xi_2. \tag{3}$$

In this case, the Gaussian images of two object points appear at the points $\pm qb/p$ from the axis. With the replacement of $b' = qb/p$, Eq. (3) is reduced to

$$I(x) = I_0^2[|K(x-b')|^2+|K(x+b')|^2+$$
$$+\mathrm{Re}\{2\gamma(b',-b')K(x-b')K^*(x+b')\}], \tag{4}$$

where Re denotes the real part, and $|K(x)|^2$ indicates the intensity diffraction pattern, due to a single object point, which is equivalent to the intensity impulse response of the imaging system.

Since the problem to be treated here is limited only to the two-dimensional annular aperture system without aberrations, the amplitude impulse response is given by [12]

$$K(x) = \frac{2J_1\left(\frac{kax}{q}\right)}{\left(\frac{kax}{q}\right)} - \varepsilon^2 \frac{2J_1\left(\frac{ka\varepsilon x}{q}\right)}{\left(\frac{ka\varepsilon x}{q}\right)}, \tag{5}$$

where $2a$ indicates the diameter of a clear circular aperture and ε is the ratio of that diameter to the diameter of a central, circular obstruction ($0 \leqslant \varepsilon \leqslant 1$). The parameter ε specifies the size of the central obstruction when the outer circular aperture is fixed to have the diameter $2a$. GUPTA *at al.* [2] have used the normalized amplitude impulse response for Eq. (5) in such a way that at the center $x = 0$ for any values of ε $K(x)$ is always 1. Therefore, their study could not show the actual intensity in the diffraction pattern for a variation of the central obstruction ε, even though it reveals the effect of partially coherent light on the two-point resolution. However, the loss of light due to the central obstruction in the aperture is very important in the imaging system, since the detectability by various detectors is very often influenced by the level of the absolute intensity reaching the detector. Consequently, the intensity variation in the partially coherent two-point image should be investigated as a function of the central obstruction. That is, why Eq. (5) is used in this paper for the amplitude impulse response of the annular aperture system. By substituting Eq. (5) for Eq. (4) and putting $X = \frac{kax}{q}$ and $B - \frac{kab'}{q}$, Eq. (4) finally becomes

$$I(x) = \left\{\frac{2J_1(x-B)}{(X-B)} - \varepsilon^2 \frac{2J_1(\varepsilon X-\varepsilon B)}{(\varepsilon X+\varepsilon B)}\right\}^2 +$$
$$+\left\{\frac{2J_1(X+B)}{(X+B)} - \varepsilon^2 \frac{2J_1(\varepsilon X+\varepsilon B)}{(\varepsilon X+\varepsilon B)}\right\}^2 +$$
$$+2\gamma(b',-b')\left\{\frac{2J_1(X-B)}{(X-B)} - \varepsilon^2 \frac{2J_1(\varepsilon X-\varepsilon B}{\varepsilon X-\varepsilon B}\right\}\times$$
$$\times\left\{\frac{2J_1(X+B)}{(X+B)} - \varepsilon^2 \frac{2J_1(\varepsilon X+\varepsilon B)}{(\varepsilon X+\varepsilon B)}\right\}, \tag{6}$$

where a trivial constant I_0^2 is omitted.

Equation (6) gives the intensity distribution of the two-point object illuminated by partially coherent light. It is to be noted that the intensity distribution $I(x)$ in Eq. (6) does not take a normalized form although each term in the brackets is normalized to 1. In the limiting in-phase coherent ($\gamma = 1$) and incoherent ($\gamma = 0$) cases of illumination, Eq. (6) becomes

$$I_{co}(X) = \left|\frac{2J_1(X-B)}{(X-B)} + \frac{2J_1(X+B)}{(X+B)} - \right.$$
$$\left. - \varepsilon^2 \frac{2J\varepsilon_1(X-\varepsilon B)}{(\varepsilon X-\varepsilon B)} - \varepsilon^2 \frac{2J_1(\varepsilon X+\varepsilon B)}{(\varepsilon X+\varepsilon B)}\right|^2, \tag{7a}$$

$$I_{in}(X) = \left\{\frac{2J_1(X-B)}{(X-B)} - \varepsilon^2 \frac{2J_1(\varepsilon X-\varepsilon B)}{(\varepsilon X-\varepsilon B)}\right\}^2 +$$
$$+\left\{\frac{2J_1(X+B)}{(X+B)} - \varepsilon^2 \frac{2J_1(\varepsilon X+\varepsilon B)}{(\varepsilon X+\varepsilon B)}\right\}^2. \tag{7b}$$

The critical Sparrow resolution for these two limiting cases was already studied in detail by Asakura [16] as a function of the central obstruction ε.

3. Results and Discussion

3.1 Intensity Distribution

First of all, the intensity distribution of the two-point image has been evaluated by using Eq. (6) with fixed diameter $2a$ of the annular aperture. In an evaluation of Eq. (6), three parameters of γ (partially coherent condition of illumination), ε (size of a central obstruction) and $2B$ (two-point separation) can be varied. For various fixed values of the point separation $2B$, the effect of varying γ on the two-point image can be studied. This effect can also be studied for a variety of values of the central obstruction γ. Figure 1 shows the intensity distribution of the two-point image as a function of the partially coherent condition of illumination for three different values $\varepsilon = 0.2$, 0.4, 0,6, and a fixed two-point separation $2B = 3.6$. The chosen value $2B = 3.6$ is a separation situated between the incoherent Sparrow limit ($2B = 2.976$) and the coherent Sparrow limit ($2B = 4.600$) for the clear circular aperture [1, 16, 18].

In Fig. 1 the position of two Gaussian image points is indicated by the dotted straight lines. (a), (b) and (c) of this figure clearly show that two points are well resolved in the incoherent limit $\varepsilon = 0$, while two coherent points of the same separation are not resolved (see the curve of $\varepsilon = 1$). With the increse of ε, the maxima of resultant intensity distributions move closer together and a central dip of intensity finally disappears, the two points consequently being not resolved. In other words, the resolution decreases with the increase of ε. However, the resolution increases with increasing ε (compare, for example, the curves of $\gamma = 0.2$ in (a), (b) and (c)). There obviously appears a gradual loss of light reaching the image plane with the increase of ε (compare (a), (b) and (c)). The interesting point to note in Fig. 1 is that two Gaussian image points indicated by the dotted lines do not always correspond to the two peaks of the resultant intensity distribution. This means that the measurable separation from the two peaks is not always the same with the real separation of two object points.

3.2 Critical Sparrow Resolution

The two-point resolution has been generally discussed in a Rayleigh sense. Though the Rayleigh criterion is surely useful, it is only a criterion without

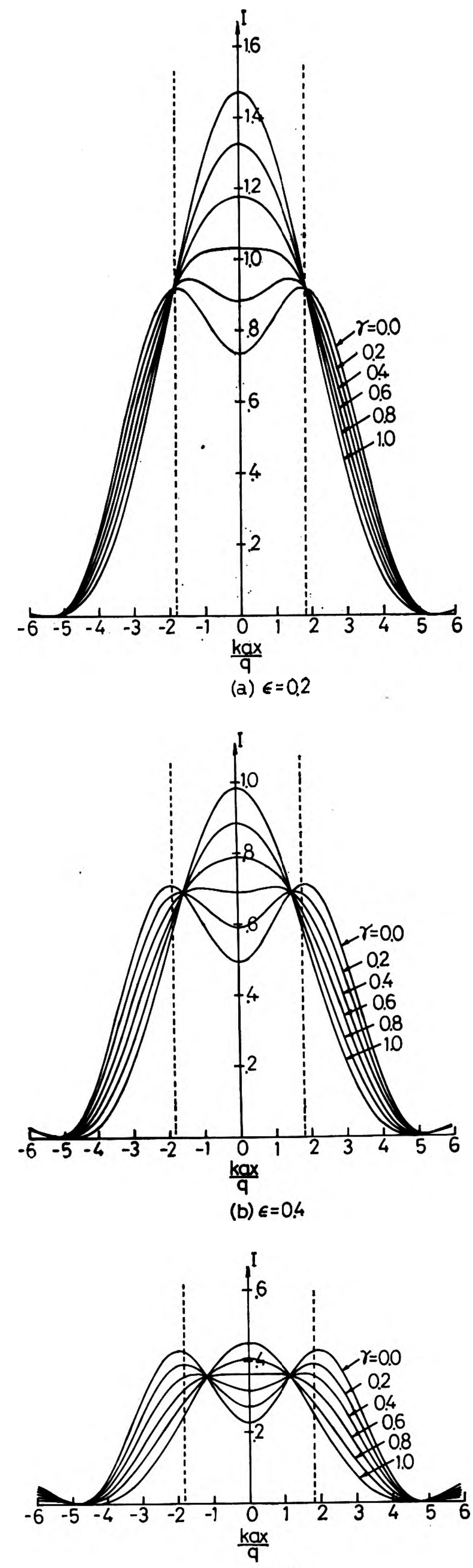


Fig. 1. Image intensity distribution for various values of γ in the three different annular aperture systems when the separation of two Gaussian image points is $2B = 3.6$: (a) $\varepsilon = 0.2$, (b) $\varepsilon = 0.4$, and (c) $\varepsilon = 0.6$

any theoretical and, especially, physical backgrounds so that it can not be considered as a basic resolution law [18]. Compared with the Rayleigh criterion, the Sparrow criterion is certainly based on the ultimate limit of two object points in the image plane and holds a theoretical background. By this reason, the Sparrow criterion has been used extensively in the study of partially coherent two-point resolution.

The Sparrow criterion states that two points are just resolved if the second derivative of the resultant image intensity distribution vanishes at the middle point between two Gaussian image points. This criterion is obtained by finding the point separation $2B = \delta$ which satisfies the equation

$$\left.\frac{\partial^2 I(X)}{\partial X^2}\right|_{x=0} = 0. \tag{8}$$

By inserting Eq. (6) into Eq. (8) and using two mathematical relations of Bessel functions, i. e.

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x),$$

$$J_{n-1}(x)+J_{n+1}(x) = \frac{2n}{x}J_n(x),$$

the condition of Eq. (8) becomes

$$(1-\gamma)\left\{\frac{J_2(B)}{B}-\varepsilon^3\frac{J_2(\varepsilon B)}{(\varepsilon B)}\right\}^2 -$$
$$-(1+\gamma)\left\{\frac{J_1(B)}{B}-\varepsilon^2\frac{J_1(\varepsilon B)}{(\varepsilon B)}\right\}\times\left[\frac{J_1(B)}{B}-3\frac{J_2(B)}{B^2}-\right.$$
$$\left.-\varepsilon^4\left\{\frac{J_1(\varepsilon B)}{(\varepsilon B)}-3\frac{J_2(\varepsilon B)}{(\varepsilon B)^2}\right\}\right] = 0. \tag{9}$$

The value B satisfying Eq. (9) determines the critical Sparrow resolution of two object points whose Gaussian image points have a separation $2B = \delta$.

From Eq. (9) the critical Sparrow resolution δ has been evaluated by an electronic computer using the iteration method as functions of γ and ε. The result is plotted in Fig. 2 as a function of ε for six values $\gamma = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$. The critical resolution is seen to increase gradually with the increase of the central obstruction for any conditions of partially coherent light. The critical separation also decreases nearly monotonically with increasing γ (this point can be understood from the fact that there is a nearly constant interval between curves). As a conclusion (Fig. 2), it is clearly noted that, similarly to the Rayleigh criterion, the partially coherent two-point resolution is increased in a Sparrow sense by using the annular aperture. Figure 3 shows the central

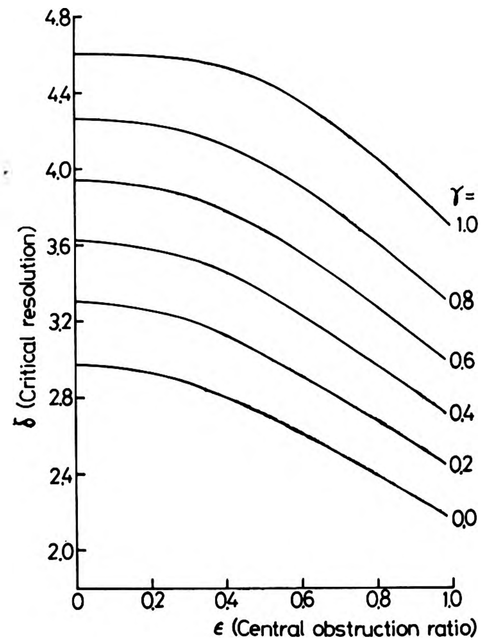


Fig. 2. Critical Sparrow resolution δ as a function of the central obstruction ratio ε for various values of γ

intensity of resultant images due to the two object points whose Gaussian image points of separation

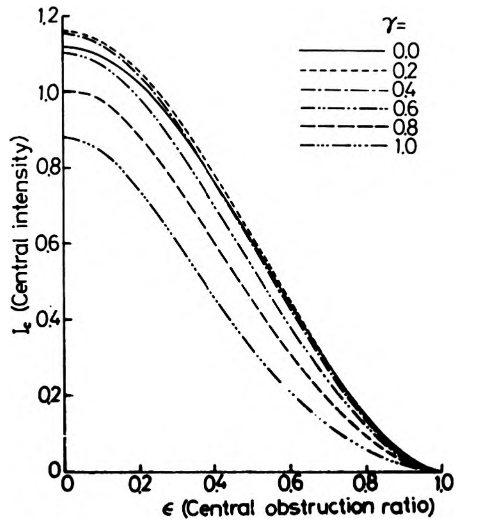


Fig. 3. Central intensity at the states of critical Sparrow resolution δ as a function of the central obstruction ratio ε for various values of γ

$2B$ satisfy the critical Sparrow resolution with the condition of Eq. (9). The central intensity is plotted as a function of ε for a variety of the partially coherent illumination γ. From Fig. 3 it is obvious that the central intensity gradually decreases with increasing ε and takes a low value for γ greater than 0.2. This means that, for any fixed values of the central obstruction, the central intensity of resultant images satisfying the Sparrow resolution always decreases with increasing γ (i. e. as the light approaches the coherent limit). The central intensity in incoherent light $\gamma = 0.0$ bekaves in a slightly different way for small values of ε when compared with the other partially and fully coherent cases, respectively. It becomes lower than the central intensity of partially coherent light characterized by $\gamma \leqslant 0.5$. Finally, Fig. 4 is plotted to show the actual image intensity distribution at the states of critical Sparrow resolution for various values of the central obstruction ε and two different coherence conditions $\gamma = 0.4$ and 0.8. The dotted lines in Fig. 4 indicate the two Gaussian image points which satisfy the critical Sparrow resolution. The results shown in Figs. 2 and 3 are explicitly verified by Fig. 4.

3.3 Measurable Point Separation and Peak Intensity

The only measurable quantities, which are also useful in the two-point resolution problem, are the separation of two peaks and the peak intensity in the resultant image intensity distribution. As it is evident from Fig. 1 and was already reported by Grimes and Thompson [1] for the clear circular aperture, the separation of two peaks, which is normally considered to be the real separation of two object points, does not always correspond to the real separation. Therefore, the ratio R_s of the measurable to the real point separation is studied in relation to the two parameters of γ and ε and plotted in Fig. 5 as a function of the real separation expressed in terms of the dimensionless parameter $2B$. For a perfect system without diffraction effects, R_s always becomes unity. The curves shown in Fig. 5(a) are given for various values of γ ranging from the incoherent to the coherent limit, while the central obstruction is held constant as $\varepsilon = 0.4$. This figure indicates that the measurable point separation from two peaks oscillates about the value $R_s = 1.0$ with about 16% overshoot in the worst case. This oscillation is generally reduced as γ decreases. However, for great values of $2B$, the measurable separation in the incoherent case $\gamma = 0.0$ exhibits an oscillation weeker than those for partially coherent cases (compare the curves for $\gamma = 0.0$ and 0.2). When compared with the case of the clear circular aperture [1], the oscillation in the measurable separation is further enhanced by using the annular aperture which blocks of the light contribution in the central part of the pupil in the imaging system.

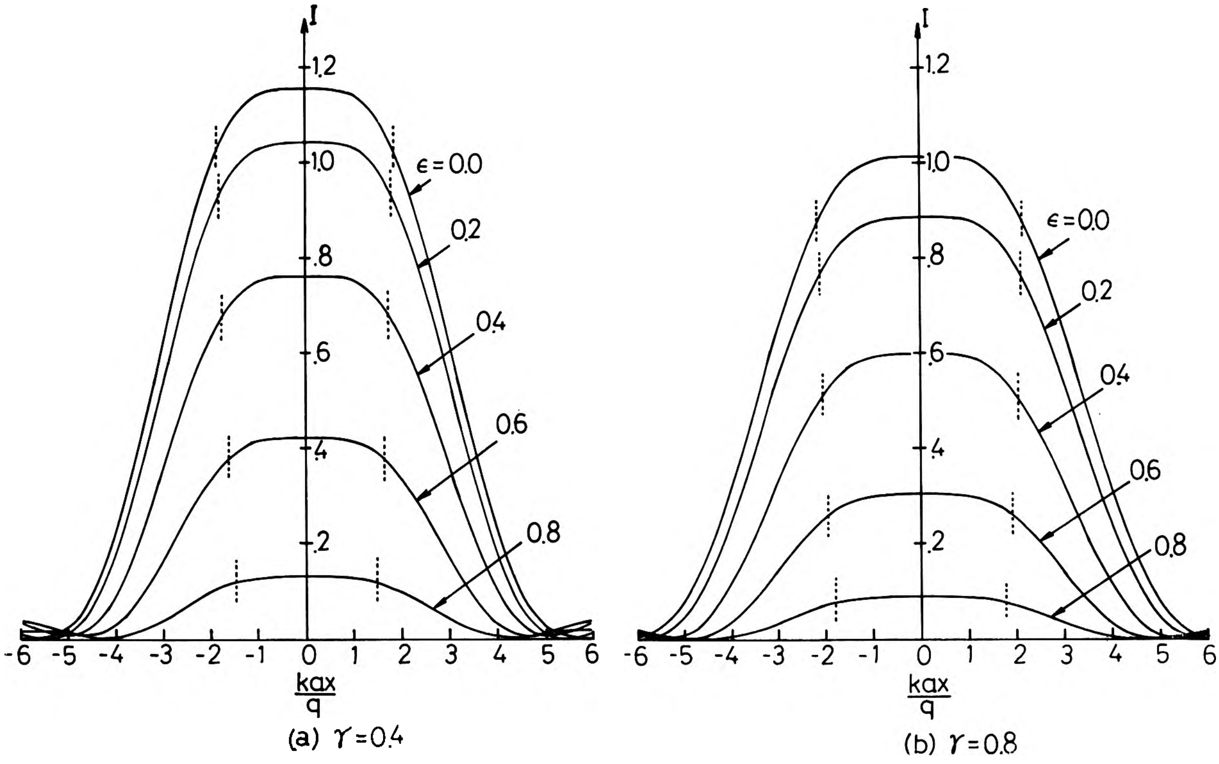
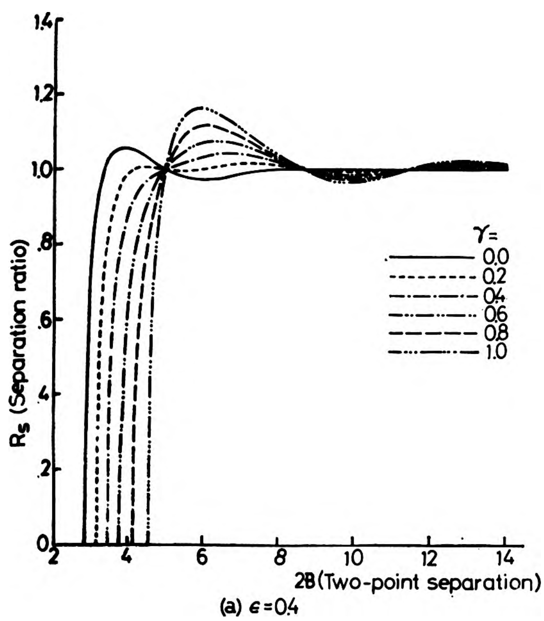
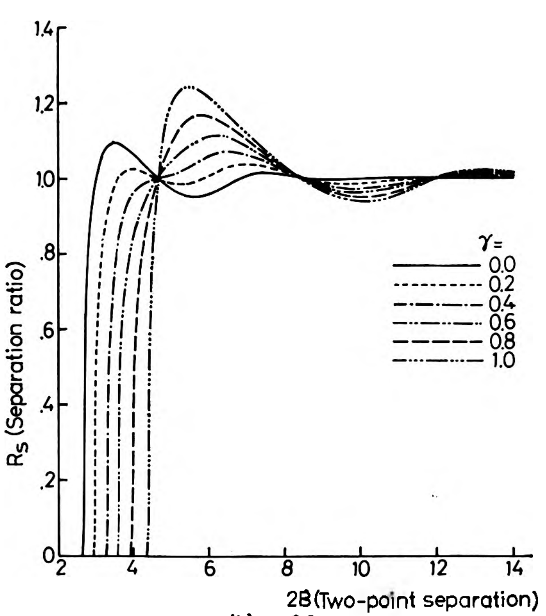


Fig. 4. Image intensity distribution for various values of ε at the states of critical Sparrow resolution δ under two different coherence conditions of illumination: (a) $\gamma = 0.4$ and (b) $\gamma = 0.8$

Figure 5(b) and (c) show the curves of R_s for various values of γ when the central obstruction is $\varepsilon = 0.6$ and 0.8, respectively. An oscillating behaviour similar to Fig. 5 (a) is also observed in these two figures. Comparison of Figs. 5 (a), (b) and (c) reveals that the oscillation in the measurable separation is enhanced with the increase of ε and that the curves move toward the left side with increasing ε. In other words, the increase of the central obstruction produces a larger difference between the measurable and real separations. In the worst case i.e $\varepsilon = 0.8$ (see the curve $\gamma = 1.0$ of Fig. 5 (c)), a nearly 26% deviation of the measurable separation from the real one is produced. However, the movement of the curves to the left side indicates the increase of the Sparrow resolution with the increase





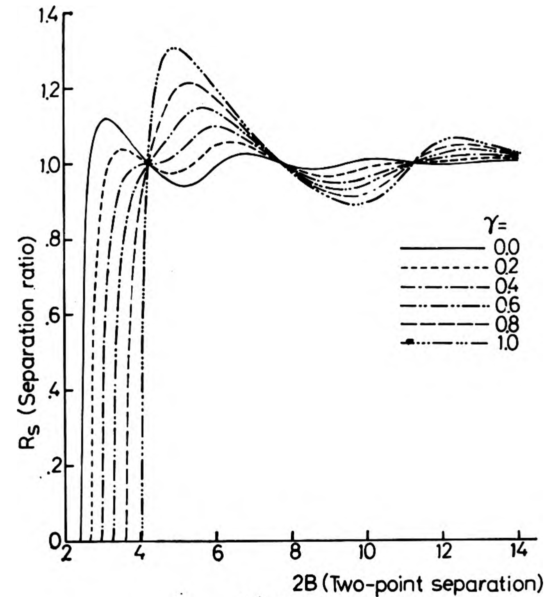


Fig. 5. The ratio R_s as a function of the actual separation $2B$ for various values of γ in the three different annular aperture systems: (a) $\varepsilon = 0.4$, (b) $\varepsilon = 0.6$, and (c) $\varepsilon = 0.8$

of ε. It is further noted from Fig. 5 (c) that R_s oscillates considerably even under completely incoherent illumination (see the curve of $\gamma = 0.0$).

The other measurable quantity, peak intensity, which is also normally considered to be proportional to the real brightness of the object point, does not always correspond to the real object intensity. This point has been recently discussed by Asakura [21] in the circular aperture imaging system. The study of the peak intensity is very important for detecting the real intensity of object points being situated very closely from each other. Hence, the ratio R_i of the measurable peak intensity to the real point object intensity is examined as a function of the real Gaussian image point separation $2B$ for two parameters of γ and ε. It is obvious that, as ε increases, i.e. the central obstruction becomes larger, the total intensity reaching the image plane decreases. Therefore, the absolute value of R_i decreases with the increase of ε. The curves of Fig. 6 (a) show a variation in the peak intensity as a function of the point separation $2B$ for a fixed value $\varepsilon = 0.4$. The end value of curves at the left side indicates the intensity ratio R_i corresponding to the two object points satisfying the critical Sparrow resolution. Below this critical separation, two Gaussian image points converge into a single peak, consequently, R_i is increasing monotonically with the decrease of $2B$. It is noted from Fig. 6 (a) that R_i oscillates about a certain constant value proportional to the real object intensity. This oscillation is enhanced with increasing γ. This means that the measurable peak intensity differs from the real object intensity as the illumination approaches the coherent limit.

Figures 6 (b) and (c) show also a variation of R_i as a function of the two-point separation $2B$ for various values of γ when the central obstruction is fixed as $\varepsilon = 0.6$ and 0.8. In these two figures an oscillating behaviour similar to that presented in Fig. 6 (a) is observed. These figures show, when compared with Fig. 6 (a), that the oscillating amplitude in the first loop region gradually diminishes with increasing ε while in the second loop region it raises with the increase of ε. It is interesting to note that the type of oscillation varies depending upon the value ε of the central obstruction (cf Fig. 6 (a,) (b) and (c)).

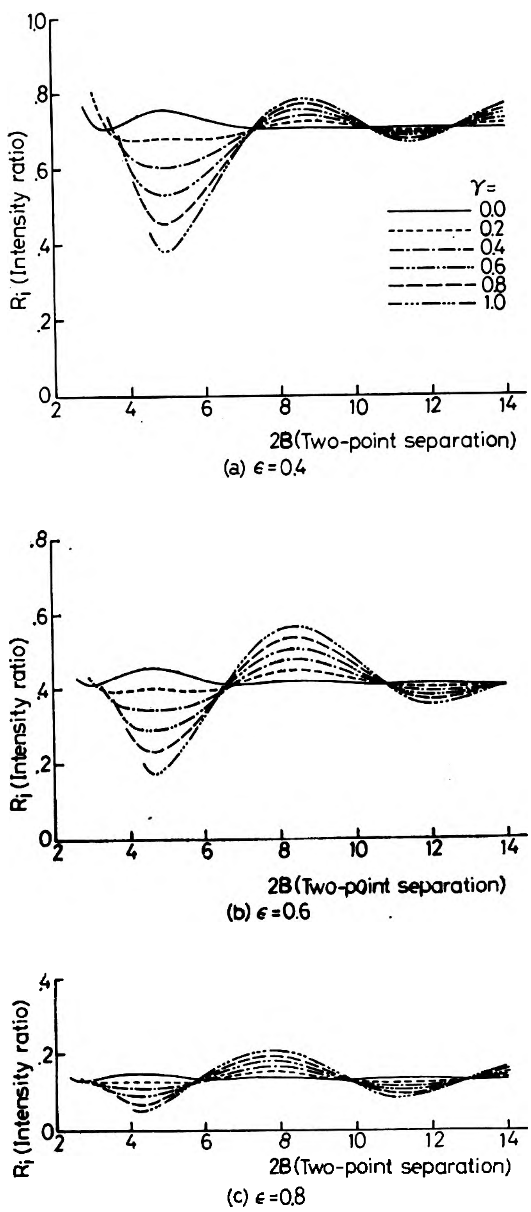


Fig. 6. The ratio R_i as a function of the actual separation $2B$ for various values of γ in the three different annular aperture systems: (a) $\varepsilon = 0.4$, (b) $\varepsilon = 0.6$, and (c) $\varepsilon = 0.8$

4. Conclusion

In this paper two-point resolution using the Sparrow criterion has been studied for the annular aperture under the partially coherent illumination. The resolution strongly depends on the coherence condition of illumination for any values of the central obstruction. The central obstruction affects the measurable quantities of the point separation and the peak intensity. The difference between these measurable quantities and the true values increases gradually as the illuminating light approaches the coherent limit and the central obstruction increases. This difference is usually extremely reduced in incoherent illumination.

The results obtained in this paper indicate the effects that can be produced in an imaging system when coherent or partially coherent light is used for illumination. Hence, the effect of the coherence condition of illumination and of the aperture variation in the imaging system can be well understood from the present two-point resolution study which treats imaging performance for the simple two--point object under various coherence conditions of illumination.

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The authors wish to thank Mr. T. Ueno for assistance in carrying out the computational work.

Pouvoir de résolution à deux points dans la lumière partieuement cohérente pour les systèmes à l'ouverture annulaire

En utilisant le critère de Sparrov on a examiné le pouvoir de résolution à deux points dans la lumière partiellement cohérente pour les systèmes à l'ouverture annulaire. A été examinée également la dépendence entre le pouvoir de résolution et les deux paramètres suivants la cohérence spatiale d'éclairage et le degré de diaphragmation centrale dans l'ouverture annulaire. On a considéré les rapports entre les grandeurs mesurées et les grandeurs réelles pour la distance d'entre-les-deux-points et pour les grandeurs extrêmes d'intensité, pris dans leur dépendance de paramètres mentionnés ci-dessus.

Двухточечная разрешающая способность в частично когерентном свете для систем с кольцевой апертурой

Исследовалась двухточечная разрешающая способность в частично когерентном свете для систем с кольцевой апертурой на основе критерия разрешающей способности Спаррова. Обсуждена зависимость разрешающей способности от двух параметров: пространственной когерентности освещения и степени центрального диафрагмирования в кольцевой апертуре. Исследовано соответствие измеренных и действительных величин как для расстояния между точками, так и для вершинных интенсивностей в зависимости от обоих параметров.

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Received, September 17, 1974