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Fourth Luneberg Apodization Problem in Partially Coherent Light

A fourth Luneberg apodization problem of increasing the Sparrow resolution limit under the partially coherent illumination is formulated for the slit aperture via the calculus of variations. The required pupil functions obtained by solving a homogeneous Fredholm integral equation have been evaluated under various spatial coherence conditions of illumination.

1. Introduction

Apodization is applied to determine the light distribution over the exit pupil of an optical system in order to obtain a desired distribution of light over a given plane in the image space. Until now many papers involved in both the theoretical analysis and experimental technique of apodization have appeared. A comprehensive review of apodization was given by JACQUINOT and ROIZEN-DOSSIER [1]. In the past the research work on apodization was conducted either under the completely coherent or incoherent illumination. No paper however, treated the apodization problem in partially coherent light. The fourth Luneberg apodization problem [3] under the partially coherent illumination has been for the first time studied by ASAKURA and UENO in [2]. This problem was studied by BARAKAT [4] under the two extreme conditions of completely coherent and incoherent illumination. Since in [2] the problem was restricted to the circular aperture only, it might be interesting to extend the study to the case of the slit aperture of an optical system, which is the subject of the present paper. Of course, the general formulation for the slit aperture is formally the same as that for the circular aperture, but both cases differ in detailed development of analysis. For this reason, this paper presents a basic solution for the fourth Luneberg apodization problem in the case of the slit aperture illuminated by partially coherent light.

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2. Formulation of the problem

In previous paper [2] a theoretical study of apodization has been performed in the case of a circular aperture of the optical system, in order to increase two-point in the contex of Sparrow resolution cryterion, provided that the point objects are illuminated by partially coherent light. The paper [2], being frequently referred to in the present paper, will be denoted by I for abbreviation. The subject of the present paper is to study the above problem in the case of a slit aperture of the optical system. The optical system under discussion is assumed to be free from aberrations. The complex amplitude distribution in the receiving plane, due to a point object located on the optical axis of the system, is given by

$$D(v) = D(v,\delta_0) = \int_{-1}^{1} T(\delta_0, x)\exp(ivx)\,dx, \tag{1}$$

where $T(\delta_0, x)$ is the apodized pupil function which satisfies

$$T(\delta_0, x) \leqslant T_0. \tag{2}$$

If this inequality is true the optical system is a passive one and δ_0 is a parameter which will be discussed shortly. In Eq. (2), T_0 is the non--apodized, uniform pupil function with a constant value. In view of the Parseval's theorem for the Fourier transform relationship of Eq. (1), the total light flux passing through the pupil may be described by the relation:

$$\int_{-\infty}^{\infty} |D(v,\ \delta_0)|^2 dv = \int_{-1}^{1} |T(\delta_0, x)|^2 dx, \quad (3)$$

where the constant $1/2\pi$ at the right hand side of this equation is omitted for the sake of simplicity. If the total light flux passing through the non-apodized aperture is set be equal to one, the non-apodized pupil function T_0 must be

$$T_0 = \frac{1}{\sqrt{2}}, \quad (4)$$

which corresponds to a maximum constant value of Eq. (2). Then, the total light flux passing through the apodized aperture is given in a normalized form by

$$q = \frac{\int_{-1}^{1} |T(\delta_0, x)|^2 dx}{\int_{-1}^{1} |T_0|^2 dx} = \int_{-1}^{1} |T(\delta_0, x)|^2 dx, \quad (5)$$

where q ranges within the interval $0 < q \leq 1$.

In accordance with the statement of I, the partially coherent Sparrow resolution criterion is given, using Eq. (1) at the Sparrow resolution limit

$$\left[D\left(\frac{\delta_0}{2}\right) = D\left(\frac{\delta_0}{2},\ \delta_0\right)\right],$$

by

$$\left\{\frac{\partial^2 D\left(\frac{\delta_0}{2}\right)}{\partial v^2}\right\} D\left(\frac{\delta_0}{2}\right) + a\left\{\frac{\partial D\left(\frac{\delta_0}{2}\right)}{\partial v}\right\}^2 = 0, \quad (6)$$

where δ_0 is the Sparrow limit of resolution corresponding to the resolvable limiting separation of two point objects situated at the same distance from the optical axis, and a is

$$a = \frac{1-\gamma}{1+\gamma}. \quad (7)$$

In Eq. (7), γ is the complex degree of coherence characterizing the coherence condition of illumination, and having a value in 0.1 interval $(0 \leq \gamma \leq 1)$ in which $\gamma = 1$ and 0 correspond to the completely coherent and incoherent illuminations, respectively. If $T(\delta_0, x)$ is a real, even function, then Eq. (6) becomes in terms of Eq. (1)

$$\int_{-1}^{1} T(\delta_0, x)\cos\left(\frac{\delta_0 x}{2}\right) \times dx \int_{-1}^{1} x^2 T(\delta_0, x)\cos\left(\frac{\delta_0 x}{2}\right)dx - -a\left|\int_{-1}^{1} xT(\delta_0, x)\sin\left(\frac{\delta_0 x}{2}\right)dx\right|^2 = 0 \quad (8a)$$

or

$$\int_{-1}^{1}\int_{-1}^{1} T(\delta_0, s)T(\delta_0, t)H(\delta_0, a; s, t)ds\,dt, \quad (8b)$$

where

$$H(\delta_0, a; s, t) = -t^2\cos\left(\frac{\delta_0 s}{2}\right)\cos\left(\frac{\delta_0 t}{2}\right) + +ast\sin\left(\frac{\delta_0 s}{2}\right)\sin\left(\frac{\delta_0 t}{2}\right). \quad (9)$$

We are now in a position to formulate the apodization problem for the slit aperture. The problem is to determine a pupil function $T(\delta_0, x)$ which satisfies the two conditions of Eqs. (5) and (8), such that the central intensity of the diffraction image due to a single point object

$$|D(0,\ \delta_0)|^2 = \left|\int_{-1}^{1} T(\delta_0, x)dx\right|^2 \quad (10)$$

be a maximum (this equation is obtained by putting $v = 0$ in Eq. (1) and then squaring the resultant equation). Condition (8) is the partially coherent Sparrow resolution criterion. Condition (5) states that the total light flux passing through the apodized aperture takes a certain value less than the one for the non-apodized aperture. This later condition comes from a passivity of the optical system denoted by Eq. (2). As in the case of a circular aperture the calculus of variations is used to obtain an integral equation for the desired pupil function in a sense of the Sparrow resolution criterion. By using Eqs. (5), (8) and (10), the variation problem is

$$V = \left\{\int_{-1}^{1} T(\delta_0, x)dx\right\}^2 + +\mu\left\{\int_{-1}^{1} |T(\delta_0, x)|^2 dx - q\right\} + +\lambda\int_{-1}^{1} T(\delta_0, s)T(\delta_0, t)H(\delta_0, a; s, t)ds\,dt, \quad (11)$$

where μ and λ are the unknown Lagrange multipliers which are determined from the constraint equations. Assume a solution of the form

$$T(\delta_0, x) + \varepsilon B(x), \quad (12)$$

where ε is a small parameter which is ultimately to be made zero, and $B(x)$ is an arbitrary function, with continuous first and second derivatives, which vanishes at the end points $(-1, 1)$. Substituting Eq. (12) into Eq. (11) and solving the following equation

$$\left.\frac{\partial V}{\partial \varepsilon}\right|_{\varepsilon=0} = 0, \tag{13}$$

we have

$$\int_{-1}^{1} B(x)dx \Big\{2\int_{-1}^{1} T(\delta_0, x)dx + 2\mu T(\delta_0, x) + \lambda\int_{-1}^{1} T(\delta_0, s)H(\delta_0, a; s, x)ds + \lambda\int_{-1}^{1} T(\delta_0, t)H(\delta_0, a; x, t)dt\Big\} = 0. \tag{14}$$

A necessary and sufficient condition for the above integral to vanish is that the bracketed terms become zero. Rewriting the bracketed terms, we have

$$2\mu T(\delta_0, x) + \int_{-1}^{1} T(\delta_0, s)\Big[2 + \lambda\big\{H(\delta_0, a; s, x) + H(\delta_0, a; x, s)\big\}\Big]ds = 0, \tag{15}$$

where

$$H(\delta_0, a; s, x) = -x^2\cos\left(\frac{\delta_0 s}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right) + axs\sin\left(\frac{\delta_0 x}{2}\right)\sin\left(\frac{\delta_0 s}{2}\right),$$

$$H(\delta_0, a; x, s) = -s^2\cos\left(\frac{\delta_0 s}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right) + axs\sin\left(\frac{\delta_0 x}{2}\right)\sin\left(\frac{\delta_0 s}{2}\right). \tag{16}$$

Equation (15) is a secondary homogeneous Fredholm equation which must be solved to obtain the desired pupil function to increase the Sparrow resolution criterion under the partially coherent illumination.

3. Solution of integral equation

We are now going to solve the homogeneous Fredholm equation (15). Rewriting Eq. (15), we have

$$T(\delta_0, x) = -\frac{1}{2\mu}\int_{-1}^{1} T(\delta_0, s)K(\delta_0, a; s, x)ds \tag{17}$$

with a kernel given by

$$K(\delta_0, a; s, x) = 2 - \lambda s^2\cos\left(\frac{\delta_0 s}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right) + 2\lambda axs\sin\left(\frac{\delta_0 s}{2}\right)\sin\left(\frac{\delta_0 x}{2}\right) - \lambda x^2\cos\left(\frac{\delta_0 s}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right). \tag{18}$$

Since the kernel is separable by two variables x and s, as is seen from Eq. (18), it is put in the following form

$$K(\delta_0, a; s, x) = \sum_{i=1}^{4} a_i(x)b_i(s), \tag{19}$$

where

$$a_1(x) = 1,$$

$$b_1(S) = 2 = 2a_1(s),$$

$$a_2(x) = \cos\left(\frac{\delta_0 x}{2}\right),$$

$$b_2(s) = -\lambda s^2\cos\left(\frac{\delta_0 s}{2}\right) = \lambda a_4(s),$$

$$a_3(x) = -x\sin\left(\frac{\delta_0 x}{2}\right),$$

$$b_3(s) = -2\lambda as\sin\left(\frac{\delta_0 s}{2}\right) = 2\lambda a a_3(s), \tag{20}$$

$$a_4(x) = -x^2\sin\left(\frac{\delta_0 x}{2}\right),$$

$$b_4(s) = \lambda\cos\left(\frac{\delta_0 s}{2}\right) = \lambda a_2(s).$$

Substituting Eq. (19) into Eq. (17), we have

$$T(\delta_0, x) = \sum_{i=1}^{4} c_i a_i(x), \tag{21}$$

where

$$c_1 = -\frac{1}{2\mu}\int_{-1}^{1} T(\delta_0, s) b_1(s) ds$$
$$= -\frac{1}{\mu}\int_{-1}^{1} T(\delta_0, s) a_1(s) ds,$$
$$c_2 = -\frac{1}{2\mu}\int_{-1}^{1} T(\delta_0, s) b_2(s) ds$$
$$= -\frac{\lambda}{2\mu}\int_{-1}^{1} T(\delta_0, s) a_4(s) ds, \tag{22}$$
$$c_3 = -\frac{1}{2\mu}\int_{-1}^{1} T(\delta_0, s) b_3(s) ds$$
$$= -\frac{\lambda a}{\mu}\int_{-1}^{1} T(\delta_0, s) a_3(s) ds,$$
$$c_4 = -\frac{1}{2\mu}\int_{-1}^{1} T(\delta_0, s) b_4(s) ds$$
$$= -\frac{\lambda}{2\mu}\int_{-1}^{1} T(\delta_0, s) a_2(s) ds.$$

The solution of the integral equation (15) now reaches Eq. (21) in which the four parameters c_1, c_2, c_3 and c_4 must be determined via Eq. (22) by using the constraint equations (5) and (8). Unfortunately, only two constraint equations (5) and (8) exist in spite of the fact that four equations are required to determine the four unknown parameters c_1, c_2, c_3 and c_4. The additional two equations can be derived from Eq. (22):

$$\frac{c_2}{c_4} = \frac{\int_{-1}^{1} T(\delta_0, s) a_4(s) ds}{\int_{-1}^{1} T(\delta_0, s) a_2(s) ds}, \tag{23}$$

$$\frac{c_3}{c_4} = \frac{2a\int_{-1}^{1} T(\delta_0, s) a_3(s) ds}{\int_{-1}^{1} T(\delta_0, s) a_2(s) ds}. \tag{24}$$

By referring to Eqs. (20) and (22), the constraint equation (8) can be expressed by

$$\frac{\int_{-1}^{1} T(\delta_0, s) a_4(s) ds}{\int_{-1}^{1} T(\delta_0, s) a_2(s) ds} + a\left\{\frac{\int_{-1}^{1} T(\delta_0, s) a_3(s) ds}{\int_{-1}^{1} T(\delta_0, s) a_2(s) ds}\right\}^2 = 0. \tag{25}$$

Thus, there are four equations (5), (23), (24), (25) from which c_1, c_2, c_3 and c_4 can be determined. Unfortunately, these four equations are nonlinear. There, we must solve a set of four nonlinear equations simultaneously. In order to derive four nonlinear equations in a more explicit form, we set the pupil function $T(\delta_0, x)$ of Eq. (21) in the form

$$T(\delta_0, x) = c_4\sum_{i=1}^{4} k_i a_i(x), \tag{26}$$

where

$$k_i = \frac{c_i}{c_4};$$
$$(k_4 = 1)$$

and $a_i(x)$ is also given by Eq. (20). Consequently, the task of determining c_1, c_2, c_3 and c_4 becomes that of determining k_1, k_2, k_3 and k_4, because c_4 is simply a constant. Before proceeding further, we define a set of integrals given by

$$F_{ij} = \int_{-1}^{1} a_i(x) a_j(x) dx, \tag{27}$$

which will appear in the subsequent analysis and, therefore, has been evaluated in Appendix A by using Eq. (20) for $a_i(x)$ and $a_j(x)$. By means of Eq. (26), the four constraint equations (5), (23), (24) and (25) become

$$\begin{gathered} c_4^2\sum_{i,j=1}^{4} k_i k_j F_{ij} = q, \\ k_2\sum_{i=1}^{4} k_i F_{i2} - \sum_{i=1}^{4} k_i F_{i4} = 0, \\ k_3\sum_{i=1}^{4} k F_{i2} - 2a\sum_{i=1}^{4} k_i F_{i3} = 0, \\ 4ak_2 + k_3^2 = 0. \end{gathered} \tag{28}$$

The nonlinear algebraic equations given by Eq. (28) must be solved for each k_1, k_2, k_3 and k_4. The method of solving these algebraic equations can be found in I and, therefore, the detailed treatment of that method is omitted here. Once the parameters k_1, k_2, k_3 and k_4 are determined, the pupil function of Eq. (26), we are looking for, together with the present apodization problem, finally takes the form

$$T(\delta_0, x) = k + k_2 \cos\left(\frac{\delta_0 x}{2}\right) - -k_3 \sin\left(\frac{\delta_0 x}{2}\right) - k_4 x^2 \cos\left(\frac{\delta_0 x}{2}\right), \quad (29)$$

where Eq. (20) is used for $a_i(x)$ and a trivial constant c_4 is omitted without loss of generality. This form of Eq. (29) was already derived by BARAKAT [4] in the fourth Luneberg apodization problem under the incoherent illumination. In his study [4] the parameters k_1, k_2, k_3 and k_4 are a function of δ_0 alone, but at the present study they are functions of both δ_0 and γ (the coherence condition of illumination). Consequently, these parameters vary, when the coherence of illumination is changed.

4. Results and discussion

The four parameters k_1, k_2, k_3 and k_4, which are the coefficients of the pupil function in Eq. (29), where computed for various values of the coherence condition of illumination with accuracy to five decimals. These parameters are listed in Table for $q = 1$. In this table, the passive condition of an optical system given by Eq. (2) has been taken into account. As it was already discussed in I, a variation of the total light flux q passing through the apodized aperture does not influence the form of pupil functions obtained under the present apodization scheme. This means that the pupil functions corresponding to various values of q become equivalent to those corresponding to $q = 1$ by a normalization. By this reason, the data for the case of $q = 1$ are only tabulated in Table.

The pupil functions $T(\delta_0, x)$ obtained by using the above parameters are illustrated in a normalized form by Fig. 1 for various values of δ_0 at different coherence states of illumination. This normalization is taken in such a way that the pupil function of uniform amplitude distribution over aperture becomes equal to one under various coherence conditions of illumination. The resultant pupil functions of Fig. 1 are, of course, qualitatively similar to those for the circular aperture [2]. The general behaviours of the pupil functions as functions of δ_0 and γ are clearly seen in Fig. 1. The non-apodized values $\delta_0 = 2.606, 2.903, 3.196, 3.494, 3.809, 4.163$ for the coherence conditions of illumination $\gamma = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$ are the values of the Sparrow resolution limits for the uniform amplitude distribution over the aperture, under various coherence conditions of illumination. As δ_0 is progressively made to decrease below non-apodized value under each coherence condition of illumination, the pupil function $T(\delta_0, x)$ is such as to weigh against the centre of the aperture. This phenomenon appears up to a certain value δ_0 different for each of the coherence conditions of illumination. As δ_0 is further decreased below this value, there is inversely a weighting at the edge of the aperture. Note that the maximum transmittance lies at the centre of the aperture except for the case of the completely coherent illumination $\gamma = 1.0$ (see Fig. 1f). When the illumination approaches the completely coherent light and δ_0 is decreased, there appears a weighing in the ring-shape region of the aperture. Under the completely coherent illumination, the maximum transmittance is produced at the edge of the aperture for $3.8 < \delta_0$. It is obviously recognized from Fig. 1 that the pupil function investigated under the present apodization scheme is largely affected by the coherence condition of illumination. Thus it is concluded that the coherence condition of illumination must be known beforehand in order to obtain an appropriate pupil function fitting the apodization purpose.

The total intensity distribution for the slit aperture due to the two point objects illuminated by partially coherent light is [5, 6]

$$I\left(v, \frac{\delta_0}{2}\right) = \frac{1}{|D_0(0)|^2}\left\{\left|D\left(v - \frac{\delta_0}{2}\right)\right|^2 + \left|D\left(v + \frac{\delta_0}{2}\right)\right|^2 + +2\gamma\left|D\left(v - \frac{\delta_0}{2}\right)\right|\left|D\left(v + \frac{\delta_0}{2}\right)\right|\right\}, \quad (30)$$

where $|D_0(0)|^2$ is the central intensity ($v = 0$) for the uniform amplitude distribution, due to the single point object, which is given by

$$|D_0(0)|^2 = \left|\int_{-1}^{1} T_0 dx\right|^2 = 1, \quad (31)$$

Pupil function parameters ($q = 1$)

a) $\gamma = 0$

δ_0	k_1	k_2	k_3	k_4
2.606	0.70711	0	0	0
2.5	0.70908	−0.08567	−0.32585	0.30986
2.4	0.71401	−0.19322	−0.68898	0.61420
2.3	0.72102	−0.32368	−1.07827	0.89800
2.2	0.73044	−0.46758	−1.45073	1.12527
2.1	0.74274	−0.61235	−1.76485	1.27163
1.8	1.13948	−0.00140	0.12063	2.60030
1.4	1.11038	−0.00025	0.04751	2.24000
1.0	1.08110	−0.00003	0.01490	1.99597
0.2	1.06142	0	0.00010	1.77638

b) $\gamma = 0.2$

δ_0	k_1	k_2	k_3	k_4
2.903	0.70711	0	0	0
2.8	0.70968	−0.07272	−0.24011	0.29730
2.6	0.73209	−0.28773	−0.83033	0.89857
2.4	0.78432	−0.58367	−1.45283	1.35611
2.2	0.87727	−0.90524	−1.92757	1.53916
2.0	1.16476	−0.00202	0.12520	2.91049
1.6	1.11994	−0.00042	0.05233	2.42895
1.0	1.08111	−0.00002	0.00995	1.99864
0.2	1.06142	0	0.00007	1.77638

c) $\gamma = 0.4$

δ_0	k_1	k_2	k_3	k_4
3.196	0.70711	0	0	0
3.0	0.71897	−0.14230	−0.38355	0.60305
2.8	0.76752	−0.38701	−0.89664	1.21182
2.4	1.23484	−0.00471	0.17527	3.80471
2.0	1.16544	−0.00134	0.08268	2.97060
1.6	1.12009	−0.00028	0.03399	2.44709
1.2	1.09116	−0.00004	0.01192	2.11543
0.2	1.06142	0	0.00004	1.77638

d) $\gamma = 0.6$

δ_0	k_1	k_2	k_3	k_4
3.494	0.70711	0	0	0
3.2	0.74102	−0.22128	−0.47506	1.01988
3.0	0.83758	−0.51864	−0.92594	1.65309
2.8	1.07098	−0.98585	−1.44349	2.11358
2.4	1.23678	−0.00286	0.10614	3.94377
2.0	1.16600	−0.00080	0.04924	3.01788
1.6	1.12020	−0.00016	0.01999	2.46094
1.2	1.09118	−0.00002	0.00697	2.11876
0.2	1.06142	0	0.00002	1.77638

e) $\gamma = 0.8$

δ_0	k_1	k_2	k_3	k_4
3.809	0.70711	0	0	0
3.6	0.72121	−0.09296	−0.18467	0.82545
3.4	0.79007	−0.30635	−0.49476	1.79787
3.2	1.00867	−0.74729	−0.91390	2.51474
2.8	1.33383	−0.00134	0.05943	5.91820
2.4	1.24045	−0.00134	0.04933	4.09051
2.0	1.16781	−0.00038	0.02269	3.07379
1.6	1.12116	−0.00008	0.00915	2.48243
1.2	1.09173	−0.00001	0.00319	2.12773
0.2	1.06149	0	0.00002	1.77722

f) $\gamma = 1.0$

δ_0	k_1	k_2	k_3	k_4
4.163	0.70711	0	0	0
4.0	0.71523	0	0	0.83878
3.8	0.76306	0	0	2.50601
3.6	0.88497	0	0	4.78936
3.2	1.27290	0	0	7.73044
2.0	1.16688	0	0	3.00746
1.2	1.09121	0	0	2.12346
0.2	1.06142	0	0	1.77639

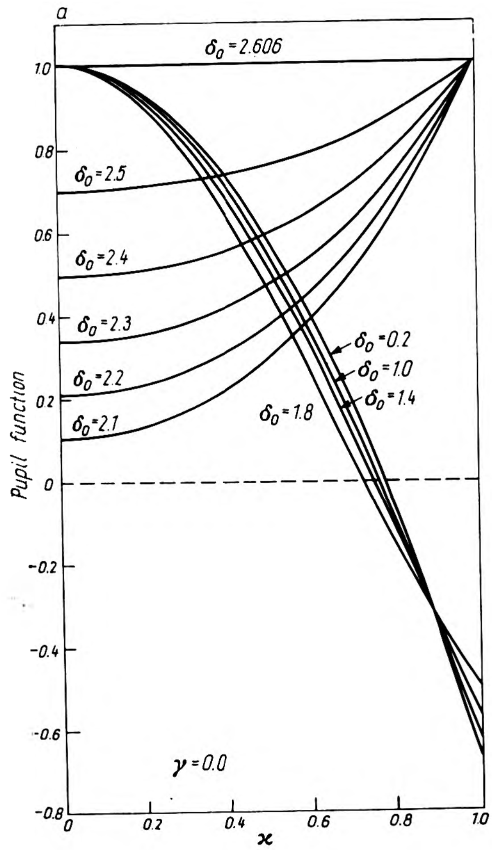

Fig 1a

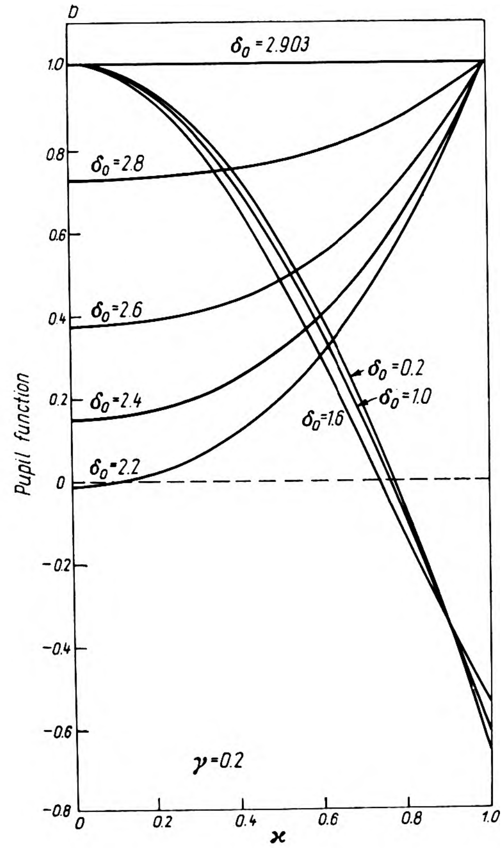

Fig 1b

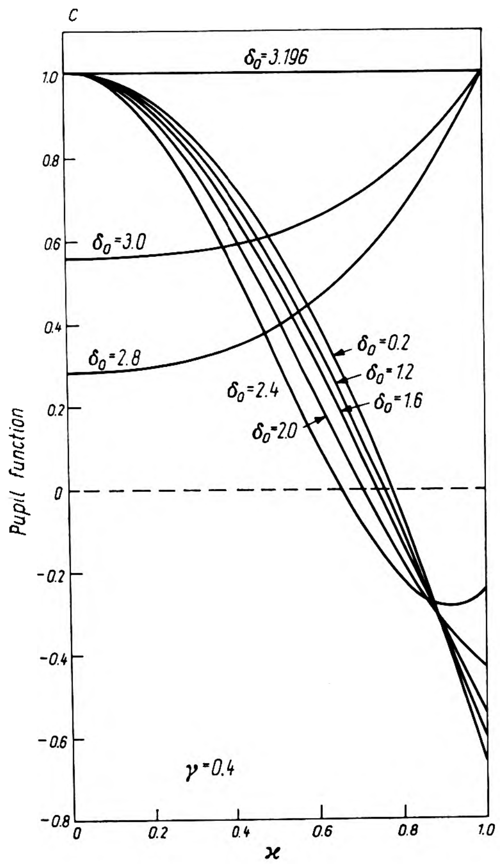

Fig 1c

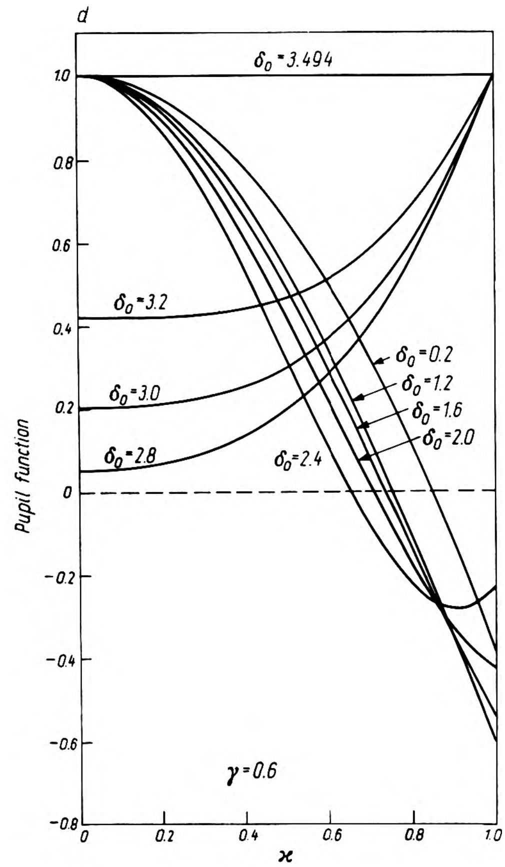

Fig 1d

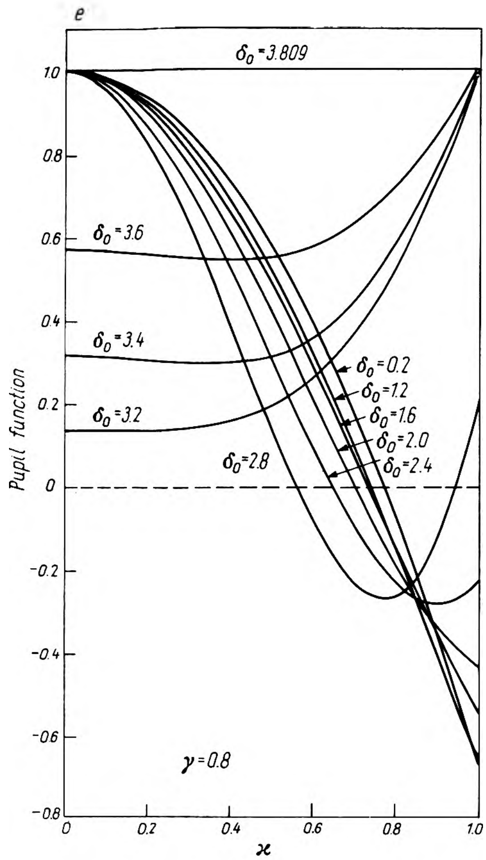

Fig 1e

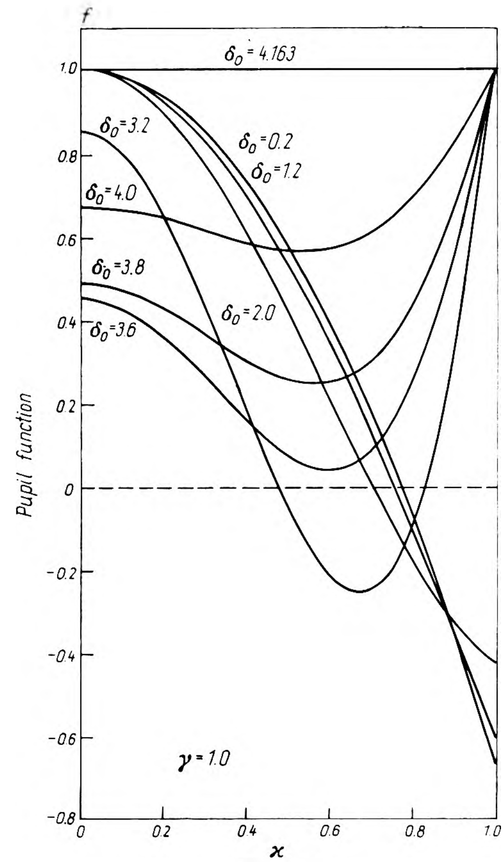

Fig 1f

Fig. 1. Pupil functions $T(\delta_0, x)$ for various values of the Sparrow resolution limit δ_0 under various states of the coherence condition γ of illumination

T_0 being given by Eq. (4). In Eq. (30), the amplitude distribution $D\left(v \pm \frac{\delta_0}{2}\right)$ in the image space due to the single point object situated at $\pm\frac{\delta_0}{2}$ separated from the axis is, using Eqs. (1) and (21), given by

$$D\left(v \pm \frac{\delta_0}{2}\right) = \int_{-1}^{1} T(\delta_0, x)\cos\left\{\left(v \pm \frac{\delta_0}{2}\right)x\right\}dx = \sum_{i=1}^{4} c_i \int_{-1}^{1} a_i(x)\cos\left\{\left(v \pm \frac{\delta_0}{2}\right)x\right\}dx = \sum_{i=1}^{4} c_i G_i\left(v \pm \frac{\delta_0}{2}\right), \tag{32}$$

where

$$G_i\left(v \pm \frac{\delta_0}{2}\right) = \int_{-1}^{1} a_i(x)\cos\left\{\left(v \pm \frac{\delta_0}{2}\right)x\right\}dx. \tag{33}$$

In Appendix B, $G_i\left(v \pm \frac{\delta_0}{2}\right)$ has been evaluated by using various properties of Appendix A The intensity distributions corresponding to the pupil functions of Fig. 1a–f at various states of the coherence condition of illumination, are shown in Fig. 2 as a function of δ_0. The present results are very similar to those obtained for the circular aperture [2]. The central intensity $I\left(0, \frac{\delta_0}{2}\right)$ for various coherence conditions γ of illumination, according to the present apodization scheme, at first decreases until δ_0 is reduced to a certain value, and then increases above that value, as the two point objects are brought closer together. As the loss of central intensity is increased, the side-lobe intensity is increased. With the decrease of γ, the intensity distribution at the central area is broadend and, at the same time, the central intensity is decreased. As a conclusion, the behaviours of the intensity distribution vary at various states of the coherence condition of illumination.

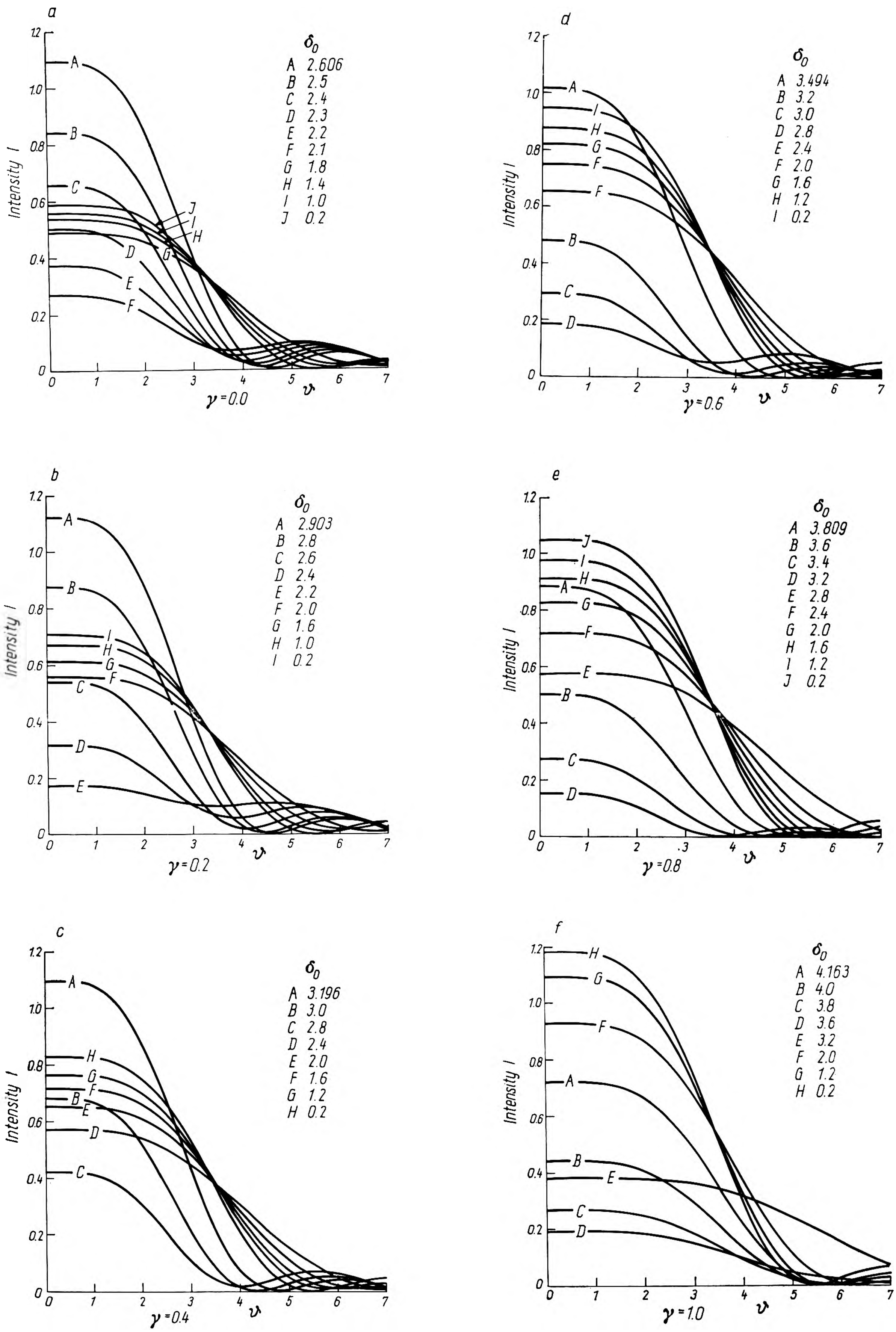

Fig. 2. Intensity distribution $I(v, \delta_0/2)$ of two-point image for various values of the Sparrow resolution limit δ_0 under various states of the coherence condition γ of illumination

Appendix A

The integrals of Eq. (27) are evaluated by using Eq. (20) for $a_i(x)$ and $a_j(x)$. Before evaluating these integrals, we use an integral of the following form

$$P(n, k) = \int_{-1}^{1} x^n \exp(ikx)\,dx. \qquad (A1)$$

This integral has a relation, except for the case $k = 0$,

$$P(n, k) = \frac{1}{ik}\{\exp(ik) + (-1)^{n+1}\exp(-ik)\} - \frac{n}{ik} P(n-1, k) = \frac{i}{k}\{nP(n-1, k) - T_n(k)\}, \qquad (A2)$$

where

$T_n(k) = 2\cos k$, when n is an odd number
$= 2i\sin k$, when n is an even number.

By the relation of Eq. (A2), we have the following relations

$$P(0, k) = \int_{-1}^{1} \exp(ikx)\,dx = \frac{2\sin k}{k} = 2\bar{P}(0, k),$$

$$P(1, k) = \frac{i}{k}\{P(0, k) - T_1(k)\} = \frac{2i}{k}\{\bar{P}(0, k) - \cos k\} = 2i\bar{P}(1, k),$$

$$P(2, k) = \frac{i}{k}\{2P(1, k) - T_2(k)\} = -\frac{2}{k}\{2P(1, k) - \sin k\} = 2\bar{P}(2, k), \qquad (A3)$$

$$P(3, k) = \frac{i}{k}\{3P(2, k) - T_3(k)\} = \frac{2i}{k}\{3\bar{P}(2, k) - \cos k\} = 2i\bar{P}(3, k),$$

$$P(4, k) = \frac{i}{k}\{4P(3, k) - T_4(k)\} = -\frac{2}{k}\{4\bar{P}(3, k) - \sin k\} = 2\bar{P}(4, k),$$

where

$$\bar{P}(0, k) = \frac{\sin k}{k},$$

$$\bar{P}(1, k) = \frac{1}{k}\{\bar{P}(0, k) - \cos k\},$$

$$\bar{P}(2, k) = -\frac{1}{k}\{2\bar{P}(1, k) - \sin k\}, \qquad (A4)$$

$$\bar{P}(3, k) = \frac{1}{k}\{3\bar{P}(2, k) - \cos k\},$$

$$\bar{P}(4, k) = -\frac{1}{k}\{4\bar{P}(3, k) - \sin k\}.$$

Using the integral of Eq. (A1), we have

$$P(n, u+v) + P(n, u-v) = \int_{-1}^{1} x^n \exp\{i(u+v)x\}\,dx + \int_{-1}^{1} x^n \exp\{i(u-v)x\}\,dx = 2\int_{-1}^{1} x^n \exp(iux)\cos(vx)\,dx, \qquad (A5)$$

from which we can derive the relations:

$$\frac{1}{2}\mathrm{Re}\{P(n, u+v) + P(n, u-v)\} = \int_{-1}^{1} x^n \cos(ux)\cos(vx)\,dx,$$

$$\frac{1}{2}\mathrm{Im}\{P(n, u+v) + P(n, u-v)\} = \int_{-1}^{1} x^n \sin(ux)\cos(vx)\,dx, \qquad (A6)$$

where Re and Im indicate the real and imaginary parts of the bracketed term, respectively. Using Eq. (A6) and referring to Eqs. (A3) and (A4), we evaluate a set integrals F_{ij} in which $a_i(x)$ and $a_j(x)$ are given by Eq. (20).

$$F_{11} = \int_{-1}^{1} dx = P(0, 0) = 2,$$

$$F_{12} = \int_{-1}^{1} \cos\left(\frac{\delta_0 x}{2}\right) dx = 2\bar{P}\left(0, \frac{\delta_0}{2}\right),$$

$$F_{13} = -\int_{-1}^{1} x\sin\left(\frac{\delta_0 x}{2}\right) dx = -\frac{1}{2}\mathrm{Im}\left\{P\left(1, \frac{\delta_0}{2}\right) + P\left(1, \frac{\delta_0}{2}\right)\right\} = -\mathrm{Im}P\left(1, \frac{\delta_0}{2}\right) = -2\bar{P}\left(1, \frac{\delta_0}{2}\right),$$

$$F_{14} = -\int_{-1}^{1} x^2\cos\left(\frac{\delta_0 x}{2}\right)dx$$

$$= -\frac{1}{2}\,\mathrm{Re}\left\{P\left(2, \frac{\delta_0}{2}\right) + P\left(2, \frac{\delta_0}{2}\right)\right\}$$

$$= -\mathrm{Re}P\left(2, \frac{\delta_0}{2}\right) = -2\bar{P}\left(2, \frac{\delta_0}{2}\right), \quad \text{(A7)}$$

$$F_{22} = \int_{-1}^{1} \cos\left(\frac{\delta_0 x}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right)dx$$

$$= \frac{1}{2}\,\mathrm{Re}\{P(0, \delta_0) + P(0, 0)\} = \bar{P}(0, \delta_0) + 1,$$

$$F_{23} = -\int_{-1}^{1} x\sin\left(\frac{\delta_0 x}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right)dx$$

$$= -\frac{1}{2}\,\mathrm{Im}\{P(1, \delta_0) + P(1, 0)\} = -\bar{P}(1, \delta_0),$$

$$F_{24} = -\int_{-1}^{1} x^2\cos\left(\frac{\delta_0 x}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right)dx$$

$$= -\frac{1}{2}\,\mathrm{Re}\{P(2, \delta_0) + P(2, 0)\}$$

$$= -\left\{\bar{P}(2, \delta_0) + \frac{1}{3}\right\},$$

$$F_{34} = \int_{-1}^{1} x^3\sin\left(\frac{\delta_0 x}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right)dx$$

$$= \frac{1}{2}\,\mathrm{Im}\{P(3, \delta_0) + P(3, 0)\} = \bar{P}(3, \delta_0),$$

$$F_{44} = \int_{-1}^{1} x^4\cos\left(\frac{\delta_0 x}{2}\right)\cos\left(\frac{\delta_0 x}{2}\right)dx$$

$$= \frac{1}{2}\,\mathrm{Re}\{P(4, \delta_0) + P(4, 0)\} = \bar{P}(4, \delta_0) + \frac{1}{5}.$$

The integral F_{33} which remains still unsolved can be solved by referring to the following relation

$$P(n, u+v) - P(n, u-v)$$

$$= 2i\int_{-1}^{1} x^n \exp(iux)\sin(vx)\,dx$$

$$= -2\int_{-1}^{1} x^n \sin(ux)\sin(vx)\,dx +$$

$$+2i\int_{-1}^{1} x^n \cos(ux)\sin(vx)\,dx, \quad \text{(A8)}$$

from which we have

$$-\frac{1}{2}\,\mathrm{Re}\{P(n, u+v) - P(n, u-v)\}$$

$$= \int_{-1}^{1} x^n \sin(ux)\sin(vx)\,dx. \quad \text{(A9)}$$

Consequently, F_{33} reaches

$$F_{33} = \int_{-1}^{1} x^2 \sin\frac{\delta_0 x}{2}\,\sin\frac{\delta_0 x}{2}\,dx$$

$$= -\frac{1}{2}\,\mathrm{Re}\{P(2, \delta_0) - P(2, 0)\}$$

$$= -\left\{\bar{P}(2, \delta_0) - \frac{1}{3}\right\}. \quad \text{(A10)}$$

Appendix B

The integral $G_i(v \pm \delta_0/2)$ of Eq. (33) is evaluated in the following. Setting $t = v \pm \delta_0/2$ in Eq. (33) and using Appendix A, we have

$$G_1(t) = \int_{-1}^{1} a_1(x)\cos(tx)\,dx$$

$$= \int_{-1}^{1} \cos(tx)\,dx = \mathrm{Re}P(0, t) = 2\bar{P}(0, t),$$

$$G_2(t) = \int_{-1}^{1} a_2(x)\cos(tx)\,dx$$

$$= \int_{-1}^{1} \cos\left(\frac{\delta_0 x}{2}\right)\cos(tx)\,dx$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{P\left(0, \frac{\delta_0}{2} + t\right) + P\left(0, \frac{\delta_0}{2} - t\right)\right\}$$

$$= \bar{P}\left(0, \frac{\delta_0}{2} + t\right) + \bar{P}\left(0, \frac{\delta_0}{2} - t\right),$$

$$G_3(t) = \int_{-1}^{1} a_3(x)\cos(tx)\,dx$$

$$= -\int_{-1}^{1} x\sin\left(\frac{\delta_0 x}{2}\right)\cos(tx)\,dx$$

$$= -\frac{1}{2}\,\mathrm{Im}\left\{P\left(1, \frac{\delta_0}{2} + t\right) + P\left(1, \frac{\delta_0}{2} - t\right)\right\}$$

$$= -\left\{\bar{P}\left(1, \frac{\delta_0}{2} + t\right) + \bar{P}\left(1, \frac{\delta_0}{2} - t\right)\right\},$$

$$G_4(t) = \int_{-1}^{1} a_4(x)\cos(tx)\,dx$$

$$= -\int_{-1}^{1} x^2 \cos\left(\frac{\delta_0 x}{2}\right)\cos(tx)\,dx$$

$$= -\frac{1}{2}\,\mathrm{Re}\left\{P\left(2, \frac{\delta_0}{2}+t\right)+P\left(2, \frac{\delta_0}{2}-t\right)\right\}$$

$$= -\left\{\overline{P}\left(2, \frac{\delta_0}{2}+t\right)+\overline{P}\left(2, \frac{\delta_0}{2}-t\right)\right\}.$$

Le quatrième problème de l'apodisation de Luneberg sous lumière partiellement cohérente

Le quatrième problème de l'apodisation de Luneberg, se rapportant à l'augmentation de la limite de séparation de Sparrow sous éclairage partiellement, cohérent, à etet formulé pour appareillage a fente à l'aide du calcul des variations. Les fonctions de lentille recherchées, obtenues par la sulution de l'équation intégrale aux limites fixes de Fredholm, ont été évaluées pour différentes conditions de cohérence spatiale de la lumière.

Четвертый вопрос аподизации Люнеберга в частично когерентном свете

Четвертый вопрос аподизации Люнеберга, касающийся увеличения предела разрешающей способности Спаррола при частично когерентном освещении, сформулирован для щелевой апертуры при помощи вариационного исчисления. Требуемые зрачковые функции, полученные решением однородного интегрального уравнения Фредгольма, определены при разных условиях пространственной когерентности освещения.

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