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Teaching optics

Problems related to numerical determination of the image given by aberrated optical system under the coherent illumination. Educational aspects

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A comparison of the advantages and disadvantages of the convolution and harmonic approaches to the image evaluation is presented from the student's point of view. On the basis of a simple object it has been shown that in some cases the convolution method gives better insight into the imaging process.

1. Introduction

The determination of the image complex amplitude distribution given by an optical system is the basic problem of the imaging theory. There are two ways of finding the image distribution, namely, the convolution and the harmonic approaches. In the education process the harmonic approach is usually emphasised due to its clear interpretation of the imaging limits. The aim of this paper is to call attention to the advantages of the convolution method, which in some cases provides better explanation of image distortion.

2. Convolution approach

The basic equation allowing determination of the complex amplitude distribution $V'(\mathrm{A})$ of the image of an arbitrary object under coherent illumination is [1]

$$V'(\mathrm{A}) = C_1 V(\mathrm{A}) \otimes V'_\delta(\mathrm{A}) \tag{1}$$

where $V(\mathrm{A})$ is the complex amplitude distribution on the object surface, $\otimes$ – the convolution operator, V'_δ – the complex amplitude distribution of the point image (the response function of an optical system), $\mathrm{A}(A_x, A_y)$ – the parameterised coordinate in vector form at the object or image surface, A_x, A_y – Cartesian components of the vector $\mathbf{A}$. The point spread function is determined as the Fourier transform of the pupil function $V_z(\varrho)$ (field distribution at the pupil generated by the point object), which can be written in the following form:

$$V'_\delta(\mathrm{A}) = C_2 \mathrm{FT}^- [V_z(\varrho)], \tag{2}$$

$\varrho(\rho_x, \rho_y)$ – radial coordinate, in the vector form, at the pupil plane.

The respective normalisation of the object and image coordinates induces the same notation of both, however, the normalisation problem as well as the form of the constants C_1 and C_2 are out of scope of this paper.

The complex amplitude distribution $V'_\delta(\mathrm{A})$ of the point image can be determined in a limited area only. Even in some cases, when the pupil function is known analytically (for example, aberration-free and nonapodized optical system with rotational symmetry), the integration (1) for a more complex object can be calculated in a limited area, too. In other words, the integral relation (1) can be performed always with a limited precision. The problem becomes important when it is necessary to estimate the exactness of the wavefront imaging in interferometric systems with high measurement precision.

3. Harmonic approach

The determination of the image complex amplitude distribution $V'(\mathrm{A})$ can be performed otherwise with the aid of the harmonic approach. Taking the Fourier transformation of both sides of Eq. (1) we have [1]

$$v'(\boldsymbol{\alpha}) = v(\boldsymbol{\alpha})t(\boldsymbol{\alpha}) \tag{3}$$

where: $v'(\boldsymbol{\alpha})$ and $v(\boldsymbol{\alpha})$ are complex amplitudes of the image and object harmonics, respectively, $\boldsymbol{\alpha}(\alpha_x, \alpha_y)$ – circular spatial frequency of the harmonic in the vector form, and the optical transfer function $t(\boldsymbol{\alpha})$ is equal to the transmittance of the pupil of the optical system at the point

$$\boldsymbol{\varrho} = \boldsymbol{\alpha}. \tag{4}$$

The main advantage of relation (3) is its linearity. This feature is especially appreciated for periodical and linear (one-dimensional) objects, because according to (4), the individual harmonics are located at discrete points in the pupil along a straight line perpendicular to the linear structure of the object. The analyses are particularly simple in the case of systems without aberrations and apodization. All harmonics located in the pupil are transferred into the image space without any changes. The image complex amplitude distribution is the sum of all transferred harmonics, or in the case of nonperiodic objects, the Fourier integral of the image spectrum. This means that

$$V'(\mathrm{A}) = C_3 \mathrm{FT}^+ [v'(\boldsymbol{\alpha})] \tag{5}$$

where FT^+ is the inverse Fourier operator.

4. Remarks on some educational aspects

In Figure 1, a diagram showing the relations for two approaches mentioned above is presented. The advantages of the harmonic approach in comparison to the convolution method are usually emphasized in the education process. Namely, due to the linear relation of the harmonic transfer the properties of the optical systems and the objects can be separated. On the contrary, both functions connected by the

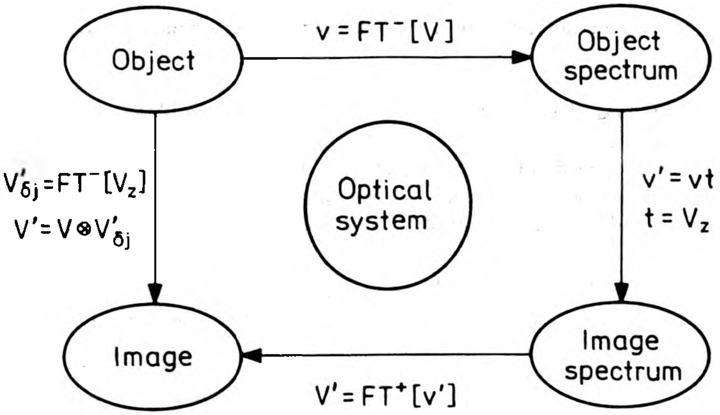

Fig. 1. Two approaches to image determination.

convolution procedure have to be known simultaneously. On the basis of relation (1) the imaging quality of the optical system can be determined for a given object only.

5. Substantial remarks and conclusions

In real cases there are no periodical and no one-dimensional objects. Every object has to be limited, so its spatial spectrum is spread over the whole pupil area. The problem of how to precisely determine the two-dimensional spectrum appears once more. Instead of determining the point image using the convolution approach (see Eq. (1)), we determine the object spectrum in the harmonic approach. Both ways require the same procedure of the Fourier transformation, so they induce errors of the same character.

The spatial frequencies of transferred harmonics are limited sharply by the pupil dimensions, which makes the calculation easier. When lecturing on this topic, this fact is especially emphasized to students. However, for the spread object spectrum and the aberrated or apodized systems due to different values of transfer function $t(\alpha)$ for different spatial frequencies α, various harmonics are transferred differently. Moreover, the return from the image spectrum to the image of an object requires the inverse Fourier transformation procedure.

In the Table, characteristics of the method and the calculation problems are listed. In real cases both ways require two two-dimensional integrals. The essential differences reduce to: 1) the limited object area and the limitless image of the point object for convolution approach, and 2) the limitless object spectrum and the limited image spectrum for the harmonics approach. For exact calculations the point image has to be known over the whole object area, because it is necessary to take into account the influence of one edge of the object on the image of the second one. Similarly, the image spectrum has to be known over the whole pupil area.

The harmonic approach appears to be more convenient for calculations due to the FFT procedure. After all, the convolution is a time consuming relation. However, the density of the Fourier transform samples depends on the sample density of the

Table. Method characteristics and calculation problems in the two approaches to image determination

Methods	Method characteristics	Calculation problems
Convolution approach $V'(\mathrm{A}) = CV(\mathrm{A}) \otimes V'_{\delta j}(\mathrm{A})$	Accentuated influences of the various object distributions as well as the aberrations and the apodization of the optical system on the image.	Time consuming operation due to the two-dimensional integral relation. Numerical errors related to the limited data of the point spread function and chosen sampling distribution. Numerical integration errors.
Harmonic approach $v'(\alpha) = v(\alpha)\, t(\alpha)$	Direct conclusions related to periodical objects, and especially to the limited resolution of optical systems. Accentuated influences of the spatial frequency filtration.	Different influences of optical system aberrations and the apodization upon the transfer of individual harmonics. Numerical errors of the object spectrum determination. Numerical errors of the FFT procedure.

transformed function, which not always allows the samples to be determined at desirable points. Moreover, for the harmonic approach the reasons of image changes are latent in the spatial spectrum. On the other hand, the convolution method gives the essential image changes directly.

The two methods have different advantages and disadvantages. The choice of the method depends on our goal, the form of the object or the form of the pupil function. Generally, we cannot indicate *a priori* the predominance of one of the methods.

On the basis of previous considerations we can conclude that the separation of the properties of objects and optical systems, expressed by Eq. (3), seems to be virtual only. The exactness of the wavefront shape transfer depends on the optical system parameters and the shape of the transferred wavefront together.

It is worth emphasising that the problems with determination of the image of an arbitrary object under coherent illumination are similar to the determination of the transfer function of optical systems in incoherent illumination. In this case one has to choose between the correlation method and the transformation one [2].

6. Example of determining the image of a simple object

To confirm the conclusions formulated above that the choice of the convolution or harmonic approach depends on the object character, let the object be in the form of a small two-element segment on the dark background (see Fig. 2). Such a choice has been inspired by the simplicity of the consideration only. Moreover, let the segment

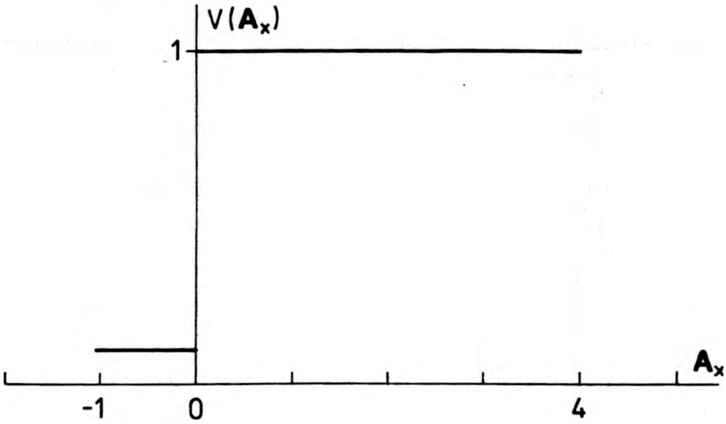

Fig. 2. Object amplitude distribution in the *x*-section.

width in the *y*-direction be small in comparison with dimension of the central part of the diffraction point image such that the two-dimensional convolution operator in (1) can be reduced to one-dimensional integral only. Both elements of the segment differ between themselves considerably in the amplitude values. The values are given in the figure.

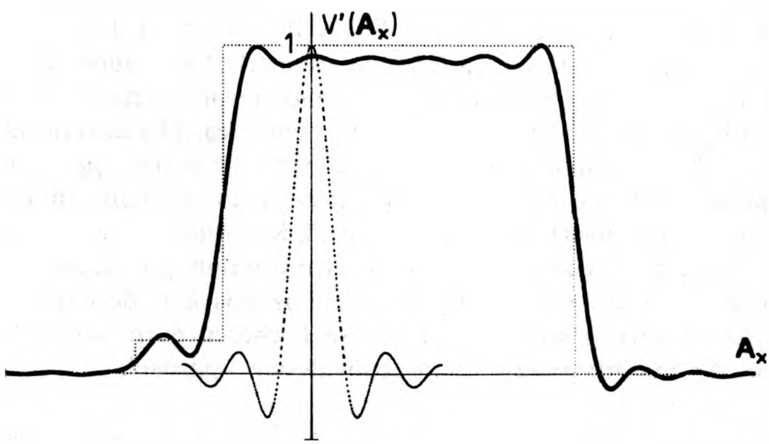

Fig. 3. Image amplitude distribution of the object shown in Fig. 2. The object amplitude distribution and Airy point spread function are marked by the thin and dashed lines, respectively.

The results of the convolution of the object with the Airy point spread function (the image of the object from Fig. 2) in the *x*-section are shown in Fig. 3. The corresponding object amplitude distribution and Airy function are marked with the thin and dashed lines, respectively. It is clearly seen from the diagram that imaging of the object fragment with small amplitude is greatly distorted. The side lobes of the

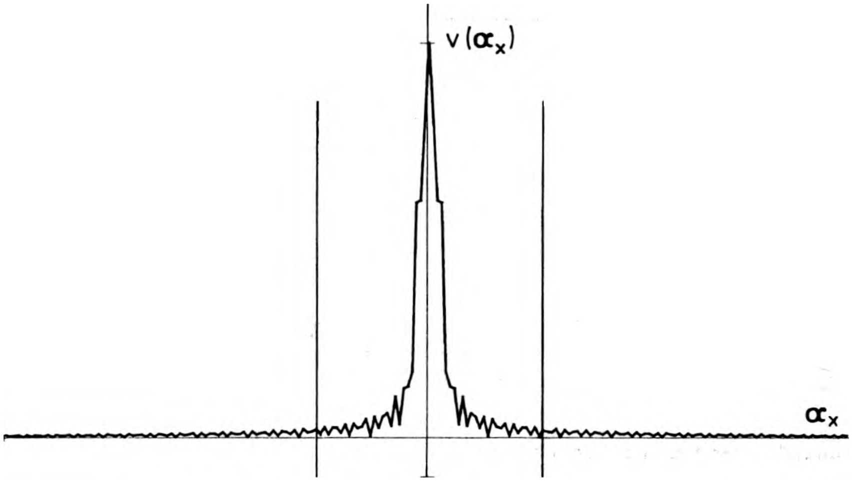

Fig. 4. Spatial spectrum of the x-section of the object shown in Fig. 2. The pupil edges are marked with double lines.

point images derived from the object fragment with high amplitude are the direct reason of the distortion.

Choosing the harmonic approach the object space spectrum has to be determined. The amplitude distribution of the space spectrum in the section related to Fig. 3 is shown in Fig. 4. The pupil edges corresponding to the relations between the dimensions of the object and the point spread function are marked by double lines. The image shown in Fig. 3 can be found by the two-dimensional inverse Fourier transformation of the object spectrum transferred by the pupil area. The final result is the same as in Fig. 3. However, following the analyses of the harmonic approach the influence of the spatial frequency filtration is accentuated, and in the convolution approach – the influence of the object shape is emphasised. Returning to Fig. 3, one can conclude that without the knowledge about the convolution procedure the explanation of the image distortion using the harmonic approach is doubtful.

Evidently the object example described in the paper is chosen purposefully in order to call attention to the advantages of the convolution approach.

References

[1] Jóźwicki R., *Theory of Optical Imaging*, (in Polish), [Ed.] PWN, Warszawa 1988, Chap. 7.
[2] MacDonald J., Optica Acta **18** (1971), 269.

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