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Gaussian beams with optical vortex of charge 2- and 3-diffraction by a half-plane and slit

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In the paper, the half-plane diffraction of the Gaussian light beam containing double and triple optical vortex is examined. The analysis is based on scalar theory of diffraction in Fresnel–Kirchhoff approximation. Special attention is paid to the dynamics of the optical vortex within the diffracted beam. Some results for the case of diffraction by single slit are also presented.

1. Introduction

Optical vortex (OV) is a point singularity in phase distribution of the electromagnetic wave [1], [2]. Optical wave fronts with phase singularities reveal some charateristic and useful features. They have a helical geometry and non-zero angular momentum [3], which is different from polarization (photon spin) part of the total angular momentum of the electromagnetic field (it is called orbital momentum) [4] – [6]. The orbital angular momentum is quantified in respect of both quantum and geometry. From the point of view of quantum its z-projection (for single photon) takes values $L_z = m\hbar$, where m is a positive integer characterizing the OV. The non-zero angular momentum makes the OV a stable structure within the wave front. From geometrical (topological) point of view the integral calculated along the closed curve including OV could have a value $2\pi m$ [7]. Because of the above properties the value of m is often called the topological charge of the OV. The distribution of the OV belonging to the same wave front is limited by some topological rules [7], [8]. Because of the special features of the OV, they have been widely studied in the last decade. A number of papers were devoted to the possible application of the OV [9] – [13].

In this paper, the diffraction pattern of the Gaussian beam containing OV (with topological charge $m = 2$ and $m = 3$) diffracted by half-plane and single slit is examined. The analysis is performed on the basis of the scalar diffraction theory (Fresnel–Kirchhoff diffraction integral [14]). The half-plane diffraction of pure Gaussian beam has been examined in [15].

2. Optical beam with OV

For the purpose of present analysis, a Gaussian beam with multicharge OV is considered. Such a beam can be described by the equation [16] (in parabolic approximation [17])

$$E(x,y,z_D) = E_z(x+i\,sgn\,y)^m \exp\{-r^2 A\} \tag{1a}$$

where:

$$E_z = E_0 \frac{\omega_0}{(\omega(z_D))^{m+1}} \exp\{-ikz\} \exp\left\{-(m+1)\,i\,\mathrm{atan}\left(\frac{2z_D}{b}\right)\right\}, \tag{1b}$$

$$A = \frac{1}{\omega(z_D)^2} + \frac{ik}{2R(z_D)}, \tag{1c}$$

$$\omega(z_D) = \omega_0\sqrt{1+\left(\frac{2z_D}{b}\right)^2}, \tag{1d}$$

$$R(z_D) = z_D\left[1+\left(\frac{b}{2z_D}\right)^2\right], \tag{1e}$$

$$b = \frac{2\pi\omega_0^2}{\lambda}, \tag{1f}$$

$$r^2 = x^2+y^2, \tag{1g}$$

$k = 2\pi/\lambda$ is the wave vector, λ – the wavelength, b – the beam confocal parameter, z_D – the distance from the beam waist to plane of interest, $\omega(z_D)$ – the radius of the beam, $R(z_D)$ – the radius of curvature of the wave front.

The topological charge value $m > 1$ means that the wave front singularity can be decomposed into m single OV [7], [18]. The sign of the parameter *sgn* determined the sign of the topological charge.

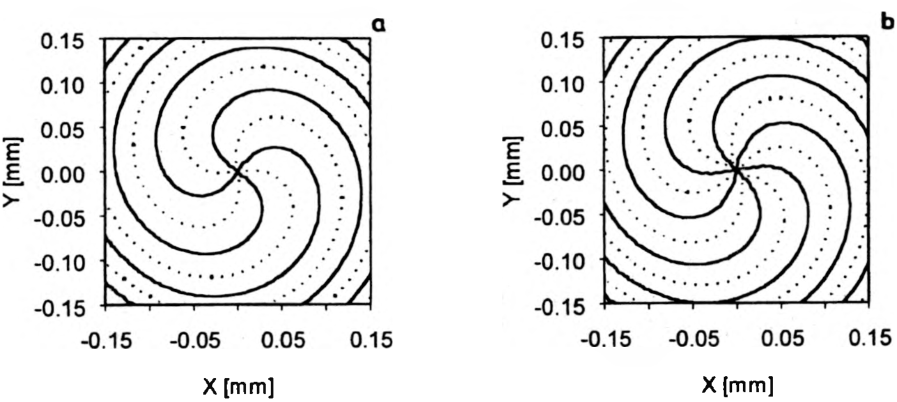

Fig. 1. Phase map of Gaussian beam with double OV (a) and triple OV (b). The solid line corresponds to line $\mathrm{Re}(E) = 0$, the dotted line corresponds to the line $\mathrm{Im}(E) = 0$.

Formula (1a) can be rewritten as

$$E(x,y,z_D) = E_z \exp\{-r^2 A\} \sum_{t=0}^{m} \binom{m}{t} sgn^t i^t x^{m-t} y^t. \tag{2}$$

Figure 1 shows an example of the phase map (*i.e.*, the plot of the line $\mathrm{Re}(E) = 0$ and $\mathrm{Im}(E) = 0$) of converging Gaussian beam carrying double and triple OV. The lines intersect at a singular point.

3. Diffraction integral

To calculate the diffraction pattern we follow the method described in [19], where the same problem was solved for the case of Gaussian beam with single OV. The diffraction by single slit can be described by the formula (Fresnel–Kirchhoff diffraction integral)

$$E(x_i,y_i,z_i) = T \int_{q}^{h} \int_{-\infty}^{\infty} [\exp\{-r^2 A\} \sum_{t=0}^{m} \binom{m}{t} sgn^t i^t x^{m-t} y^t$$
$$\times \exp\{-iK(x^2+y^2) - 2x_i x - 2y_i y\}]\mathrm{d}x\mathrm{d}y \tag{3}$$

where:

$$T = E_z \frac{\exp\{-ikz_i\}}{i\lambda z_i} \exp\{-iK(x_i^2+y_i^2)\}, \tag{3a}$$

$$K = \frac{k}{2z_i} \tag{3b}$$

q, h are the coordinates of slit edges. In the case of half-plane $h \to \infty$.

To calculate the integral (3) the following relation was used:

$$\int_q^h y^{n+2} \exp\{-Gy\}\mathrm{d}y = \frac{iSy}{2F} \int_q^h y^{n+1} \exp\{-Gy\}\mathrm{d}y + \frac{n+1}{2F} \int_q^h y^n$$
$$\times \exp\{-Gy\}\mathrm{d}y - \frac{1}{2F}(h^{n+1} Nh - q^{n+1} Nq) \tag{4}$$

where:

$$Sy = 2Ky_i, \tag{4a}$$

$$Gy = -y^2 F + iSy \cdot y, \tag{4b}$$

$$Nh = \exp\{-h^2 F + iSy \cdot h\}, \tag{4c}$$

$$Nq = \exp\{-q^2 F + iSy \cdot q\}. \tag{4d}$$

Let: $Sx = 2Kx_i$, $\quad$ (4e)

$$Gx = -x^2 F + iSx \cdot x. \tag{4f}$$

3.1. Diffraction by a slit

In the case of Gaussian beam with double charge OV the integral (3) can be written as:

$$E(x_i, y_i, z_i) = T(E2_x E_y + 2i\,sgn\,E1_x E1_y - E2_y E_x), \tag{5}$$

$$E2_x = \int_{-\infty}^{\infty} x^2 \exp\{-Gx\}dx = -\frac{\sqrt{\pi}}{4} Mx \frac{(Sx^2-2F)}{F^{5/2}}, \tag{5a}$$

$$E_y = \int_q^h \exp\{-Gy\}dy = \frac{1}{2}\sqrt{\frac{\pi}{F}}\, My\,(\mathrm{erf}(H)-\mathrm{erf}(Q)), \tag{5b}$$

$$E1_x = \int_{-\infty}^{\infty} x\exp\{-Gx\}dx = \frac{i\sqrt{\pi}}{2} Mx\cdot Sx\cdot F^{-3/2}, \tag{5c}$$

$$E1_y = \int_q^h y\exp\{-Gy\}dy = \frac{i\sqrt{\pi}}{4} F^{-3/2}\, Sy\cdot My(\mathrm{erf}(H)-\mathrm{erf}(Q)) + \frac{Nq-Nh}{2F}, \tag{5d}$$

$$E_x = \int_q^h \exp\{-Gx\}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{F}}\, Mx, \tag{5e}$$

$$E2_y = \int_q^h y^2 \exp\{-Gy\} = -\frac{Nh}{2F}\left(h+\frac{iSy}{2F}\right)+\frac{Nq}{2F}\left(q+\frac{iSy}{2F}\right)$$

$$+\frac{\sqrt{\pi}}{4} My\cdot F^{-3/2}\left(1-\frac{Sy^2}{2F}\right)(\mathrm{erf}(H)-\mathrm{erf}(Q)), \tag{5f}$$

$$Q = \frac{1}{2}\frac{2qF-iSy}{\sqrt{F}}, \tag{5g}$$

$$H = \frac{1}{2}\frac{2hF-iSy}{\sqrt{F}}, \tag{5h}$$

$$Mx = \exp\left\{\frac{-Sx^2}{4F}\right\}, \tag{5i}$$

$$My = \exp\left\{\frac{-Sy^2}{4F}\right\}. \tag{5j}$$

In the case of Gaussian beams with triple OV, expression (3) can be written as

$$E(x_i, y_i, z_i) = T(E3_x E_y + 3i\,sgn\,E2_x E1_y - 3E1_x E2_y - i\,sgn\,E_x E3_y) \tag{6}$$

where:

$$E3_x = \int_{-\infty}^{h} x^3 \exp\{-Gx\}\mathrm{d}x = \frac{i\sqrt{\pi}Mx\cdot Sx}{4}F^{-5/3}\left(3-\frac{Sx^3}{2F}\right), \tag{6a}$$

$$E3_y = \int_{q}^{h} y^3 \exp\{-Gy\}\mathrm{d}y = Nh\left(\frac{-h^2}{2F}-\frac{hSy}{4F^2}+\frac{Sy^2}{8F^3}-\frac{1}{2F^2}\right)$$
$$-Nq\left(\frac{-q^2}{2F}-\frac{qSy}{4F^2}+\frac{Sy^2}{8F^3}-\frac{1}{2F^2}\right)$$
$$+i\sqrt{\pi}My\left(\frac{3}{8}Sy\cdot F^{-5/2}-\frac{1}{16}Sy\cdot F^{-7/2}\right)(\mathrm{erf}(H)-\mathrm{erf}(Q)). \tag{6b}$$

3.2. Diffraction by a half-plane

If the condition: $\arg(H) < \left|\frac{\pi}{4}\right|$ is satisfied [20] one can evaluate the limit $h \to \infty$. Formulas (5b), (5d), (5f), (6b) take the form:

$$E_y = \int_{q}^{\infty} \exp\{-Gy\}\mathrm{d}y = \frac{1}{2}\sqrt{\frac{\pi}{F}}My\,\mathrm{erfc}(Q), \tag{7a}$$

$$E1_y = \int_{q}^{\infty} y\exp\{-Gy\}\mathrm{d}y = \frac{i\sqrt{\pi}}{4}F^{-3/2}\,Sy\cdot My\,\mathrm{erfc}(Q)+\frac{1}{2}FNq, \tag{7b}$$

$$E2_y = \int_{q}^{\infty} y^2\exp\{-Gy\} = \frac{My\sqrt{\pi}}{4}\,\mathrm{erfc}(Q)\left(F^{-3/2}-\frac{1}{2}Sy^2F^{-5/2}\right)+\frac{Nq}{2F}\left(q+\frac{Sy}{2F}\right), \tag{7c}$$

$$E3_y = \int_{q}^{\infty} y^3\exp\{-Gy\}\mathrm{d}y = \frac{iMy\sqrt{\pi}}{}\,\mathrm{erfc}(Q)\left(3F^{-5/2}Sy-\frac{1}{2}F^{-7/2}Sy^3\right)$$
$$+\frac{Nq}{2F}\left(q^2+\frac{iSy\cdot q}{2F}-\frac{Sy^2}{4F^2}+\frac{1}{2F}\right). \tag{7d}$$

4. Examples

As an example some diffraction patterns of Gaussian beam (1) diffracted by half-plane and single slit are calculated. The beam parameters are: $E_z = 1$, $b = 15$,

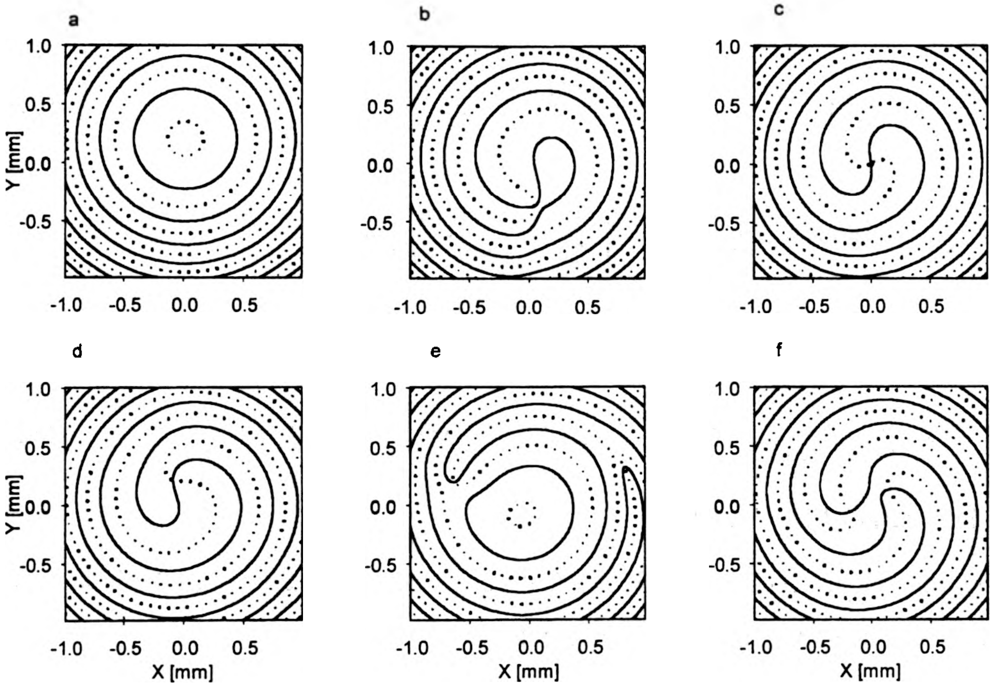

Fig. 2. Phase map of Gaussian beam with double OV diffracted by a half-plane. **a** — $q = 0.2$ mm, **b** — $q = 0.01$ mm, **c** — $q = 0$ mm, **d** — $q = 0.01$ mm, **e** — $q = -0.07$ mm, **f** — $q = -0.18$ mm.

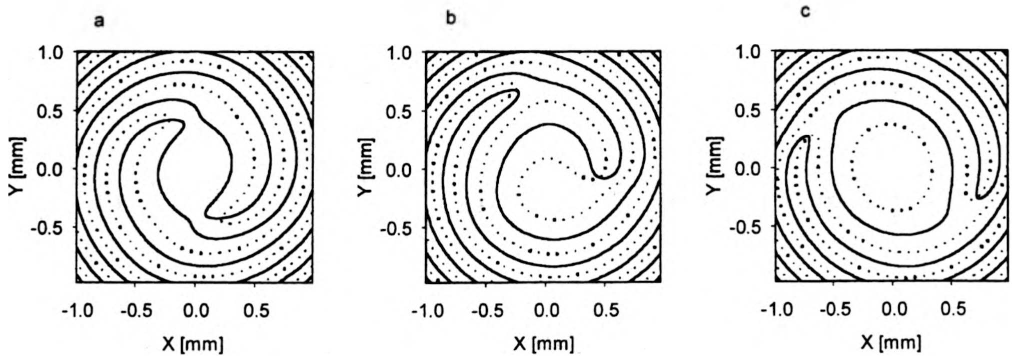

Fig. 3. Phase map of Gaussian beam with double OV diffracted by a single slit: **a** — slit located symmetrically — slit width 0.24 mm; **b** — the same slit shifted, $q = -0.1$ mm, $h = 0.14$ mm; **c** — slit located symmetrically — slit width 0.18 mm.

$z_D = 15$. These parameters correspond to strongly converging beam [15]. In all cases the image plane is at $z_i = 500$ mm from the object plane.

Figure 2 shows the results in the case of Gaussian beam with double OV diffracted by a half-plane. For $q = 0.2$ (q determines the half-plane edge position) there are no intersection points in the phase maps in the area considered (calculated at image plane, Fig. 2**a**). If $q = 0.01$ one intersection point arises (Fig. 2**b**), and changes its position when the plane edge is shifted down (Figs. **b**–**d**). For $q = -0.07$ the second intersection point appears (Fig. 2**e**). Both singular points change their relative position while the edge shifts (Fig. 2**e**,**f**). Figure 3 shows the diffraction image in the case of single slit illuminated by the same beam as in the previous example. In Fig. 3**a**, the beam is incident to a slit of width 0.24 mm located symmetrically against optical axes. If the slit center moves, the diffraction pattern symmetry is broken (Fig. 3**b**). In Figure 3**c**, the slit is at central position and its width is 0.18 mm. As one can see the intersection points are more distant from each other (compared to Fig. 3**a**).

Figure 4 shows the diffracted image of the half-plane illuminated by Gaussian beam with triple OV (Fig. 1**b**). As in the case of beam with double OV the number of singular points at the image and their position change with the half-plane edge position.

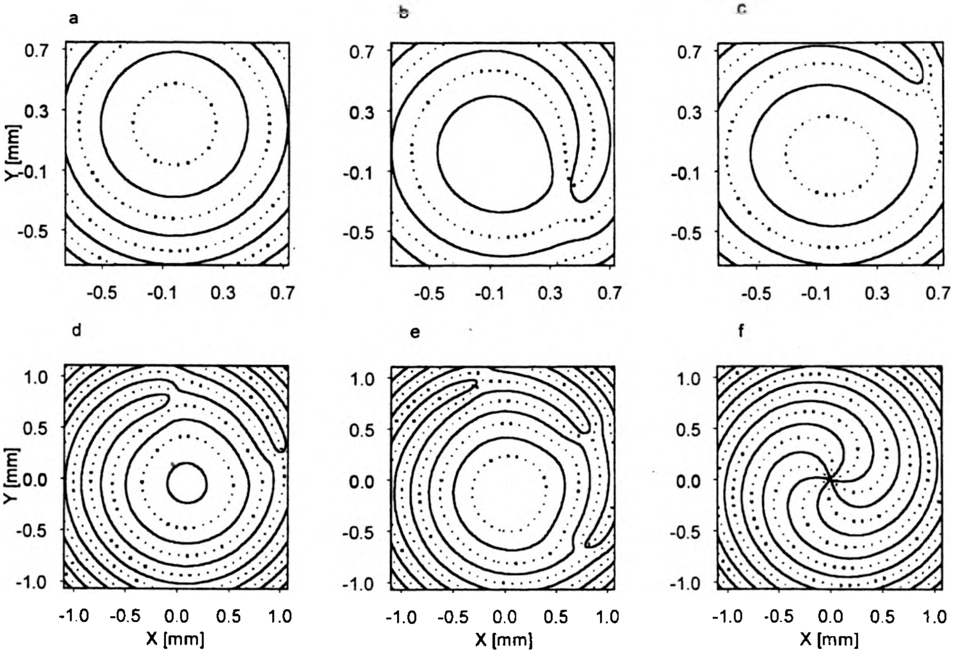

Fig. 4. Phase map of Gaussian beam with triple OV diffracted by a half-plane: **a** – $q = 0.25$ mm, **b** – $q = 0.03$ mm, **c** – $q = 0$ mm, **d** – $q = -0.05$ mm, **e** – $q = -0.12$ mm, **f** – $q = -1$ mm.

Figure 5 shows the diffraction image of single slit illuminated by the same beam as in the previous example. Changing the slit position one can change the number of singular points at the phase map (at the plotted area), Fig. 7**a**,**b**.

It should be noted that the number of observed singular points depends on the observation area. Figure 6 shows part of the phase map corresponding to that shown in Fig. 4**c**, but plotted in larger area. One can see one more singular point. Figure 7 shows the intensity distribution for the same case. Comparing Figs. 4**c**, 6 and 7 it is easy to notice that one of the singular points lies deep in the dark area of the diffraction pattern. For that reason it can be hardly detected. The author admits that

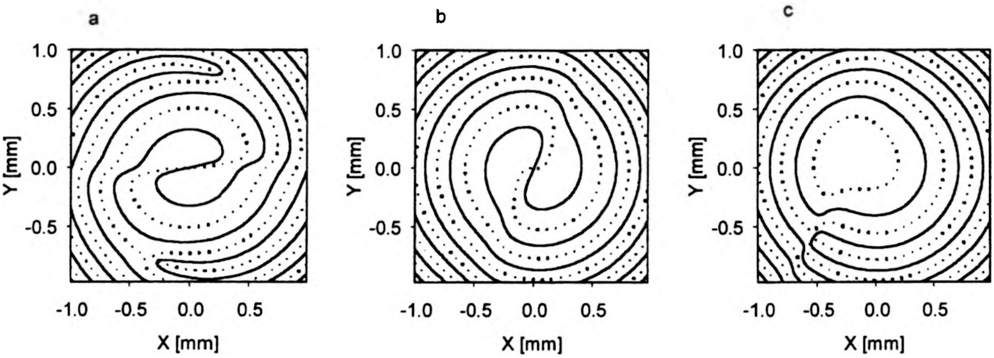

Fig. 5. Phase map of Gaussian beam with triple OV diffracted by a single slit: **a** — slit located symmetrically, slit width 0.24 mm, **b** — slit located symmetrically, slit width 0.1 mm, **c** — the same slit shifted, $q = -0.04$ mm, $h = 0.06$ mm.

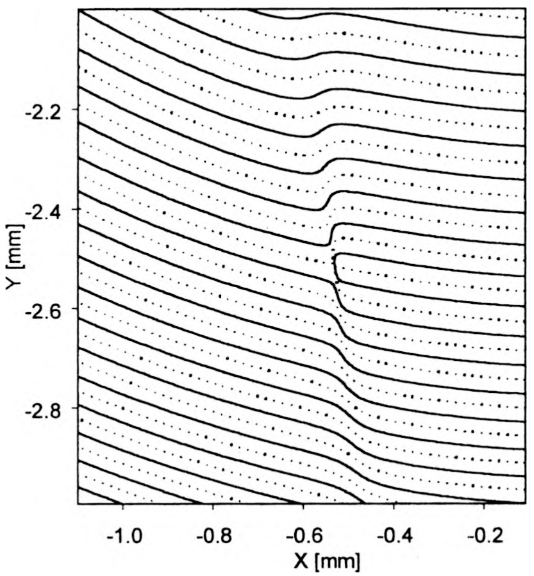

Fig. 6. Different part of the phase map shown in Fig. 4c ($q = 0$ mm). The intersection point shows the location of second OV.

similar results presented in paper [19] were misinterpreted. The conclusion was that for $q > 0$ no singular point survives the diffraction process, which is not exactly true. As regards the present case, for an observation area being large enough one can find singular point for $q > 0$, however, deep in the dark part of the diffraction patterns.

5. Conclusions

The results obtained using Fresnel–Kirchhoff diffraction integral were in very good agreement with numerical calculations based on known diffraction algorithm [21],

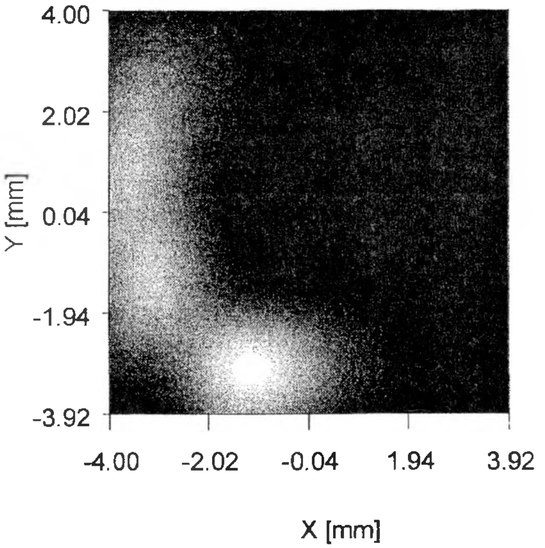

Fig. 7. Light intensity distribution of the beam with triple OV diffracted by a half-plane ($q = 0$ mm).

not presented in the paper. Unfortunately, the author is not ready to perform experimental verification. The method used in [19] is too poor to produce the multicharge OV beam of sufficient quality. For this purpose, more sophisticated methods have to be applied, which are not available to the author yet.

The author hopes that the present results are convenient even in the absence of experiments. The stability of the OV and their characteristic dynamical behaviour in diffraction process make them useful as beam markers that enable precise optical measurements. The dynamics of the compound OV is more characteristic and complicated than that of a single one [19]. This enables more complex measurements. Successful application of the OV to optical measurement demands a well constructed theoretical model describing them (model of diffraction and scattering process). This work is a contribution to developing such a model.

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