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Homographic wavelet analysis in identification of characteristic image features

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Wavelet transformations connected with subgroups $SL(2, \mathrm{C})$, performed as homographic transformations of a plane have been applied to identification of characteristic features of two-dimensional images. It has been proven that wavelet transformations exist for symmetry groups $SU(1, 1)$ and $SL(2, \mathrm{R})$.

1. Introduction

In the present work, the problem of an analysis, processing and recognition of a two-dimensional image has been studied by means of wavelet analysis connected with subgroups of $GL(2, \mathrm{C})$ group, acting in a plane by homographies $h_A(z) = \dfrac{az+b}{cz+d}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathrm{C})$. The existence of wavelet reversible transformations has been proven for individual subgroups in the group of homographic transformations. The kind of wavelet analysis most often used in technical applications is connected with affine subgroup $h(z) = az+b$, a case for $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, of the symmetry of plane $\bar{\mathrm{C}} \simeq S^2$ maintaining points at infinity [1], [2]. Adoption of the wider symmetry group means rejection of the invariability of certain image features, which is reasonable if the problem has a certain symmetry or lacks affine symmetry. The application of wavelet analysis connected with a wider symmetry group is by no means the loss of information. On the contrary, the information is duplicated for additional symmetries or coded by other means. Changing the symmetry group of wavelet transform we obtain another interpretation of its value. For instance, by filtering the homographic transform of a suitable wavelet we can cut off the lines of certain curvature and length.

Considering the problem from the point of view of the wider symmetry group, makes it possible to choose another symmetry subgroup characteristic of a given problem. This is the case, for instance, in recognition of objects from air and satellite photographs distorted by atmosphere or when distortion of shape occurs in stereographic projection of images obtained by means of cameras with hyperbolic or elliptic lenses, which are used in security systems or sight systems.

From the theoretical point of view, the wavelet analysis for locally dense Lie group relies on the existence of Haar measure for this group (the measure invariable to right or left multiplication in the group) and construction of integrable unreducible unitary representation [2]–[4]. The existence of the measure independent of locally dense groups is univocally guaranteed by Haar statement exact to the constant factor (see [5]). On the other hand, the integrability of a representation (in the sense of Statement 1) for the definite case sometimes does not occur [6], [7]. This takes place, *e.g.*, for the whole $GL(2, \mathrm{C})$ group acting on the C plane through homographies.

Studying a function with an image in complex numbers introduces additional difficulties from a technical point of view. It is natural to interpret $|f(z)|$ in terms of signal intensity. For $f(z)$ phase one can give an interpretation of polarization for problems with coherent light or as a colour of a point in periodic colour scale. Imposing some additional limitation on the function, *e.g.*, holomorphity, causes also limitations of the interpretation of the function value.

2. Wavelet analysis for affine group

Below the well known results of wavelet theory for affine group of the R^2 plane are presented in short. These results are described in terms of complex function analysis.

The R^2 plane with affine symmetry group $A(\mathrm{R}^2)$ can be accomplished as one-dimensional straight line C with affine group $A(\mathrm{C}) = \mathrm{C}^{*} \dot{+} \mathrm{C}$ topologically isomorphic with $S^1 \times \mathrm{R}_+ \times \mathrm{R}^2$. With $A(\mathrm{C}) \ni g = (a, b) : z \to a(z+b)$ acting on C, we obtain natural unreducible unitary representation T in a set of functions integrable with the square $L^2(\mathrm{C}, d\mu)$, $d\mu = d\bar{z}, dz$

$$T_g f(z) = |a|^{-1} f(a^{-1}(z-b)).$$

Defining the Fourier transform

$$Ff(k) = \hat{f}(k) = \frac{1}{2\pi} \int\limits_{\mathrm{C}} e^{-ikz} f(z) d\mu$$

one can obtain the representation $\hat{T}$ of the group $A(\mathrm{C})$ in frequency variables

$$\hat{T}_g \hat{f}(k) = |a| e^{-ibz} \hat{f}(\bar{a}z).$$

On the other hand, the $A(\mathrm{C})$ group operates also on itself, *e.g.*, by the left multiplication $L_g(h) = gh$, giving its representation over $L^2(A(\mathrm{C}), d\mu)$, with left-invariable measure $d\mu_A = \frac{1}{|a|^2} da d\bar{a} db d\bar{b}$.

Proposition 2.1. *T is the unreducible representation integrable with the square, i.e., there exists $\psi \in L^2(\mathrm{C}, d\mu)$, called the basic wavelet, such that*

$$\langle T_g \psi | \psi \rangle \in L^2(A(\mathrm{C}), d\mu_A).$$

In such a case the dense set of wavelets $\{T_g \psi\}$ also exists. The basic wavelet can be conveniently defined by the equivalent condition

$$(2\pi)^2 \int_{\mathrm{C}} |\hat{\psi}(k)|^2 \frac{d\bar{k}dk}{|k|^2} = 1$$

which cuts off the lower frequencies.

Proposition 2.2 *The linear mapping $W_\psi : L^2(\mathrm{C}, d\mu) \to L^2(A(\mathrm{C}), d\mu_A)$*

$$W_\psi f(g) := \langle T_g \psi | f \rangle$$

called wavelet transformation is the isomorphism of Hilbert space

$$\langle W_\psi f | W_\psi f \rangle = \langle f | f \rangle.$$

The reciprocal transformation of the $W_\psi(L^2(\mathrm{C}, d\mu)$ image is given by the following formula:

$$W_\psi^{-1} f(z) = \int_{A(\mathrm{C})} f(g) T_g \psi(z) d\mu_A.$$

Corollary 2.3. *The image of the transformation $W_\psi(L^2(\mathrm{C}, d\mu)) \subset L^2(A(\mathrm{C}), d\mu_A)$ is a Hilbert subspace with reproducing kernel*

$$L(g', g) = W_\psi \psi(g'^{-1} g).$$

Hence, for any $f, g \in L^2(\mathrm{C}, d\mu)$ and any wavelet ψ one can obtain the following distribution of the unity in $L^2(\mathrm{C}, d\mu)$

$$\langle f | h \rangle = \int_{A(\mathrm{C})} \langle f | T_g(\psi) \rangle \langle T_g(\psi) | h \rangle d\mu_A.$$

Proposition 2.4. *There exists a discrete symmetry subgroup $\Lambda < A(\mathrm{C})$ which generates a base $\{T_\lambda \psi\}$ in $L^2(\mathrm{C})$, in the following sense: there are $a, b \in \mathrm{R}_+$ for which*

$$a \|f\|^2 \leqslant \sum_{\lambda \in \Lambda} |\langle \lambda | f \rangle|^2 \leqslant b \|f\|^2.$$

For instance, $\Lambda_{(\sqrt{p}, q)} = ((\sqrt{p})^l, qk) \simeq \mathrm{Z}_L \times \mathrm{Z}^2$ where $l \in \mathrm{Z}_L$, $k \in \mathrm{Z}^2$ is a subset.

The above statements have been proved elsewhere [3], [7]. The following examples illustrate continuous wavelets:

Mexican hat, for any positive operators $A, B > 0$ is defined in Fourier coordinates as:

$$\hat{\psi}(k) = \langle k | Ak \rangle e^{-\langle k | Bk \rangle}.$$

Morlet wavelet, for any positive and reversible operator C we define
$\psi(z) = (e^{2z_0 z} - e^{-\langle z_0|C^{-1}z_0\rangle})e^{-\langle z|Cz\rangle}$.

3. Homographic wavelet analysis

The requirement that symmetry group should preserve linear structure of a space is not generally justified in problems especially connected with a recognition and decoding. It is necessary in such a case to include a wider symmetry group, consistent with a problem given. For instance, in case of processing and recognition of images deformed by elliptical and hyperbolic lenses, homographies are the proper symmetry groups.

3.1. Properties of a homography

Let us remind of the essential properties of homographic transformations of a plane:

Lemma 3.1. *If* (z_1,z_2,z_3) *and* (w_1,w_2,w_3) *are two threes of different points in condensed plane* $\bar{\mathrm{C}} \simeq S^2$, then there exists exactly one homography h transforming those points on themselves: $h(z_1) = w_1$, $h(z_2) = w_2$, $h(z_3) = w_3$.

Lemma 3.2. *Homographies preserve an intersection angle between curves.*

Group $GL(2,\mathrm{C})$ has got two-sided Haar measure, which in coordinates $g = \begin{pmatrix} a_1, & a_2 \\ a_3, & a_4 \end{pmatrix}$ equals $d\mu_{GL} = |\det g|^{-2} da_1 d\bar{a}_1 \ldots da_4 d\bar{a}_4$, but its representation as homographic transformations on a plane $h_g(z) = \dfrac{a_1 z + a_2}{a_3 z + a_4}$ is not integrable.

Let us classify subgroups $GL(2,\mathrm{C})$, acting at C as homographies, according to geometries described by them [5]:

Affine geometry: $\mathrm{C} \simeq \mathrm{R}^2$ with the volume measure $d\mu = d\bar{z}dz$ and affine symmetry group $A(\mathrm{C})$ of elements h_A generated by matrix $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. The right Haar measure for this group is $d\mu_{A(\mathrm{C})} = \dfrac{1}{|a|^2} da d\bar{a} db d\bar{b}$.

Spherical geometry: $S^2 \simeq \bar{\mathrm{C}}$ with volume measure in stereographic coordinates $d\mu_{\bar{\mathrm{C}}} = \dfrac{d\bar{z}dz}{(1+\bar{z}z)^2}$ and symmetry groups of homography h_A generated by matrices $A \in SU(2)$. Haar measure for this group, expressed in coordinates called Euler angles equals $d\mu_{SU(2)} = \dfrac{1}{2\pi^2} \sin 2\theta d\theta d\varphi d\psi$, where $\theta \in [0,\pi/2)$, $\varphi \in [0,\pi)$, $\psi \in [-\pi,\pi)$. Group elements are represented in those coordinates by

$$A = \begin{pmatrix} \cos\theta e^{i(\varphi+\psi)} & i\sin\theta e^{i(\varphi-\psi)} \\ i\sin\theta e^{i(\psi-\varphi)} & \cos\theta e^{-i(\varphi-\psi)} \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

creating the dense set.

Elliptic geometry: $\mathsf{D} = \{z \in \mathsf{C}; |z| < 1\}$ with volume measure $d\mu_{\mathsf{D}} = \dfrac{d\bar{z}dz}{(1-\bar{z}z)^2}$ and symmetry groups of homographies of groups generated $SU(1,1)$. For this geometry Haar measure is $d\mu_{SU(1,1)} = \dfrac{1}{\pi}\sin 2\tau d\tau d\varphi d\psi$, where: $\tau \in \mathsf{R}$, $\varphi \in [0, \pi)$, $\psi \in [-\pi, \pi)$ are Euler angles as coordinates of $SU(1,1)$. The elements of the following form:

$$A = \begin{pmatrix} \cosh\tau e^{i(\varphi+\psi)} & \sinh\tau e^{i(\varphi-\psi)} \\ \sinh\tau e^{i(\varphi-\psi)} & \cosh\tau e^{-i(\varphi-\psi)} \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

are dense in the group. Other orbits of $SU(1,1)$ group acting at $\bar{\mathsf{C}}$ are $\mathsf{C}\backslash\bar{\mathsf{D}}$ and $\partial\mathsf{D} \cup \{\infty\}$. Due to the local isomorphism $SL(2,\mathsf{R}) \simeq SU(1,1)$ and Caley transformation

$$\mathsf{D} \ni z \to \frac{z+i}{iz+1} \in \mathsf{C}_+$$

we can perform elliptical geometry above the half-plane $\mathsf{C}_+ = \{z \in \mathsf{C}; \mathrm{Im}\, z > 0\}$ with measure $d\mu_{\mathsf{C}_+} = \left(\dfrac{z-\bar{z}}{2i}\right)^2 dz d\bar{z}$ and symmetry group of homography h_A, where $\mathsf{A} \in SL(2, \mathsf{R})$. In this case, Haar measure is $d\mu_{SL(2,R)} = \dfrac{1}{\pi} e^{2t} d\varphi dt ds$, where $t \in \mathsf{R}$, $s \in \mathsf{R}$, $\varphi \in [-\pi, \pi)$ are Euler angles for $SL(2, \mathsf{R})$. In these coordinates any group element can be approximated by elements of the following form:

$$A = \begin{pmatrix} \cos\varphi e^t, & s\cos\varphi e^t + \sin\varphi e^{-t} \\ -\sin\varphi e^t, & -s\sin\varphi e^t + \cos\varphi e^{-t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

C_+ is one of the orbits C_+, C_-, $\mathsf{C}_0 \cup \{\infty\}$ acting in this group on the whole Riemann plane $\bar{\mathsf{C}}$, classified by a sign of the imaginary part.

3.2. Elliptic wavelets

The $SU(1,1)$ group consists of elements of form $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, where $|a|^2 - |b|^2 = 1$. As the subgroup $GL(2, \mathsf{C})$ it acts a natural way at $\bar{\mathsf{C}}$ through homographies $h_g(z) = \dfrac{az+b}{\bar{b}z+\bar{a}}$. One of this action orbits is disc D. Going back to this action at functions over disc D, we can obtain the whole family of actions parameterized by $\lambda > 0$ in the form

$$T_g f(z) = \frac{1}{(\bar{b}z+\bar{a})^\lambda} f\left(\frac{az+b}{\bar{b}z+\bar{a}}\right). \tag{2}$$

Let $\mathscr{D}_\lambda$ be the Hilbert space of analytical functions over the disc D integrable with the square relative to the measure $d\mu_{\mathsf{D}} = \dfrac{\lambda-1}{\pi}(1-\bar{z}z)^{\lambda-2} d\bar{z}dz$ invariant to the above action. Representations $SU(1,1)$ described by Eq. (2) in Hilbert spaces $\mathscr{D}_\lambda$ with scalar product

$$\langle f|g\rangle = \int_{\mathsf{D}} \overline{f(z)}\, g(z) d\mu_{\mathsf{D}}$$

are unitary and unreducible for natural $\lambda > 1$. It is shown below that the above representations of $SU(1,1)$ group are integrable in the sense of Statement 1.1.

Let $\hat{z}$ be an operator of multiplication by an argument $\hat{z}f(z) := zf(z)$. Its conjugation in the sense of a scalar product $\langle\cdot|\cdot\rangle_\lambda$ is an operator of the following form

$$\delta := \hat{z}^+ = \partial_z \frac{1}{\lambda - 1 + z\partial_z}.$$

Eigenstates of this operator, called coherent states or generalized exponents, are candidates for being the basic wavelets. There are holomorphic functions $K_v(z) = \frac{1}{(1-vz)^\lambda}$, well known in the theory of coherent states, numbered by eigenvalues $v \in \mathscr{D}$. Thes create a dense set in $\mathscr{D}_\lambda$. Moreover, a unity distribution takes place

$$f(v) = \int_{\mathsf{D}} f(z) \frac{1}{(1-\bar{z}v)^\lambda} d\mu_{\mathsf{D}}. \tag{3}$$

Proposition 3.3. *The unreducible and unitary representations T of the $SU(1,1)$ group defined by Eq. (2) are integrable, i.e., there exist $\psi \in \mathscr{D}_\lambda$ such that*

$$\langle T_g\psi|\psi\rangle \in L^2(SU(1,1), d\mu_{SU(1,1)}).$$

Proof. Let us take $\psi = K_0 \equiv 1$ a constant function on the disc for the basic wavelet. In such a case

$$T_g\psi(z) = (\bar{b}z + \bar{a})^{-\lambda} = \bar{a}^{-\lambda} K_{\bar{b}/\bar{a}}(z),$$

and after using unity distribution (3) one can find

$$\langle T_g\psi|\psi\rangle = \int_{\mathsf{D}} 1 \cdot \frac{1}{(b\bar{z}+a)^\lambda} d\mu_{\mathsf{D}} = a^{-\lambda}.$$

Using Euler coordinates one can calculate

$$\int_{SU(1,1)} |\langle T_g\psi|\psi\rangle|^2 d\mu_{SU(1,1)} = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\tau \int_0^\pi d\varphi \int_{-\pi}^{\pi} d\psi \sinh(2\tau)\cosh^{-2\lambda}(\tau)$$
$$= 4\pi \int_1^\infty x^{-2\lambda+1} dx = \frac{2\pi}{\lambda-1} < \infty.$$

∎

Of course, other basic wavelets can also exist, *e.g.*,

$$\psi(z) = \theta(\arg z - \alpha)\theta(\arg z)$$

which cuts a window viewing angle α. If ψ is a holomorphic function such that $\langle T_g\psi|\psi\rangle$ is integrable over $SU(1,1)$ to unity, then the so-called wavelet transformation exists: $W_\psi : \mathscr{D}_\lambda \to L^2(SU(1,1), d\mu_{SU(1,1)})$

$$(W_\psi f)(a,b) = \frac{\lambda-1}{\pi}\int_{\mathsf{D}} f(z)\psi\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)\frac{(1-\bar{z}z)^{\lambda-2}}{(b\bar{z}+a)^\lambda}\,d\bar{z}dz.$$

The inverse transformation becomes

$$(W_\psi^{-1}g)(z) = \int_{SU(1,1)} g(a,b)\overline{\psi\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)}\frac{1}{(\bar{b}z+\bar{a})^\lambda}\,d\mu_{SU(1,1)}.$$

For instance, if the basic wavelet is in a coherent state $\psi_v = \sqrt{\frac{\lambda-1}{\pi}}K_{\bar{v}}$, then the corresponding wavelet transform W_v acts in the following way:

$$W_v f(a,b) = \sqrt{\frac{\lambda-1}{\pi}}(a-v\bar{b})^{-\lambda} f\left(\frac{v\bar{a}-b}{a-v\bar{b}}\right),$$

and its reverse

$$W_v^{-1}g(z) = \sqrt{\frac{\lambda-1}{\pi}}\int_{SU(1,1)}\frac{g(a,b)}{(\bar{a}-\bar{v}b+(\bar{b}-\bar{v}a)z)^\lambda}\,d\mu_{SU(1,1)}.$$

So we have a family wavelet transforms for $SU(1,1)$ group, parameterized by points of a disc D and natural numbers $\lambda > 1$.

3.3. Representations of $SL(2,\mathsf{R})$ over C_+

Let C_+ be an orbit of $SL(2,\mathsf{R})$ acting on $\bar{\mathsf{C}}$. For each $\lambda > 0$ the operation

$$T_g f(z) = \frac{1}{(cz+d)^\lambda} f\left(\frac{az+b}{cz+d}\right), \quad g \in SL(2,\mathsf{R}) \tag{4}$$

is determined in the function set over C_+. The measure of invariance of this operation is $d\mu_{\mathsf{C}_+} = \frac{\lambda-1}{\pi}\left(\frac{z-\bar{z}}{2i}\right)^{\lambda-2} dz d\bar{z}$. Let us denote by $\mathscr{F}_\lambda$ the Hilbert space of analytical functions with scalar product

$$\langle f|g\rangle = \int_{\mathsf{C}_+}\overline{f(z)}g(z)d\mu_{\mathsf{C}_+}.$$

For natural λ, T are unitary unreducible representations in Hilbert spaces $\mathscr{F}_\lambda$. Similarly as in the former case, one can find the coupling $\delta = \hat{z}^*$ of multiplying operator by an argument $\hat{z}f(z) = zf(z)$, which is as follows:

$$\delta f(z) = (\lambda-1)\int_i^z f(\tau)d\tau + zf(z).$$

Its eigenvalues are $v \in \mathrm{C}_-$, while corresponding eigenvectors are holomorphic functions

$$K_v(z) = \left(\frac{2i}{z-v}\right)^\lambda$$

known to us as coherent states for $SU(1,1)$ after Caley transform. According to the construction of the representation in the Hilbert space $\mathscr{F}_\lambda$, the following unity distribution takes place

$$f(v) = \int_{\mathrm{C}_+} f(z)\left(\frac{2i}{v-\bar{z}}\right)^\lambda d\mu_{\mathrm{C}_+}. \tag{5}$$

The basic wavelet ψ is defined as an element such that the integer of the function $\langle T_g\psi|\psi\rangle$ over $SL(2,\mathrm{R})$ relative to Haar measure is equal to 1.

Proposition 3.4. *Representations (4) of the $SL(2,\mathrm{R})$ group are integrable, i.e., basic wavelet exists.*

Proof. Let us consider the following wavelet $\psi = K_{-i} = \left(\frac{2i}{z+i}\right)^\lambda$. In this case $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms ψ to $T_g\psi = \left(\frac{2i}{(a+ic)z+b+id}\right)^\lambda$. Using unity distribution in $\mathscr{F}_\lambda$ one can calculate

$$\langle T_g\psi|\psi\rangle = \int_{\mathrm{C}_+}\left(\frac{2i}{z+i}\right)^\lambda\left(\frac{2i}{id-b-(a-ic)\bar{z}}\right)^\lambda d\mu_{\mathrm{C}_+} = \frac{1}{(a-ic)^\lambda}\psi\left(\frac{id-b}{a-ic}\right)$$

$$= \left(\frac{2i}{i(d+a)+c-b}\right)^\lambda = \left(\frac{2ie^{-i\varphi}e^t}{-se^{2t}+i(e^{2t}+1)}\right)^\lambda$$

where: t, s, φ are Euler angles described by Eq. (1). Afterwards one can verify that (for $\lambda > 1$)

$$\int_{SL(2,\mathrm{R})} |\langle T_g\psi|\psi\rangle|^2 d\mu_{SL(2,\mathrm{R})} = \frac{1}{\pi}\int_{\mathrm{R}} dt \int_{\mathrm{R}} ds \int_{-\pi}^{\pi} d\varphi e^{2t}\left(\frac{2e^{2t}}{s^2e^{4t}+(e^{2t}+1)^2}\right)^\lambda =$$

$$= \int_0^\infty dx \int_R ds \left(\frac{2x}{s^2x^2+(1+x)^2}\right)^\lambda = \int_0^\infty 2^\lambda \frac{x^{\lambda-1}}{(1+x)^{2\lambda-1}} dx \int_R \frac{1}{(1+s^2)^\lambda} ds < \infty.$$

Dividing ψ by the corresponding constant one can obtain the basic wavelet. ∎

Wavelet transform $W_\psi: \mathscr{F}_\lambda \to L^2(SL(2,\mathrm{R}), d\mu_{SL(2,\mathrm{R})})$ for each wavelet ψ has the following form

$$(W_\psi f)(a,b,c,d) = \frac{\lambda-1}{\pi} \int_{\mathrm{C}_+} f(z)\psi\left(\frac{az+b}{cz+d}\right) \frac{1}{(c\bar{z}+d)^\lambda}\left(\frac{z-\bar{z}}{2i}\right)^{\lambda-2} d\bar{z}dz$$

where: a, b, c, $d \in \mathrm{R}$; $ad-bc=1$. On the other hand, the reverse transform has the form

$$(W_\psi^{-1} g)(z) = \int g(a,b,c,d)\,\psi\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^\lambda} d\mu_{SL(2,R)}.$$

For wavelet $\psi_v = K_{\bar{v}}$, $v \in \mathrm{C}_+$ transformation generated is proportional in action to the following transformation:

$$W_v f(a,b,c,d) = \frac{1}{(a-vc)^\lambda} f\left(\frac{vd-b}{a-vc}\right),$$

and reverse to transformation

$$W_v^{-1} g(z) = \int_{SL(2,\mathrm{R})} \frac{g(a,b,c,d)}{((a-\bar{v}c)z+b-\bar{v}d)^\lambda} d\mu_{SL(2,\mathrm{R})}.$$

3.4. Wavelets for $SU(2)$

In this case, situation is quite simple, because $SU(2)$ is a compact group. As a result, all its unreducible unitary representations are integrable [1], and any normalized vector of space can be considered as the basic wavelet. Examples of such a representation are given below.

We give analogous expressions for transforms of this subgroup.

The $SU(2)$ group consisting of matrices of the following form $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $|a|^2+|b|^2=1$, topologically equivalent to S^3, acts on Riemann plane through homographies in a transitive way and has only one orbit $\bar{\mathrm{C}}$. This operation is transferred to the function set over $\bar{\mathrm{C}}$ according to the formula

$$T_g f(z) = \frac{1}{(-\bar{b}z+\bar{a})^\lambda} f\left(\frac{az+b}{-\bar{b}z+\bar{a}}\right), \quad \lambda > 0.$$

For discrete λ, measures $d\mu_{\mathrm{C}} = \frac{\lambda-1}{\pi^2}(1+\bar{z}z)^{\lambda-2}d\bar{z}dz$ invariant in relation to the above expression define Hilbert spaces S_λ of analytical functions with scalar product

$$\langle f|g\rangle = \int_{\mathrm{C}} \overline{f(z)}g(z)d\mu_{\mathrm{C}},$$

on which T acting as unitary unreducible representations.

According to integrability of representations of dense groups for any function $\psi \in S_\lambda$ normalized by a condition $\langle T_g\psi|\psi\rangle_{SU(2)} = 1$ one can define the transform

$$(W_\psi f)(a,b) = \int_{\mathrm{C}} f(z)\,\psi\left(\frac{az+b}{-\bar{b}z+\bar{a}}\right)\frac{(1-\bar{z}z)^{\lambda-2}}{(-b\bar{z}+a)^\lambda}\,d\bar{z}dz$$

where $(a,b)\in \mathrm{C}^2$; $|a|^2+|b|^2 = 1$. The inverse transform has the following form:

$$(W_\psi^{-1}g)(z) = \int_{S^3} g(a,b)\overline{\psi\left(\frac{az+b}{-\bar{b}z+\bar{a}}\right)}\frac{1}{(-\bar{b}z+\bar{a})^\lambda}\,d\mu_{SU(2)}.$$

4. Summary

The transformations proposed can be adopted as additional useful tools in solving problems of an analysis and recognition of objects obtained from air photographs. In general, the above wavelet transforms have justified application in all the cases in which an image is conformally deformed, without change of angles and with simultaneous change of the curvature. Because inverse wavelet transformation is not used for the recognition of images by means of extraction of characteristic features, therefore identification of characteristic features can be performed practically by any analyzing function ψ.

The wavelet transformation for $SU(1,1)$, $SL(2,\mathrm{R})$, $SU(2)$ can be adopted for sight systems with very wide viewing angle. In such systems the deformation of image makes it impossible to identify an object by application of the wavelet transformation corresponding to the affine symmetry group. The application of the transformation connected with a suitable symmetry subgroup of $GL(2,\mathrm{C})$ (depending of the lens shape), enables us to avoid the problem of reconstruction of the deformed image. It is especially important when one does not know in what way the image has been deformed. For instance, the same object observed by the spherical lens will have different shape if located in different places. It means that there is no symmetry for shifts. The application of an ordinary affine wavelet analysis in such a case will make it impossible to identify the same objects located in different positions with respect to the lens axis. Therefore, it is necessary "to straighten" the image before wavelet analysis. Another possibility is to apply the other wavelet analysis (in this example it should correspond to spherical geometry) instead of "straightening".

Moreover, the present work enhances the method of searching basic wavelets for a given representation as common eigenvectors of the family of operators coupled with operators of multiplying by a certain family of functions. We used here one operator, namely multiplying by an argument.

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