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On the Application of the Theory of Normal Congruences to the Examination of both Transforming Properties of the Light Beams, and Imaging Quality in Noncentric Optical Systems

In this paper the transforming formulae for coefficients of quadratic form for the ray-bundles of arbitrary divergence, which are treated as normal congruences, have been derived for the case of noncentric optical systems.

Two measures of image quality have been proposed basing on study of spreading degree of middle surfaces for beam-congruence: some examples of numerical calculation of the proposed measures are given.

1. Introduction

From the geometrical viewpoint any light beam of arbitrary divergence, emerging from an arbitrary object point may be treated as a normal rectilinear congruence.

A beam-congruence passing through the optical system, when refracted or reflected at the surfaces of the system, as well as in the intersurface spaces, is subjected to consecutive transformations.

In the present paper an attempt has been made to examine the transforming properties of the quantities characterizing the congruence in the course of imaging a single object point by a noncentric optical system. The considerations are restricted to the systems composed of spherical and plane surfaces interfacing uniform optical media. New measures of imaging quality in noncentric optical systems based on examination of the so-called central congruence surface [1] in the image surface have been moreover, suggested.

The application of the theory of normal congruences, and spealing widely, of differential geometry, to various problems of geometrical optics is by no means new.

General properties of a beam of rays resulting from the *Fermat Principle* and the *Laws of Malus* may be found in the fundamental works in the field of geometrical optics by Czapski and Eppenstein [2], Tudorowski [3], Luneburg [4], Born and Wolf [5], Herzberger [6], Stavroudis [7] and others. The congruence through an optical system of rotational symmetry has been traced by Herzberger [8] and [9] with the approximation up to fifth order of expansion of the magnitudes characterizing the congruence with respect to the parameters determining the congruence. Kneisly [10], Stavroudis [7] and Parker [11] discuss the properties of the wavefronts (in the sense of geometrical optics) and calculate the caustic surfaces. Hofmann and Klebe in [12] and [13] derive the formulae from which the position of the principal curvature centres of an arbitrary beam passing through the single refracting surface can be calculated. The method given by the authors quoted requires that the angles between the main direction of the wave front and the main directions of the imaging surface be estimated for each ray of beam, separately. Thus, none of the so far published papers deals with transformation of the quantities characterizing the congruence as a beam rays passing through a noncentric optical system.

The problem of the imaging quality in noncentric optical systems appears when the tolerances on the noncentricity of elements in the centric optical systems are to be determined, as well as when optical systems of great in-

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tentional noncentricity are designed. In such systems the geometric aberrations defined in classical way, as well as physical measures and cryteria may be used to approximate evaluation of the imaging quality only because the definitions of those measures include the assumption of rotational symmetry of the optical system. In concentric systems it is difficult, moreover, to establish a plane, which would correspond to the Gaussian plane for ideally centred systems. In the first order of approximation the problem was discussed by Bartkowska [14], [15] and Sands [16], [17]. The geometrical aberrations defined for ideally centric systems (after a slight modification of the respective formulae) were applied to an imaging quality evaluation of noncentric systems e.g. in [18] and [19]. It should be noticed, however, that the greater the noncentricity the more inaccurate is such an evaluation. In the region of third order optics the formulae for aberrations caused by decentrations were given by Slevogt [20], Bartkowska [21] and Gerlovin [22], while the influence of decentration on the changes in wave aberration was considered by Hopkins and Tiziani [23] and Rimmer [24].

The calculation of the position of the astigmatic foci in the systems without the plane of symmetry is a separate problem considered among other in papers [12], [13] and [25].

In the present paper a new method of optical imaging evaluation is proposed for noncentric systems. This method is independent of the plane of reference (i.e. of the plane corresponding to the Gaussian plane for the centric systems) and does not require any approximations.

2. Fundamental quantities characterizing the congruence

In order to examine the properties of a beam passing through an optical system by employing the formalism of the theory of congruence the beam should be described by its direction vector $\boldsymbol{l}$ as a function of two parameters u^1, u^2 and by an arbitrary surface $\boldsymbol{R}$ (as a function of the same parameters, called the reference surface, through which the beam passes. The choice of the type of parametrization in the concrete cases depends on the type of problem (see for instance, [4], [7–13]).

The purpose of the present paper is to examine the transforming properties of the beam of rays (and not of the wavefront). Hence, directional cosines p and q between the ray and the x- and y-axes of the Cartesian coordinate system have been chosen as parameters. In the assumed coordinate system the x-axis represents a fixed direction, used to determine the position of the curvature centres of the successive surfaces of the optical imaging system. Due to this parametrization all the quantities characterizing the beam in the course of its transformation on the successive imaging surfaces can be expressed with respect to one coordinate system. From the mathematical viewpoint the parallel beam presents a particular case, as then the directional vector of the congruence is independent of the parameters (singular congruence). However, a strictly parallel beam is a theoretical idealization and may appear only in the object space. Thus it is sufficient for the beam to pass the first imaging surface, and the singularity disappears.

In the theory of congruence the so-called focal surfaces and the middle surface are considered as characteristic reference surfaces of the congruence. The focal surfaces represent from the viewpoint of the geometric optics the locus of the astigmatic foci of the infinitely thin beams of rays composing the beams of finite divergence, and are known as caustic surfaces, while the middle surface is a surface intermediate between the focal surfaces, at which the beam becomes the narrowest. Therefore the determination of the middle surface position is important for the evaluation of the imaging quality from the viewpoint of geometrical optics. In order to find the middle and focusing surfaces of congruence (beam) the coefficients G_{ij} and B_{ij}, as well as the respective discriminants G and B of the first and second quadratic forms of this congruence must be known [1]. The coefficients G_{ij} and B_{ij} are defined as follows:

$$G_{ij} = \boldsymbol{l}_i \boldsymbol{l}_j, \tag{1}$$

$$B_{ij} = \boldsymbol{l}_i \boldsymbol{R}_j, \tag{2}$$

where – according to denotation assumed in differential geometry – the indices i and j, ascribed to $\boldsymbol{l}$ and $\boldsymbol{R}$, denote the derivatives of these vectors with respect to parameters used to determine the above vectors. Moreover, the derivatives with respect to the given parameters

p or q (in chapter 5.5. y or z) will be denoted by the index p or q, respectively (in chapter 5.5. y or z). Other literal or numerical indices will not denote the derivatives.

Hereafter, the coefficients g_{ij} and discriminant g of the first quadratic form of the congruence reference surfaces will also appear. (By the reference surfaces we mean here the imaging surfaces of the optical system.) The coefficients g_{ij} are defined as follows:

$$g_{ij} = \boldsymbol{R}_i \boldsymbol{R}_j. \tag{3}$$

If the vector functions $\boldsymbol{l} = \boldsymbol{l}(p, q)$ are known and $\boldsymbol{R} = \boldsymbol{R}(p, q)$ then the middle surface $\boldsymbol{R}_c$ of the congruence is defined as

$$\boldsymbol{R}_c = \boldsymbol{R} - T\boldsymbol{l}, \tag{4}$$

while the focal surfaces as

$$\begin{aligned} \boldsymbol{R}_{t1} &= \boldsymbol{R} + t_1 \boldsymbol{l}, \\ \boldsymbol{R}_{t2} &= \boldsymbol{R} + t_2 \boldsymbol{l}. \end{aligned} \tag{5}$$

The quantity T is a length of a segment measured from the reference surface along the rectilinear congruence (along the ray), where

$$2T = \frac{G_{11}B_{22} + G_{22}B_{11} - 2G_{12}B_{12}}{\sqrt{G}}. \tag{6}$$

The quantities t_1 and t_2 being also the segment measured along the ray are the roots of a quadratic equation

$$t^2 + 2Tt + \frac{B}{G} = 0;$$

thus

$$\begin{aligned} t_1 &= -T + \sqrt{T^2 - \frac{B}{G}}, \\ t_2 &= -T - \sqrt{T^2 - \frac{B}{G}}. \end{aligned} \tag{7}$$

From the optical viewpoint t_1 and t_2 are the positions of the astigmatic foci of a infinitely thin beam of rays around the ray of direction $\boldsymbol{l}$.

3. Scheme of ray tracing in noncentric optical systems

Before numerical calculation of the congruences of the light beams are performed, the way of determining the decentration must be accepted and scheme of ray tracing in noncentric optical systems developed.

The way of determining the decentration appearing during production of the optical elements, as well as during assembling of those elements was discussed by HOFMANN in [26–30] and by HOPKINS and TIZIANI in [23]. After having analyzed these works we have decided to take the vector $\Delta\boldsymbol{c}\{0, \Delta y, \Delta z\}$. As practical measure of decentration to be used in the course of ray-tracing. The length of the vector is equal to the distance between the curvature centre of the decentred surface in the optical system and the reference axis i.e. the x-axis. For the tilted plane surfaces, for which the vector $\Delta\boldsymbol{c}$ does not exhibit any physical sense, its components may be calculated from the following formulae:

$$\begin{aligned} \Delta y &= d A_y, \\ \Delta z &= d A_z, \end{aligned} \tag{8}$$

where d is an axial distance of the considered plane surface from the preceding surface of the optical system, and $\boldsymbol{A}_y$ and $\boldsymbol{A}_z$ are components of a unity vector normal to the tilted plane surface. The vector $\Delta\boldsymbol{c}$ originates from the point of intersection of x-axis with the given surface.

ALLEN and SNYDER [31] and STAVROUDIS [7] give schemes of ray tracing in noncentric optical systems. A common feature of those schemes is that the coordinates of the intersection points of the ray with the succesive surfaces of the optical system, as well as directional cosines of the rays are described in a coordinate system, in which one axis changes its direction with respect to the consecutive surfaces, according to the direction of the so-called ray-base. Since the directional cosines are the parameters of the congruences with respect to the reference system of steady direction of the axis, a new ray-tracing scheme being a modification of the Feder's scheme for ideally centric systems has been elaborated [32].

To calculate the ray passage from one surface to the next one and to estimate the directional vector $\boldsymbol{l}$ after refraction or reflection the following procedure is necessary

$$\boldsymbol{K} = d\boldsymbol{A}_0 - \boldsymbol{R} + \Delta\boldsymbol{c}^*, \tag{9a}$$

$$e = \boldsymbol{K} \cdot \boldsymbol{l}, \tag{9b}$$

$$\boldsymbol{M}^2 = \boldsymbol{K}^2 - e^2, \tag{9c}$$

$$M_A = (e\boldsymbol{l} - \boldsymbol{K})\boldsymbol{A}^* \tag{9d}$$

$$\cos i^* = \sqrt{(\boldsymbol{l}\boldsymbol{A}^*)^2 - C^*(C^*\boldsymbol{M}^2 - 2M_A)}, \tag{9e}$$

$$\cos i' = \sqrt{1-\gamma^2(1-\cos^2 i)}, \tag{9f}$$

$$a = \cos i' - \gamma \cos i, \tag{9g}$$

$$e_1^* = \frac{C^* \boldsymbol{M}^2 - 2M_A}{(\boldsymbol{l}\boldsymbol{A}^*)+\cos i^*} \tag{9h}$$

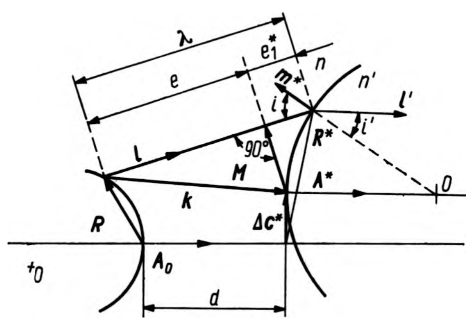
$$\lambda = e + e_1^*, \tag{9i}$$

$$\boldsymbol{R}^* = \boldsymbol{R} + \lambda \boldsymbol{l} - d\boldsymbol{A}_0, \tag{9j}$$

$$\boldsymbol{m}^* = -C^*(\boldsymbol{R}^* - \Delta \boldsymbol{c}^*) + \boldsymbol{A}^*, \tag{9k}$$

$$\boldsymbol{l}' = \gamma \boldsymbol{l} + a\boldsymbol{m}, \tag{9l}$$

where M_A is a projection of the vector $\boldsymbol{M}$ onto the direction of the vector $\boldsymbol{A}$; C is the reciprocal of the curvature radius of the surface; and γ is a ratio of refractive indices in front of and behind the surface considered. The other values, occurring in the formulae (9a)–(9l) have been marked in Figure. The quantities denoted by



stars refer to the next surface during the passage from one surface to the next one, while those primed denote the quantities after reflection or reflection of the ray. These notation will be used hereafter. The vector $\boldsymbol{A}_0$ has always components $\{1, 0, 0\}$; vector $\boldsymbol{A}$ for the spherical surface has also components $\{1, 0, 0\}$, while those for the plane surface are calculated from the given value and the tilting azimuth [19].

4. Fundamental transforming properties

Hereafter we will use the following formulae from the theory of surfaces and congruences:

$$\boldsymbol{l} = \frac{1}{\sqrt{G}}(\boldsymbol{l}_p \times \boldsymbol{l}_q), \tag{10}$$

$$\boldsymbol{m} = \frac{1}{\sqrt{g}}(\boldsymbol{R}_p \times \boldsymbol{R}_q), \tag{11}$$

where $\boldsymbol{m}$ is a unity vector normal to the surface $\boldsymbol{R}$ (which in particular may be an imaging surface of the optical system).

A scalar multiplication of the both sides of equation (10) by the respective sides of eq. (11) yields the cosine of the incidence angle of the ray in the surface on the left-hand side and the product $(\boldsymbol{l}_p \times \boldsymbol{l}_q)(\boldsymbol{R}_p \times \boldsymbol{R}_q)$ equal to the discriminant of the second quadratic form of the congruence. Hence,

$$\cos i = \frac{B}{\sqrt{Gg}}. \tag{12}$$

From the Malus law [1], [4], [7] it is well known that the normal congruence (beam of rays) preserves its normality in the course of reflection or refraction. This means that the scalar product of the directional cosine and the vector tangent to the imaging surface $\boldsymbol{R}_p$ or $\boldsymbol{R}_q$ multiplied by the refractive index n of the respective medium is an invariant of the congruence at refraction or reflection. Thus two invariants can be written

$$\begin{aligned} n\boldsymbol{R}_p\boldsymbol{l} &= n'\boldsymbol{R}_p\boldsymbol{l}', \\ n\boldsymbol{R}_q\boldsymbol{l} &= n'\boldsymbol{R}_q\boldsymbol{l}', \end{aligned} \tag{13}$$

where n and n' denote the respective refractive indices of the media in front of an behind the imaging surface $\boldsymbol{R}$, and $\boldsymbol{l}$, $\boldsymbol{l}'$ are the congruence directional vectors in front of and behind the refraction and reflection.

The correlation between the products $\boldsymbol{R}_i\boldsymbol{l}$ and $\boldsymbol{R}_i^*\boldsymbol{l}$ for the neighbouring imaging surfaces may be obtained by differentiating both sides of the formula (9j) with respect to each parameter, and a scalar multiplication of both sides by $\boldsymbol{l}$. Hence,

$$\boldsymbol{R}_i^*\boldsymbol{l} = (\boldsymbol{R}_i + \lambda_i \boldsymbol{l} + \lambda \boldsymbol{l}_i)\boldsymbol{l}$$

and after avoiding the brackets on the right--hand side

$$\boldsymbol{R}_i^*\boldsymbol{l} = \boldsymbol{R}_i\boldsymbol{l} + \lambda_i. \tag{14}$$

The relations (13) and (14) will be exploited in the course of this work. In order to simplify the notation the following abbreviation will be introduced

$$\begin{aligned} \boldsymbol{R}_p\boldsymbol{l} &= \xi, \ \boldsymbol{R}_p\boldsymbol{l}' = \xi', \ \boldsymbol{R}_p^*\boldsymbol{l} = \xi^*, \\ \boldsymbol{R}_q\boldsymbol{l} &= \eta, \ \boldsymbol{R}_q\boldsymbol{l}' = \eta', \ \boldsymbol{R}_q^*\boldsymbol{l} = \eta^*, \end{aligned} \tag{15}$$

then the relations (13) and (14) take the form

$$\gamma\xi = \xi', \tag{16}$$

$$\gamma\eta = \eta',$$

$$\xi^* - \xi = \lambda_p,$$

$$\eta^* - \eta = \lambda_q, \tag{17}$$

where

$$\gamma = \frac{n}{n'}.$$

5. Transformation of the coefficients of the quadratic forms of congruence

5.1. Transformation of the coefficients B_{ij} of the second quadratic form

The transforming formulae for the coefficients of the second quadratic form of congruence describing the refraction or reflection at a given imaging surface, and the passage from one surface to the next one can be derived from the definition (2) and equations (9j) and (91) after having differentiated their both sides consecutively with respect to the parameters p and q. By calculating the respective products $\boldsymbol{R}_i^* \boldsymbol{l}_j$ and taking account of (1) and (10) we get the transforming formulae for the coefficients of the second quadratic form of congruence at the passage from the surfaces $\boldsymbol{R}$ to $\boldsymbol{R}^*$:

$$B_{ij}^* = B_{ij} + \lambda G_{ij}. \tag{18}$$

On the other hand, the products of $\boldsymbol{R}_i \boldsymbol{l}_j'$ yield in the transforming formulae for the coefficients of the second quadratic form of congruence at is transformation by the surface $\boldsymbol{R}$

$$B_{ij}^* = \gamma B_{ij} - a g_{ij}. \tag{19}$$

The transforming formulae (18) and (19) are identical with the respective transforming formulae for the coefficients of the second quadratic form of wavefront, given in [7] and [11]. According to the theory of the surface and congruence normal to this surface, such a similarity had to be expected [1].

5.2. Transforming of the G_{ij} coefficients of the first quadratic form of the congruence

If a beam passes through a uniform medium from one surface to the next surface of the optical system, then the values of the coefficients G_{ij} remain unchanged. The transforming formulae of these coefficients for reflection or refraction of the beam can be obtained from (16). If the quantities γ and ξ and η are given in front of the surface, then from these relations the quantities ξ' and η' may be calculated after imaging. By taking account of (19) and (11) the squares of the quantities ξ' and η' as well as the product $\xi'\eta'$ may be expressed by the sought coefficients G_{ij} and by the coefficients B_{ij}' obtained from the transformation (19). We get the following three expressions

$$\begin{aligned}
\xi'^2 &= \frac{1}{G'}[\boldsymbol{R}_p(\boldsymbol{l}_p' \times \boldsymbol{l}_q')]^2 \\
&= \frac{1}{G'}\begin{vmatrix} g_{11} & B_{11}' & B_{12}' \\ B_{11}' & G_{11}' & G_{12}' \\ B_{12}' & G_{12}' & G_{22}' \end{vmatrix}, \\
\eta'^2 &= \frac{1}{G'}[\boldsymbol{R}_q(\boldsymbol{l}_p' \times \boldsymbol{l}_q')]^2 \\
&= \frac{1}{G'}\begin{vmatrix} g_{22} & B_{12}' & B_{22}' \\ B_{12}' & G_{11}' & G_{12}' \\ B_{22}' & G_{12}' & G_{22}' \end{vmatrix}, \\
\xi'\eta' &= \frac{1}{G'}[\boldsymbol{R}_p(\boldsymbol{l}_p' \times \boldsymbol{l}_q')][\boldsymbol{R}_q(\boldsymbol{l}_p' \times \boldsymbol{l}_q')] \\
&= \frac{1}{G'}\begin{vmatrix} g_{12} & B_{12}' & B_{22}' \\ B_{11}' & G_{11}' & G_{12}' \\ B_{12}' & G_{12}' & G_{22}' \end{vmatrix}.
\end{aligned} \tag{20}$$

The relations (20) present a system of three inhomogeneous linear equations with respect to coefficients G_{ij}. Finally, after some rearrangements the following solution of the system of linear equations (20) is obtained

$$\begin{aligned}
G_{11}' &= \frac{1}{g\cos^2 i'}[2B_{11}'B_{12}'(\xi'\eta' - g_{12}) - \\
&\quad - B_{12}'^2(\xi'^2 - g_{11}) - B_{11}'(\eta'^2 - g_{22})], \\
G_{12}' &= \frac{1}{g\cos^2 i'}[B_{12}'B_{22}'(\xi'^2 - g_{11}) + \\
&\quad + B_{11}'B_{12}'(\eta'^2 - g_{22}) - (B' + 2B_{12}')(\xi'\eta' - g_{12})], \\
G_{22}' &= \frac{1}{g\cos^2 i}[2B_{12}'B_{22}'(\xi'\eta' - g_{12}) - \\
&\quad - B_{22}'(\xi'^2 - g_{11}) - B_{12}'^2(\eta'^2 - g_{22})].
\end{aligned} \tag{21}$$

5.3. Transformation of the quantity T

If we know the way in which the coefficients G_{ij} and B_{ij} are transformed during the passage from one surface to the other in the optical system, as well as at refraction or reflection of the beam on certain surface, then by the same means we may also determine the quantity T.

However, it is also possible to derive the transforming formulae immediately for this quantity.

The transforming formula for the quantity T determining the position of the middle surface of the beam at its passage from the surface $\boldsymbol{R}$ to the next surface $\boldsymbol{R}^*$ has the following form

$$T^* = T+\lambda, \tag{22}$$

where:

T^* or T denotes the position of the beam middle surface with respect to the surface $\boldsymbol{R}$ (or $\boldsymbol{R}^*$),

λ — the distance between the surfaces under consideration measured along the straight line (ray), for which the values T and T^* are determined.

By virtue of the expressions (20) it may be seen that the squares of the quantities ξ, η, ξ', and η' or the respective products of these quantities can be expressed by the coefficients G_{ij}, B_{ij}, G'_{ij} and B'_{ij}, respectively, and by the coefficients g_{ij}. After having risen to the second power both sides of each invariant (16) and taking account of (6) we get the transforming formula for the quantity T at the reflection or refraction of the beam:

$$\gamma^2\left(g_{ij}-2TB_{ij}+\frac{B}{G}G_{ij}\right) = g_{ij}-2T'B'_{ij}+\frac{B'}{G'}G'_{ij}. \tag{23}$$

5.4. Calculation of the coefficients g_{ij} of the imaging surfaces

The coefficients g_{ij} for the consecutive imaging surfaces can be determined indirectly after the quantities ξ^* and η^* are determined. The latter are calculated from a system of equations to be determined below.

First equation has been obtained owing to a suitable rearrangement of the product $(\boldsymbol{l}\times\boldsymbol{m})(\boldsymbol{l}\times\boldsymbol{m}^*)$, where the quantities $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{m}^*$ are determined according to (9).

Finally, the obtained linear equation with respect to the quantities ξ^* and η^* has the form:

$$\sqrt{gg^*}(\boldsymbol{l}\times\boldsymbol{m})(\boldsymbol{l}\times\boldsymbol{m}^*) = E\xi^*+F\eta^*, \tag{24}$$

where

$$E = (g_{22}+\lambda B_{22})\,\xi-(g_{12}+\lambda B_{12})\,\eta,$$

$$F = (g_{11}+\lambda B_{11})\,\eta-(g_{12}+\lambda G_{12})\,\xi.$$

The second equation is obtained by a suitable rearrangement of the relation

$$\cos^2 i^* = (\boldsymbol{lm}^*)^2,$$

which finally results in a quadratic equation

$$g^*\sin^2 i^* = P\xi^{*2}+Q\eta^{*2}+S\xi^*\eta^*, \tag{25}$$

where

$$P = 2T^*B^*_{22}-\frac{B^*}{G}G_{22},$$

$$Q = 2T^*B^*_{11}-\frac{B^*}{G}G_{11},$$

$$S = 2T^*B^*_{12}-\frac{B^*}{G}G_{12}.$$

The squares of the quantities ξ^* and η^* obtained from the solutions of the equations (24) and (25) include the coefficients g^*_{11} and g^*_{22}. The coefficient g^*_{12} can be calculated from the definition of the discriminant g^*

$$g^* = |g^*_{ij}|. \tag{26}$$

Thus, with the parametrization chosen the calculation of the coefficients of the first quadratic form for the imaging surface is not a simple procedure. Many attempts have been made to find a simpler method, so far however, without success. Nevertheless it cannot be excluded that such a method does exist.

5.5. Calculation of the coefficients of a quadratic form of congruence and the reference surface in the object space

The first reference surface of the congruence is reduced to the object point (if the object is located at an arbitrary distance) or is an arbitrary surface perpendicular to the x-axis (if the beam comes from an infinitely distant point). In the case of a beam emerging from a point of finite distance the coefficients of the first quadratic form G_{ij} are calculated from the definition (1) after computing the partial derivatives of the directional vectors $\boldsymbol{l}\{r, p, q\}$ with respect to the parameters p and q.

The coefficients of the second quadratic form of the congruence can be calculated from the transforming formulae (18) bearing in mind that the coefficients B_{ij} for an object point are equal to zero.

The coefficients g_{ij} of the first imaging surface cannot be estimated from the system of equations (24) and (25), because both the sides of those equations are equal to zero.

Thus two cases should be considered:

a) An equation of the sphere in the form

$$(\boldsymbol{R}-\boldsymbol{\rho}_c)^2 = \varrho^2, \tag{27}$$

(where $\boldsymbol{\rho}_c$ is the vector of the centre of the sphere and ϱ is its radius) together with the straight line equation

$$\boldsymbol{A} = \text{const.} \tag{28}$$

b) An equation of the plane in the form

$$\boldsymbol{R} = \boldsymbol{R}_0 + \lambda \boldsymbol{l} \tag{29}$$

together with (28).

By substitution of (29) for (27) in case of a spherical surface, or considering the condition

$$\boldsymbol{R}\cdot\boldsymbol{A} = 0$$

in case of a plane surface the following equations are obtained

$$(\boldsymbol{R}_0+\lambda\boldsymbol{l}-\boldsymbol{\rho}_c)^2 = \varrho^2, \tag{30}$$

$$\boldsymbol{R}_0\boldsymbol{A}+\lambda\boldsymbol{l}\boldsymbol{A} = 0. \tag{31}$$

After differentiating both sides of the last equations and a suitable transformation, the quantities for the first imaging surface are given by the formulae

$$\lambda_i = \lambda \frac{\boldsymbol{m}\boldsymbol{l}_i}{\cos i} \tag{32}$$

for the spherical surface, and by

$$\lambda_i = \lambda \frac{\boldsymbol{l}_i\boldsymbol{A}}{\cos i} \tag{33}$$

for the plane surface, respectively.

The values λ_i for the first surface of the optical system are equal to ξ and η, respectively because of [14]. On the other hand, the squares ξ and η as well as their product $\xi\eta$ contain the coefficients g_{ij}.

In the case of parallel beam (singular congruence) parametrization must be changed. Namely, the coordinates y and z of the point of intersection of the light beam with a plane $\boldsymbol{R}_0$ distant by d_0 from the first surface of the system and perpendicular to the axis of reference x are introduced as the parameters. The components of the directional vector $\boldsymbol{l}$ do not depend on those parameters and therefore the coefficients G_{ij} and B_{ij} are equal to zero in the object space.

The coefficients of the first quadratic form of the first surface of the system may be calculated from the definition, after differentiating both the sides of formulae (9j) with respect to the parameters y and z. Calculation of those derivatives gives the following relations

$$\boldsymbol{R}_{i'} = \boldsymbol{R}_{0i'}+\lambda_i\, \boldsymbol{l},$$

where the sign "prime" above the index i indicates that the derivatives are calculated with respect to the parameters y and z and not p and q.

When calculating the products $\boldsymbol{R}_{i'j'}$, and taking account of the fact that $\boldsymbol{R}_{0y}$ has the components $\{0, 1, 0\}$ and $\boldsymbol{R}_{0z}-\{0, 0, 1\}$ we obtain

$$\begin{aligned} g'_{11} &= 1+2p\lambda y+\lambda_y^2, \\ g'_{12} &= q\lambda_y+p\lambda_z+\lambda_z\lambda_z, \\ g'_{22} &= 1+2q\lambda_z+\lambda_z^2. \end{aligned} \tag{34}$$

The values λ_y and λ_z are calculated by differentiating the relations (30) or (31). Thus, generally

$$\lambda_{i'} = \lambda \frac{\boldsymbol{m}\boldsymbol{R}_{0i'}}{\cos i},$$

$$\lambda_{i'} = \lambda \frac{\boldsymbol{A}\boldsymbol{R}_{0i'}}{\cos i}.$$

After imaging on the first surface of the system the congruence ceases to be singular, therefore the coefficients $G_{i'j'}$, and $B_{i'j'}$, are different from zero and can be calculated from the transforming formulae (21) and (19). After suitable transformations the following transforming formulae may be derived:

$$\begin{aligned} G'_{11} &= a^2C^2g'_{11}+D^2C^2(p+\lambda_y)^2, \\ G'_{12} &= a^2C^2g'_{12}+D^2C^2(p+\lambda_y)(q+\lambda_z), \\ G'_{22} &= a^2C^2g'_{22}+D^2C^2(q+\lambda_z)^2, \end{aligned} \tag{35}$$

where: a is calculated from (9g) and the quantity D is determined as follows:

$$D = \gamma\left(\gamma \frac{\cos i}{\cos i'} -1\right).$$

Here, the sign "prime" above the coefficients $G_{i'j'}$ and $g_{i'j'}$ denotes that they are expressed in the parametrization (y, z).

The transition to the parametrization p, q is made by applying the transforming equations in the following form:

$$l_i = l_{i'} \frac{\partial u^{i'}}{\partial u_j},$$

$$R_i = R_{i'} \frac{\partial u^{i'}}{\partial u_j}. \tag{36}$$

After having performed those transformations the transforming formulae (18), (19), (21), (24), (25) derived above may be applied to evaluation of the coefficients of the quadratic formes of congruence and of the reference surface, when the beam is passing the optical system.

6. An application of the theory of congruences to the examination of the imaging quality in the concentric optical systems

As a result of calculation of the congruences, represented by the properly chosen rays emerging from one object point, the quantities T, t_1 and t_2 are obtained. These quantities determine the points of the middle surface and the beam focus surfaces, respectively.

In order to examine the quality of imaging it is sufficient to consider the middle surface, because the more diffuse is such a surface the more diffuse are the focus surfaces. In the case of an ideal imaging all the three surfaces are reduced to one point, being the Gaussian image point. This point is a centre of curvature of the ideal spherical surface in the exit pupil of the system. If the radius R of the sphere is known, then the evaluation of two quantities ΔT_a and δT_a characterizing the degree of deviation of the aberrated imaging from the ideal one is possible, and

$$\Delta T_a = \overline{T}_a - R, \tag{37}$$

$$\delta T_a = \sqrt{\frac{\sum_{k=1}^{N} (\overline{T}_a - T_{ak})^2}{N(N-1)}},$$

where

$\overline{T}_a$ – is the average value of the calculated quantities,

T_{ak} – is the quantity referred to the sphere of an ideal wave surface,

N – is the number of rays being calculated.

The calculation of the proposed quality measures (37) have been programmed on a computer. The results of the measurement are presented in the Tables.

Table 1

Photographic objective
$f' = 340$ mm, $N = 5.6$, $2\omega = 40°$

	Field	ΔT_a	δT_a
Centered optical system	0° 20°	−1.34 5.18	0.27 0.32
Decentered optical system Lens 1 $\Delta c = 0.07$ $\theta = 0°$ Lens 3 $\Delta c = 0.07$ $\theta = 90°$ Lens 5 $\Delta c = 0.07$ $\theta = 225°$	0° 20°	−2.30 7.10	0.59 0.76

Table 2

Microscope objective 10×
$f' = 15$ mm, $A = 0.3$

	Field	ΔT_a	δT_a
Centered optical system	0 0.7 mm	−0.042 −0.32	0.17 0.24
Decentered optical system Lens 1 $\Delta c = 0.05$ $\theta = 45°$ Lens 3 and 4 $\Delta c = 0.05$ $\theta = 135°$ $\delta = 0.003$	0 0.7 mm	−25.9 −27.9	6.4 7.6

Table 3

Microscope ocular 8×
$N = 10$, $2\omega = 40°$

	Field	ΔT_a	δT_a
Centered optical system	0° 15°	0.26459 1.7938	0.03103 0.02378
Decentered optical system Lens I $\Delta c = 0.05$ $\theta = 90°$ Lens II $\Delta c = 0.05$ $\theta = 225°$	0° 15°	0.26531 1.6932	0.03426 0.02733
Decentered optical system Lens I $\Delta c = 0.1$ $\theta = 45°$ Lens II $\Delta c = 0.1$ $\theta = 165°$	0° 15°	0.26658 1.5830	0.03946 0.03455

Table 4

Telescopis objective

$f = 100$ mm, $N = 4$, $2\omega = 10°$

	Field	ΔT_a	δT_a
Centered optical system	0°	0.09626	0.03433
	5°	−0.21236	0.07625
Decentered optical system Lens I $\Delta c = 0.1$ surface 1 $\theta = 0°$ surface 2 $\theta = 90°$ Lens II $\Delta c = 0.1$ $\theta = 225°$	0°	0.09700	0.04215
	5°	−0.23129	0.08135
Decentered optical system Lens I $\Delta c = 0.05$ surface 1 $\theta = 45°$ surface 2 $\theta = 105°$ Lens II surface 3 $\theta = 165°$ surface 4 $\theta = 225°$ surface 5 $\theta = 330°$	0°	0.09644	0.03627
	5°	−0.21240	0.08172

7. Concluding remarks

In this paper the transforming formulae for the coefficients of the quadratic form of the beam — congruence describing the beam passing through an arbitrary noncentric optical system have been derived.

The proposed measures of the decentration aberration after the advantage of being independent of the image surface position in the first order region. They do not contain, moreover, any approximations while including all the geometrical aberrations of a given beam, this being often more convenient than the discussion of single aberrations.

In view of the computation data presented in this paper it is clear that the decentration exerts an essential influence on the actual values of the measures, though the results obtained being too fragmentary, cannot be used to formulate a general criterion of imaging quality.

Применение теории нормальных конгруенций для оценки трансформационных свойств светового пучка и качества отображения в нецентрированных оптических системах

Трансформационные формулы для коэффициентов квадратной формы выведены для пучков лучей с произвольной расходимостью, рассматриваемых в качестве нормальных конгруэнций, в случае нецентрированных оптических систем.

Предложенные меры качества отображения на основе изучения степени распространения серединных поверхностей для пучка-конгруэнции. Приведено несколько примеров численного расчета для предложенных мер.

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