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Image of a Line in the Case of Defocusing and Asymmetrical Apodization in Incoherent Light

The image of a line for defocusing and asymmetrical apodization in incoherent light, and for systems with rectangular pupil is determined. Numerical calculations for the exponential function of apodization are given. It is shown that the displacement of the intensity in the image plane is consistent with displacement determined on the ground of geometrical optics.

1. Introduction

In the previous paper [1] we have reported on the phase of optical transfer function for systems free of aberrations, but with defocusing and asymmetrical apodization. The starting point of that report was the analysis of a parallactic error for visual instruments. The eye being an optical system with axial apodization (Stiles-Crawford's phenomenon [2]), the transversal displacement of the eye with respect to the exit pupil of the instrument produces a combined optical system (the observed and the instrument) with asymmetrical type of apodization. We have proved that the phase of optical transfer function, for the defocusing different from 0 and asymmetrical apodization, depends on the spatial frequency of intensity. Moreover, for sufficiently high values of defocusing the harmonics with different spatial frequencies may be shifted a l over the image plane in different directions. Given an object with the complex spatial spectrum the structure of the image cannot be easily forseen. For this purpose it is necessary to add up all the harmonics transferred to the image plane, and thus to estimate the whole image, which, in general, will not be similar to the object. In case of a simple object form with complex spatial spectrum the harmonic analysis is not a convenient procedure. The straightforward calculation of the image intensity may lead quicker to obtaining the final result.

An object in the form of a single line has both a very simple form and complex spectrum. It can be readily applied to the adjustment of optical instruments to eliminate the parallactic error. Therefore it is interesting to know its image when defocusing the optical system and introducing an asymmetrical apodization. For the sake of simplicity we have restricted our considerations (like in [1]) to the rectangular pupil with coefficient of apodization varying in one of the axial directions, and with object line perpendicular to the direction of variation of apodization.

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2. General consideration

Let u'_ξ, u'_η, be the angular coordinates of exit pupil in a rectangular form (Fig. 1). π' — Gaussian image plane; π'_1—defocused image plane; ξ', η' — plane of exit pupil; ξ' — direction of variable apodization.

The amplitude and phase distribution T in the image of a point ([3] Eqs. 3.81 and 3.83) is given by

$$T = A \iint\limits_{-\infty}^{\infty} V(u'_\xi, u'_\eta)\exp[-ik(u'_\xi x' + u'_\eta y')]du'_\xi du'_\eta, \qquad (1)$$

where: $V(u'_\xi, u'_\eta)$ — pupil function describing amplitude and phase distribution in the pupil plane; $k = 2\pi/\lambda$; λ — wavelength;

In our case

$$V(u'_\xi, u'_\eta) = V_0 f(u'_\xi)\exp(ik\Delta'), \qquad (2)$$

where

$f(u'_\xi)$ — function of apodization representing the changes of amplitude in pupil plane (real function) with regard to $u'_\xi = 0$ $(f(0) = 1)$; Δ' — wave aberration of the optical system; V_0 — the amplitude in the point $u'_\xi = 0$, $u'_\eta = 0$;

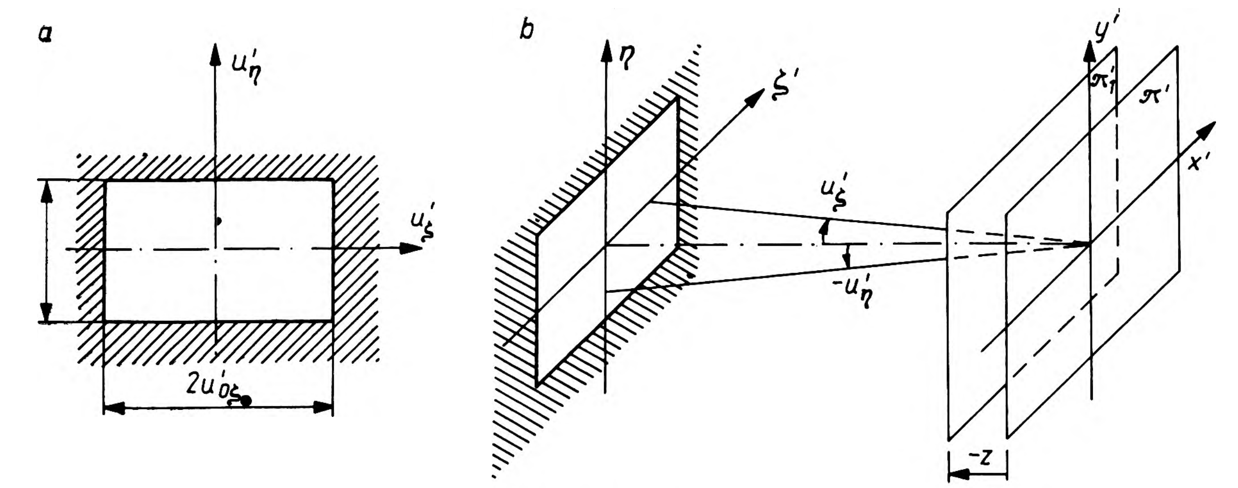

Fig. 1

For sufficiently small aperture angle and the system free of aberrations we can write

$$\Delta' = -\frac{(u_\xi'^2+u_\eta'^2)z}{2}; \tag{3}$$

z — linear value of defocusing;

After substituting to (1) we have

$$T = AV_0C_1C_2 \tag{4}$$

with

$$C_1(y', z) = \int_{-u'_{0\eta}}^{u'_{0\eta}} \exp\left(-i\frac{ku_\eta'^2 z}{2}\right)\exp(-iku'_\eta y')\,du'_\eta$$
$$= F_\eta\left[\exp\left(-i\frac{ku_\eta'^2 z}{2}\right)\right], \tag{5}$$

$$C_2(x', z) = \int_{-u'_{0\xi}}^{u'_{0\xi}} f(u'_\xi)\exp\left(-i\frac{ku_\xi'^2 z}{2}\right)\times$$
$$\times\exp(-iku'_\xi x')\,du'_\xi$$
$$= F_\xi\left[\exp\left(-i\frac{ku_\xi'^2 z}{2}\right)\right], \tag{6}$$

where F_i $(i = \xi, \eta)$ denotes Fourier transforms with regard to respective coordinates (the value of pupil function differs from 0 only within a limited area, and the integration can be taken in the range $-\infty, \infty$).

As the intensity distribution in the image of a point is determined by $D = TT^*$, the intensity distribution I_l in the image of a line parallel to y' is given by

$$I_l(x', z) = AA^*V_0V_0^*C_2C_2^*\int_{-\infty}^{\infty} C_1C_1^*\,dy'.$$

It is convenient to normalize this value by letting $I_{ln}(0, 0) = 1$ which occurs if

$$I_{ln}(x', z) = \frac{I_l(x', z)}{I_l(0, 0)}$$

$$= \frac{C_2C_2^*}{[C_2C_2^*]_{\substack{x'=0\\ z=0}}}\cdot\frac{\int_{-\infty}^{\infty} C_1C_1^*\,dy'}{\left[\int_{-\infty}^{\infty} C_1C_1^*\,dy'\right]_{z=0}}.$$

But from Parseval's theorem [4] and in the face of (5) the value

$$\int_{-\infty}^{\infty} C_1C_1^*\,dy' = \int_{-\infty}^{\infty} |C_1|^2\,dy' = a\int_{-\infty}^{\infty} |1|^2\,du_\eta'$$

is independent of z (a — a coefficient of proportionality introduced for our parameters). Hence, the normalized intensity distribution in the image of a line can be expressed by

$$I_{ln}(x', z) = \frac{C_2C_2^*}{[C_2C_2^*]_{\substack{x'=0\\ z=0}}}. \tag{7}$$

We introduce new variables by letting

$$\Phi = \frac{ku_{0\xi}'^2 z}{2}, \tag{8}$$

$$Z = ku'_{0\xi}x', \tag{9}$$

$$u' = su'_{0\xi}. \tag{10}$$

For a given width $2u'_{0\xi}$ of the pupil the quantity Φ characterizes the degree of defocusing, Z is a normalized coordinate of the image plane, and s — new variable of integration.

The substitution of (6), and (8)–(10) for (7), and a simple calculation, yield finally

$$I_{ln}(Z,\Phi) = \frac{\left[\int\limits_{-1}^{1} f(su'_{0\xi})\cos(\Phi s^2+Zs)\,ds\right]^2+}{\left[\int\limits_{-1}^{1} f(su'_{0\xi})\,ds\right]^2}$$

$$\frac{+\left[\int\limits_{-1}^{1} f(su'_{0\xi})\sin(\Phi s^2+Zs)\,ds\right]^2}{\left[\int\limits_{-1}^{1} f(su'_{0\xi})\,ds\right]^2}. \quad (11)$$

From this equation some general conclusions can be drawn:

1. If the function of apodization is symmetrical, as it occurs for $u'_\xi = 0$, i.e. if $f(-u'_\xi) = f(u'_\xi)$, then we have $I_{ln}(-Z,\Phi) = I_{ln}(Z,\Phi)$, independently of the quantity of defocusing (independently of Φ), this means that the indensity distribution is symmetrical with regard to point $Z = 0$. Such a conclusion is evident from physical point of view.

In order to prove it, it should be noted that

$$\cos(\Phi s^2+Zs) = \cos\Phi s^2\cos Zs - \sin\Phi s^2\sin Zs,$$

$$\sin(\Phi s^2+Zs) = \sin\Phi s^2\cos Zs + \cos\Phi s^2\sin Zs,$$

and as for symmetrical limits of integration the integrals of odd functions disappear the remaining terms in (11) will be expressed by functions cos Zs.

2. For $\Phi = 0$ we have also $I_{ln}(-Z, 0) = I_{ln}(Z, 0)$ independently of kind of function $f(u'_\xi)$. This results immediately from (11).

3. To calculate the intensity distribution I_{ln} in a range of defocusing $(-\Phi, \Phi)$ it suffices to estimate it within the interval $(0, \Phi)$, because $I_{ln}(-Z, -\Phi) = I_{ln}(Z, \Phi)$. This means that the changes of intensity distribution are the same on both sides of the Gaussian plane, but their directions in the image plane are opposite.

3. Exponential function as an example of asymmetrical apodization function

Similarly to [1] let the function of apodization be

$$f(u'_\xi) = \exp(bu'_\xi), \quad (12)$$

where b is the parameter depending on the degree of apodization.

According to (10)

$$f(u'_\xi) = \exp\left(\frac{Es}{2}\right) \quad (13)$$

with ([1] Eq. (27))

$$E = 2bu'_{0\xi}. \quad (14)$$

Now, from (11)

$$I_{ln}(Z,\Phi) = \frac{\left[\int\limits_{-1}^{1}\exp\left(\frac{Es}{2}\right)\cos(\Phi s^2+Zs)\,ds\right]^2+}{N(E)}$$

$$\frac{+\left[\int\limits_{-1}^{1}\exp\left(\frac{Es}{2}\right)\sin(\Phi s^2+Zs)\,ds\right]^2}{N(E)}, \quad (15)$$

where

$$N(E) = 4\left[\frac{sh(0.5E)}{0.5E}\right]^2.$$

For $E = 0$, 1, and 2 we have $N(E) = 4$, 4.345 and 5.524, respectively. If E is sufficiently small then we can put $N(E) = 4+E^2/3$.

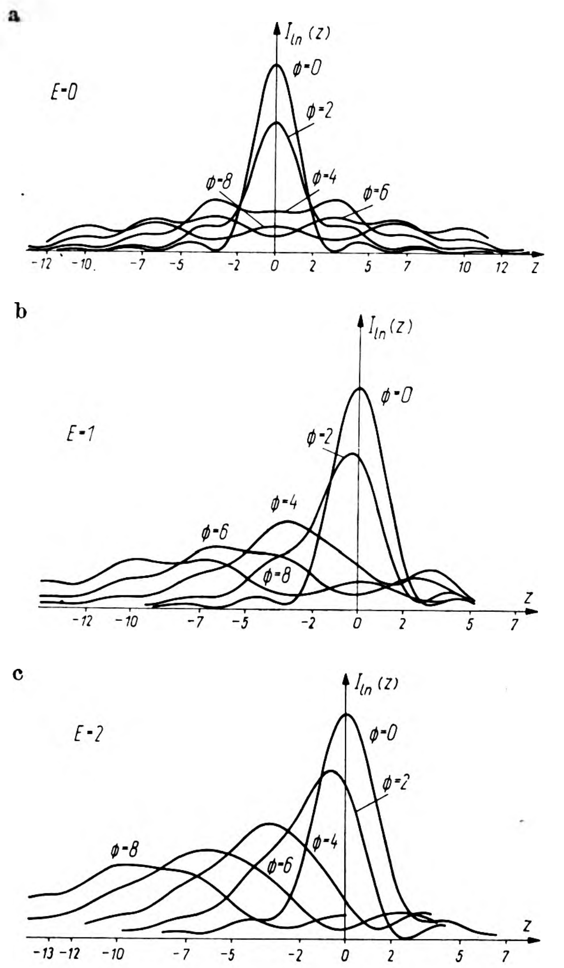

Fig. 2

By means of the computer we calculated the intensity distribution $I_{ln}(Z,\Phi)$ for $E = 0$, 1, and 2. The results of this calculation are presented in Figs 2a, b, c.

As it would be expected the intensity distributions are rather complex so that too precise conclusions are impossible to deduce. It is evident that for exponential function of apodization the increase of defocusing Φ causes the displacement of the image intensity with regard to point $Z = 0$.

To compare it with geometrical consideration we can repeat the argumentation of the paper [1] and rewrite here from the Eq. (24)

$$l'_g = -\frac{2}{\pi \tilde{x}_g} T_0 \Phi, \tag{16}$$

which describes the image displacement l'_g from geometrical point of view, where $\tilde{x}_g$ — the limiting frequency, and

$$T_0 = \frac{\int_{-1}^{1} f^2(u'_{0\xi} s) s ds}{\int_{-1}^{1} f^2(u'_{0\xi} s) ds}. \tag{17}$$

Using (9) of the present paper, because

$$l'_g \equiv x' \text{ and } \tilde{x}_g = \frac{2u'_{0\xi}}{\lambda}$$

we obtain

$$Z_g = -2T_0 \Phi, \tag{18}$$

where Z_g is the normalized displacement of the centre of the image intensity distribution determined on the ground of geometrical optics.

As for $E = 0$, 1, and 2 we have successively $T_0 = 0$, 0.3130 and 0.5373 [1], then the values of Z_g for different E can be calculated from the expressions given in the Table

E	0	1	2
Z_g	0	-0.626Φ	-1.075Φ

For sufficiently small value of E (in this case $T_0 = E/3$) we can put simply

$$Z_g = -\frac{2}{3}\Phi E. \tag{19}$$

The results obtained from the Table and Fig. 2 are consistent but cannot be the same. Geometrical consideration supplies correct conclusions for the structures of sufficiently low frequencies [1]. The difference between the results obtained on the ground of geometrical and wave optics is due to the harmonics with higher frequencies being transfered to the image plane. The coincidence of the results is not suprising, because for $\Phi > 3$ the values of contrast transfer function are significant for the frequencies much lower than the limiting frequency [1].

We have not calculated the centre area of $I_{ln}(Z, \Phi)$, because for such complex and wide intensity distribution this notion is physically useless. It would be more advisable to consider the centre of area for the significant values of the intensity, i.e. to use the approximative method depending on a detector.

Moreover, from Fig. 2 it results that for high value of Φ the image of line is more distinct for $E = 2$ than for $E = 0$, for example. It is clear because the increase of the value of E reduces the influence of the pupil area with low coefficient of transmission and, in a sense, is equivalent to the decrease of the pupil width.

Изображение линии в случае разрегулированной и асимметрической расфокусировки

Определено изображение линии для разрегулированной и асимметрической расфокусировки в некогерентном свете и для систем с прямоугольным зрачком линзы. Приводятся численные расчеты для экспоненциальной функции расфокусировки. Показано, что смещение интенсивности соответствует смещению, определенному на основе геометрической оптики.

References

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