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Determination of wave changes along propagation path

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Determination of complex amplitude changes of a wave propagating between two planes using two and three terms in the series expansion is presented. The third order spherical aberration is taken into consideration. Proposition of the estimation of correctness of the two terms method is given. The investigation is illustrated with numerical examples.

1. Introduction

Propagating waves undergo changes along their propagation path. In the simplest case, a plane wave with uniform amplitude distribution maintains its planar wavefront shape, but its phase changes linearly. A similar situation is encountered in the case of a spherical wave with uniform amplitude distribution. However, its wavefront curvature and amplitude values change. If a complex field distribution is known at a plane, then the complex field distribution at another plane depends on a distance between both planes and on the character of complex amplitude distribution of the propagating wave.

In the previous paper [1], the analytical solution for determination of the complex amplitude changes along the propagation path of a wave was found. Such a solution was possible due to the relation between the complex amplitude distribution of the wave under study and its Fourier transform [2]. The concentric displacement of the reference sphere along the wave propagation introduces relevant phase factor into the Fourier transform distribution. Expanding this factor into the power series and taking the inverse Fourier transform of the Fourier distribution the propagating wave change between both reference spheres can be found. So the relation describing the field change takes the series form. The individual terms are proportional to the partial derivatives of the respective degrees of the initial field distribution and the respective power index of the propagation distance normalised by the wavelength and the diameter of the beam. The derivative and power degrees increase with the series index.

When analysing of small wave aberrations and sufficiently small propagation distances it is enough to consider the first two series terms only due to negligibly small values of the remaining terms. Such an approach has been used to determine the influence of aberrations of interferometer elements on measurement errors [3], [4].

The aim of the paper is to determine the validity range of the relations with the first two and three terms of the series. For simplicity the study is applied to the wave propagation between two reference planes, which is a typical configuration in the case of the Fourier spectrometer. Conclusions will be formulated considering a wave with a third order spherical aberration without any apodisation.

2. General relations

Let $V(\varrho)$ be a field distribution on the reference plane Σ, where ϱ is the radial position vector (Fig. 1). The field distribution at the plane Σ', as a result of the propagation of the field from Σ, can be described by the following equation [1]:

$$V'(\varrho_n) = V(\varrho_n) + \sum_{s=1}^{\infty} \frac{(iZ)^s}{s!} \left[\sum_{j=0}^{s} \binom{s}{j} \frac{\partial^{(2s)} V(\varrho_n)}{\partial \rho_{xn}^{(2s-2j)} \partial \rho_{yn}^{(2j)}} \right] \tag{1}$$

where the parameterised distance Z between the planes Σ and Σ' is given by

$$Z = \frac{\lambda z}{4\pi \rho_m^2}, \tag{2a}$$

λ is the wavelength, ρ_m – maximum linear dimension at the plane Σ, ρ_{xn} and ρ_{yn} – Cartesian components of the parameterised vector ϱ, where

$$\varrho_n = \frac{\varrho}{\rho_m}. \tag{2b}$$

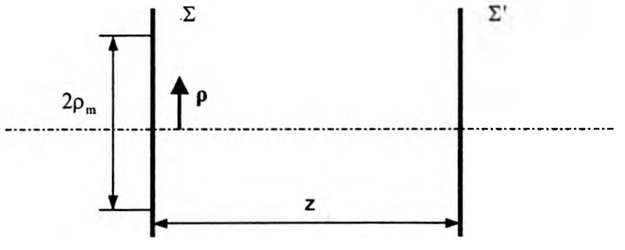

Fig. 1. Wave propagation between two planes.

It is worth noting that due to the assumed parameterisation ((Eq. (2)) the quantities and Z and $|\rho_n| = \rho_n$ are dimensionless ($0 \leqslant \rho_n \leqslant 1$). For simplicity, in Eq. (1) the linear phase factor $\exp(ikz)$ has been neglected. It has to be taken into consideration in interferometric analyses only.

For the first three terms of the sum in Eq. (1) and the case with the rotational symmetry we can write

$$V'(\rho_n) \approx V(\rho_n) + iZ\left(\frac{\partial^2 V}{\partial \rho_n^2} + \frac{2}{\rho_n}\frac{\partial V}{\partial \rho_n}\right) - \frac{Z^2}{2}\left(\frac{\partial^4 V}{\partial \rho_n^4} + \frac{2}{\rho_n}\frac{\partial^3 V}{\partial \rho_n^3} - \frac{1}{\rho_n^2}\frac{\partial^2 V}{\partial \rho_n^2} + \frac{1}{\rho_n^3}\frac{\partial V}{\partial \rho_n}\right)$$

$$-i\frac{Z^3}{6}\left(\frac{\partial^6 V}{\partial \rho_n^6} + \frac{3}{\rho_n}\frac{\partial^5 V}{\partial \rho_n^5} - \frac{3}{\rho_n^2}\frac{\partial^4 V}{\partial \rho_n^4} + \frac{6}{\rho_n^3}\frac{\partial^3 V}{\partial \rho_n^3} - \frac{9}{\rho_n^4}\frac{\partial^2 V}{\partial \rho_n^2} + \frac{9}{\rho_n^5}\frac{\partial V}{\partial \rho_n}\right). \tag{3}$$

Let us assume that the initial field distribution has a uniform amplitude distribution (the wave free of apodisation). This means that

$$V(\rho_n) = V_0 \exp[i2\pi W(\rho_n)] \tag{4}$$

where the wave amplitude $V_0 = \text{const}$ and the aberration $W(\rho_n)$ is defined as a fraction of λ.

Substituting (4) into (3) we can write

$$V'(\rho_n) \approx V(\rho_n)[1 + \Delta v_r(\rho_n) + i\Delta v_i(\rho_n)] \tag{5}$$

where the real quantities Δv_r and Δv_i have the following forms

$$\Delta v_r(\rho_n) = -2\pi Z\left(\frac{\partial^2 W}{\partial \rho_n^2} + \frac{1}{\rho_n}\frac{\partial W}{\partial \rho_n}\right)$$

$$-(2\pi Z)^2\left[2\left(\frac{\partial W}{\partial \rho_n}\right)^n - 1.5\left(\frac{\partial^2 W}{\partial \rho_n^2}\right)^2 - 2\frac{\partial W}{\partial \rho_n}\frac{\partial^3 W}{\partial \rho_n^3}\right.$$

$$\left.-3\frac{1}{\rho_n}\frac{\partial W}{\partial \rho_n}\frac{\partial^2 W}{\partial \rho_n^2} + 0.5\frac{1}{\rho_n^2}\left(\frac{\partial W}{\partial \rho_n}\right)^2\right]$$

$$+\frac{Z^3}{6}\pi\left\{2\frac{\partial^6 W}{\partial \rho_n^6} + 480\pi^4\left(\frac{\partial W}{\partial \rho_n}\right)^4\frac{\partial^2 W}{\partial \rho_n^2} - 120\pi^2\frac{\partial^4 W}{\partial \rho_n^4}\left(\frac{\partial W}{\partial \rho_n}\right)^2\right.$$

$$-480\pi^2\frac{\partial^3 W}{\partial \rho_n^3}\frac{\partial^2 W}{\partial \rho_n^2}\frac{\partial W}{\partial \rho_n} - 120\pi^2\left(\frac{\partial^2 W}{\partial \rho_n^2}\right)^3$$

$$+\frac{3}{\rho_n}\left[2\frac{\partial^5 W}{\partial \rho_n^5} + 32\pi^4\left(\frac{\partial W}{\partial \rho_n}\right)^5 - 120\pi^2\left(\frac{\partial^2 W}{\partial \rho_n^2}\right)^2\frac{\partial W}{\partial \rho_n} - 80\pi^2\frac{\partial^3 W}{\partial \rho_n^3}\left(\frac{\partial W}{\partial \rho_n}\right)^2\right]$$

$$-\frac{3}{\rho_n^2}\left[2\frac{\partial^4 W}{\partial \rho_n^4} - 48\pi^2\left(\frac{\partial W}{\partial \rho_n}\right)^2\frac{\partial^2 W}{\partial \rho_n^2}\right] + \frac{6}{\rho_n^3}\left[2\frac{\partial^3 W}{\partial \rho_n^3} - 8\pi^2\left(\frac{\partial W}{\partial \rho_n}\right)^3\right]$$

$$\left.-\frac{18}{\rho_n^4}\frac{\partial^2 W}{\partial \rho_n^2} + \frac{18}{\rho_n^5}\frac{\partial W}{\partial \rho_n}\right\}, \tag{6a}$$

$$\Delta v_i(\rho_n) = -(2\pi)^2 Z\left(\frac{\partial W}{\partial \rho_n}\right)^2$$

$$-\pi Z^2\left\{\frac{\partial^4 W}{\partial \rho_n^4} - 24\pi^2\left(\frac{\partial W}{\partial \rho_n}\right)^2\frac{\partial^2 W}{\partial \rho_n^2} + \frac{2}{\rho_n}\left[\frac{\partial^3 W}{\partial \rho_n^3} - 4\pi^2\left(\frac{\partial W}{\partial \rho_n}\right)^3\right] - \frac{1}{\rho_n^2}\frac{\partial^2 W}{\partial \rho_n^2} + \right.$$

$$+\frac{1}{\rho_n^3}\frac{\partial W}{\partial \rho_n}\Big\}+\frac{Z^3}{6}\pi^2\Big\{64\pi^4\left(\frac{\partial W}{\partial \rho_n}\right)^6+24\frac{\partial^5 W}{\partial \rho_n^5}\frac{\partial W}{\partial \rho_n}+60\frac{\partial^4 W}{\partial \rho_n^4}\frac{\partial^2 W}{\partial \rho_n^2}$$

$$+40\left(\frac{\partial^3 W}{\partial \rho_n^3}\right)^2-720\pi^2\left(\frac{\partial^2 W}{\partial \rho_n^2}\right)^2\left(\frac{\partial W}{\partial \rho_n}\right)^2-320\pi^2\frac{\partial^3 W}{\partial \rho_n^3}\left(\frac{\partial W}{\partial \rho_n}\right)^3$$

$$+\frac{3}{\rho_n}\left[20\frac{\partial^4 W}{\partial \rho_n^4}\frac{\partial W}{\partial \rho_n}+40\frac{\partial^3 W}{\partial \rho_n^3}\frac{\partial^2 W}{\partial \rho_n^2}-160\pi^2\left(\frac{\partial W}{\partial \rho_n}\right)^3\frac{\partial^2 W}{\partial \rho_n^2}\right]$$

$$-\frac{3}{\rho_n^2}\left[16\frac{\partial^3 W}{\partial \rho_n^3}\frac{\partial W}{\partial \rho_n}+12\left(\frac{\partial^2 W}{\partial \rho_n^2}\right)^2-16\pi^2\left(\frac{\partial W}{\partial \rho_n}\right)^4\right]$$

$$+\frac{72}{\rho_n^3}\frac{\partial^2 W}{\partial \rho_n^2}\frac{\partial W}{\partial \rho_n}-\frac{36}{\rho_n^4}\left(\frac{\partial W}{\partial \rho_n}\right)^4\Big\}. \tag{6b}$$

According to (5) the propagation of the aberrated wave generates some changes of its phase and amplitude distributions simultaneously, even for the uniform initial state of the wave amplitude distribution.

Putting for the propagating wave

$$V'(\rho_n) = |V'(\rho_n)|\exp[i2\pi W'(\rho_n)], \tag{7}$$

the phase and intensity distributions of the propagating wave can be described by the following relations:

$$W'(\rho_n) = W(\rho_n)+\Delta W(\rho_n), \tag{8a}$$

$$I'(\rho_n) = V'(\rho_n)V'^{*}(\rho_n) = I_0\{[1+\Delta v_r(\rho_n)]^2+\Delta v_i^2(\rho_n)\} \tag{8b}$$

where $I_0 = V_0^2 = \text{const}$. According to (8a) and (5) the change of the wave aberration induced by the wave propagation can be determined by the formula

$$\Delta W(\rho_n) = \frac{1}{2\pi}\arctan\frac{\Delta v_i(\rho_n)}{1+\Delta v_r(\rho_n)}. \tag{9}$$

3. Third order spherical aberration

The aberration analyses still require considering the defocusing as a compensation term. According to Hopkin's notation [5] we can write

$$W(\rho_n) = W_{20}\rho_n^2+W_{40}\rho_n^4. \tag{10}$$

In this case the relations (6a) and (6b) reduce to the following forms:

$$\begin{aligned}\Delta v_r(\rho_n) = {}& 8\pi Z(4W_{40}\rho_n^2+W_{20})-64(\pi Z)^2[2\pi^2\rho_n^4(2W_{40}\rho_n^2+W_{20})^4\\ &-(8W_{40}\rho_n^2+W_{20})^2+30W_{40}^2\rho_n^4]\\ &+512(\pi Z)^3[2\pi^2\rho_n^4(2W_{40}\rho_n^2+W_{20})^4(16W_{40}\rho_n^2+3W_{20})\\ &-352W_{40}^3\rho_n^6-246W_{40}^2W_{20}\rho_n^4-42W_{40}W_{20}^2\rho_n^2-W_{20}^3],\end{aligned} \tag{11a}$$

$$\begin{aligned}\Delta v_i(\rho_n) = & -16\pi^2 Z \rho_n^2 (2W_{40}\rho_n^2 + W_{20})^2 - 64\pi Z^2 [W_{40} \\ & - 4\pi^2 \rho_n^2 (20 W_{40}^3 \rho_n^6 + 24 W_{40}^2 W_{20} \rho_n^4 + 9 W_{40} W_{20}^2 \rho_n^2 + W_{20}^3)] \\ & + 2048\pi^2 Z^3 [\pi^4 \rho_n^6 (2W_{40}\rho_n^2 + W_{20})^6 / 3 \\ & - 1.5\pi^2 \rho_n^2 (2W_{40}\rho_n^2 + W_{20})^2 \ (34 W_{40}^2 \rho_n^4 + 14 W_{40} W_{20} \rho_n^2 + W_{20}^2) \\ & + 0.75 (8 W_{40}^2 \rho_n^2 + W_{40} W_{20})]. \end{aligned} \tag{11b}$$

It is worth repeating that according to (1) the components of Eqs. (11a) and (11b) with the factor Z^S ($S = 2, 3$) belong to the second and third term of the series (1), respectively.

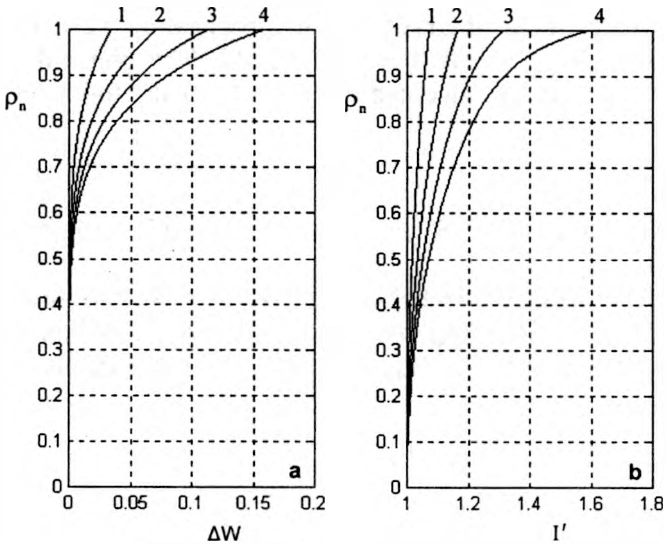

Fig. 2. Phase distributions (**a**) and intensity distributions (**b**) for $W_{40} = 1$ and various distances $Z/Z_{max} = 0.25$, 0.5, 0.75 and 1. The curves are denoted by 1, 2, 3, and 4, respectively.

The first exemplification of the wave changes along the propagation path is presented in Fig. 2 for $W_{40} = 1.0$, $W_{20} = 0$ and 4 parametrised propagation distances ($Z/Z_{max} = 0.25$, 0.5, 0.75 and 1). The parameters $\lambda = 4$ µm, $\rho_m = 5$ mm and $z_{max} = 100$ mm have been assumed, for which the maximum value of Z_{max} is 0.00127. According to Eqs. (8a) and (9) the changes of the wave aberration $\Delta W(\rho_n)$ and the intensity distribution $I(\rho_n)$ of the propagating wave are presented in Fig. 2**a** and Fig. 2**b**, respectively. The individual curves are denoted by the values Z/Z_{max}.

An increase of intensity of the propagating wave (see Fig. 2**b**) is induced by the focusing influence of the aberrated wave. The axial intensity changes in the case under consideration are too small to be noticed in the diagram. Their maximum value for $Z/Z_{max} = 1$ is 10^{-7}.

It is worth noting that the axial phase change

$$\Delta W(0) = \frac{1}{2\pi} \arctan \frac{\Delta v_i(0)}{1+\Delta v_r(0)} \tag{12}$$

where:

$$\Delta v_i(0) = -64\pi Z^2 W_{40} + 1536\pi^2 Z^3 W_{40} W_{20}, \tag{13a}$$

$$\Delta v_r(0) = -8\pi Z W_{20} + 64(\pi Z)^2 W_{20}^2 - 512(\pi Z)^3 W_{20}^3 \tag{13b}$$

is related to the phase change of the plane wave. The whole axial phase change $\Delta W_f(0)$ of the propagating wave is given by

$$\Delta W_f(0) = kz + 2\pi \Delta W(0). \tag{14}$$

Changes of the axial phase induced by the aberration of the propagating wave are small. The maximum value for Z/Z_{max} is $\Delta W(0) \approx -5.2 \cdot 10^{-5} \lambda$. The phase distribution counted with respect to its axial value can be found from the following relation:

$$W'(\rho_n) = \Delta W(\rho_n) - \Delta W(0) \approx \Delta W(\rho_n). \tag{15}$$

The change $\Delta W(\rho_n)$ of the wave aberration $W_{40} = 1$ for $Z/Z_{max} = 0.25$, 0.5, 0.75 and 1 is presented in Fig. 2**a**.

Taking into account the defocusing (coefficient W_{20} in Eq. (10)), we can choose its value to minimize the maximum value of $|\Delta W(\rho_n)|$ over the whole area. According to Eqs. (12) and (11) it can be shown that for $W_{40} = 1$ the optimum defocusing

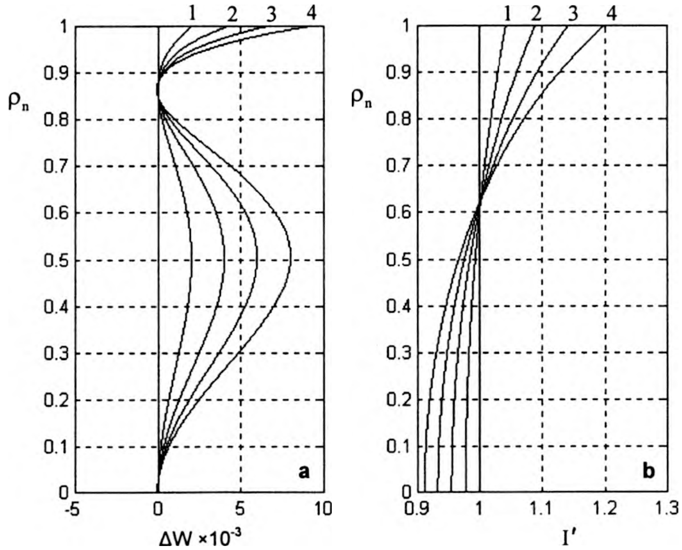

Fig. 3. Phase distributions (**a**) and intensity distributions (**b**) for $W_{40} = 1$, $W_{20} = -1.5$ and various distances $Z/Z_{max} = 0.25$, 0.5, 0.75 and 1. The curves are denoted by 1, 2, 3 and 4, respectively.

is $W_{20} = -1.5$. For this case and $Z/Z_{max} = 1$ the maximum value of $|\Delta W|_{max}$ decreases from $|\Delta W|_{max} = 0.16$ (for $W_{40} = 1$) to 0.008. Analogously to Fig. 2, the aberration and intensity changes (changes of $\Delta W(\rho_n)$ and $I'(\rho_n)$) are presented in Fig. 3.

As it is seen from Figures 2 and 3, the aberration and intensity changes of the propagating wave increase with the propagation distance and, in general, they cannot be neglected in the analysis. Moreover, the choice of proper defocusing can significantly decrease changes of both factors.

4. Correctness of the limited number of the series terms

As it was mentioned earlier (see Section 1) for small wave aberrations and sufficiently small propagation distances even the first two series terms can be sufficient due to negligibly small values of the remaining ones. The main problem for any case under consideration is to find how many terms are indispensable to determine the field changes correctly. The relations concerning the formulated problem are too complex to solve it in a general way

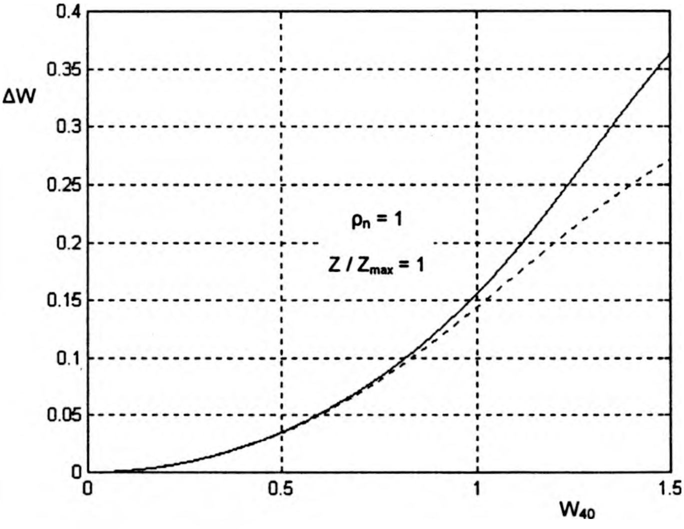

Fig. 4. Phase differences ΔW versus the aberration coefficient W_{40} for $\rho_n = 1$ using two (dashed line) and three (solid line) series terms.

We have exemplified the relations for some chosen cases. The propagating wave phase changes $\Delta W(\rho_n)$ versus the third order spherical aberration W_{40} for $\rho_n = 1$ and $Z/Z_{max} = 1$ are presented in Fig. 4 and the same phase changes versus the normalized propagation distance Z/Z_{max} for various aberration values W_{40} are shown in Fig. 5. The dashed and solid lines are related to the choice of two and three series terms, respectively.

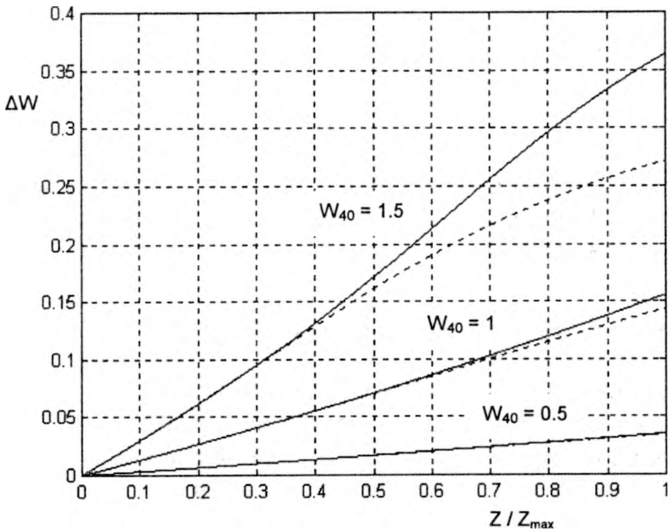

Fig. 5. Phase differences ΔW versus the propagation distance Z/Z_{max} for $\rho_n = 1$ and three values of aberration coefficient W_{40} using two (dashed line) and three (solid line) series terms.

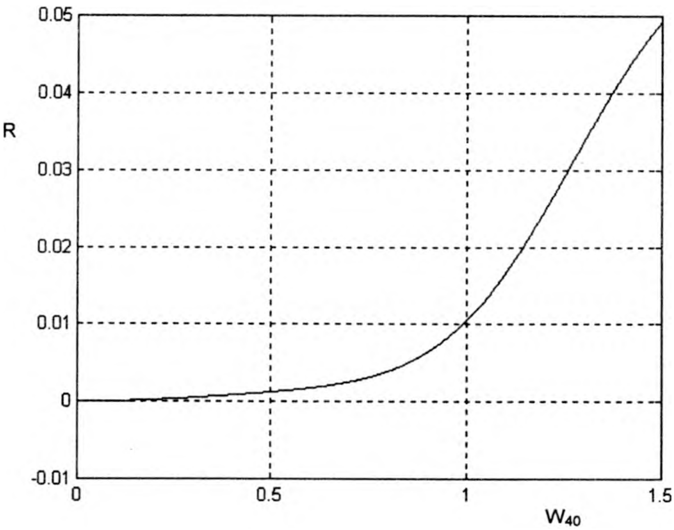

Fig. 6. Relative error R as a function of the aberration coefficient W_{40} for $\rho_n = 1$ and $Z/Z_{max} = 1$.

The admissible error introduced by the use of two series terms only depends on the final aberration value. If the initial aberration W_{40} is accepted, then the correctness criterion can be formulated as

$$R = \left| \frac{d(\Delta W)}{W_{40} + \Delta W} \right| < a \tag{16}$$

where $d(\Delta W)$ is the phase difference between two and three term cases, $W_{40}+\Delta W$ is the aberration value for the final plane, a is a permissible value of the relative error. The diagram of R versus W_{40} for $\rho_n = 1$ and $Z/Z_{max} = 1$ is given in Fig. 6. If, for example, $a = 2\%$ then, according to Fig. 6, the correct results based on two series terms are obtained for $W_{40} < 1.2$. The same consideration can be applied to determination of the permissible distance Z/Z_{max} for a given value of the aberration W_{40}. For example, for $W_{40} = 1.5$ and $a = 2\%$ the condition is $Z/Z_{max} < 0.68$.

5. Conclusions

The changes of the propagating wave can be determined analytically using some number of terms in the series (1). The required number depends on the aberration value, the propagation distance and the precision of considerations. It is not possible to a give general criterion. The choice depends on the individual problem. The paper presents the method of estimation and the application range for the third order spherical aberration using two or three series terms. For small aberrations and small distances of the propagating wave two series terms can be sufficient to solve some wave problems.

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